Function spaces of variable smoothness and integrability

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2-microlocal F and B spaces

Definition of the spaces

Resolution of Unity

$$\varphi_0(x) = \begin{cases} 1 \text{ , if } |x| \leq 1 \\ 0 \text{ , if } |x| \geq 2 \end{cases} .$$

We set $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and $\varphi_j(x) = \varphi(2^{-j}x)$ for all $j \in \mathbb{N}$. Then we obtain

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \qquad \text{for all } x \in \mathbb{R}^n.$$

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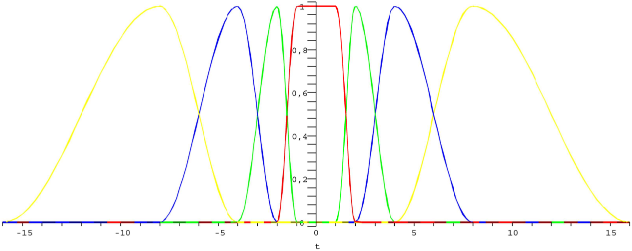

Admissible weight sequence

Definition 1

Let $\alpha \geq 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \leq \alpha_2$. We say that a sequence of non-negative measurable functions $\boldsymbol{w} = \{w_j\}_{j=0}^{\infty}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ if and only if

1. *There exists a constant $\mathrm{C} > 0$ such that*

$$0 < w_j(x) \leq \mathrm{C} w_j(y) \left(1 + 2^j|x-y|\right)^{\alpha} \quad j \in \mathbb{N}_0 \text{ and } x, y \in \mathbb{R}^n.$$

2. *For all $j \in \mathbb{N}_0$ we have*

$$2^{\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Such a system $\{w_j\}_{j=0}^{\infty} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ is called admissible weight sequence.

Definition 2

Let $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *and* $\{\varphi_j\}_{j\in\mathbb{N}_0}$ *be a Resolution of Unity. Furthermore let* $0 < q \le \infty$*. For* $0 < p \le \infty$ *we define*

$$B^{\boldsymbol{w}}_{pq}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^{\boldsymbol{w}}_{pq}(\mathbb{R}^n)\|_{\varphi} < \infty \right\} \quad \textit{with}$$

$$\|f|B^{\boldsymbol{w}}_{pq}(\mathbb{R}^n)\|_{\varphi} = \left(\sum_{j=0}^{\infty} \|(\varphi_j \hat{f})^{\vee}|L_p(w_j)\|^q \right)^{1/q} .$$

For $0 < p < \infty$ *we define*

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Properties

- Independent of the resolution of unity
- Lift operator
- Invariance under Diffeomorphisms
- Local means characterization
- Characterization by decomposition in atoms, molecules and wavelets
- Characterization by differences

Lebesgue spaces of variable integrability

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and

$$p_- = \operatorname*{ess-inf}_{x \in \Omega} p(x) \text{ and } p_+ = \operatorname*{ess-sup}_{x \in \Omega} p(x) \ .$$

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For $f : \Omega \to \mathbb{C}$ measurable, we define the **convex modular**

$$\varrho_p(f) = \int_{\Omega_0} |f(x)|^{p(x)} dx + \operatorname*{ess-sup}_{x \in \Omega_\infty} |f(x)| \,.$$

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$$\varrho_p(f) = \int_{\Omega_0} |f(x)|^{p(x)} dx + \operatorname*{ess-sup}_{x\in\Omega_\infty} |f(x)| \;.$$

Then $L_{p(\cdot)}(\Omega)$ is the collection of all f such that there exists an $\lambda > 0$ with $\varrho_p(f/\lambda) < \infty$.

Properties

- $L_{p(\cdot)}(\Omega)$ is a Banach space with the norm

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$$\int_\Omega |f(x)g(x)|dx \leq c_p \left\| f \right| L_{p(\cdot)}(\Omega) \right\| \left\| g \right| L_{p'(\cdot)}(\Omega) \right\|$$

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- Dual space $(L_{p(\cdot)}(\Omega))' = L_{p'(\cdot)}(\Omega) \Leftrightarrow p(\cdot) \in L_\infty(\Omega)$
- $\left\| f | L_{p(\cdot)}(\Omega) \right\| \sim \sup_{\varrho_{p'}(g) \leq 1} \int_\Omega f(x)g(x)dx$

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$$p(x) = \begin{cases} r & \text{, for } x \in [0,1) \\ s & \text{, for } x \in (-1,0) \end{cases} \quad \text{and}$$

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Then $f \in L_{p(\cdot)}(\Omega)$ but $f(x+h) \notin L_{p(\cdot)}(\Omega)$ for all $h \in (0, 1)$.

