2-microlocal Besov spaces

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2-microlocal Besov spaces

- Known spaces (Bony, Jaffard, Moritoh)
- Local Regularity of functions
- The 2-microlocal Besov spaces $B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n)$

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Properties

- Lifting Property
- Embeddings
- Local Means

Resolution of Unity

$$\varphi_0(x) = \begin{cases} 1 \text{ , if } |x| \leq 1 \\ 0 \text{ , if } |x| \geq 2 \end{cases} .$$

We put $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and $\varphi_j(x) = \varphi(2^{-j}x)$ for all $j \in \mathbb{N}$, then

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \qquad \text{for all } x \in \mathbb{R}^n.$$

Then $\operatorname{supp} \varphi_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and we have the Littlewood-Paley Analysis of $f \in \mathcal{S}'(\mathbb{R}^n)$

$$f = \sum_{j=0}^{\infty} (\varphi_j \hat{f})^{\vee} .$$

The spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$

Definition 1 (J. M. Bony 1984)

Let $x_0 \in \mathbb{R}^n$ and $s, s' \in \mathbb{R}$. A function $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to $C^{s,s'}_{x_0}$ if there exists a constant $C > 0$ s.t.

$$|(\varphi_j \hat{f})^\vee(x)| \leq C 2^{-js}(1 + 2^j|x - x_0|)^{-s'} \quad \text{for all } j \in \mathbb{N} \text{ and } x \in \mathbb{R}^n.$$

This can be rewritten in

$$\sup_{j \in \mathbb{N}_0} \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \left(2^{js}(1 + 2^j|x - x_0|)^{s'} |(\varphi_j \hat{f})^\vee(x)| \right) \leq C \, .$$

The spaces $H^{s,s'}_{x_0}(\mathbb{R}^n)$

Definition 2 (J. M. Bony 1984)

Let $x_0 \in \mathbb{R}^n$ and $s, s' \in \mathbb{R}$. A function $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to $H^{s,s'}_{x_0}$ if

$$\left\| 2^{js}(1+2^j|x-x_0|)^{s'}(\varphi_j \hat{f})^\vee(x) \,\Big|\, L_2 \right\| \leq c_j \quad \textit{and} \quad \sum_{j=0}^{\infty} |c_j|^2 < \infty \; .$$

This can be rewritten in

$$\sum_{j=0}^{\infty} 2^{2js} \left\| (1+2^j|x-x_0|)^{s'}(\varphi_j \hat{f})^\vee(x) \,\Big|\, L_2 \right\|^2 < \infty \; .$$

Interludium: Local Regularity of functions

Definition 3

Let $x_0, s \in \mathbb{R}$. *Then* $f : \mathbb{R} \to \mathbb{R}$ *belongs to the Hölder space* $C^s_{x_0}$ *if and only if there exist a constants* $C, r > 0$ *and a polynomial* P *with* $\deg P \leq \lfloor s \rfloor$ *such that*

$$|f(x) - P(x - x_0)| \leq |x - x_0|^s \quad \textit{for all } |x - x_0| \leq r.$$

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The pointwise Hölder exponent of f at x_0 is

$$\alpha_p(x_0) = \sup\{s : f \in C^s_{x_0}\} \ .$$

Definition 4

Let f be a function in $L_1^{loc}(\mathbb{R})$ and denote by $f^{(-l)}$ the primitive of order l. Then f is called a (h, β_c)-type chirp at $x_0 \in \mathbb{R}$ if for all $n \in \mathbb{N}$

$$f^{(-n)} \in C_{x_0}^{h+n(1+\beta_c)} .$$

β_c is called the chirp exponent.

