THE BINET-LEGENDRE ELLIPSOID IN FINSLER GEOMETRY

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Abstract. In this paper we introduce a new construction that associates a Riemannian metric $g_F$ (called the Binet-Legendre metric) to a given Finsler metric $F$ on a smooth manifold $M$. The transformation $F \mapsto g_F$ is $C^0$-stable and has good smoothness properties, in contrast to previous constructions. The Riemannian metric $g_F$ also behaves nicely under conformal or bilipshitz deformation of the Finsler metric $F$. These properties makes it a powerful tool in Finsler geometry and we illustrate that by solving a number of named Finslerian geometric problems. We also generalize and give new and shorter proofs of a number of known results. In particular we answer a question of M. Matsumoto about local conformal mapping between two Minkowski spaces, we describe all possible conformal self maps and all self similarities on a Finsler manifold. We also classify all compact conformally flat Finsler manifolds, solve a conjecture of S. Deng and Z. Hou on the Berwaldian character of locally symmetric Finsler spaces, and extend the classic result of H.C. Wang about the maximal dimension of the isometry groups of Finsler manifolds to manifolds of all dimensions.

Our methods apply even in the absence of the strong convexity assumption usually assumed in Finsler geometry. The smoothness hypothesis can also be replaced to that of partial smoothness, a notion that we introduce in the paper. Our results apply therefore to a vast class of Finsler metrics not usually considered in the Finsler literature.

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1. Introduction

In the present paper, a Minkowski space\(^1\) will be a pair \((E, F_0)\) where \(E\) is a finite dimensional real vector space and \(F_0 : E \rightarrow [0, \infty)\) is a Minkowski norm, that is \(F_0\) is positively homogenous and convex and it vanishes only at \(\xi = 0:\)

(a) \(F_0(\lambda \cdot \xi) = \lambda \cdot F_0(\xi)\) for any \(\lambda \geq 0.\)
(b) \(F_0(\xi + \eta) \leq F_0(\xi) + F_0(\eta).\)
(c) \(F_0(\xi) = 0 \Rightarrow \xi = 0.\)

Observe that \(F_0\) is a norm in the usual sense if and only if it is symmetric: \(F(-\xi) = F(\xi).\)

A Finsler metric on a smooth manifold \(M\) of dimension \(n \geq 2\) is given by a continuous function \(F : TM \rightarrow [0, \infty)\) such that for every point \(x \in M\) the restriction \(F_x = F|_{T_xM}\) is a Minkowski norm.

The Finsler metric is called reversible if \(F_x\) is actually a norm, that is if it satisfies \(F(x, -\xi) = F(x, \xi)\) for all point \(x\) and any tangent vector \(\xi \in T_xM.\) More generally, we will say that the Finsler metric is quasireversible if there exists a constant \(c\) such that

\[F(x, -\xi) \leq c \cdot F(x, \xi)\]

for any \((x, \xi) \in TM.\)

The Finsler metric is said to be of class \(C^k\) if the restriction of \(F\) to the slit tangent bundle \(TM^0 = TM \setminus \{\text{the zero section}\}\) is a function of class \(C^k.\)

It is customary in Finsler geometry to require the Finsler metric to be of class at least \(C^2\) and strongly convex, that is the Hessian of the restriction of \(F^2\) to \(T_xM \setminus \{0\}\) is assumed to be positive definite for any \(x \in M.\) However we shall avoid these hypothesis as they exclude from the theory some interesting and important examples. In fact our basic assumption throughout the paper is that \(M\) is a smooth manifold and the Finsler metric \(F\) is continuous. Whenever more smoothness will be needed, it will be explicitly mentioned.

A number of concepts of Riemannian geometry naturally extend to Finsler geometry, in particular one defines the length of a smooth curve \(\gamma : [0, 1] \rightarrow M\) as

\[\ell(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))dt.\]

We can then define the distance \(d(x, y)\) between points as the infimum of the length of all smooth curves \(\gamma : [0, 1] \rightarrow M\) joining these two points (i.e., \(\gamma(0) = x, \gamma(1) = y\)). This distance satisfies the axioms of a metric except perhaps the symmetry, i.e. the condition \(d(x, y) = d(y, x)\) is not satisfied, unless the Finsler metric is reversible. Together with the distance comes the notion of completeness: the Finsler Manifold \((M, F)\) is said to be forward complete if every forward Cauchy sequence converges. A sequence \(\{x_i\} \subseteq M\) is forward Cauchy if for any \(\varepsilon > 0,\) there exists an integer \(N\) such that \(d(x_i, x_{i+k}) < \varepsilon\) for any \(i \geq N\) and \(k \geq 0\) (we similarly define backward Cauchy sequences by the condition \(d(x_{i+k}, x_i) < \varepsilon,\) and the corresponding notion of backward completeness).

\(^1\)The terminology has also another meaning (not used in the present paper): a Minkowski space also means a real vector space equipped with a non degenerate bilinear form of signature \((1, n - 1).\) This ambiguity is unfortunate, but the context always makes it clear which notion of Minkowski space is being used.
Our main goal in this paper is to construct a natural Riemannian metric $g_F$ associated to an arbitrary Finsler metric $F$ on a smooth manifold $M$. The correspondence $F \rightarrow g_F$ will satisfy the following regularity and proportionality properties:

- The Riemannian metric $g_F$ has the same (or better) regularity as the Finsler metric $F$.
- If $F$ is Riemannian, i.e. if $F(x, \xi) = \sqrt{g_x(\xi, \xi)}$ for a some Riemannian metric $g$, then $g_F = g$.
- If two Finsler metrics $F_1$ and $F_2$ are conformally equivalent, i.e., if $F_1(x, \xi) = \lambda(x) \cdot F_2(x, \xi)$ for some function $\lambda : M \rightarrow \mathbb{R}$, then the corresponding Riemannian metrics are also conformally equivalent with essentially the same conformal factor: $g_{F_1} = \lambda^2 \cdot g_{F_2}$.
- If $F_1$ and $F_2$ are $C^0$-close, then so are $g_{F_1}$ and $g_{F_2}$.
- If $F_1$ and $F_2$ are bilipschitzly equivalent, then so are $g_{F_1}$ and $g_{F_2}$.
- If $F$ is quasireversible, then $F$ and $g_F$ are bilipschitzly equivalent.

More precise statements and additional properties will be given in section 6. This construction is a powerful tool to reduce some problems of Finsler geometry to the results and techniques of Riemannian geometry. We shall give concrete applications of this to the following geometric problems:

1. A generalization of the result of Wang [60] about the possible dimensions of the isometry groups of Finsler manifolds to manifolds of all dimensions.
2. The description of local conformal maps between Minkowski spaces, thus answering a question raised by Matsumoto in [40].
3. The description of all Finsler spaces admitting a non trivial self-similarity.
4. The topological classification of conformally flat compact Finsler manifolds.
5. The description of all conformal self maps in a Finsler manifold.
7. A positive solution to the conjecture of S. Deng and Z. Hou [17] stating that a locally symmetric Finsler space is Berwald.

We will also construct a family of new scalar invariants of Finsler manifolds extending the well known Minkowski functionals from convex geometry.

The paper is organized as follows. In section 2 we recall some useful facts from convex geometry and in section 3, we describe the most natural construction (from the viewpoint of convex geometry) of a Riemannian metric associated to a Finsler one. This construction is based on the so called John ellipsoid, and we show by an example that this generally gives us a non smooth Riemannian metric, even if the metric we start with is real-analytic. Our construction of a regular Riemannian metric is given in section 4. We will call the obtained metric the Binet-Legendre metric associated to the Finsler metric, since its construction is similar to the construction of the Binet and Legendre ellipsoids. In section 5 we introduce partially smooth Finsler metrics and show that the corresponding Binet-Legendre metrics are smooth.

In section 6, we establish the basic properties of the Binet-Legendre construction. The remaining sections 7 to 14 are devoted to the solutions of the aforementioned geometric problems.

### 2. A DETOUR BY CONVEX GEOMETRY

A Finsler metric on the manifold can also be seen as a “field of convex domains” sitting in the tangent bundle $TM$ of the manifold $M$. 

Indeed, it is well known that a convex bounded open set $\Omega$ in $\mathbb{R}^n$ containing the origin is essentially the same as a Minkowski norm on $V$. Given $\Omega$, we define its radial function $R_\Omega : \mathbb{R}^n \rightarrow (0, \infty]$ as follows:

$$R_\Omega(\xi) = \sup\{t \in \mathbb{R} \mid t \cdot \xi \in \Omega\}.$$

Observe that $R(\xi) > 0$ for any $\xi \in \mathbb{R}^n$ because $\Omega$ contains the origin and $R(\xi) = \infty$ if and only if $\xi = 0$ since $\Omega$ is bounded.

The inverse of the radial function is called the Minkowski functional associated to $\Omega$:

$$F_\Omega(\xi) = \frac{1}{R_\Omega(\xi)} = \inf\{t > 0 \mid \frac{1}{t} \xi \in \Omega\}.$$

It is easy to check that the Minkowski functional is positively homogenous and vanishes only at $\xi = 0$, and the convexity of $\Omega$ implies that of $F_\Omega$, see [32, §14.3]. In other words $F_\Omega$ satisfies the conditions (a),(b) and (c) of a Minkowski norm.

Conversely, the convex domain $\Omega$ can always be reconstructed from its Minkowski functional using the formula

$$\Omega = \{\xi \in \mathbb{R}^n \mid F_\Omega(\xi) < 1\},$$

this implies in particular that the Minkowski space $(\mathbb{R}^n, F)$ can also be thought of as a pair $(\mathbb{R}^n, \Omega)$ where $\Omega \subset \mathbb{R}^n$ is a convex bounded and open set containing the origin.

Observe that the Minkowski space $(\mathbb{R}^n, F)$ is euclidean, that is $F$ is the norm associated to a scalar product, if and only if $\Omega$ is an open ellipsoid centered at the origin.

The previous considerations can be trivially generalized to the context of Finsler manifolds: a Finsler metric $F$ on a smooth manifold $M$ gives rise to an open subset of the tangent bundle defined as

$$\Omega = \{(x, \xi) \in TM \mid F(x, \xi) < 1\} \subset TM.$$

The set $\Omega \subset TM$ contains the zero section $TM_0$ and its restriction $\Omega_x = \Omega \cap T_x M$ to each tangent space is bounded and convex. The Finsler metric $F : TM \rightarrow \mathbb{R}$ can be recovered from $\Omega \subset TM$ via the formula

$$F(x, \xi) = \inf\{t > 0 \mid \frac{1}{t} \xi \in \Omega\}.$$

Let us give a few examples, more can be found in [46]:

**Example 2.1.** Given a bounded open set $\Omega_0 \subset \mathbb{R}^n$ that contains the origin, one naturally defines a Finsler structure on $M$ by parallel transporting $\Omega_0$. That is $\Omega \subset T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ is defined as

$$\Omega = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \in \Omega_0\}.$$ 

The space $\mathbb{R}^n$ equipped with this Finsler structure is characterized by the property that its Lagrangian $F$ is invariant with respect to the standard translations of $\mathbb{R}^n$, i.e. $F$ is independent of the point $x$:

$$F(x, \xi) = F_0(\xi).$$

Such a Finsler manifold is of course essentially the same as a Minkowski space.

**Example 2.2.** Assume that the manifold $M$ is itself a bounded open convex domain in $\mathbb{R}^n$. One can then define a Finsler structure $\Omega \subset TM = M \times \mathbb{R}^n$ as

$$\Omega = \{(x, \xi) \in M \times \mathbb{R}^n \mid (\xi + x) \in M\}.$$
This structure is called the \textit{tautological Finsler structure} on $M$ because the unit ball $\Omega_x \subset T_x M$ is simply a copy of the domain $M$ itself with the point $x$ as its center. The associated Lagrangian is given by
\begin{equation}
F(x, \xi) = \inf \{ t > 0 \mid \frac{1}{t} \xi \in (M - x) \},
\end{equation}
and it is explicitly computable for simple enough domains (see [46, 47]). As explained in [46], the associated distance is the well known \textit{Funk metric} given by
\[
\rho(x, y) = \log \frac{|x - a|}{|y - a|},
\]
where $a$ is the intersection of the ray with origin $x$ and direction $(y - x)$ with the boundary of $M$:
\[
a = \partial M \cap (x + \mathbb{R}_+(y - x)).
\]

\textbf{Example 2.3.} If $\Omega \subset TM$ is a Finsler structure with Lagrangian $F$ and $A_x : T_x M \to T_x M$ is a field of invertible endomorphisms of the tangent bundle (i.e. $A$ is an invertible $(1, 1)$ tensor field), then a new Finsler structure can be defined by
\[
\Omega_A = A^{-1} \Omega.
\]
The corresponding Lagrangian is $F_A(x, \xi) = F(x, A_x(\xi))$. (Observe that any Riemannian metric on a domain in $\mathbb{R}^n$ can be obtained from the euclidean metric by this procedure.)

\textbf{Example 2.4.} Let $\Omega \subset TM$ be an arbitrary Finsler structure with Lagrangian $F$. If a vector field $Z : M \to TM$ such that $F(x, Z(x)) < 1$ for any point $x$, then a new Finsler structure can be defined as
\begin{equation}
\Omega_Z = \{(x, \xi) \in TM \mid \xi \in (\Omega_x - Z(x))\}.
\end{equation}
The corresponding Lagrangian is given by
\begin{equation}
F_Z(x, \xi) = \inf \{ t > 0 \mid \frac{1}{t} \xi \in (\Omega_x - Z(x)) \},
\end{equation}
and for $\xi \neq 0$, it is computable from the identity
\begin{equation}
F\left( x, \frac{\xi}{F_Z(x, \xi)} + Z(x) \right) = 1.
\end{equation}
This new Finsler structure $\Omega_Z$ is called the \textit{Zermelo transform} of the structure $\Omega \subset TM$ with respect to the vector Field $Z$. The Zermelo transform of a Riemannian metric is called a Randers metric and its geodesics are the solution to the so called “Zermelo navigation problem”, see [5].

3. \textbf{The John ellipsoid and the John Metric}

Recall that our aim is to define a correspondence that associates a Riemannian metric to a Finsler one.

There are many constructions of an ellipsoid associated to an open bounded convex domain. The most famous one is probably the one suggested by F. John in 1948 [28], where he proved that each convex body $\Omega \subset \mathbb{R}^n$ contains a unique ellipsoid $J[\Omega] \subseteq \Omega$ of largest volume. A careful study of the uniqueness proof shows that the John ellipsoid depends continuously on the convex body $K$. If $\Omega$ is symmetric with respect to the origin (i.e. $-\Omega = \Omega$), then $J[\Omega]$ is centered in the origin and we have
\begin{equation}
J[\Omega] \subseteq \Omega \subseteq \sqrt{n} \cdot J[\Omega],
\end{equation}
see [6] and [57, Section 3.3].