The maximal operator

The Hardy-Littlewood maximal operator $\mathcal{M}$ for $f \in L_1^{loc}(\mathbb{R}^n)$ is defined as

$$(\mathcal{M}f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy \ ,$$

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We have for $1 < p \leq \infty$

$$\| Mf | \, L_p(\mathbb{R}^n)\| \leq c \, \| f | \, L_p(\mathbb{R}^n)\|$$

Definition 1

Let $g \in C(\mathbb{R}^n)$. We say that g is *locally* log*-Hölder continuous*, abbreviated $g \in C^{\log}_{loc}(\mathbb{R}^n)$, if there exists $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\log(e + 1/|x - y|)}$$

holds for all $x, y \in \mathbb{R}^n$.

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We say, that g is *globally* $\log$-*Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if g is locally $\log$-Hölder continuous and there exists $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

holds for all $x \in \mathbb{R}^n$.

Boundedness of the maximal operator on $L_{p(\cdot)}(\mathbb{R}^n)$

Theorem 2 (Cruz-Uribe, Diening, Fiorenza, Harjulehto, Hästö, Mizuta, Nekvinda, Neugebauer, Shimomura)

If $p \in C^{\log}(\mathbb{R}^n)$ *and* $1 < p_- \leq p_+ \leq \infty$, *then* $\mathcal{M} : L_{p(\cdot)}(\Omega) \to L_{p(\cdot)}(\Omega)$ *is bounded; ie.*

$$\left\| Mf \,|\, L_{p(\cdot)}(\mathbb{R}^n) \right\| \leq c \left\| f \,|\, L_{p(\cdot)}(\mathbb{R}^n) \right\| \qquad (1)$$

We say that $p \in \mathcal{B}(\mathbb{R}^n)$ if (1) holds.

Vector-valued version

Theorem 3 (Cruz-Uribe et al.)

If $p \in \mathcal{B}(\mathbb{R}^n)$*, then for all* $1 < q \leq \infty$*,*

$$\left\| \mathcal{M} f_j \,|\, L_{p(\cdot)}(\ell_q) \right\| \leq c \left\| f_j \,|\, L_{p(\cdot)}(\ell_q) \right\| \; .$$

Local means characterization for B and F

Let $p : \mathbb{R}^n \to (0, \infty]$ with $0 < p_- \leq p_+ \leq \infty$, $0 < q \leq \infty$ and $\{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$.

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1. The space $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ is defined as

$$B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \big\| f \big| \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \big\|_{\varphi} < \infty \right\}$$

$$\big\| f \big| \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \big\|_{\varphi} = \left(\sum_{j=0}^{\infty} \left\| w_j (\varphi_j \hat{f})^{\vee} \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^{q} \right)^{1/q} .$$

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2. If $p^+ < \infty$ then $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ is defined by

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The Peetre maximal function

Given a system $\{\psi_k\}_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ we set $\Psi_k = \hat{\psi}_k \in \mathcal{S}(\mathbb{R}^n)$ and define the Peetre maximal operator for every $f \in \mathcal{S}'(\mathbb{R}^n)$ and $a > 0$ as

$$(\Psi_k^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|(\Psi_k * f)(y)|}{1 + |2^k(y-x)|^a} \ , \quad x \in \mathbb{R}^n \text{ and } k \in \mathbb{N}_0 \ .$$

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We start with two given functions $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\psi_j(x) = \psi_1(2^{-j+1}x)\ , \quad \text{for } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}.$$

Furthermore, for all $j \in \mathbb{N}_0$ we write $\Psi_j = \hat{\psi}_j$.

Conditions

Let $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < q \leq \infty$, $p \in \mathcal{P}(\mathbb{R}^n)$ and let $a \in \mathbb{R}$, $R \in \mathbb{N}_0$ with $R > \alpha_2$. Further, let ψ_0, ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$ with

$$D^{\beta}\psi_1(0) = 0\ , \quad \text{for } 0 \leq |\beta| < R\ ,$$

and

$$\begin{aligned} |\psi_0(x)| > 0 \quad &\text{on} \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \\ |\psi_1(x)| > 0 \quad &\text{on} \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \end{aligned}$$

for some $\varepsilon > 0$.