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Definition 5

Let $C^\alpha(\Omega)$ be the usual Hölder spaces on Ω. Then $\alpha_l(f,x_0,\eta) = \sup\{\alpha : f \in C^\alpha(B_\eta(x_0))\}$; and the local Hölder exponent is defined by

$$\alpha_l(x_0) = \lim_{\eta \to 0} \alpha_l(f,x_0,\eta) \ .$$

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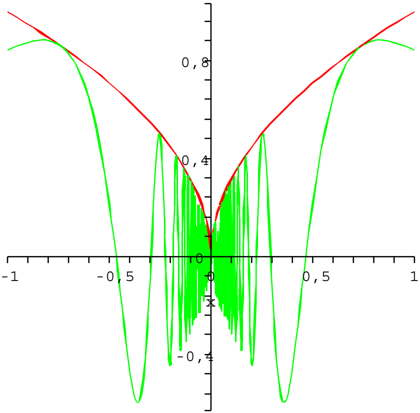
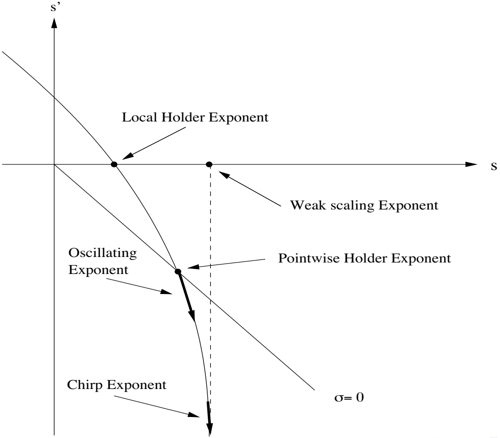
other exponents: weak scaling exponent, oscillating exponent

Cusp and Chirp



2-microlocal frontier and regularity exponents



The Idea

We define generalized Besov spaces with an "admissible" sequence of weight functions $\boldsymbol{w} = \{w_j\}_{j \in \mathbb{N}_0}$ with the norm:

$$\big\| f \big| \, B^{s,\boldsymbol{w}}_{pq} \big\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \| w_j (\varphi_j \hat{f})^{\vee} | L_p \|^q \right)^{1/q}$$

for $0 < p, q \leq \infty$.

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for $0 < p, q \leq \infty$.

- What are the conditions for our admissible weight sequence?
- Example weights: $w_j(x) = (1 + 2^j |x - x_0|)^{s'}$
- Is the definition independent of the Resolution of Unity?

Definition 6

Let $\alpha, \alpha_1, \alpha_2 \geq 0$. *We say that a sequence of non-negative measurable functions* $\boldsymbol{w} = \{w_j\}_{j=0}^{\infty}$ *belongs to the class* $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *if and only if*

1. *There exists a constant* $\mathrm{C} > 0$ *such that*

$$0 < w_j(x) \leq \mathrm{C} w_j(y) \left(1 + 2^j|x - y|\right)^{\alpha} \quad j \in \mathbb{N}_0 \text{ and } x, y \in \mathbb{R}^n.$$

2. *For all* $j \in \mathbb{N}_0$ *we have*

$$2^{-\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Such a system $\{w_j\}_{j=0}^{\infty} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *is called admissible weight sequence.*

Definition 7

Let $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *and* $\{\varphi_j\}_{j\in\mathbb{N}_0}$ *be a Resolution of Unity. Furthermore let* $0 < p \leq \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$, *then we define*

$$B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n)\|_{\varphi} < \infty \right\} \quad \textit{with}$$

$$\|f|B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n)\|_{\varphi} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^{\vee}|L_p(w_j)\|^q \right)^{1/q} .$$

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Remark 8

This definition is independent of the function φ, *so we can suppress it in the notation of the norm.*

Weight sequences

- $w_j(x) = (1 + 2^j \operatorname{dist}(x, U))^{s'}$, where $U \subset \mathbb{R}^n$
  $U \subset \mathbb{R}^n$ is an open subset $\rightarrow$ [Moritoh, Yamada 2004]

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- Let $w : \mathbb{R}^n \rightarrow [0, \infty)$ be a measurable function with the following properties:
  There are constants $\mathcal{C}_1$, $\mathcal{C}_2 \geq 1$ and $\beta \geq 1$, such that for all $x, y \in \mathbb{R}^n$

$$0 \leq w(x) \leq \mathcal{C}_1 w(y) + \mathcal{C}_2 |x - y|^\beta \ .$$

  We define now for $s' \in \mathbb{R}$ and $j \in \mathbb{N}_0$

$$w_j(x) = \left(1 + 2^j w(x)\right)^{s'/\beta} \quad \text{for all } x \in \mathbb{R}^n.$$