In general, the John ellipsoid is not symmetric and its center $Q_\Omega$ is not the origin. We call this center the John point of $\Omega$, and we denote by

$$J^*[\Omega] = J[\Omega] - Q_\Omega$$

the centered John ellipsoid of $\Omega$. It was also proved in [28, Theorem III], that for an arbitrary open bounded convex set $\Omega \subseteq \mathbb{R}^n$, we have

$$\Omega - Q_\Omega \subseteq n \cdot J^*[\Omega].$$

Recall that $\Omega$ contains the origin, thus the above inclusion together with the fact that $J^*[\Omega]$ is centrally symmetric implies

$$(0 - Q) \in n \cdot J^*[\Omega] = -n \cdot J^*[\Omega],$$

that is $Q \in n \cdot J^*[\Omega]$. It then follows from (3.2) that

$$\Omega \subseteq Q + n \cdot J^*[\Omega] \subseteq 2n \cdot J^*[\Omega].$$

Note however that in general $J^*[\Omega] \not\subseteq \Omega$. Moreover, one can construct for any given (large) $\kappa > 0$ an open convex sets $\Omega$ containing 0 such that the centered John ellipsoid is not contained in $\kappa \cdot \Omega$, see figure 1.

The centered John ellipsoid allows us to construct a natural continuous Riemannian metric on any Finsler manifold. More precisely we have the following Theorem.

**Theorem 3.1.** Any Finsler manifold $(M, F)$ carries a well defined Riemannian metric $j_F$ of class $C^0$ whose unit ball at any point $x \in M$ is the centered John ellipsoid $J^*[\Omega_x] \subseteq T_x M$ of the Finsler unit ball $\Omega_x \subseteq T_x M$. Furthermore the following inequality hold:

$$\frac{1}{2n} \sqrt{j_F(\xi, \xi)} \leq F(x, \xi)$$

for any $(x, \xi) \in T_M$. If the Finsler metric $F$ is reversible, then we have the better estimates

$$\frac{1}{\sqrt{n}} \sqrt{j_F(\xi, \xi)} \leq F(x, \xi) \leq \sqrt{j_F(\xi, \xi)}.$$  

In particular a reversible Finsler metric $F$ is bilipschitz equivalent to the Riemannian metric $j_F$. 

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**Figure 1.** If one of the vertices of the triangle $\Omega$ is close to the origin of the coordinate system, the constant $\kappa$ such that $\kappa \cdot \Omega \supset Q_\Omega$ is very large.
This Riemannian metric \( j_F \) will be called the \textit{John metric} associated to the Finsler metric. In a certain sense, the John metric is the “most natural” for convex geometers. When we discussed our ideas with specialists in convex geometry, they always suggested \( j_F \) as a good candidate for a Riemannian metric satisfying our requirements. However, \( j_F \) does not always have the same or a better smoothness as \( F \). Example 3.2 below shows that the John metric may fail to be \( C^1 \), even if the Finsler metric is analytic.

\textbf{Proof.} The centered John ellipsoid \( J^*[\Omega_x] \subseteq T_x M \) is the unit ball of a uniquely defined positive definite bilinear form \( j_x \) on \( T_x M \). By continuity of the John construction, these bilinear forms give us a \( C^0 \)-Riemannian metric \( j_F \) on \( M \), that is naturally associated to the Finsler metric \( F \). Observe that the norm associated to the metric \( j_F \) is the Minkowski functional \( F \circ J^*[\Omega] \) of \( J^*[\Omega_x] \), thus the inclusion (3.3) can be rewritten as

\[
\frac{1}{2n} F_j^*[\Omega](\xi) \leq F_{\Omega}(\xi),
\]

which the inequality (3.4). In the reversible case, \( \Omega_x \subseteq T_x M \) is symmetric around the origin and the inclusion (3.1) can be written as

\[
\frac{1}{\sqrt{2n(n+1)}} \sqrt{j_F}(\xi, \xi) \leq F(x, \xi).
\]

The proof of the last assertion is straightforward. \( \Box \)

\textbf{Remark.} Arguing as in [28] one can slightly improve the inequality (3.3):

\[
\Omega \subseteq \sqrt{2n(n+1)} \cdot J^*[\Omega].
\]

It follows that the inequality (3.4) can also be improved as

\[
\frac{1}{\sqrt{2n(n+1)}} \sqrt{j_F}(\xi, \xi) \leq F(x, \xi).
\]

\textbf{Example 3.2.} Consider the following Finsler metric \( F \) on \( M = \mathbb{R}^n \):

\[
F(x, \xi) = ||\xi||_{p(x)}^p = \left( \sum_{i=1}^{n} |\xi_i|^{p(x)} \right)^{1/p(x)},
\]

where \( p(x) = 1 + e^{x^1} \). If one identifies \( T_x M \) with \( \mathbb{R}^n \), the Finsler unit ball is

\[
\Omega_x = \left\{ \xi \in \mathbb{R}^n \mid \sum_{i=1}^{n} |\xi_i|^{p(x)} < 1 \right\}.
\]

It is easy to see that the John ellipsoid of \( \Omega_x \) is an euclidean ball centered at the origin. Indeed, each \( \Omega_x \) is invariant with respect to the symmetries \( \sigma_i : (\ldots, \xi_i, \ldots) \mapsto (\ldots, -\xi_i, \ldots) \) and \( \sigma_{ij} : (\ldots, \xi_i, \ldots, \xi_j, \ldots) \mapsto (\ldots, \xi_j, \ldots, \xi_i, \ldots) \), and since the John ellipsoid \( J[\Omega_x] \) of \( \Omega_x \) is unique, it must be \( \sigma_i \) and \( \sigma_{ij} \)-invariant for all \( i, j = 1, \ldots, n \). Since the euclidean balls centered at the origin are the only ellipsoid invariant with respect to all such symmetries, the John ellipsoid must be such a ball.

The radius \( r \) of \( J[\Omega_x] \) only depends on \( p = p(x) \) and a calculation shows that

\[
r(x) = \min \left\{ 1, n^2 \frac{1}{p} \right\} = \left\{ \begin{array}{ll}
\frac{1}{n^2} & \text{if } 1 < p \leq 2 \\
1 & \text{if } p \geq 2.
\end{array} \right.
\]
Indeed, for \( p \leq 2 \) a common point of the boundary of \( J[\Omega_x] \) of \( \Omega_x \) is given by \( \xi = (1, 0, \cdots, 0) \), while for \( p \geq 2 \) a common point of the boundary of \( J[\Omega_x] \) of \( \Omega_x \) is \( \xi = \left( \left( \frac{1}{n} \right)^{\frac{1}{p}}, \cdots, \left( \frac{1}{n} \right)^{\frac{1}{p}} \right) \) (see figure 2).

It is elementary to check that the function \( r(x) \) is not differentiable when \( x_1 = \log(2) \), i.e. \( p = 2 \) (see also figure 3). Therefore the ellipsoid \( J[\Omega_x] \) does not depend smoothly on \( x \). Moreover, the metric \( j_F \) has a discontinuous curvature and therefore cannot be made smooth by a \( C^0 \)-change of the coordinates.

We have thus constructed an analytical Finsler metric \( F \) on \( \mathbb{R}^n \) such that the associated John metric is given by

\[
j_x(\xi, \eta) = \frac{1}{r(x)^2} \langle \xi, \eta \rangle,
\]

where \( r(x) \) is not differentiable.

Our next result says that the volume form of the John metric is comparable to the canonical (Busemann-Hausdorff) measure of the Finsler metric \( F \). Let us first recall what this measure is: Remember that a density on the differentiable manifold \( M \) is a Borel measure \( d\nu \) such that on any coordinate chart \( \phi : U \subset M \rightarrow \mathbb{R}^n \), the measure \( \phi_*d\nu \) is absolutely continuous with respect to the Lebesgue measure, that is it can be written as

\[
\phi_*d\nu = a(x)dx_1dx_2\cdots dx_n
\]

(3.8)

where \( x_1, x_2, \ldots, x_n \) are the coordinates defined by the chart \( \phi \) and \( a(x) \) is a positive measurable function. Such a measure is essentially unique: any other density being a scalar multiple of \( d\nu \).

A density on the manifold \( M \) naturally induces a Lebesgue measure \( d\tau_x \) on (almost) each tangent space \( T_x M \), this measure is given by

\[
d\tau_x = a(x)d\xi_1d\xi_2\cdots d\xi_n,
\]
where $\xi_1, \xi_2, \ldots, \xi_n$ are the natural coordinates on $T_xM$ associated to $x_1, x_2, \ldots, x_n$ and $a(x)$ is given by (3.8).

To define the Busemann-Hausdorff measure $d\mu_F$ on the Finsler Manifold $(M, F)$, one chooses an arbitrary density $d\nu$ on $M$ and sets

$$d\mu_F = \frac{\omega_n}{\sigma(x)} d\nu,$$

where

$$\sigma(x) = \text{Vol}_r(\Omega_x) = \int_{\Omega_x} d\tau_x$$

and $\omega_n$ is the volume of the unit ball $B^n \subset \mathbb{R}^n$ in the standard euclidean space. Clearly, the function $\sigma(x)$ is a continuous never vanishing function, so $d\mu_F$ is well defined. Recall that $\Omega_x = \{ \xi \in T_xM \mid F(x, \xi) < 1 \}$ is the Finsler unit ball in $T_xM$.

It is obvious that the Busemann-Hausdorff measure $d\mu_F$ is independent of the chosen density $d\nu$. It is also clear that in the special case where $F = \sqrt{g}$ is Riemannian, the Busemann-Hausdorff measure coincides with the Riemannian volume measure, i.e. $d\mu_F = d\text{vol}_g$. In fact $d\mu_F$ is the normalized density on $M$ for which the associated volume of the unit ball $\Omega_x$ is equal to $\omega_n = \text{Vol}(B^n)$ for any point $x$ in $M$.

It is also known, but somewhat delicate to prove, that if $F$ is a reversible Finsler metric on $M$, then $d\mu_F$ coincides with the $n$-dimensional Hausdorff measure of the metric space associated to the Finsler structure, see [2, 9, 11, 12]

**Proposition 3.3.** Let $(M, F)$ be a Finsler manifold with Busemann-Hausdorff measure $d\mu_F$. Then the following inequalities hold:

$$d\mu_F \leq d\text{vol}_{j_F} \leq n^n \cdot d\mu_F,$$

where $d\text{vol}_{j_F}$ is the Riemannian density of the John metric $j_F$ associated to the Finsler metric.

**Proof** Choose $d\nu = d\text{vol}_{j_F}$ as initial density. Since $J[\Omega_x] \subset \Omega_x \subset T_xM$ for any point $x$ in $M$, we have

$$\omega_n = \text{Vol}_{j_F}(J^*[\Omega_x]) = \text{Vol}_{j_F}(J[\Omega_x]) \leq \text{Vol}_{j_F}(\Omega_x) = \sigma(x).$$

Conversely, using (3.2), we have

$$\sigma(x) = \text{Vol}_{j_F}(\Omega_x) = \text{Vol}_{j_F}(\Omega_x - Q_x) \leq \text{Vol}_{j_F}(n \cdot J^*[\Omega_x]) = n^n \text{Vol}_{j_F}(J^*[\Omega_x]) = n^n \omega_n,$$

where $Q_x$ is the John point of $\Omega_x$. The two previous inequalities, together with the relation $d\text{vol}_{j_F} = \frac{\sigma(x)}{\omega_n}$ imply the Theorem. \qed

Note that, due to (3.1), the second inequality in the previous Proposition can be improved as follows in the case of a reversible Finsler metric:

$$d\text{vol}_{j_F} \leq n^{n/2} \cdot d\mu_F.$$
4. The Binet-Legendre Riemannian metric

The lack of smoothness of the John metric has its origin in the way the John ellipsoid was defined: as the solution to a maximization problem. To smoothly associate a Riemannian metric to a Finsler one, we suggest to use an averaging procedure rather than a maximization procedure. We are mainly interested in the following construction:

**Proposition 4.1.** Let $V$ be an $n$-dimensional real vector space and $F : V \to \mathbb{R}$ be a Minkowski norm on $V$. There exists a unique scalar product $g_F$ on $V$ such that

$$
g_F(\xi, \xi) = \frac{(n+2)}{\lambda(\Omega)} \int_{\Omega} g_F(\xi, \eta)^2 d\lambda(\eta),$$

where $\Omega = \{ \xi \in V \mid F(\xi) < 1 \}$ is the unit sphere associated to $F$ and $\lambda$ is a Lebesgue measure on $V$.

Recall that a Lebesgue measure on a real vector space of finite dimension is a positive translation invariant Borel measure. It is unique up to a constant and the identity (4.1) is independent of the chosen Lebesgue measure.

**Proof.** Let us first prove the uniqueness of $g_F$ assuming its existence. The scalar product $g_F$ defines a natural scalar product $g_F^*$ on the dual space $V^*$ by the requirement that

$$
g_F^*(\xi^*, \xi^*) = g_F(\xi, \xi),$$

for any $\xi \in V$, where $\xi^* \in V^*$ is defined as $\xi^* = g_F(\xi, \cdot)$. Applying the formula (4.1) to $\theta = \xi^* \in V^*$ gives then

$$
g_F^*(\theta, \theta) = \frac{(n+2)}{\lambda(\Omega)} \int_{\Omega} \theta(\eta)^2 d\lambda(\eta).$$

This means that the scalar product $g_F^*$ is directly defined from the $F$-unit ball $\Omega \subseteq V$. The scalar product $g_F$ on $V$ is then uniquely defined from $\Omega$ through the formulæ (4.3) and (4.2). To prove the existence, we observe that

$$
g_F^*(\theta, \varphi) = \frac{(n+2)}{\lambda(\Omega)} \int_{\Omega} (\theta(\eta) \cdot \varphi(\eta)) d\lambda(\eta).$$

defines a scalar product on $V^*$ (in fact this is up to a constant the $L^2$-scalar product of the linear functions $\theta$ and $\varphi$ restricted to $\Omega$). This scalar product induces by duality (4.2) a scalar product on $V$ which satisfies (4.1). □

**Remark.** It is interesting to interpret this construction in the language of convex geometry. Formula (4.3) associates an ellipsoid in $V^*$ (the unit ball of the metric $g_F^*$) to an arbitrary convex body containing the origin $\Omega \subseteq V$. This ellipsoid is called the *Binet ellipsoid* of $\Omega$. The polar dual of the Binet ellipsoid is called the *Legendre ellipsoid* of $\Omega$, it lives in $V$ and it is simply the unit ball $B$ of the metric $g_F$. The Legendre ellipsoid satisfies

$$
\int_B \theta^2(\xi) d\xi = \int_{\Omega} \theta^2(\xi) d\xi
$$

for any $\theta \in V^*$, and it is the unique ellipsoid with this property. This claim can be proved from property (a) of Proposition 4.4 below.

The integral (4.5) is called the *moment of inertia* of $\Omega$ in the codirection $\theta$. Thus the Legendre ellipsoid is the unique ellipsoid having the same moment of inertia as $\Omega$ in all possible
codirections and it has the following mechanical interpretation: The motion of a homogenous rigid body $\Omega$ which freely moves in 3-space and is subjected to no external force is dynamically equivalent to a similar motion of its Legendre ellipsoid (see [34]).

The correspondence $F \mapsto g_F$ will be called the Binet-Legendre construction, let us give some examples.

