ON SUBMAXIMAL DIMENSION OF THE GROUP OF ALMOST ISOMETRIES OF FINSLER METRICS.

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Abstract. We show that the second greatest possible dimension of the group of (local) almost isometries of a Finsler metric is $\frac{n^2-n}{2} + 1$ for $n = \dim(M) \neq 4$ and $\frac{n^2-n}{2} + 2 = 8$ for $n = 4$. If a Finsler metric has the group of almost isometries of dimension greater than $\frac{n^2-n}{2} + 1$, then the Finsler metric is Randers, i.e., $F(x,y) = \sqrt{g_x(y,y)} + \tau(y)$. Moreover, if $n \neq 4$, the Riemannian metric $g$ has constant sectional curvature and, if in addition $n \neq 2$, the 1-form $\tau$ is closed, so (locally) the metric admits the group of local isometries of the maximal dimension $\frac{n(n+1)}{2}$.

In the remaining dimensions 2 and 4, we describe all examples of Finsler metrics with 3 resp. 8-dimensional group of almost isometries.

1. Definition and results

Let $(M,F)$ be a connected Finsler manifold of dimension $n \geq 2$. We assume that all objects in our paper are sufficiently smooth. We require that $F$ is strictly convex but allow $F$ to be not reversible.

Definition ([2]). A diffeomorphism $\phi : U \rightarrow V$, where $U,V \subseteq M$, is called an almost isometry, if $T(p,q,r) = T(\phi(p),\phi(q),\phi(r))$ for all $p,q,r \in U$, where $T$ is the “triangular” function given by

$$T(p,q,r) := d(p,q) + d(q,r) - d(p,r),$$

where $d$ is the (generally, nonsymmetric) distance corresponding to $F$, i.e.

$$d(p,q) = \inf \left\{ \int_0^1 F(c(t),c(t)')dt \mid c \in C^1([0,1];M) \text{ with } c(0) = p, c(1) = q \right\}.$$

Similarly, a vector field $K$ is an almost Killing, if its local flow acts by almost isometries.

Example 1. Let $\phi : U \rightarrow V$ be a diffeomorphism such that $\phi_*(F) = F + df$, where $f$ is a smooth function (the metric transformation $F \mapsto$...
$F + df$ was called trivial projective change in [5]). Then, $\phi$ is an almost isometry.

Indeed, the transformation $F \mapsto F + df$ adds $f(q) - f(p)$ (resp. $f(r) - f(q)$, resp. $f(p) - f(r)$) to the length of every curve connecting $p$ and $q$ (resp. $q$ and $r$, resp. $r$ and $p$) and therefore does not change the value of $T(p, q, r)$.

**Fact 2** ([2], Proposition 3.2). Every almost isometry is as in Example 1.

In our paper we answer the following natural question: what is the submaximal dimension of the group of almost isometries, and when it is greater than the submaximal dimension of the group of isometries? Since in view of Fact 2 every almost isometry of $F$ is an isometry of the symmetrized Finsler metric $F_{\text{sym}}(x, y) = \frac{1}{2}(F(x, y) + F(x, -y))$, the dimension of the group of almost isometries can not be greater than the dimension of the group of isometries of $F_{\text{sym}}$ which, in view of [1, Theorem 3.3] and [6, §3], is at most $\frac{n(n+1)}{2}$-dimensional. It is easy to construct examples of (nonriemannian) reversible Finsler metrics whose group of isometries (and therefore the group of almost isometries) has dimension $\frac{n^2-n}{2} + 1$, see [6, §3].

All our considerations are local, so we actually speak not about the group of almost isometries, but about the linear vector space of almost Killing vector fields. Most examples and statements survive or could be generalized for the global setup though.