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<u>Special case:</u> $w : \mathbb{R}^n \rightarrow [0, \infty)$ subadditiv, that means

$$0 \leq w(x + y) \leq \tilde{c}_1 \left(w(x) + w(y)\right) \quad \text{and} \quad w(x) \leq \tilde{c}_2 |x|^\beta \ .$$

Lifting Property

Let $\sigma \in \mathbb{R}$. Then the Liftoperator I_σ is defined through

$$I_\sigma : f \to \left(\langle \xi \rangle^\sigma \hat{f} \right)^\vee \quad \text{where } \langle \xi \rangle^\sigma := (1 + \xi^2)^{\sigma/2}.$$

Then we have the following: I_σ is an isomorphic mapping from $B^{s,\boldsymbol{w}}_{pq}$ onto $B^{s-\sigma,\boldsymbol{w}}_{pq}$ and

$$\left\| I_\sigma f \middle| B^{s-\sigma,\boldsymbol{w}}_{pq} \right\| \sim \left\| f \middle| B^{s,\boldsymbol{w}}_{pq} \right\| \ .$$

Theorem 1

Let $s \in \mathbb{R}$, $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < p, q \leq \infty$ *and let* $m \in \mathbb{N}_0$. *Then*

$$\sum_{|\beta|\leq m} \left\| D^\beta f \right| B^{s-m,\boldsymbol{w}}_{pq}(\mathbb{R}^n) \Big\|$$

and

$$\left\| f \right| B^{s-m,\boldsymbol{w}}_{pq}(\mathbb{R}^n) \Big\| + \sum_{i=1}^{n} \left\| \frac{\partial^m f}{\partial x_i^m} \right| B^{s-m,\boldsymbol{w}}_{pq}(\mathbb{R}^n) \Bigg\|$$

are equivalent quasi-norms on $B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n)$.

Embeddings

Let $s \in \mathbb{R}$ and $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $\boldsymbol{\varrho} \in \mathcal{W}^{\beta}_{\beta_1,\beta_2}$.

1. If $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$ and $\frac{w_j(x)}{\varrho_j(x)} \leq c$, then

$$B^{s,\boldsymbol{\varrho}}_{pq_1}(\mathbb{R}^n) \hookrightarrow B^{s,\boldsymbol{w}}_{pq_2}(\mathbb{R}^n) \ .$$

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2. If $0 < p \leq \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $\frac{w_j(x)}{\varrho_j(x)} \leq c$, then for all $\varepsilon > 0$

$$B^{s,\boldsymbol{\varrho}}_{pq_1}(\mathbb{R}^n) \hookrightarrow B^{s-\varepsilon,\boldsymbol{w}}_{pq_2}(\mathbb{R}^n)\ .$$

3. If $0 < p_1 \leq p_2 \leq \infty$, $0 < q \leq \infty$ and $s_1, s_2 \in \mathbb{R}$ with

$$\frac{w_j(x)}{\varrho_j(x)} \leq c 2^{j\left(s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} \tag{1}$$

for all $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, then we have

$$B^{s_1,\boldsymbol{\varrho}}_{p_1 q}(\mathbb{R}^n) \hookrightarrow B^{s_2,\boldsymbol{w}}_{p_2 q}(\mathbb{R}^n) \ .$$

Local Means

Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball and $k \in \mathcal{S}(\mathbb{R}^n)$ a function with support in B. For a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ the corresponding local means are defined for $x \in \mathbb{R}^n$ and $t > 0$ by

$$k(t,f)(x) = \int_{\mathbb{R}^n} k(y) f(x+ty)dy = t^{-n} \int_{\mathbb{R}^n} k\left(\frac{y-x}{t}\right) f(y)dy \ .$$