Example 4.2. Let $F : \mathbb{R}^n \to \mathbb{R}$ be the max-norm $F(\xi) = \max_i |\xi_i|$. Its unit ball is the cube $\Omega_F = [-1,1]^n$ and we claim that the Binet-Legendre scalar product is given by

$$g_F(\xi, \eta) = \frac{3}{n+2} \langle \xi, \eta \rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product. To prove this identity, one may argue as in Example 3.2. The Binet-Legendre metric $g_F$ is clearly invariant under the symmetries $\sigma_i, \sigma_{ij}$ defined in Example 3.2 and therefore $g_F$ is a multiple of the standard scalar product. To check the coefficient, one may use Formula (4.3) with an arbitrary covector. Choose the covector $\varepsilon_1$ dual to the first basis vector and defined by $\varepsilon_1(\xi) = \xi_1$. Using (4.3) one obtains

$$g^*_F(\varepsilon_1, \varepsilon_1) = \frac{n+2}{2^n} \int_{[-1,1]^n} \xi_1^2 \, d\xi_2 \cdots d\xi_n = \frac{n+2}{3}.$$ 

This implies by duality that the Binet-Legendre norm of the first basis vector $e_1$ is given by

$$g_F(e_1, e_1) = \frac{3}{n+2} = \frac{3}{n+2} \langle e_1, e_1 \rangle.$$

Example 4.3. Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is a Minkowski norm and $\zeta \in \mathbb{R}^n$ is a vector such that $F(\zeta) < 1$. Define a new Minkowski norm $F' = F'_{\zeta}$ as follows:

$$F'(x, \xi) = \inf \{ t > 0 \mid F\left(\frac{1}{t} \xi + \zeta\right) \leq 1\}$$

(compare with (2.4)). Then the Binet-Legendre scalar products of $F$ and $F'$ are related by

$$g_{F'}(\xi, \xi) = g_F(\xi, \xi) + (n+2) \cdot g_F(\xi, \zeta)^2.$$ 

To prove this identity, let $\Omega'$ and $\Omega'$ be the unit balls of $F$ and $F'$ respectively. Since $\Omega' = \Omega - \zeta$, we have

$$g^*_{F'}(\theta, \theta) = \frac{n+2}{\lambda(\Omega')} \int_{\Omega'} \theta(\eta)^2 \, d\eta = \frac{(n+2)}{\lambda(\Omega)} \int_{\Omega} \theta(\eta + \zeta)^2 \, d\eta = \frac{(n+2)}{\lambda(\Omega)} \left( \int_{\Omega} \theta(\eta)^2 \, d\eta + 2\theta(\zeta) \int_{\Omega} \theta(\eta) \, d\eta + \theta(\zeta)^2 \int_{\Omega} \, d\eta \right).$$

Since $-\Omega = \Omega$, we have $\int_{\Omega} \theta(\eta) \, d\eta = 0$ and therefore

$$g^*_{F'}(\theta, \theta) = \frac{n+2}{\lambda(\Omega')} \int_{\Omega'} \theta(\eta)^2 \, d\eta + (n+2)\theta(\zeta)^2 = g^*_F(\theta, \theta) + (n+2) \cdot g^*_F(\theta, \zeta)^2.$$

We now prove a few basic properties of the scalar product $g_F$.

Proposition 4.4. The transformation $F \mapsto g_F$ satisfies the following properties
(a) If \( F \) is euclidean, i.e. \( F(\xi) = \sqrt{g(\xi, \xi)} \) for some scalar product \( g \), then \( g_F = g \).

(b) If \( A \in GL(V) \), then \( g_{A^*F} = A^*g_F \).

(c) \( g_F = \kappa^2 g_F \) for any \( \kappa > 0 \).

(d) if \( \frac{1}{c} \cdot F_1 \leq F_2 \leq c \cdot F_1 \) for some constant \( c > 0 \), then
\[
\frac{1}{2\pi} \cdot g_{F_1} \leq g_{F_2} \leq c^{2n} \cdot g_{F_1}.
\]

**Proof.** a) Suppose \( F = \sqrt{g} \) is euclidean and let \( e_1, e_2, \ldots, e_n \) be an orthonormal basis on \((V, g)\) and \( x_1, x_2, \ldots, x_n \) be the corresponding coordinate system. The convex set \( \Omega \) coincides with the unit ball \( \Omega = B^n = \{ x \in V \mid \sum x_i^2 < 1 \} \) and formula (4.3) gives
\[
g_F^*(\varepsilon_i, \varepsilon_i) = \frac{(n+2)}{Vol(\mathbb{B}^n)} \int_{\mathbb{B}^n} x_i^2 dx,
\]
where \( \varepsilon_i = e_i^\flat \). Now the integral on the left hand side computes as follows:
\[
\int_{\mathbb{B}^n} x_i^2 dx = \frac{1}{n} \int_{\mathbb{B}^n} \sum_{i=1}^n x_i^2 dx = \frac{1}{n} \int_{S^{n-1}} \int_0^1 r^{n+1} dr d\sigma = \frac{\text{Area}(S^{n-1})}{n(n+2)}.
\]
But \( \text{Area}(S^{n-1}) = n \cdot \text{Vol}(\mathbb{B}^n) \) and we thus have
\[
g_F^*(\varepsilon_i, \varepsilon_i) = \frac{(n+2)}{Vol(\mathbb{B}^n)} \cdot \frac{\text{Area}(S^{n-1})}{n(n+2)} = 1.
\]
If \( j \neq i \), then
\[
g_F^*(\varepsilon_i, \varepsilon_j) = \frac{(n+2)}{Vol(\mathbb{B}^n)} \int_{\mathbb{B}^n} x_i x_j dx = 0
\]
because the function \( x_i x_j \) is antisymmetric with respect to the orthogonal transformation \( x_i \mapsto -x_i \). It follows that \( \varepsilon_1, \ldots, \varepsilon_n \) is an orthonormal basis of \( V^* \) for the scalar product \( g_F^* \). By duality, \( e_1, \ldots, e_n \) is also an orthonormal basis of \( V \) for the scalar product \( g_F \) and therefore \( g_F = g \).

We now prove property (b). If \( A \in GL(V) \), then the unit ball \( \Omega_A \) associated to \( A^*F = A \circ F \) is the set \( A^{-1} \cdot \Omega \), indeed
\[
\Omega_A = \{ \xi \in V \mid F(A\xi) < 1 \} = \{ A^{-1}\eta \in V \mid F(\eta) < 1 \} = A^{-1} \cdot \Omega.
\]
Therefore
\[
g_{A^*F}^*(\theta, \theta) = \frac{(n+2)}{\lambda(A^{-1}\Omega)} \int_{A^{-1}\Omega} \theta(\eta)^2 d\lambda(\eta) = \frac{(n+2)}{\lambda(\Omega)} \int_{A^{-1}\Omega} \theta(\eta)^2 d\lambda(\eta).
\]
Setting \( \xi = A\eta \), we have from the change of variable formula
\[
\int_{A^{-1}(\Omega)} \theta(\eta)^2 d\lambda(\eta) = \int_{\Omega} \theta(A^{-1}\xi)^2 |\det(A^{-1})| d\lambda(\xi),
\]
and thus
\[
g_{A^*F}^*(\theta, \theta) = \frac{(n+2)}{\lambda(\Omega)} \int_{\Omega} \theta(A^{-1}\xi)^2 d\lambda(\xi) = g_F^*(\theta \circ A^{-1}, \theta \circ A^{-1}).
\]
This is the relation between \( g_{A^*F}^* \) and \( g_F^* \). In the space \( V \), we then have by duality
\[
g_{A^*F}(\xi, \xi) = g_F(A\xi, A\xi).
\]
Property (c) is the special case of property (b) corresponding to scalar matrices. To prove (d), let $F_1, F_2$ be two Minkowski norms satisfying $\frac{1}{c} \cdot F_1 \leq F_2 \leq c \cdot F_1$, then the corresponding unit balls also satisfy
$$\frac{1}{c} \cdot \Omega_1 \subset \Omega_2 \subset c \cdot \Omega_1.$$ This implies in particular that
$$\frac{1}{\lambda(\Omega_2)} \leq \frac{c^n}{\lambda(\Omega_1)}.$$ We also have
$$\int_{\Omega_2} \theta(\eta)^2 d\lambda(\eta) \leq \int_{\Omega_1} \theta(\eta)^2 d\lambda(\eta) = c^n \cdot \int_{\Omega_1} \theta(\xi)^2 d\lambda(\xi)$$ (set $\xi = c\eta$). Therefore
$$\int_{\Omega_2} \theta(\eta)^2 d\lambda(\eta) \leq \frac{(n+2)}{\lambda(\Omega_2)} \cdot \frac{c^n \cdot (n+2)}{\lambda(\Omega_1)} \cdot \int_{\Omega_1} \theta(\eta)^2 d\lambda(\eta),$$ that is
$$g_{F_2}(\theta, \theta) \leq c^n \cdot g_{F_1}(\theta, \theta).$$ The dual scalar product satisfies then
$$g_{F_1}(\xi, \xi) \leq c^n \cdot g_{F_2}(\xi, \xi).$$

**Definition.** Let $(M, F)$ be a Finsler manifold. At every point $x \in M$, one has a Minkowski norm $F_x$ and hence an associated Binet-Legendre scalar product $g_{F_x}$. This field of scalar products is called the **Binet-Legendre metric** of $(M, F)$.

Note that beside the John metric and the Binet Legendre metric, we could use other procedures that associate an ellipsoid to a given convex body, such as the one considered in [39, 43, 48, 55, 56]. However, none of these constructions would give a metric with the desired properties listed in the introduction. We will show in the following sections of the present paper that the Binet-Legendre metric does satisfy the required properties and we will illustrate the usefulness of that metric.

## 5. Partially smooth Finsler metrics

A **homogeneous diffeomorphism** of a finite-dimensional vector space $V$ is a diffeomorphism $A_0 : V \setminus \{0\} \to V \setminus \{0\}$ such that for every $\lambda > 0$ and for every $v \in V, v \neq 0$ we have $A(\lambda v) = \lambda A(v)$.

A field of homogeneous diffeomorphisms of $TM$ is a diffeomorphism $A : TM^0 \to TM^0$, where $TM^0 = TM \setminus \text{(the zero section)}$ such that for every $x \in M$ we have that $A_x = A|_{T_xM}$ is a homogeneous diffeomorphism of $T_xM$.

**Definition 5.1.** Let $(M, F)$ be a Finsler manifold and $U \subseteq M$ the domain of some coordinate system $(x_1, ..., x_n)$. Then $F$ is said to be $C^k$-**partially smooth in the coordinates** $x_i$ if there exists a $C^k$-smooth field of homogeneous diffeomorphisms $A : TU^0 \to TU^0$ such that the function $x \mapsto F(x, A_x(\xi))$ is of class $C^k$ in $U$ for any fixed $\xi \in \mathbb{R}^n$.

In this definition, we use the identification $TU = U \times \mathbb{R}^n$ defined by the coordinate system. The vector field $\xi$ is thus “constant in the coordinate system $x_i$”. 

Lemma 5.2. Let $U$ be some domain in the a Finsler manifold $(M, F)$. If $F$ is $C^k$-partially smooth in some coordinate system on $U$, then it is partially smooth in any coordinate system on $U$.

Remark. In this lemma and also in definition 5.3 below we assume that the manifold itself is at least $C^{k+1}$-smooth.

Partial smoothness in some coordinate domain is thus in fact an intrinsic notion, and we are led to the following global definition.

Definition 5.3. A Finsler manifold is $C^k$-partially smooth if it is $C^k$-partially smooth in some neighborhood of any of its point.

Proof of Lemma 5.2. From the local nature of the concept, on may assume that $M$ is a domain $U \subset \mathbb{R}^n$ and that the Finsler metric $F$ is $C^k$-partially smooth in the natural coordinates of $\mathbb{R}^n$. We consider a field $A$ of homogeneous diffeomorphisms as in the the definition 5.1: $A$ is $C^k$-smooth and the mapping $x \mapsto F(x, A_x(\xi))$ is of class $C^k$ in $U$ for any constant vector field $\xi$. Let $y_j$ be another coordinate system on $U$, specifically, let $\phi : V \to U$ be a diffeomorphism from some domain $V$ onto $U$ and set $x = \phi(y)$. The Finsler structure $F$ on $U$ transforms into the Finsler structure $\tilde{F}$ on $V$ defined as

$$\tilde{F}(y, \xi) = F(\phi(y), d\phi_y(\xi)).$$

Define now the field of homogeneous diffeomorphism $\tilde{A}$ as $\tilde{A}_y = d\phi_y^{-1} \circ A_{\phi(y)}$. For any fixed vector $\xi \in \mathbb{R}^n$, the function

$$V \ni y \mapsto \tilde{F}(y, \tilde{A}_y(\xi)) = F(\phi(y), d\phi_y \circ \tilde{A}_y(\xi)) = F(\phi(y), A_{\phi(y)}(\xi))$$

is the composition of the $C^k$ functions $\phi : V \to U$ and $x \mapsto F(x, A_x(\xi))$, therefore

$$y \mapsto \tilde{F}(y, \tilde{A}_y(\xi))$$

is of class $C^k$ for any constant vector $\xi$ and we conclude that $\tilde{F}$ is partially smooth in the coordinates $y_j$.

$\square$

5.1. Examples of partially smooth Finsler metrics.

(a) Every smooth Finsler metric is partially smooth

(b) A Minkowski norm $F(\xi)$ on $\mathbb{R}^n$ (in the sense of section 2) is partially smooth. Indeed, we canonically identify $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ and look at $F$ as a “function of 2 variables which is constant in the first variable”: $F(x, \xi) = F(\xi)$, i.e., the field $A$ of the homogeneous diffeomorphisms consists of identities $id_{\xi} : T_xM \to T_xM$.

(c) Let $F_1$ and $F_2$ be Finsler metrics on the same manifold such that $F_1$ is partially smooth and $F_2$ is smooth. If $f : M \to [0, 1]$ is a smooth function, then the following interpolation

$$F(x, \xi) = f(x)F_1(x, \xi) + (1 - f(x))F_2(x, \xi)$$

is again a partially smooth Finsler metric.

(d) As a special case of the previous example, consider the Finsler metric on $M = \mathbb{R}^2$ given by

$$F(x_1, x_2, \xi_1, \xi_2) = (1 - f(x_1)) \cdot (|\xi_1| + |\xi_2|) + f(x_1) \cdot \sqrt{\xi_1^2 + \xi_2^2}$$

where $f : \mathbb{R} \to [0, 1]$ is a smooth function such that $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$. The Finsler metric $F$ is partially smooth, it is independent of the variable $x_2$ and
it interpolates from the $L^1$ norm on the plane to the euclidean norm as $x_1$ varies from 0 to 1.

(e) Choose a fixed Minkowski norm $F_0$ on $\mathbb{R}^n$ and take a smooth matrix-valued function $A : \mathbb{R}^n \rightarrow GL(\mathbb{R}^n)$. The following metric $F(x, \xi) = F_0(A^{-1}(x) \cdot \xi)$ is partially smooth and it is smooth if and only if the Minkowski norm $F_0$ is smooth on $\mathbb{R}^n \setminus \{0\}$.

(f) The tautological Finsler metric from Example 2.2 is partially smooth.

(g) If the metric $F$ is partially smooth, then the metrics from example 2.3, and example 2.4 are also partially smooth.