Theorem 4 (Local means characterization)

1. *If there exists* $0 < p_0 < p^-$ *with* $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$, *then for* $a > \frac{n}{p_0} + \alpha$

$$\big\| f \,|\, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \big\| \sim \big\| (\Psi_k * f) w_k \,|\, \ell_q(L_{p(\cdot)}) \big\| \sim \big\| (\Psi_k^* f)_a w_k \,|\, \ell_q(L_{p(\cdot)}) \big\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

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holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$*.*

2. *If* $p^+ < \infty$ *and if there exists* $p_0 < \min(p^-, q)$ *with* $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$*, then for* $a > \frac{n}{p_0} + \alpha$

$$\big\| f \big| F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \big\| \sim \big\| (\Psi_k * f) w_k \big| L_{p(\cdot)}(\ell_q) \big\| \sim \big\| (\Psi_k^* f)_a w_k \big| L_{p(\cdot)}(\ell_q) \big\|$$

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Lemma 5

If $G_\nu(x) = \sum_{k=0}^{\infty} 2^{-|\nu-k|\delta} g_k(x)$ *then there exist constants* $C_1, C_2 \geq 0$ *such that*

$$\| G_k \,|\, \ell_q(L_p) \| \leq C_1 \, \| g_k \,|\, \ell_q(L_p) \|$$

and

$$\| G_k \,|\, L_p(\ell_q) \| \leq C_2 \, \| g_k \,|\, L_p(\ell_q) \| \quad .$$

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and

$$\left\| G_k \,|\, L_{p(\cdot)}(\ell_q) \right\| \leq C_2 \left\| g_k \,|\, L_{p(\cdot)}(\ell_q) \right\| \; .$$

Definition of $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

Definition 6

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $\{\varphi_j\}_{j\in\mathbb{N}_0}$ a resolution of unity. Further, let $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \le p^+ < \infty$ and $0 < q^- \le q^+ \le \infty$. The space $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined by

$$F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \left\{ f \in S' : \left\| f \right| F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \big\|_{\varphi} < \infty \right\}$$

$$\left\| f \right| F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \big\|_{\varphi} = \left\| \left(\sum_{j=0}^{\infty} \left| (\varphi_j \hat{f})^{\vee}(x) w_j(x) \right|^{q(x)} \right)^{1/q(x)} \right| L_{p(\cdot)}(\mathbb{R}^n) \right\| .$$

There is no vector-valued MI if q is variable

- Suppose: $\left\| \mathcal{M} f_j | L_p(\ell_{q(\cdot)}) \right\| \leq c \left\| f_j | L_p(\ell_{q(\cdot)}) \right\|$

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- Suppose: $\left\| \mathcal{M} f_j | L_p(\ell_{q(\cdot)}) \right\| \leq c \left\| f_j | L_p(\ell_{q(\cdot)}) \right\|$
- $n = 1$ and $\Omega_1 = (-1, 0)$, $\Omega_2 = (0, 1)$, $\Omega = \Omega_1 \cup \Omega_2$ and $q|_{\Omega_j} = q_j$ for $j = 0, 1$

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- Set $f_k(x) = a_k \chi_{\Omega_0}(x)$

Replacement for the vector-valued MI

Define for $\nu, m \in \mathbb{N}_0$ the function $\eta_{\nu,m}(x) = 2^{\nu n}(1 + |2^\nu x|)^{-m}$.

Lemma 7 (DHR)

Let $p, q \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$. Then the inequality

$$\left\| \left\| \eta_{\nu,m} * f_\nu \right| \ell_{q(\cdot)} \right\| \left| L_{p(\cdot)}(\mathbb{R}^n) \right\| \leq c \left\| f_\nu \right| L_{p(\cdot)}(\ell_{q(\cdot)}) \right\|$$

holds for every sequence $\{f_\nu\}$ of $L_1^{loc}(\mathbb{R}^n)$ functions and constant $m > n$.

Lemma 8

Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $0 < q^- \le q^+ \le \infty$ and $0 < p^- \le p^+ \le \infty$. For any sequence $\{g_j\}_{j \in \mathbb{N}_0}$ of nonnegative measurable functions on $\mathbb{R}^n$ denote for some $\delta > 0$

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Then with constant $c = c(p, q, \delta)$ we have

$$\left\| \{G_j\}_{j \in \mathbb{N}_0} \middle| L_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \le c \left\| \{g_j\}_{j \in \mathbb{N}_0} \middle| L_{p(\cdot)}(\ell_{q(\cdot)}) \right\| .$$

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