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Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ be two functions satisfying

$$\operatorname{supp} k_0 \subseteq B \ , \qquad \operatorname{supp} k^0 \subseteq B \ ,$$

and

$$\hat{k_0}(0) \neq 0 \ , \qquad \hat{k^0}(0) \neq 0 \ .$$

For $N \in \mathbb{N}_0$ we define the iterated Laplacian

$$k(y) := \Delta^N k^0(y) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} \right)^N k^0(y) \ , \qquad y \in \mathbb{R}^n \ .$$

It follows easily that

$$\check{k}(x) = |x|^{2N} \check{k^0}(x) \qquad \text{and that implies}$$

$$D^\beta \check{k}(0) = 0 \qquad \text{for} \qquad 0 \leq |\beta| < 2N \ .$$

Theorem 2

Let $\boldsymbol{w} = \{w_j\}_{j \in \mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1, \alpha_2}$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$. *Furthermore, let* $N \in \mathbb{N}_0$ *with* $2N > s + \alpha_2$ *and let* $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ *and the function* k *be defined as above. Then*

$$\left\| k_0(1, f) w_0 \right| L_p(\mathbb{R}^n) \| + \left(\sum_{j=1}^{\infty} 2^{jsq} \left\| k(2^{-j}, f) w_j \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \sim \left\| f \right| B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n) \right\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Daubechies Wavelets

Daubechies Wavelets

Theorem 3

For any $k \in \mathbb{N}$ there is a real compactly supported scaling function $\psi_F \in C^k(\mathbb{R})$ and a real compactly supported associated wavelet $\psi_M \in C^k(\mathbb{R})$ such that $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$ and

$$\int_{\mathbb{R}} x^l \psi_M(x) dx = 0 \qquad \text{for all } l \in \{0, \dots, k-1\} .$$

We have, that $\{\psi_{jm} : j \in \mathbb{N}_0, m \in \mathbb{Z}\}$ is an orthonormal basis in $L_2(\mathbb{R})$, where

$$\psi_{jm}(t) := \begin{cases} \psi_F(t-m), & \text{if } j = 0, m \in \mathbb{Z} \\ 2^{\frac{j-1}{2}} \psi_M(2^{j-1}t - m), & \text{if } j \in \mathbb{N}, m \in \mathbb{Z} \end{cases} .$$

Daubechies Wavelets in $\mathbb{R}^n$

We define

$$G^0 = \{F, M\}^n \qquad \text{and} \qquad G^j = \{F, M\}^{n*} \quad \text{if } j \geq 1 \ ,$$

where the * indicates, that at least one G_i of $G = (G_1, \ldots, G_n) \in \{F, M\}^{n*}$ must be an M. We set for $j \in \mathbb{N}_0$, $G \in G^j$ and $m \in \mathbb{Z}^n$

$$\Psi^j_{Gm}(x) = 2^{j\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^j x_r - m_r) \ .$$

Then $\{\Psi^j_{Gm} : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$.

Definition 9

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ *and* $0 < p, q \leq \infty$. *Then*

$$\tilde{b}^{s,\boldsymbol{w}}_{pq} := \left\{ \lambda = \{\lambda^j_{Gm}\} \subset \mathbb{C} : \left\| \lambda | \, \tilde{b}^{s,\boldsymbol{w}}_{pq} \right\| < \infty \right\}$$

where

$$\left\| \lambda | \, \tilde{b}^{s,\boldsymbol{w}}_{pq} \right\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^j_{Gm}|^p w^p_j(2^{-j}m) \right)^{q/p} \right)^{1/q} .$$

Wavelet Decomposition Theorem I

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and $\{\Psi^j_{Gm}\}$ be the Daubechies Wavelets with

$$k > \max(\sigma_p - s + \alpha_1, s + \alpha_2) \ . \qquad (2)$$

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^j_{Gm} 2^{-j\frac{n}{2}} \Psi^j_{Gm} \qquad \text{with } \lambda \in \tilde{b}^{s,\boldsymbol{w}}_{pq} \ ,$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{\sigma,\boldsymbol{\varrho}}_{pq}(\mathbb{R}^n)$ with $\sigma < s$ and $\frac{\varrho_j(x)}{w_j(x)} \to 0$ for $|x| \to \infty$.