(h) Consider smooth functions $f_1, ..., f_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that for every $x \in \mathbb{R}^2$ the points $f_1(x), ..., f_n(x)$ are the vertices of a convex polygon $P_x$ such that the point 0 lies in its interior. We identify $T\mathbb{R}^2$ with $\mathbb{R}^2 \times \mathbb{R}^2$ and consider the Finsler metric whose $\Omega_x = P_x$ at every $x \in \mathbb{R}^2$. Then, this metric is partially smooth.

The latter example was in fact one of our original motivations for introducing the notion of partially smooth Finsler metric. This example also suggests the following remark: Finsler geometry can be used to describe certain phenomena in natural sciences (such as light prolongation in crystals or certain diffusion processes in organic cells), but to use Finsler geometry in such context, one needs to accept non-smooth metrics and the class of partially smooth Finsler metrics seems quite appropriate. Indeed, the cells or crystals can be viewed as a field of convex bodies at every point of $\mathbb{R}^3$ or of $\mathbb{R}^2$, which is very close to our definition of the Finsler structure. In particular the Finsler metric in example (h) above should be relevant in describing crystal structures.

The notion of partially smooth Finsler metrics is mainly motivated by the following result.

**Theorem 5.4.** Let $(M, F)$ be a Finsler manifold and $g_F$ the corresponding Binet-Legendre metric. If $F$ is $C^k$-partially smooth, then both the Buseman-Haussorff density $d\mu_F$ and the Binet-Legendre metric $g_F$ are of class $C^k$.

**Proof.** Let $U \subseteq M$ be the domain of some coordinate system $x_i$. Then the Buseman-Haussorff density is given on $U$ by

$$d\mu_F = \frac{\omega_n}{\text{Vol}(\Omega_x)} dx_1 \cdots dx_n$$

where $\text{Vol}(\Omega_x)$ is the euclidean volume of the unit ball $\Omega_x \subseteq T_xU = \mathbb{R}^n$ computed in the coordinate system $x_i$. We thus need to prove the smoothness of $x \mapsto \text{Vol}(\Omega_x)$. By hypothesis, there exists a $C^k$ field of homogeneous diffeomorphism. Let us define $\Omega'_x = A_x^{-1}(\Omega_x)$. Writing $\xi' = A_x(\xi)$, we have

$$\Omega'_x = \{\xi' \in \mathbb{R}^n \mid F(x, A_x(\xi')) < 1\},$$
and
\[ \text{Vol}(\Omega_x) = \int_{\Omega} d\xi = \int_{F(x,A_x(\xi')) < 1} \text{Jac}(A_x)(\xi') d\xi'. \]

Using polar coordinates \( \xi' = r \cdot u \), with \( u \in S^{n-1} \), this gives
\[ \text{Vol}(\Omega_x) = \int_{S^{n-1}} \left( \int_{r=0}^{1/F(x,A(u))} \text{Jac}(A_x)(r \cdot u) r^{n-1} dr \right) du, \]
where \( du \) stands for the spherical measure on \( S^{n-1} \) and \( \text{Jac}(A_x) \) is the Jacobian determinant \( \det \left( \frac{\partial \xi}{\partial r} \right) \). Since the functions \( \text{Jac}(A_x) \) and the bound \( 1/F(x,A(u)) \) \( C^k \) smoothly depend on \( x \), the integral
\[ I(x,u) = \int_{r=0}^{1/F(x,A(u))} \text{Jac}(A_x)(r \cdot u) r^{n-1} dr \]
also smoothly depends on \( x \). Then,
\[ \text{Vol}(\Omega_x) = \int_{S^{n-1}} I(x,u) \]
smoothly depends on \( x \) as we claimed.

The proof for the Binet-Legendre metric is similar. It suffices to prove that the dual metric \( g^* \) \( F \) is smooth, i.e., that \( x \mapsto (g_F)_x^*(\theta, \theta) \) is smooth in \( U \) for any fixed covector \( \theta : \mathbb{R}^n \to \mathbb{R} \).

We denote by \( \Theta(\xi) \) the function \( \theta(\xi)^2 \) and by \( \tilde{\Theta} \) the function \( \Theta \circ A_x \). Arguing as above and using formula (4.3), we have
\[
\frac{\text{Vol}(\Omega_x)}{(n+2)!} \cdot g_F^*(\theta, \theta) = \int_{\Omega} \Theta(\xi) d\xi = \int_{F(x,A_x(\xi')) < 1} \Theta(A_x(\xi)) \text{Jac}(A_x)(\xi') d\xi' = \int_{S^{n-1}} \left( \int_{r=0}^{1/F(x,A(u))} \tilde{\Theta}(r \cdot u) \text{Jac}(A_x)(r \cdot u) r^{n-1} dr \right) du
\]
This is again a \( C^k \) function of \( x \in U \), which completes the proof.

6. FURTHER PROPERTIES OF THE BINET-LEGENDRE METRIC

In the next theorem, we collect some of the basic properties of the Binet-Legendre metric.

**Theorem 6.1.** Let \( (M,F) \) be a \( C^k \)-partially smooth Finsler manifold and let \( g_F \) be the associated Binet-Legendre metric. The following properties hold:

a) \( g_F \) is a Riemannian metric of class \( C^k \).
b) If \( F \) is Riemannian, i.e., if \( F(x,\xi) = \sqrt{g_2(\xi,\xi)} \) for some Riemannian metric \( g \), then \( g_F = g \).
c) If \( A \in \text{Aut}(TM) \) is a \( C^k \)-field of automorphisms of the tangent bundle of \( M \), then \( g_{A^*F} = A^*g_F \).
d) If \( F_1(x,\xi) = \lambda(x) \cdot F_2(x,\xi) \) for some function \( \lambda : M \to \mathbb{R}_+ \), then \( g_{F_1} = \lambda^2 \cdot g_{F_2} \).
e) If \( \frac{1}{\lambda} \cdot F_1 \leq F_2 \leq \lambda \cdot F_1 \) for some function \( \lambda : M \to \mathbb{R}_+ \), then
\[
\frac{1}{\lambda^{2n}} \cdot g_{F_1} \leq g_{F_2} \leq \lambda^{2n} \cdot g_{F_1}.
\]

**Proof.** The first property is Theorem 5.4 and the other properties (b)–(e) directly follow from Proposition 4.4. \( \square \)
Remark. Property (e) of the Theorem implies in particular the Binet-Legendre construction is $C^0$-stable, i.e. two $C^0$-close Finsler metrics have $C^0$-close Binet-Legendre metrics. It also implies that two Finsler metrics which are Bilipschitz equivalent have Bilipschitz equivalent Binet-Legendre metrics (take $\lambda$ to be a constant in property (e)).

If the manifold $M$ carries an almost complex structure $J \in \text{Aut}(TM)$ (i.e. $J^2 = -\text{id}$), then one says that the Finsler structure is almost complex if $F(x, J\xi) = F(x, \xi)$ for any $(x, \xi) \in TM$. It follows immediately from property (c) above that the Binet-Legendre metric $g_F$ of an almost complex Finsler metric $F$ is hermitian, i.e. $g_F(J\xi, J\xi) = g_F(\xi, \xi)$.

In the quasireversible case, more can be said:

**Theorem 6.2.** Let $(M, F)$ be a quasireversible $C^k$-partially smooth Finsler manifold, then

a) $F$ and $g_F$ are bilipschitzly equivalent.

b) If $F$ is complete, then so is $g_F$.

c) The Riemannian volume density of $g_F$ is comparable to the Busemann-Hausdorff density.

**Proof.** (a) By hypothesis, there exists a constant $c$ such that $F(x, -\xi) \leq c \cdot F(x, \xi)$ for any $(x, \xi) \in TM$. Let us set $F'(x, \xi) = \frac{1}{2}(F(x, \xi) + F(x, -\xi))$, then $F'$ is reversible and bilipschitz equivalent to $F$:

$$\frac{1}{c} \cdot F'(x, \xi) \leq F(x, \xi) \leq c \cdot F'(x, \xi).$$

Let $j_{F'}$ be the John metric of $F'$ and recall the inequality (3.5) from Theorem 3.1:

$$\frac{1}{\sqrt{n}} \sqrt{j_{F'}(\xi, \xi)} \leq F'_H(x, \xi) \leq \sqrt{j_{F'}(\xi, \xi)}.$$

The John metric is in general not smooth, but locally one can find a smooth Riemannian metric $h$ such that $\frac{1}{2}h \leq j_F \leq 2h$. It follows from the previous two inequalities that $F$ and $h$ are bilipschitz equivalent. From property (c) in Theorem 6.1, we see that $g_F$ is bilipschitz equivalent to the Binet-Legendre metric of $h$, which coincides with $h$ itself by property (a) in Theorem 6.1. We conclude that $F$ and $g_F$ are bilipschitz equivalent.

The property (b) is an immediate consequence of (a) since completeness is a bilipschitz property.

Property (c) is also a consequence of (a): Let us denote by $d\nu$ the Riemannian volume density of $g_F$, then the Busemann-Hausdorff density is

$$d\mu_F = \frac{\omega_n}{\text{Vol}_\nu(\Omega_x)} d\nu.$$

Because $g_F$ is bilipschitz equivalent to $F$, we have $\frac{1}{k} \cdot B_x \subset \Omega_x \subset k \cdot B_x$ for some constant $k$ where $B_x \subseteq T_xM$ is the unit ball for the metric $g_F$. It follows at once that

$$\omega_n k^{-n} d\nu \leq d\mu_F \leq \omega_n k^n d\nu.$$

**Remark.** The quasireversibility assumption of the previous theorem is necessary. For instance the Funk metric is forward complete but not backward complete, hence it cannot be bilipschitz equivalent to any Riemannian metric.

\[2\]In complex Finsler geometry, one often assumes the stronger Rizza condition [51], which says that $F(x, \cos(\theta)\xi + \sin(\theta)J\xi) = F(x, \xi)$ for any $(x, \xi) \in TM$ and any $\theta \in \mathbb{R}$. 


As an immediate application of the Binet-Legendre construction, we have the following Proposition whose proof is obvious.

**Proposition 6.3.** Let \( f : (M_1, F_1) \to (M_2, F_2) \) be a \( C^1 \) map between two partially smooth Finsler manifold. If \( f \) is either an isometry, a similarity, a conformal or quasiconformal map, then the map \( f : (M_1, g_{F_1}) \to (M_2, g_{F_2}) \) is also an isometry, respectively a similarity, a conformal or quasiconformal map. Moreover, if \( f \) is a conformal map with respect to the Finsler metrics \( F_1, F_2 \), and is furthermore an isometry or a similarity with respect to the Riemannian structures \( g_{F_1}, g_{F_2} \), then it is in fact an isometry, respectively a similarity, with respect to the Finsler structures.

In particular, the group of isometries of a partially smooth Finsler manifold \((M, F)\) is a subgroup of the group of isometries of \((M, g_F)\). It is a closed subgroup and therefore it is a Lie group and its dimension is at most \( \frac{1}{2} n(n+1) \).

The next sections are devoted to more applications of the Binet Legendre construction.

### 7. On the number of Killing vector fields

In 1947, H.C. Wang proved that a smooth and strongly convex \( n \)-dimensional Finsler manifold of dimension \( n \neq 2, 4 \) is Riemannian if its group of isometries has dimension greater than \( \frac{n(n-1)}{2} + 1 \), see [60, 61]. Our next result extends Wang’s theorem to all dimensions. Our proof is more direct and also works for partially smooth metrics and without the strong convexity condition. This theorem gives a positive answer to a question raised by S. Deng and Z. Hou in [18, page 660].

A vector field \( K \) on a Finsler manifold \((M, F)\) is said to be a Killing vector field if it generates a local flow \( \phi^t_K \) of local isometries for the metric \( F \). Proposition 6.3 implies that any Killing vector field of \((M, F)\) is also a Killing vector field for the Binet-Legendre metric \( g_F \).

**Theorem 7.1.** Let \((M^n, F)\) be a partially \( C^2 \)-smooth connected Finsler manifold. If the dimension of the space of Killing vector fields of \((M, F)\) is greater than \( \frac{n(n-1)}{2} + 1 \), then \( F \) is actually a Riemannian metric.

Observe that the bound given in the Theorem is sharp: The (non Riemannian) Minkowski space \( \mathbb{R}^n \) with smooth and strongly convex norm

\[
F(\xi) = \left( \sum_{i=1}^{n} \xi_i^2 + \xi_n^4 \right)^{1/4}
\]

has \( r = n + \text{dim} \ SO(n-1) = \frac{n(n-1)}{2} + 1 \) linearly independent complete Killing vector fields.

**Proof.** Let \( r > \frac{n(n-1)}{2} + 1 \) be the dimension of the space of Killing vector fields. Take a point \( x \) and choose \( r - n \) linearly independent Killing vector fields \( K_1, \ldots, K_{r-n} \) vanishing at \( x \), this is possible because the dimension \( T_x M \) is \( n \). The point \( x \) is then a fixed point of the corresponding local flows \( \phi^t_{K_1}, \ldots, \phi^t_{K_{r-n}} \). By Proposition 6.3, any Killing vector field for \( F \) is also a Killing vector field of \( g_F \). In particular, for every fixed \( t \), the differentials of \( \phi^t_{K_1}, \ldots, \phi^t_{K_{r-n}} \) at \( x \) are linear isometries of \( (T_x M, g_F) \). Let us denote by \( \Phi_t \in \text{End}(T_x M) \) the differentials \( \Phi_t = \left( \frac{d}{dt} \vert_{t=0} \phi^t \right) \). We claim that \( \Phi_1, \ldots, \Phi_{r-n} \) are linearly independent.
Indeed, assume that $\sum_{i=1}^{r-n} a_i \Phi_i = 0$ for some constants $a_i \in \mathbb{R}$ and consider the Killing field $K = \sum_{i=1}^{r-n} a_i K_i$. Let us denote by $\phi^K_t$ the (local) flow generated by $K$; because $\phi^K_t \circ \exp_x = \exp_x \circ d_x \phi^K_t$, we have for $y = \exp_x(\xi)$:

$$K_y = \frac{d}{dt} \bigg|_{t=0} \phi^K_t(y) = \frac{d}{dt} \bigg|_{t=0} \exp_x(d\phi^K_t(\xi)) = 0$$

since $(\frac{d}{dt}d_x \phi^K_t)_{t=0} = \sum_{i=1}^{r-n} a_i \Phi_i = 0$. It follows that $K = 0$ in an open neighborhood of the point $x$ implying $K \equiv 0$ on the whole manifold. Because $K_i$ are assumed to be linearly independent, we have $a_i = 0$ for all $i$ and $\Phi_i$ are thus linearly independent as claimed.