**Example 3.** Let $g$ be a Riemannian metric of constant sectional curvature (locally it has $\frac{n(n+1)}{2}$-dimensional space of Killing vector fields). Consider a closed 1-form $\tau$ such that its $g$-norm $|\tau|_g < 1$ at all points. Then, any Killing vector field of $g$ is an almost Killing vector field of the Randers Finsler metrics $F_{g, \tau}$ defined by $F_{g, \tau}(x, y) = \sqrt{g_x(y, y) + \tau(y)}$, and vice versa.

Indeed, the implication "$\Rightarrow$" follows from Example 1 and the implication "$\Leftarrow$" follows from the above mentioned observation that almost isometries are isometries of the symmetrized metric (which in the case of Randers metrics is essentially the initial metric $g$).

The main results of our papers are Theorems 4, 5, 8, 9 and Examples 6, 7 below.

**Theorem 4.** Assume the vector space of almost Killing vector fields on $(M, F)$ is more than $\frac{n^2-n}{2} + 1$- dimensional. Then, $F$ is a Randers metric, i.e., $F(x, y) = \sqrt{g_x(y, y) + \tau(y)}$ for a Riemannian metric $g$ and for a 1-form $\tau$ with $|\tau|_g < 1$. 

Theorem 5. Let $F = F_{g,\tau}$ be a Randers Finsler metric. Assume $n = \dim(M) \neq 2, 4$. Suppose the space of almost Killing vector fields is more than $\frac{n^2-n}{2} + 1$-dimensional. Then, $F$ is as in Example 3, i.e., $g$ has constant sectional curvature and the form $\tau$ is closed.

The above two theorems answer our question for all dimensions except of $n = 2, 4$, and, in a certain sense, tell us that for metrics with many Killing resp. almost Killing vector fields there is no big difference between isometries and almost isometries. The case $n = 2$ will be considered in Example 6 and Theorem 8; we will see that in this case the submaximal dimension of the space of almost Killing vector fields is still $\frac{n^2-n}{2} + 1 = 2$, but the description of Finsler metrics with the space of almost Killing vector fields of the highest dimension 3 is slightly more complicated (the metric is still of constant curvature but the form $\tau$ may be not closed).

The 4-dimensional case (considered in Example 7 and Theorem 9) is much more interesting: remarkably, there exist examples of Randers Finsler metrics such that they are not as in Example 3 and such that the dimension of the space of almost Killing vector fields is 8 which is greater than $\frac{n^2-n}{2} + 1 = 7$; we construct them all.

Example 6. Consider a Riemannian $(n = 2)$-dimensional manifold $(M^2, g)$ of constant sectional curvature. Let $\omega \equiv Vol_g = \sqrt{\det(g)}dx \wedge dy$ be the volume form of $g$ and $\tau$ be a 1-form such that for a certain constant $c$ we have $d\tau = c \cdot \omega$. Assume that $|\tau|_g < 1$ so $F_{g,\tau}$ is a Finsler metric. Then, locally, the space of almost Killing vector fields for this metric is $\frac{n^2-n}{2} + 1 + 1 = 3$-dimensional.

Note that Example 6 is essentially local and can not live on closed manifolds, since the volume form on a closed manifold can not be differential of a 1-form by Stokes’ theorem.

Example 7. Consider a Kähler 4-manifold $(M^4, g, \omega)$ of constant holomorphic curvature. Take a 1-form $\tau$ such that $d\tau = c \cdot \omega$ for a certain constant $c$. Assume that $|\tau|_g < 1$ so $F_{g,\tau}$ is a Finsler metric. Then, locally, the space of almost Killing vector fields for this metric is $\frac{n^2-n}{2} + 1 + 1 = 8$-dimensional.

Indeed, consider the space of Killing vector fields preserving the volume form $\omega$ in dimension 2 or the Kähler symplectic form $\omega$ in dimension 4. This space is 3-dimensional in dimension 2 (since every orientation-preserving isometry preserves the volume form) and 8-dimensional in dimension 4 (since a Kähler space of constant holomorphic curvature is a locally symmetric space with $\omega$-preserving-isotropy subgroup isomorphic to the 4-dimensional $U_2$). Every such Killing vector field is an
almost Killing vector field of $F_{g,\tau}$. Indeed, its flow is an isometry of $g$ and sends $\tau$ to another 1-form such that its differential is still $c \cdot \omega$ (because this isometry preserves $\omega$).