Wavelet Decomposition Theorem II

The representation

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^j_{Gm} 2^{-j\frac{n}{2}} \Psi^j_{Gm} \qquad \text{with } \lambda \in \tilde{b}^{s,\boldsymbol{w}}_{pq} \; ,$$

is unique,

$$\lambda^j_{Gm} = \lambda^j_{Gm}(f) = 2^{j\frac{n}{2}} \left\langle f, \Psi^j_{Gm} \right\rangle$$

and

$$I : f \mapsto \{2^{j\frac{n}{2}} \left\langle f, \Psi^j_{Gm} \right\rangle\}$$

is an isomorphic map from $B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n)$ onto $\tilde{b}^{s,\boldsymbol{w}}_{pq}$. Moreover, if in addition $\max(p,q) < \infty$ then $\{\Psi^j_{Gm}\}$ is in unconditional basis in $B^{s,\boldsymbol{w}}_{pq}(\mathbb{R}^n)$.

The 2-microlocal spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$

Let $s, s' \in \mathbb{R}$, $U \subseteq \mathbb{R}^n$ bounded and $0 < p, q \leq \infty$. Then

$$\left\| f \middle| B^{s,s'}_{pq}(\mathbb{R}^n, U) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j \hat{f})^{\vee} (1 + 2^j \operatorname{dist}(\cdot, U))^{s'} \middle| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q}$$

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and

$$\left\| \lambda \middle| \tilde{b}^{s,s'}_{pq}(U) \right\| = \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^j_{Gm}|^p (1 + 2^j \operatorname{dist}(2^{-j}m, U))^{ps'} \right)^{q/p} \right)^{1/q} .$$

Wavelet decomposition of $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ I

Let $U \subset \mathbb{R}^n$ bounded, $s' \in \mathbb{R}$ and $w_j(x) = (1 + 2^j \operatorname{dist}(x, U))^{s'}$. Further, let $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and $\{\Psi^j_{Gm}\}$ be the Daubechies Wavelets with

$$k > \max(\sigma_p - s - \min(0, s'), s + \max(0, s')) \ . \qquad (3)$$

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^j_{Gm} 2^{-j\frac{n}{2}} \Psi^j_{Gm} \qquad \text{with } \lambda \in \tilde{b}^{s,s'}_{pq}(U) \ ,$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{t,t'}_{pq}(\mathbb{R}^n, U)$ with $t < s$ and $t' < s'$.

Wavelet decomposition of $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ II

The representation

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^j_{Gm} 2^{-j\frac{n}{2}} \Psi^j_{Gm}$$

is unique,

$$\lambda^j_{Gm} = \lambda^j_{Gm}(f) = 2^{j\frac{n}{2}} \left\langle f, \Psi^j_{Gm} \right\rangle$$

and

$$I : f \mapsto \{2^{j\frac{n}{2}} \left\langle f, \Psi^j_{Gm} \right\rangle\}$$

is an isomorphic map from $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ onto $\tilde{b}^{s,s'}_{pq}(U)$. Moreover, if in addition $\max(p,q) < \infty$ then $\{\Psi^j_{Gm}\}$ is in unconditional basis in $B^{s,s'}_{pq}(\mathbb{R}^n, U)$.

Waveletdecomposition for $B^{s,s'}_{\infty,\infty}(\mathbb{R}^n, x_0) = C^{s,s'}_{x_0}(\mathbb{R}^n)$

Theorem 4 (Jaffard, 91)

Let $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $w_j(x) = (1 + 2^j|x - x_0|)^{s'}$. *Then* $f \in C^{s,s'}_{x_0}(\mathbb{R}^n)$ *if and only if the wavelet coefficients satisfy*

$$|\langle f, \Psi^j_{Gm} \rangle| \le c 2^{-(n/2+s)j} (1 + 2^j|2^{-j}m - x_0|)^{-s'} \ .$$