Let us now denote by $G \in SO(T_x M, g_F)$ the smallest closed subgroup of $SO(T_x M, g_F)$ generated by the differentials of $\phi^K_1, \ldots, \phi^K_{r-n}$ at $x$. Its Lie algebra contains the linearly independent element $\Phi_1, \ldots, \Phi_{r-n}$ and we thus have dim$(G) \geq r - n$. It is known that for every $n \geq 2$, any $r - n$-dimensional subgroup of the orthogonal group $SO(n) \cong SO(T_x M, g_F)$ acts transitively on the $g_F$ unit sphere $S^{n-1} \subset T_x M$ provided $r > \frac{1}{2} n(n-1) + 1$. Indeed, for $n \neq 4$, this immediately follows for the classical result of Montgomery and Samelson [44]: they proved that for $n \neq 4$, there exists no proper subgroup of $SO(n)$ of dimension greater than $\frac{(n-1)(n-2)}{2}$. In dimension 4, the transitivity follows for example from [27, §1], where all Lie subgroups of $SO(4)$ are described.

Since the action of $G$ on $T_x M$ preserves $F$ and $g_F$ and $G$ acts transitively on the $g_F$-sphere $S^{n_1} \subset T_x M$, the ratio $F(\xi)^2 / g(\xi, \xi)$ is constant for all $\xi \in T_x M^0$ implying that $F(\xi) = \lambda(x) \cdot \sqrt{g_F(\xi, \xi)}$ for some function $\lambda : M \to \mathbb{R}_+$ and for all $\xi \in TM$. This proves that $F$ is Riemannian, furthermore, by Theorem 6.1(b), the coefficient $\lambda \equiv 1$ so that $g_F$ coincides with $F$ in the sense $g_F(\xi, \xi) = F^2(\xi)$ for all $\xi \in TM$. $\square$

Observe that hypothesis of $C^2$ partial smoothness of the metric was not really used in the proof, we only used that the flows of the Killing vector fields are of class $C^1$, which is automatically fulfilled if the metric is $C^2$-partially smooth.

**Remark 7.2.** Smooth Riemannian manifolds with large groups of isometries have been studied thoroughly, see e.g. [30] for a survey of classical results. In particular, connected Riemannian manifolds with more than $\frac{1}{4} n(n-1) + 1$ Killing vector fields are classified as follows. Let $r$ be the dimension of the space of Killing vector fields. Then

1. If $r > \frac{n(n-1)}{2} + 1$ and $n \neq 4$, then $g$ has constant sectional curvature, see [61].
2. If $n = 4$ and $r > \frac{1}{2} n(n-1) + 2 = 8$, then $g$ also has constant sectional curvature, see [27].
3. If $n = 4$ and $r > \frac{1}{4} n(n-1) + 1 = 7$ then either $M$ is Kählerian with constant holomorphic sectional curvature (in this case, $r = 8$), or $M$ has constant sectional curvature, see [27, Theorem A’].

Note that although the cited references assume the Killing vector fields to be complete, the proofs work without this hypothesis, and that, locally, a space of constant sectional curvature has $\frac{n(n+1)}{2}$-dimensional space of Killing vector fields.
8. The Liouville Theorem for Minkowski Spaces and the Solution to a Problem by Matsumoto

One of the most famous theorems of Joseph Liouville states that any conformal transformation of a domain in \( \mathbb{R}^3 \) to another such domain is either the restriction of a similarity or the composition of a similarity with an inversion, it is, in other words, the restriction of a Möbius transformation. This result has been announced in 1850 in [36], and the proof appeared as a note in the fifth edition of Monge’s book *Application de l’analyse à la géométrie* [37]. It is well known that this Theorem also holds in \( \mathbb{R}^n \) for \( n \geq 3 \). By contrast, in dimension 2 the Cauchy-Riemann equations imply that a transformation is conformal if and only if it is either holomorphic or antiholomorphic.

Our next statement says that Liouville’s Theorem still holds in non euclidean Minkowski spaces. We have in fact a stronger result.

**Theorem 8.1.** Let \((V_1, F_1)\) and \((V_2, F_2)\) be two non-euclidean Minkowski spaces of the same dimension \( n \geq 2 \). If \( f : U_1 \to U_2 \) is a conformal map between two domains \( U_1 \subset V_1 \) and \( U_2 \subset V_2 \), then \((V_1, F_1)\) and \((V_2, F_2)\) are isometric and \( f \) is (the restriction of) a similarity, that is the composition of an isometry and a homothety \( x \mapsto \text{const} \cdot x \).

**Remark.** In the last sentence of [40], Matsumoto asked whether there exists two locally Minkowski spaces which are conformal to each other. The above theorem shows that the answer to this question is negative unless the metrics are euclidean or the conformal correspondence is a similarity.

**Proof.** We will first proof the theorem for \( n \geq 3 \). Fix a point \( x \in U_1 \subset V_1 \) and let \( y = f(x) \in U_2 \subset V_2 \) be the image point. Because \( f \) is a conformal map, we have \( \frac{1}{x(y)} \cdot df_x \) is an isometry from \((T_x V_1, F_1)\) to \((T_y V_2, F_2)\), but since a Minkowski space is isometric to its tangent space at any point it follows that \((V_1, F_1)\) and \((V_2, F_2)\) are isometric.

From now on, we assume that \( V_1 = V_2 = \mathbb{R}^n \) and \( F_1 = F_2 = F \) is an arbitrary non euclidean Minkowski norm. Changing coordinates if necessary, one may also assume that the Binet-Legendre scalar product \( g_F \) of \( F \) is the standard scalar product \( \langle , \rangle \) of \( \mathbb{R}^n \). It follows that \( f \) is a conformal map in the usual sense between two domains \( U, V \subset \mathbb{R}^n \).

By the classical Liouville’s Theorem, \( f \) is the restriction of a Möbius transformation, and such a map is known to be either a similarity or the composition of a similarity and an inversion. We thus only need to prove that an inversion cannot be a conformal map of some non euclidean Minkowski norm on \( \mathbb{R}^n \). Moreover, the map \( f \) is conformal on \( \mathbb{R}^n \) (with a point removed) for the Binet-Legendre metric and therefore it is also globally conformal for the Minkowski metric.

We now prove the last assertion by contradiction. Suppose that the standard inversion \( \varphi(x) = \frac{x}{|x|^2} \) is conformal for some Minkowski norm \( F \) in \( \mathbb{R}^n \). Recall that any point \( x \) on the unit sphere \( |x| = 1 \) is a fixed point of \( \varphi \) and the differential \( d\varphi_x \) at \( x \) is the reflexion across the hyperplane \( x^\perp \). In particular \( d\varphi_x(\xi) = \xi \) for any \( \xi \perp x \) and since \( d\varphi_x \) is conformal, it must be an isometry. This shows that the Minkowski norm \( F \) is invariant under the reflexion across any hyperplane \( x^\perp \). It is therefore \( O(n) \)-invariant and \( F \) must be homothetic to the standard euclidean norm. The theorem is proved for \( n \geq 3 \).

Let us now prove it for \( n = 2 \). We again consider \( \mathbb{R}^2 \) with a fixed Minkowski metric which we denote by \( F \), and assume that \( g_F \) is the standard flat metric. Let us use the conformal
structure to construct a family of parallel lines on $\mathbb{R}^2$. Take a point $x$ and consider the unit circle $S^1_x \subset T_x \mathbb{R}^2$ in the metric $g_F$. We take a connected component $I_{max}^0$ of the ‘maximal’ set

$$I_{max} = \{ \xi \in S_1(x) \mid F(\xi) = \max_{\eta \in S^1_x} F(\eta) \}.$$ 

The set $I_{max}^0$ cannot coincide with the whole $S_1(x)$ and is therefore a connected interval. Let $\xi \in S_1(x)$ be its midpoint (with respect to the metric on $S^1_x$ induced by $g_F$).

The vector $\xi$ is not always unique (the set $I_{max}^0$ can have more than one connected components, and every connected component has its own midpoint). We choose one of it.

Note that the construction of the vector $\xi$ is conformally invariant in the following sense: if we multiply $F$ at a point $x$ by a number $\lambda$, the vector $\xi$ is divided by $\lambda$, so the direction of this vector field remains the same.

Now let us extend the vector to all points of $\mathbb{R}^2$ by parallel translations, thus obtaining a vector field that we denote by $\xi$. Let $f : U_1 \rightarrow U_2$ be a conformal (i.e. holomorphic or antiholomorphic) mapping. Then, it sends the vector field $\xi$ to another vector field $\xi' = f_* (\xi)$ that satisfies the properties by construction:

1. $\xi'$ is a smooth vector field.
2. At every point, $\xi'$ field is the mid vector of a connected component of $I_{max}^0$.

Therefore the integral curves of $\xi'$ are parallel lines in $\mathbb{R}^2$. It is well known (and easy to check) that a holomorphic or antiholomorphic map that sends a family of parallel lines to a family of parallel lines is of the type $f(z) = az + b$ or $f(z) = az + b$ with $a, b \in \mathbb{C}, a \neq 0$. Thus $f$ is a similarity and the proof is complete. □

### 9. Conformally flat compact Finsler Manifolds

A Finsler manifold $(M, F)$ is **conformally flat**, if there is an atlas whose changes of coordinates are conformal diffeomorphisms between open sets in some Minkowski space. Assuming $M$ to be non Riemannian, it follows from Theorem 8.1, that these changes of coordinates are euclidean similarities. The manifold $M$ carries therefore a similarity structure. It turns out that compact manifolds with a similarity structure have been topologically classified by N. H. Kuiper and D. Fried: they are either Bieberbach manifolds (i.e. $\mathbb{R}^n / \Gamma$, where $\Gamma$ is some crystallographic group of $\mathbb{R}^n$), or they are Hopf-manifolds i.e. compact quotients of $\mathbb{R}^n \setminus \{0\} = S^{n-1} \times \mathbb{R}$ by a group $G$ which is a semi-direct product of an infinite cyclic group with a finite subgroup of $O(n + 1)$ see [22, 31, 58]. We thus conclude:

**Theorem 9.1.** Any partially smooth connected compact conformally flat non Riemannian Finsler manifold is either a Bieberbach manifolds or a Hopf manifolds. In particular, it is finitely covered either by a torus $T^n$ or by $S^{n-1} \times S^1$.

The structure of Riemannian conformally flat manifold is more complicated, see the discussion in [33, 41, 53].

### 10. Finsler spaces with a non trivial self-similarity

A $C^1$-map $f : (M, F) \rightarrow (M', F')$ is a **similarity** if there exists a constant $a > 0$ (called the **dilation constant**) such that $F(f(x), df_x(\xi)) = a \cdot F(x, \xi)$ for all $(x, \xi) \in TM$. It is an isometry if $a = 1$. 
Clearly a similarity satisfies $d_F(f(x), f(y)) = a \cdot d_F(x, y)$ for all $x, y \in M$ and it follows from Busemann-Mayer theorem that any $C^1$-map satisfying this condition is a similarity in the previous sense.

**Theorem 10.1.** Let $(M, F)$ be a forward complete connected $C^0$-Finsler manifold. If there exists a non isometric self-similarity $f : M \rightarrow M$ of class $C^1$, then $(M, F)$ is a Minkowski space.

In the special case of a quasireversible and $C^2$-partially smooth Finsler manifold, the proof reduces via the Binet-Legendre metric to a classical Riemannian argument, since the existence of a non trivial self-similarity in a complete $C^2$-Riemannian manifold easily implies that the sectional curvature of that manifold vanishes (see [35]). The following proof in the $C^0$-case is based on a blow up argument familiar in metric geometry.

**Proof.** We first show that the map $f$ is a bijection. The injectivity follows from the fact that $d(f(x), f(y)) = a \cdot d(x, y)$, for any $x, y$ and $a > 0$. To show that $f$ is surjective, we observe that $f(M) \subset M$ is open since $f$ is an immersion and $f(M) \subset M$ is closed since it is a forward complete set. Hence $f(M) = M$ and $f$ is thus bijective.

Replacing $f$ by $f^{-1}$ if necessary, one may assume that $a < 1$. We show that $f$ has a fixed point: pick an arbitrary point $x$ and consider the sequence $y_k = f^k(x)$, we have then

$$d(y_i, y_{i+1}) = d(f^i(x), f^{i+1}(x)) = a^i d(x, f(x)),$$

which implies that the sequence is forward Cauchy. This sequence has therefore a unique limit $x_0$ and by continuity of $f$ we have

$$f(x_0) = \lim_{j \to \infty} f(y_j) = \lim_{j \to \infty} y_{j+1} = x_0,$$

we found our fixed point $x_0$. We now consider the Binet-Legendre Riemannian metric $g_F$. By Proposition 6.3, the mapping $f$ is also a similarity for $g_F$. We claim:

**Lemma 10.2.** Let $(M, g)$ be a $C^0$ Riemannian manifold. Assume that there exists a map $f : M \rightarrow M$ such that $d(f(x), f(y)) = a \cdot d(x, y)$ for some constant $0 < a < 1$ where $d$ is the distance function corresponding to the Riemannian metric $g$. If $f$ has a fixed point, then $(M, g)$ is flat, i.e., every point of $M$ has a neighborhood that is isometric to a domain in $\mathbb{R}^n$ with the standard metric.

As said before, we prove this lemma by a blow up argument\(^4\). Let $x_0 \in M$ be the fixed point of $f$ and choose $R$ small enough so that the closed $d$-ball $\mathcal{B}_R(x_0)$ is compact. It suffices to show that the restriction of the metric $d$ to this ball is flat, since for every bounded neighborhood $U \subseteq M$ there exists $m$ such that $f^m(U) \subset B_R(x_0)$.

In order to do it, we construct a sequence of flat metrics $d_m$ on $B_R(x_0)$ such that it uniformly converges to the metric of $d$, in the sense that for every $x, y \in B_R(x_0)$ we have $d_m(x, y) \rightarrow d(x, y)$ uniformly as $m \rightarrow \infty$. Choosing a smaller radius $R$ if necessary, one may assume that some coordinates $x_1, \ldots, x_n$ are defined in some neighborhood of the ball $B_R(x_0)$. Assume also that the point $x_0$ has coordinates $(0, \ldots, 0)$ and that the metric $g$ is given by the identity matrix

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\(^3\)the manifold $M$ is of class $C^1$, the metric $F$ is $C^0$

\(^4\)The proof is elementary if the metric $g$ is $C^2$: set $\kappa(x) = \max |K(\pi)|$ where $\pi$ ranges through all $2$-planes in $T_x M$ and $K$ is the sectional curvature. For a similarity $f$ with dilation constant $a$ we have $\kappa(x) = a^{2m} \kappa(f^m(x))$ thus, if $a < 1$ and $\{f^m(x)\}$ converges, we have $\kappa(x) = 0$. 
at the point $x_0$. In this neighborhood, we consider the flat (constant) Riemannian metric $g_0 = dx_1^2 + \ldots + dx_n^2$. Both metrics $g$ and $g_0$ coincide at the point $x_0$. The distance in the metric $g$ is denoted by $d$ and that in the metric $g_0$ will be denoted by $d_0$. Likewise balls in the $d$-metric are denoted by $B_r(x)$ and balls in the $d_0$-metric will be denote by $B'_r(x)$.

We take $R'$ such that $B'_R(x_0) \subset B_R(x_0)$. For every $m \in \mathbb{N}$ we define a metric $d_m$ on $B'_R(x_0)$ by

$$d_m(x, y) = \frac{1}{a_m} d_0(f^m(x), f^m(y)).$$

Let us show that the sequence of metrics $d_m$ converges to the metric $d$. Since the metric $g$ is continuous, and since at the point $x_0$ the metric $g$ coincides with the metric $g_0$, for every $\varepsilon > 0$ there exists $r(\varepsilon)$ such that for every point $x \in B'_{3r}(x_0) \cup B_{3r}(x_0)$ and for every nonzero tangent vector $\xi \in T_xM$ we have

$$\frac{1}{1 + \varepsilon} \leq \frac{\sqrt{g(\xi, \xi)}}{\sqrt{g_0(\xi, \xi)}} \leq 1 + \varepsilon.$$

These inequalities immediately give the following estimates on the length of any curve $\gamma : [0, 1] \to B_{3r}(x_0)$:

$$\frac{1}{1 + \varepsilon} L_g(\gamma) \leq L_{g_0}(\gamma) \leq (1 + \varepsilon) L_g(\gamma).$$

Assuming $\varepsilon < \frac{1}{2}$, these estimates imply that the shortest path connecting two points in $B_r(x_0)$ stays in the ball $B'_{2r}(x_0)$, and symmetrically the shortest path connecting two points in $B'_r(x_0)$ stays in the ball $B_{2r}(x_0)$. We therefore have the following inequalities for any $x, y \in B'_{r}(x_0) \cap B_r(x_0)$:

$$\frac{1}{1 + \varepsilon} d(x, y) \leq d_0(x, y) \leq (1 + \varepsilon) d(x, y).$$

Now take two arbitrary points $x, y \in B_R(x_0)$. For sufficiently large $m$, the points $f^m(x)$ and $f^m(y)$ lie in $B_r(\varepsilon)(x_0)$. By definition, the distance between $f^m(x)$ and $f^m(y)$ is the length of a shortest curve. Since this curve lies in $B_{3r}(x_0)$, the inequalities above imply that

$$\frac{1}{1 + \varepsilon} d(f^m(x), f^m(y)) \leq d_0(f^m(x), f^m(y)) \leq (1 + \varepsilon) d(f^m(x), f^m(y)).$$

Dividing this inequality by $a^m$ and using the property $d(f^m(x), f^m(y)) = a^m \cdot d(x, y)$ together with the definition of $d_m$ we obtain

$$\frac{1}{1 + \varepsilon} d(x, y) \leq d_m(x, y) \leq (1 + \varepsilon) d(x, y).$$

Since for $x, y \in B_R(x_0)$ the function $d(x, y)$ is uniformly bounded by $2R$, the metrics $d_m$ uniformly converge to the metric $d$ as $m \to \infty$. Furthermore the metrics $d_m$ are clearly flat metrics: $B_R(x_0)$ equipped with such metric is isometric to a domain in the standard euclidean space $\mathbb{R}^n$.

Is it is well known that a uniform limit of flat metrics, it is itself flat. For the sake of completeness, we give a proof of this fact in our case. We may assume that $R \geq 3$, otherwise we divide the metric by a large constant. We will prove that the metric $d$ in the ball $B_1(x_0)$ is flat.
For any $m$, we choose an isometric embedding $\phi_m : (\overline{B}_R(x_0), d_m) \to \mathbb{R}^n$ such that $\phi_m(x_0) = 0$. Let us set $x_j(m) = \phi_m^{-1}(e_i) \in \overline{B}_R(x_0)$ where $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ is the standard orthonormal basis.

Since $\overline{B}_R(x_0)$ is compact, one can find a subsequence $(x_1(m_1), \ldots, x_n(m_i))$ converging to a tuple $(x_1, \ldots, x_n) \in B_1(x_0) \times \ldots \times B_1(x_0)$. We claim that the restriction of the sequence $\phi_m$ to $B_1(x_0)$ converges to a map $\phi : B_1(x_0) \to \mathbb{R}^n$ which is an isometry.

Indeed, for any $y \in \overline{B}_R(x_0)$ the point $\phi_m(y)$ is the unique point in $\mathbb{R}^n$ such that $\|\phi_m(y)\| = d_m(x_0, y)$ and $\|\phi_m(y) - e_j\| = d_m(x_j, y)$ for any $j = 1, \ldots, n$. Since the sequence $x_j(m_i)$ converges to $x_i$ and $d_m$ converges uniformly to $d$, the sequence $\{\phi_m(y)\}$ converges to the unique point $Y \in \mathbb{R}^n$ such that $\|Y\| = d(x_0, y)$ and $\|Y - e_j\| = d(x_j, y)$ for any $j = 1, \ldots, n$.

We denote by $\phi = \lim_{i \to \infty} \phi_m(y)$ the limiting map. This is an isometry since

$$d(y, y') = \lim_{i \to \infty} d_m(y, y) = \lim_{i \to \infty} \|\phi_m(y) - \phi_m(y')\| = \|\phi(y) - \phi(y')\|.$$ 

The proof of Lemma 10.2 is complete.

The lemma just proved tells us that a neighborhood of the point $x_0 \in M$ equipped with the metric $g_{F}$ is isometric to a domain in the standard euclidean space. The next lemma (which provides the second step in the proof of Theorem 10.1) says that the metric $F$ is isometric to a Minkowskian metric in the same neighborhood.

**Lemma 10.3.** Let $F$ be a Finsler metric on a domain $U \subseteq \mathbb{R}^n$ and let $f : U \to U$ be a map which is a self-similarity with dilation constant $a < 1$ for both the Finsler metric $F$ and the standard euclidean metric $g$ on $\mathbb{R}^n$. If $f$ has a fixed point, then $F$ (the restriction of) a Minkowskian metric.

Note that in the lemma we neither suppose that $F$ is complete nor that it is quasi-reversible.

To prove this lemma, assume that $U$ contains the origin and that 0 is the fixed point. Then $f$ is the restriction of a linear similarity (still denoted by $f : \mathbb{R}^n \to \mathbb{R}^n$) and has thus the form $f(x) = a \cdot Q(x)$, for some orthogonal transformation $Q \in O(n)$. By hypothesis, we have

$$F(f(x), d_f(x, \xi)) = f^*F(x, \xi) = a \cdot F(x, \xi)$$

for any $(x, \xi) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Because $d_f(x, \xi) = a \cdot Q(\xi)$, we have

$$F(f(x), d_f(x, \xi)) = F(f(x), a \cdot Q(\xi)) = a \cdot F(f(x), Q(\xi)).$$

It follows from the two previous equalities that

$$F(x, \xi) = F(f(x), Q(\xi)) = a^n F(f^n(x), Q^n(\xi))$$

for any integer $n$. Fix an arbitrary point $x \in \mathbb{R}^n$ and choose a sequence $\{n_j\} \subset \mathbb{N}$ such that $Q^{n_j} \longrightarrow$ id in $O(n)$ as $j \to \infty$, we then have

$$F(x, \xi) = \lim_{k \to \infty} F(f^{n_k}(x), Q^{n_k}(\xi)) = F(0, \xi).$$

This shows that $F(x, \xi)$ is independent of $x$, i.e., it is a Minkowskian metric. The second lemma is proved.

We can now conclude the proof of Theorem 10.1. By Lemmas 10.2 and 10.3 the metric $F$ is a Minkowskian metric in a certain neighborhood $U$ of $x_0$. Since for bounded set $U' \subset M$ there exists $m$ such that $f^m(U') \subset U$, the metric $F$ is a Minkowskian metric in some neighborhood of every point. Clearly, $M$ is simply connected. Indeed, for every loop $\gamma$ there exists $m$ such that $f^m(\gamma)$ lies in a small neighborhood of $x_0$ and is therefore contractible. Because $f^m$ is
a homeomorphism on its image, the loop $\gamma$ is contractible as well. We established that the manifold $(M, F)$ is forward complete, simply connected and locally isometric to a Minkowski space; it is therefore globally isometric to a Minkowski space. 

Remark. In the case of smooth Finsler manifolds, Theorem 10.1 is known. A first proof was given in [26], however R. L. Lovas, and J. Szilasi found a gap in the argument and gave a new proof in [38].

11. Conformal transformations of partially-smooth Finsler metrics

In this section, we classify all conformal transformations of an arbitrary Finsler manifold.

Definition 11.1. A set $S \subseteq \text{Diff}(M)$ of transformations of the Finsler manifold $(M, F)$ is said to be essentially conformal if any $f \in S$ is a conformal transformation of $(M, F)$, but there is no conformal deformation $\lambda \cdot F$ of $F$ for which $S$ is a set of isometries. The set $S$ of conformal transformations of $M$ is termed inessential if it is not essentially conformal.

Theorem 11.2. Let $(M, F)$ be a connected $C^\infty$ partially smooth Finsler manifold, then the following conditions are equivalent.

a) There exists an essentially conformal diffeomorphism $f$ of $(M, F)$.

b) The group of conformal diffeomorphism of $(M, F)$ is essential.

c) $(M, F)$ is conformally equivalent to a Minkowski space $(\mathbb{R}^n, F)$ or to the canonical Riemannian sphere $(S^n, g_0)$.

The logic of the proof is the following: Using the Binet-Legendre construction, we reduce this theorem to the Alekseevsky-Ferrand-Schoen solution to the Riemannian Lichnerowicz-Obata conjecture (see e.g. [1, 21, 52]). We then need to prove that the Finsler metric is conformally Minkowski in the non compact case and Riemannian in the compact case. The main ideas are similar to those in [43], but here we do not work with conformal vector fields.

Remark. Note that it is obvious that $(a) \Rightarrow (b)$, but $(b) \Rightarrow (a)$ is not a priori a trivial fact because we could conceive of a Finsler manifold $(M, F)$ for which every conformal diffeomorphism would be inessential, but for which no conformal deformation $\lambda \cdot F$ of the metric would be simultaneously invariant under all conformal diffeomorphism of $(M, F)$.

Proof. As just observed, $(a)$ trivially implies $(b)$. It is also clear that $(c) \Rightarrow (a)$, since any linear contraction of a Minkowski space and any non isometric Möbius transformation of the sphere are examples of essential conformal transformations. We thus only need to prove $(b) \Rightarrow (c)$.

We know from Theorem 6.1 (d) that $f$ is also a conformal transformation for the associated Binet-Legendre metric, and $f : (M, g_F) \rightarrow (M, g_F)$ must be essential otherwise $f$ would be an inessential conformal transformation of $(M, F)$.

It follows that the full group of conformal transformations of $(M, g_F)$ is essential and by the Alekseevsky-Ferrand-Schoen Theorem, the manifold $(M, g_F)$ is either conformally equivalent to the euclidean space $\mathbb{R}^n$ or to the canonical Riemannian sphere $S^n$. Changing the Finsler metric $F$ and correspondingly the Binet-Legendre metric $g_F$ within the same conformal class, we will assume that $(M, g_F)$ is in fact isometric to $\mathbb{R}^n$ or $S^n$.

If $(M, g_F)$ is isometric to the euclidean space $(\mathbb{R}^n, g_0)$, then $f$ is a conformal transformation of $\mathbb{R}^n$ and it is therefore a map of the type $f(x) = a \cdot Q(X) + b$ with $Q \in O(n)$, $a > 0$ and
We now assume that $(M, g_F)$ is isometric to the canonical Riemannian sphere $S^n$ and $f : S^n \to S^n$ is a non isometric conformal map. It is well known that such a map has exactly either one or two fixed points.

**Case 1.** $f$ has two fixed points.

Using a stereographic projection, we identify $S^n$ with $\mathbb{R}^n \cup \{\infty\}$ and we may assume that $f(\infty) = \infty$ and $f(0) = 0$. Thus $f$ induces a conformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ which is of the type $f(x) = a \cdot Q(x)$, with $Q \in O(n)$. If $a = 1$, then $f$ is an isometry of the Finsler metric $F$; i.e. $f$ is inessential, a case that we excluded.

So we have $a \neq 1$. Consider the Finsler metric $F^+ = \rho^{-1} \cdot F$ on $\mathbb{R}^n$, its Binet-Legendre metric is the flat metric $g_0 = \rho^{-2} \cdot g_1$. The map $f(x) = a \cdot Q(x)$ is a non-isometric similarity for both the Binet-Legendre metric $g_{F^+} = g_0$ and the Finsler metric $F^+$ and we conclude from Lemma 10.3 that $F^+$ is a Minkowski metric.

Let $\varphi(x) = \frac{1}{\rho(x)}$ be the standard inversion in $\mathbb{R}^n \cup \{\infty\}$. This map exchanges the two fixed points of $f$ and the previous argument shows that $F^- = \rho^{-1} \cdot \varphi^* F$ is also a Minkowski metric. Since $\varphi$ is conformal for the Binet-Legendre metrics of $F^+$ and $F^-$, the Liouville Theorem 8.1 implies that $F^+ = \rho^{-1} \cdot F$ is an Euclidian metric on $\mathbb{R}^n$. Hence $F(\xi) = \sqrt{g_F(\xi, \xi)}$ is the standard metric on $S^n$.

**Case 2.** $f$ has exactly one fixed point.

We again identify $S^n$ with $\mathbb{R}^n \cup \{\infty\}$ and assume that $f(\infty) = \infty$. Thus $f$ induces a conformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ which is of the type $f(x) = a \cdot Q(X) + b$. Since $f$ has no fixed point in $\mathbb{R}^n$, we must have $b \neq 0$ and $a = 1$. Using Lemma 11.3 below and conjugating $f$ with a translation if necessary, we may assume that $b$ is an eigenvector of $Q$ with eigenvalue $+1$, i.e. $Q(b) = b$.

We are thus in the following situation: our map $f$ is $f(x) = Q(x) + b$ where $Q(b) = b \neq 0$ and the composition $\tilde{f} = \varphi \circ f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ is conformal for the standard metric, where $\varphi$ is the inversion. We have

$$\tilde{f}(x) = \frac{Q(x) + b}{|Q(x) + b|^2},$$

and

$$d\tilde{f}(\xi) = \frac{Q(\xi)}{|Q(x) + b|^2} - 2 \frac{Q(\xi), Q(x) + b}{|Q(x) + b|^2} Q(x) + b
= \frac{1}{|f(x)|^2} \left( Q(\xi) - 2 \frac{Q(\xi), f(x)}{|f(x)|^2} \cdot f(x) \right)
= \frac{1}{|f(x)|^2} \cdot (S_f(x) \circ Q)(\xi),$$
where $S_{f(x)}$ is the linear reflection across the hyperplane $f(x)\perp$. Since $Q(b) = b$, we have $f^n(x) = Q^n(x) + n \cdot b$, and the same calculation gives us

$$d(\tilde{f}_n)_x(\xi) = \frac{1}{|f^n(x)|^2} \cdot (S_{f(x)} \circ Q^n)(\xi),$$

for any $n \in \mathbb{N}$, where $\tilde{f}_n = \varphi \circ f^n$. The map $\tilde{f}_n$ is conformal for the Finsler metric $F$, we thus have

$$F(\tilde{f}_n(x), d(\tilde{f}_n)_x(\xi)) = \lambda_n(x) \cdot F(x, \xi)$$

for some function $\lambda_n$, therefore

$$F(x, \xi) = \frac{1}{\lambda_n(x)} \cdot F(\tilde{f}_n(x), d(\tilde{f}_n)_x(\xi)) = \mu_n(x) \cdot F(\tilde{f}_n(x), (S_{f(x)} \circ Q^n)(\xi)),$$

where $\mu_n(x) = \frac{1}{|f^n(x)|^2 \lambda_n(x)}$. Observe that $S_{f(x)}$ only depends on the direction of the vector $f^n(x)$, i.e. $S_{f(x)} = S_{f^n(x)}$, and since

$$\lim_{n \to \infty} \frac{f^n(x)}{|f^n(x)|} = \lim_{n \to \infty} \frac{Q^n(x) + n \cdot b}{|Q^n(x) + n \cdot b|} = \frac{b}{|b|},$$

we have

$$\lim_{n \to \infty} S_{f^n(x)} = S_b.$$

By the compactness of the group $O(n)$, one may find a sequence $\{n_j\} \subset \mathbb{N}$ such that $Q^{n_j} \to I$, we thus have

$$\lim_{j \to \infty} |f^{n_j}(x)|^2 \cdot d(\tilde{f}_j)_x(\xi) = \lim_{j \to \infty} (S_{f(x)} \circ Q^{n_j})(\xi) = S_b(\xi).$$

Now $\mu_n(x)$ is a bounded sequence and we may choose the subsequence $\{n_j\}$ such that $\mu_{n_j}(x)$ converges to some number $\mu(x)$. The previous considerations imply that

$$F(x, \xi) = \lim_{j \to \infty} \mu_{n_j}(x) \cdot F(\tilde{f}_{n_j}(x), (S_{f(x)} \circ Q^{n_j})(\xi)) = \mu(x) \cdot F(0, S_b(\xi))$$

for any $(x, \xi)$. It follows that $\frac{1}{\mu} F$ is a Minkowski metric.

Since the inversion $\varphi$ is conformal for the Minkowski metric $\frac{1}{\mu} F$, the Liouville Theorem 8.1 implies that $F$ is in fact a Riemannian metric and thus $F(\xi) = \sqrt{g_F(\xi, \xi)}$ is the standard metric on $\mathbb{S}^n$. □

**Lemma 11.3.** Suppose that $f(x) = Q(x) + b$ is a fixed point free transformation of $\mathbb{R}^n$ with $Q \in O(n)$, then $f$ can be decomposed as

$$f = T \circ f_1 \circ T^{-1},$$

where $T$ is a translation and $f_1(x) = Q(x) + b_1$ for some non zero vector $b_1$ such that $Q(b_1) = b_1$.

**Proof.** Let us denote by $E = \{v \in \mathbb{R}^n \mid Q(v) = v\}$. The decomposition $\mathbb{R}^n = E \oplus E^{\perp}$ is $Q$-invariant and we write $b = b_1 + b_2$ with $b_1 \in E$ and $b_2 \in E^{\perp}$. The transformation $f_2(x) = Q(x) + b_2$ has a fixed point $v_0 \in E^{\perp}$. Indeed, 1 is not an eigenvalue of the restriction $Q|_{E^{\perp}}$, therefore the equation

$$(Q - I)(v) = -b_2, \quad v \in E^{\perp}$$
has a solution \(v_0\), and we have \(Q(v_0) + b_2 = v_0\). Let us denote by \(T\) the translation \(T(x) = x + v_0\), we then have

\[
(T^{-1} \circ f \circ T)(x) = (Q(x + v_0) + b) - v_0
\]

\[
= Q(x) + (Q - I)(v_0) + (b_1 + b_2)
\]

\[
= Q(x) + b_1.
\]

It is clear that \(Q(b_1) = b_1\) since \(b_1 \in E\), and \(b_1 \neq 0\), otherwise \(f(x) = Q(x) + b_2\) would have a fixed point.

\[\square\]

12. On Berwald spaces

A \(C^k\)-Berwald space is a Finsler manifold \((M, F)\) which admits a torsion free linear connexion \(\nabla\) which is compatible with the Finsler metric. More precisely, one says that a linear connection \(\nabla\) on a smooth manifold is of class \(C^k\) if its Christoffel symbols in any coordinate system are of class \(C^k\). Recall that the parallel transport associated to a \(C^1\)-path \(\gamma: [0, 1] \to M\) from \(x = \gamma(0)\) to \(y = \gamma(1)\) is the linear map \(P_\gamma : T_x M \to T_y M\) defined as \(P_\gamma(\xi_t) = \xi_1 \in T_y M\) where \(t \to \xi_t\) is the solution to the equation \(\dot{\xi}_t = \nabla_{\dot{\gamma}(t)}\xi_t\) = 0 such that \(\xi_0 = \xi \in T_x M\). Observe that, since this ordinary differential equation is linear in \(\xi_t\), there is a unique solution for any \(t \in [0, 1]\) even when the connexion \(\nabla\) is only of class \(C^0\) (see [24]).

**Definition 12.1.** A Finsler metric \(F\) on a manifold \(M\) is said to be a \(C^k\)-Berwald metric if there exists a \(C^k\)-smooth torsion free linear connection \(\nabla\) (called an associated connection) on \(M\) whose associated parallel transport preserves the Lagrangian \(F\). That is, if \(\gamma: [0, 1] \to M\) is a smooth path connecting the point \(x = \gamma(0)\) to \(y = \gamma(1)\) and \(P_\gamma : T_x M \to T_y M\) is the associated \(\nabla\)-parallel transport, then

\[F(y, P_\gamma(\xi)) = F(x, \xi)\]

for any \(\xi \in T_x M\).

Observe that if an associated connection \(\nabla\) of a Berwald metric \(F\) is of class \(C^k\), then the metric \(F\) is \(C^k\)-partially smooth.

Note that the definition given here differs (and is more general) from that given in [4], but both definitions are equivalent for \(C^2\) and strongly convex Finsler metrics as is follows from [16, Proposition 4.3.3].

In 1981, Z.I. Szabó proved that for a smooth and strongly convex Berwald metric, there exists an associated connexion which is the Levi-Civita of some Riemannian metric on \(M\). Later, other proofs that do not require strict convexity were given in [42, 59]. Our next result, whose proof is very simple, extends Szabó’s theorem to the case of merely continuous Finsler metric.

**Theorem 12.2.** Let \((M, F)\) be a \(C^0\)-Berwald Finsler manifold. If \(\nabla\) is an associated connection, then the parallel transport associated to the connexion \(\nabla\) preserves the Binet-Legendre metric \(g_F\).

**Proof.** For any smooth path \(\gamma: [0, 1] \to M\), the parallel transport \(P_\gamma : T_x M \to T_y M\) is a linear map that sends the unit ball of \(F\) at \(x = \gamma(0)\) to the unit ball of \(F\) at \(y = \gamma(1)\).

By Proposition 4.4(b), the Binet-Legendre ellipsoid at \(x\) is sent by \(P_\gamma\) to the Binet-Legendre ellipsoid at \(y\) implying that the parallel transport preserves the Binet-Legendre metric \(g_F\) as we claim.

\[\square\]
Remark. (A) The theorem implies the following extension of Szabó’s theorem: Any partially $C^1$-Berwald metric has a unique associated linear connection $\nabla$ and this connection is the Levi-Civita connection of the Binet-Legendre metric $g_F$.

(B) One may in fact redefine a partially smooth Berwald metric as a Finsler metric for which the Levi-Civita connection of the Binet-Legendre metric preserves $F$.

(C) Observe that a Finsler manifold $(M, F)$ is flat (i.e. locally Minkowski) if and only if it is Berwald and $g_F$ is a flat Riemannian metric.

(D) It is now easy to produce examples of non Berwald metrics for which all tangent spaces $T_xM$ are isometric as Minkowski spaces (such Finsler metrics are called monochromatic in [3, §3.3]). Take a non euclidean Minkowski metric $F_0$ on $\mathbb{R}^n$ and let $A$ be a smooth field of endomorphisms such that for every point $x$ the endomorphism $A_x$ is an orthogonal transformation for the Binet-Legendre metric: $A_x \in O(\mathbb{R}^n, g_F)$. Let $\tilde{F}(x, \xi) = F_0(A_x(\xi))$, by construction $F$ and $\tilde{F}$ have the same Binet-Legendre metric. In particular $g_F$ is flat, and all tangent spaces are isometric to $F_0$, but $\tilde{F}$ is Berwald if and only if $A$ is constant.

(E) One can describe all partially smooth Berwaldian spaces by the following construction. Choose an arbitrary smooth Riemannian metric $g$ on $M$ and choose an arbitrary Minkowski norm in the tangent space at some fixed point $q$ that is invariant with respect to the holonomy group of $g$. Now extend this norm to all other tangent spaces by parallel translation with respect to the Levi-Civita connection of $g$. Since the norm is invariant with respect to the holonomy group, the extension does not depend on the choice of the curve connecting an arbitrary point to $q$, and is a partially smooth Berwaldian Finsler metric.

We see that if the holonomy group of $g_F$ acts transitively on the unit sphere in some tangent space, then the Finsler metric $F$ is actually Riemannian. When the holonomy group is not transitive, we have the following result.

**Proposition 12.3.** Let $F$ be a $C^2$ partially smooth nonriemannian Berwald metric on a connected manifold $M$. Then, either there exists another Riemannian metric $h$ which is affinely equivalent to $g_F$ but not proportional to $g_F$, or the metric $(M, g_F)$ is symmetric of rank $\geq 2$, or both.

Recall that a Riemannian symmetric space $(M, g)$ is said to be of rank $k$ if every point belongs to a subspace $E^k \subset M$ which is isometric to the euclidean space $\mathbb{R}^k$.

**Remark.** Recall that by de Rham’s splitting Theorem [19], the existence of $h$ such that it is not proportional to $g_F$, but is affine equivalent to $g_F$, implies that $(M, g_F)$ is locally decomposable, in the sense that every point of it has a neighborhood $U$ that is isometric to the direct product of two Riemannian manifolds of positive dimensions. If in addition $(M, g_F)$ is complete, the universal cover of $(M, g_F)$ is the direct product of two complete Riemannian manifolds of positive dimensions.

**Proof.** We essentially repeat the argumentation of [42, 59, 56]. Fix a point $q \in M$. For every smooth loop $\gamma(t), 0 \leq t \leq 1$ such that $\gamma(0) = \gamma(1) = q$, we denote by $P_{\gamma} : T_qM \to T_qM$ the parallel transport along that loop with respect to the Levi-Civita connection of $g$. The set

$$H_q = \{P_{\gamma} \mid \gamma : [0, 1] \to M \text{ smooth, } \gamma(0) = \gamma(1) = q\}$$

is a subgroup of the group of the orthogonal transformations of $(T_qM, g_F)$. Moreover, it is well known (see for example, [8, 54]), that at least one of the following conditions holds:

1. $H_q$ acts transitively on $S_1 = \{\xi \in T_qM \mid g(\xi, \xi) = 1\}$,
(2) the metric $g_F$ is symmetric of rank $\geq 2$,
(3) there exists another Riemannian metric $h$ such that it is non proportional to $g_F$, but is affinely equivalent to $g_F$.

In the first case, the ratio $F(\xi)/\sqrt{g_F(\xi, \xi)}$ is a constant function on the sphere $T_qM \setminus \{0\}$ implying that the metric $F$ is Riemannian, which is contrary to our hypothesis. Thus either the second or the third case hold and the Proposition is proved. \qed

13. On (locally) symmetric Finsler spaces

**Definition 13.1.** The Finsler manifold $(M, F)$ is called locally symmetric, if for every point $x \in M$ there exists $r = r(x) > 0$ and an isometry $\tilde{I}_x : B_r(x) \to B_r(x)$ (called the reflection at $x$) such that $\tilde{I}_x(x) = x$ and $d_x(\tilde{I}_x) = -\mathrm{id} : T_xM \to T_xM$. The largest $r(x)$ satisfying this condition is called the symmetry radius at $x$. The manifold $(M, F)$ is called globally symmetric if the reflection $\tilde{I}_x$ can be extended to a global isometry: $\tilde{I}_x : M \to M$.

**Theorem 13.2.** Let $(M, F)$ be a $C^2$-smooth Finsler manifold. If $(M, F)$ is locally symmetric, then $F$ is $C^\infty$-Berwald.\(^5\)

**Remark.** This theorem answers positively a conjecture stated in [17], where it has been proved for globally symmetric spaces, see also [13, §49] and [23, 29].

**Proof.** We will first prove the Theorem under the additional assumption that the metric $F$ is strongly convex. By Proposition 6.3, every local isometry for the Finsler metric $F$ is also an isometry for the Binet-Legendre metric $g_F$. It follows that $(M, g_F)$ is a Riemannian locally symmetric space.

In what follows, it will be convenient to use tilde-notation for the “Finsler” objects, and the untilded notation for the analogous objects for the Binet-Legendre metric $g_F$ (for example $B_r(x)$ will denote the $r-$ball in $g_F$, and $\tilde{B}_r(x)$ the $r-$ball in $F$; $\gamma(t)$ will denote $g_F$-geodesic and $\tilde{\gamma}(t)$ will denote $F$-geodesics). Note that a locally symmetric space is evidently reversible, so that the distance function in $F$ is symmetric, and if $t \mapsto \tilde{\gamma}(t)$ is a geodesic parametrized by arclength, the reversed curve $t \mapsto \tilde{\gamma}(-t)$ is also a geodesic parametrized by arclength.

It is known that a locally symmetric Riemannian manifold is locally isometric to a globally symmetric space [25, theorem 5.1] and is therefore real analytic. Then, for sufficiently small neighborhood $W \subset M$ and for every $x \in W$, the $g_F$-reflection $I_x$ is defined globally on $W$.

For every $x \in W$, there is also the reflection $\tilde{I}_x : \tilde{B}_r(x) \to \tilde{B}_r(x)$ for the Finsler metric. By Proposition 6.3, the Finsler reflection $\tilde{I}_x$ coincides with the restriction of the Riemannian reflection $I_x$ on $\tilde{B}_r(x) \cap W$. We do not know\(^6\) a priori whether $\tilde{I}_x$ is an $F-$isometry in the whole ball $B_{\rho}(x)$.

---

\(^5\)According to Definition 12.1, it means that the associated connexion $\nabla$ is $C^\infty$-smooth, but this does not imply that the metric $F$ itself is $C^\infty$.

\(^6\)In [7, 14] a different, less general, definition of locally symmetric Finsler manifolds was given: it was explicitly assumed that the radius of symmetry $\tilde{r}(x)$ is locally bounded below. Under this assumption, $I_x$ coincides with $\tilde{I}_x$ in the whole $\tilde{B}_r$, where $r$ can be universally chosen for all points $x$ of a sufficiently small neighborhood $W$. From Corollary 13.3 it follows, that every locally symmetric space in our definition is also a locally symmetric in the definition of [7, 14].
Claim. For every sufficiently small $g_F$-geodesically convex open set $W \subset M$ and for every $F$-geodesic $\tilde{\gamma}(t) : [-\tilde{\varepsilon}, \tilde{\varepsilon}] \to W$ parameterized by arclength, we have $I_{\tilde{\gamma}(0)}(\tilde{\gamma}(t)) = \tilde{\gamma}(-t)$ for all $t \in [-\varepsilon, \varepsilon]$.

Recall that $W$ is $g_F$-geodesically convex, if every pair of points in $W$ can be connected by a unique minimal $g_F$-geodesic and that geodesic lies in $W$. To prove the Claim, we take a $F$-geodesic $\tilde{\gamma} : [-\tilde{\varepsilon}, \tilde{\varepsilon}] \to W$, set $x = \tilde{\gamma}(0) \in W$, consider the $g_F$-reflection $I_x$ and the number \begin{equation}
 r_0(\tilde{\gamma}, x) = \text{sup}\{r' \in [0, \tilde{\varepsilon}] \mid I_x(\tilde{\gamma}(t)) = \tilde{\gamma}(-t) \text{ for all } t \in [-r', r']\}.
\end{equation}

Since the metric $F$ is strongly convex, there is a unique $F$-geodesic with any given initial vector. Then, because $I_x \equiv I_x$ in a small neighborhood of $x$, we have $r_0(\tilde{\gamma}, x) > 0$. We want to prove that $r_0(\tilde{\gamma}, x) = \tilde{\varepsilon}$. Let us assume that $r_0(\tilde{\gamma}, x) < \tilde{\varepsilon}$ and derive a contradiction. Indeed, set $x_+ = \tilde{\gamma}(r_0)$ and $x_- = \tilde{\gamma}(-r_0)$ and consider (the analytical continuation of) the $g_F$-reflections $I_{x_+}, I_x, I_{x_-}$. Consider $I_{x_-} \circ I_x \circ I_{x_+}$. It is again a $g_F$-isometry. Let us show that it coincides with $I_x$. In order to do this, we consider the $g_F$-geodesic $\gamma(t)$ containing $x_+ = \tilde{\gamma}(r_0)$ and $x_- = \tilde{\gamma}(-r_0)$. Reparameterizing this geodesic affinely if necessary, we may assume without loss of generality that $\gamma(1) = x_+$ and $\gamma(-1) = x_-$. Since the neighborhood $W$ is sufficiently small, we may assume that $\gamma$ is defined at least on $[-2, 2]$. Since $I_x(x_+) = x_- \in W$ and $I_x(x_-) = x_+ \in W$, we have that $I_x(\gamma)$ is a shortest $g_F$-geodesic connecting $x_+$ to $x_-$. By convexity of $W$, we must have $I_x(\gamma) \subset W$ and $I_x(\gamma(t)) = \gamma(-t)$. In particular, $I_x(\gamma(0)) = \gamma(0)$. By uniqueness of the fixed point of $I_x$ in a geodesically convex region, it follows that $\gamma(0) = x$. Now, \[ I_{x_+}(x) = \gamma(2), I_x(\gamma(2)) = \gamma(-2), \text{ and } I_{x_-}(\gamma(-2)) = \gamma(0) = x \]
This implies $I_{x_-} \circ I_x \circ I_{x_+}(x) = x = I_x(x)$. We next show that \[ d_x(I_{x_-} \circ I_x \circ I_{x_+}) = -\text{id}. \]
Choose a vector $\xi \in T_x M$ and extend it as parallel vector field along the geodesic $\gamma$. Since the reflection $I_{x_-}$ leaves $\gamma$ invariant and satisfies $d_{\xi_{x_-}}(\xi_{x_-}) = -\xi_{x_-}$, and since an isometry preserves parallel vector fields, we have $(I_{x_-})_*(\xi) = -\xi$ at every point of $\gamma$. The same holds for the reflections $I_x$ and $I_{x_+}$, therefore $(I_{x_-} \circ I_x \circ I_{x_+})_*(\xi) = -\xi$ (for arbitrary $\xi \in T_x M$). It follows that $d_x(I_{x_-} \circ I_x \circ I_{x_+}) = -\text{id} = d_x I_x$ and therefore $I_{x_-} \circ I_x \circ I_{x_+} = I_x$. Now, for $\delta > 0$ small enough, the mappings $I_{x_-}$ and $I_{x_+}$ are $F$-isometries in the $F$-balls $\tilde{B}_\delta(x_-)$ and $\tilde{B}_\delta(x_+)$, respectively, see figure 5. Using again the uniqueness of an $F$-geodesic with prescribed given initial vector, we see that the mapping $I_{x_-} \circ I_x \circ I_{x_+}$ sends the $F$-geodesic segment $\tilde{\gamma}|_{[r_0, r_0+\delta]}$ to the $F$-geodesic segment $\tilde{\gamma}|_{[-r_0-\delta, -r_0]}$. Replacing the isometry $I_{x_-} \circ I_x \circ I_{x_+}$ by the isometry $I_{x_-} \circ I_x \circ I_{x_-}$ in the previous argument, we obtain that $I_{x_-} \circ I_x \circ I_{x_-}$ sends the $F$-geodesic segment $\tilde{\gamma}|_{[-r_0-\delta, -r_0]}$ to the $F$-geodesic segment $\tilde{\gamma}|_{[r_0, r_0+\delta]}$. Since $I_{x_-} \circ I_x \circ I_{x_-} = I_x = I_{x_+} \circ I_x \circ I_{x_-}$, and since a locally symmetric Finsler metric is reversible, the isometry $I_x$ has the property $I_x(\gamma(t)) = \gamma(-t)$ for all $t \in [-r_0 - \delta, r_0 + \delta]$. This gives us a contradiction with (13.1) that proves the Claim.

**Figure 5.** The geodesics $\gamma$, $\tilde{\gamma}$ and the balls $\tilde{B}_\delta(x_-)$, $\tilde{B}_\delta(x_+)$
Let us now show that the metrics \( g_F \) and \( F \) are affinely equivalent in the sense of [16, p. 74], that is, for every arclength parameterised \( F \)-geodesic \( \tilde{\gamma} \) there exists a nonzero constant \( c \) such that \( \tilde{\gamma}(c \cdot t) \) is an arclength parameterised \( g_F \)-geodesic. We already saw that, for a short \( F \)-geodesic segment, the \( g_F \)-geodesic segment with same endpoints has also the same midpoint. Let us repeat the exact argument. Fix a sufficiently small \( g_F \)-geodesically convex set \( W \subset M \) and take a \( F \)-geodesic \( \tilde{\gamma} : [-\tilde{\varepsilon}, \tilde{\varepsilon}] \to W \). Let \( \gamma : [-\varepsilon, \varepsilon] \to W \) be the unique shortest \( g_F \)-geodesic such that \( \gamma(-\varepsilon) = \tilde{\gamma}(-\tilde{\varepsilon}) \) and \( \gamma(\varepsilon) = \tilde{\gamma}(\tilde{\varepsilon}) \). We assume that both geodesics are parametrised by their arclength in the metric \( F \) and \( g_F \) respectively. Let \( x = \tilde{\gamma}(0) \) be the midpoint of \( \gamma \) and let \( I_x \) be the \( g_F \) reflexion centered at \( x \). Using the previously proved claim, we find that

\[
I_x(\gamma(-\varepsilon)) = I_x(\gamma(\varepsilon)) = \tilde{\gamma}(\tilde{\varepsilon}) = \gamma(\varepsilon)
\]

and likewise \( I_x(\gamma(\varepsilon)) = \gamma(\varepsilon) \). By convexity of \( W \), we must have \( I_x(\gamma) \subset W \) and \( I_x(\gamma(t)) = \gamma(-t) \) for all \( t \in [-\varepsilon, \varepsilon] \). In particular \( I_x(\gamma(0)) = \gamma(0) \). By uniqueness of the fixed point of \( I_x \), it follows that \( \gamma(0) = x = \tilde{\gamma}(0) \). Thus, for every \( F \)-geodesic segment \( \tilde{\gamma} \) in \( W \), its middle point coincides with the middle point of the unique minimal \( g_F \)-geodesic segment with the same ends.

Replacing the geodesic segment \( \tilde{\gamma} | [-\varepsilon, \varepsilon] \) by \( \tilde{\gamma} | [0, \varepsilon] \) or by \( \tilde{\gamma} | [0, \tilde{\varepsilon}] \), we also have \( \gamma(-\frac{1}{2}\varepsilon) = \tilde{\gamma}(-\frac{1}{2}\tilde{\varepsilon}) \) and \( \gamma(\frac{1}{2}\varepsilon) = \tilde{\gamma}(\frac{1}{2}\tilde{\varepsilon}) \). Iterating this procedure, we obtain that \( \gamma(s \cdot \varepsilon) = \tilde{\gamma}(s \cdot \tilde{\varepsilon}) \) for all \( s \) in a dense subset of \([-1, 1] \). This implies that the geodesic segments \( \gamma \) and \( \tilde{\gamma} \) coincide after the affine reparameterization \( t \mapsto \frac{\tilde{\varepsilon}}{\varepsilon} t \). By [16, page 74], we obtain that \( F \) is Berwald whose associated connection is the Levi-Civita connection of \( g_F \). Thus, Theorem 13.2 is proved for strongly convex Finsler metrics.

In order to prove complete the proof for an arbitrary Finsler metrics \( F \), we consider the Finsler metric \( F_\alpha \) given by

\[
F_\alpha(\xi) = \sqrt{F(\xi)^2 + \alpha \cdot g_F(\xi, \xi)},
\]

where \( \alpha > 0 \) is some parameter. The metric \( F_\alpha \) is \( C^2 \)-smooth and strictly convex. The reflections \( I_x \) are evidently isometries of \( F_\alpha \), so that \( F_\alpha \) is locally symmetric. We then just proved that \( F_\alpha \) is Berwald and its associated connection is the Levi-Civita connection of \( g_{F_\alpha} \). Since the reflections \( I_x \) are evidently isometries of \( g_{F_\alpha} \), the metrics \( g_{F_\alpha} \) is affinely equivalent to \( g_F \) for any \( \alpha > 0 \). Then, for every \( \alpha > 0 \), the function \( F_\alpha \) is preserved by the parallel transport of the Levi-Civita connection of \( g_F \). It follows that \( F = \lim_{\alpha \to 0} F_\alpha \) is also preserved by the parallel transport of the Levi-Civita connection of \( g_F \) implying it is Berwald as we claimed.

\[\square\]

**Corollary 13.3.** Every locally symmetric \( C^2 \)-smooth Finsler manifold is locally isometric to a globally symmetric Finsler space.

**Proof.** We consider the Binet-Legendre metric \( g_F \) of our locally symmetric Finsler space \((M, F)\). Since \((M, g_F)\) is also locally symmetric, by the classical results of Cartan [25, theorem 5.1], it is locally isometric to a simply-connected globally symmetric Riemannian space \((\tilde{M}, g)\). We identify a small open set \( U \subset M \) with an open neighborhood set \( V \subset \tilde{M} \). This defines a Finsler metric \( \tilde{F} \) on \( V \). Now extend the \( \tilde{F} \) to the whole \( \tilde{M} \) using the procedure in Remark (E) from section 12, with the help of parallel transport of the Levi-Civita connection of \( g \).

Since the metric is Berwald and the manifold is simply connected, we obtain a well defined Finsler metric on \( \tilde{M} \). This metric (we denote it by \( \tilde{F} \)) is evidently locally symmetric. Since \( g \) and its isometries are real-analytic, the metric \( \tilde{F} \) is globally symmetric as we claimed. \[\square\]
**Remark.** Corollary 13.3 gives us a local description of locally symmetric ($C^2$-smooth) Finsler spaces (in special cases this description was obtained in [17, 23, 49, 50]). Indeed, take a globally symmetric simply connected Riemannian space $(M, g)$ and consider the isometry subgroup $G$ generated by all reflections. The group $G$ acts transitively on $M$. At one point $x \in M$, consider a smooth Minkowski norm $F_x : T_x M \to \mathbb{R}$ such that it is invariant with respect to the stabilizer $G_x$ of the point $x$. Next, extend $F_x$ to all points with the help of the action of $G$, i.e., for an isometry $g \in G$ with $g(x) = y$ put $F_y(d_x g(\xi)) = F_x(\xi)$. By Corollary 13.3, any $C^2$-smooth locally symmetric Finsler space is locally isometric to one constructed by this procedure.

14. The Minkowski functionals, and other conformal invariants of a Finsler manifold

The Minkowski functionals are a family of $(n+1)$ invariants associated to a bounded convex set $\Omega$ lying in an $n$-dimensional euclidean vector space $(E^n, g)$. The simplest way to define them is via the Steiner Formula:

\[ \text{Vol}^n(\Omega + t B^n) = \sum_{j=0}^{n} \binom{n}{j} W^n_j(\Omega) t^j, \]

where $B^n \subset E^n$ is the euclidean unit ball. Since the tangent space $T_x M$ of a Finsler manifold $(M, F)$ is an euclidean space (with scalar product given by the Binet-Legendre metric $g_F$), the Minkowski functionals of the $F$-unit ball $\Omega_x \subset T_x M$ are well defined. We have thus defined on the Finsler manifold $(M, F)$ a family of $n+1$ functions:

\[ w^n_k : M \to \mathbb{R}, \quad (k = 0, 1, \ldots, n) \]

(the function $w^n_0$ is in fact a constant, it is the volume of the euclidean unit ball). Observe that by construction, these functions are invariant under a conformal deformation of the Finsler metric. It is not difficult to check that if the Finsler metric $F$ is $C^k$-partially-smooth, then the Minkowski functionals $w^n_k$ are $C^k$-smooth functions on $M$.

Let us construct two additional conformal invariants: At every point $x$ one sets

\[ M(x) = \max_{0 \neq \xi \in T_x M} \frac{F(x, \xi)}{\sqrt{g(\xi, \xi)}} \quad \text{and} \quad \mu(x) = \min_{0 \neq \xi \in T_x M} \frac{F(x, \xi)}{\sqrt{g(\xi, \xi)}}. \]

It is easy to show that the functions $M$ and $\mu$ are continuous, but even if the Finsler metric is smooth, these functions may be non smooth. The Finsler metric from in example 3.2 provides an example of such a situation.

The invariants defined in the previous subsection can be used in addressing the following

**Equivalence problem for Finsler metrics.** Let $F_1$ and $F_2$ be Finsler metrics defined on the discs $U_1$ and $U_2$: Decide if $(U_1, F_1)$ is conformally equivalent to $(U_2, F_2)$, in the sense that there exists a diffeomorphism $f : U_1 \to U_2$ that sends the metric $F_1$ to the metric $\lambda \cdot F_2$ for a certain function $\lambda$ on $U_2$ ?

One may also consider the similar isometric equivalence problem. This one has been addressed by Chern in his 1948 paper [15], where he solved it by tensorial methods. His methods only works for smooth and strongly convex Finsler metrics.
For the conformal equivalence problem, we propose the following test, which only gives a necessary condition, but which works without smoothness assumptions and is quite stable and manageable from a computational viewpoint. Consider the mappings $\Phi_i : U_i \to \mathbb{R}^{n+2}$ ($i = 1, 2$) given by

$$\Phi_i(x) = (w_0^n(x), \ldots, w_{n-1}^n(x), \mu(x), M(x)).$$

If the Finsler metrics are conformally equivalent, the images of these mappings (which are in general $n$-dimensional objects in $\mathbb{R}^{n+2})$ coincide. Thus, if there exists at least one point that belongs to the first image and not the second, then the metrics are not conformally equivalent.

Note that the test may fail in some instances. In particular this test can never distinguish between two Riemannian metrics and it is in fact quite delicate to decide whether two Riemannian metrics $g_1$ and $g_2$ are locally isometric or conformally equivalent.

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