**Theorem 8.** Suppose a Randers Finsler metric is not as in Example 3. Assume $n = \dim(M) = 2$. Then, if the space of almost Killing vector fields for this metric is (at least) $\frac{n^2}{2} + 1 + 1 = 3$-dimensional, then the metric is as in Example 6 (with $c \neq 0$).

**Theorem 9.** Suppose a Randers Finsler metric is not as in Example 3. Assume $n = \dim(M) = 4$. Then, if the space of almost Killing vector fields for this metric is (at least) $\frac{n^2}{2} + 1 + 1 = 8$-dimensional, then the metric is as in Example 7. If the space of almost Killing vector fields has a higher dimension, the metric is as in Example 3.

### 2. Proofs

**Proof of Theorem 4.** Assume a metric $F$ on $M^n$ has at least $\frac{n^2}{2} - n + 2$ linearly independent almost Killing vector fields. Let us slightly improve the metric $F$, i.e., construct canonically a “better” metric $F_{\text{better}}$ such that each almost Killing vector field of $F$ is a Killing vector field of $F_{\text{better}}$. This “improvement” will take place in each tangent space independently, though of course the result will still smoothly depend on the point. Take a point $p \in M$ and consider $T_pM$. Consider the unit ball $K_F$ in the norm $F|_{T_pM}$:

$$K_F := \{ y \in T_pM \mid F(p, y) \leq 1 \}.$$  

It is a convex body in $T_pM$ containing the zero vector $\vec{0} \in T_pM$. Now, consider the dual space $T^*_pM$ and the dual(=polar) convex body there:

$$K^*_F := \{ \xi \in T^*_pM \mid \xi(y) \leq 1 \text{ for all } y \in K_F \}.$$  

Take the barycenter $b_F$ of $K^*_F$ and consider the convex body

$$K^*_{\text{better}} := K^*_F - b_F = \{ \xi - b_F \mid \xi \in K^*_F \}.$$  

The barycenter of $K^*_{\text{better}}$ lies at $\vec{0}$.

Next, consider the body $K_{\text{better}} \subset T_pM$ dual to $K^*_{\text{better}}$, and the Finsler metric $F_{\text{better}}$ such that at every point $p$ the unit ball is the corresponding $K_{\text{better}}$. Evidently, $F_{\text{better}}$ is a smooth Finsler metric.

**Example 10.** Let $F(x, y) = \sqrt{g_x(y, y)} + \tau(y)$ be a Randers metric. Then, $F_{\text{better}}(x, y) = \sqrt{g_x(y, y)}$, i.e., is essentially the Riemannian metric $g$. Moreover, if $F_{\text{better}}$ is essentially a Riemannian metric, then $F$ is a Randers metric (to see all this, it is recommended to consider a $g$-orthonormal basis in $T_pM$ and calculate everything in the corresponding coordinate system, which is an easy exercise).
Let us now show that the transformation $F \mapsto F + \tau$ (where $\tau$ is a 1-form) does not change the metric $F_{\text{better}}$ in the sense that the metrics $F_{\text{better}}$ constructed for $F$ and for $F + \tau$ coincide. This fact is well known in the convex geometry, we prove it for convenience of the reader.

First, since the sets $K_F$, $K_{F + \tau}$ are convex and compact, the maximum of every 1-form $\xi \in T^*_pM$ over $K_F$ resp. $K_{F + \tau}$ is achieved at a point of the unit sphere $S_F = \{ y \in T_pM \mid F(p, y) = 1 \}$ resp. $S_{F + \tau} = \{ y \in T_pM \mid F(p, y) + \tau(y) = 1 \}$.

Consider the bijection

$$f : S_F \rightarrow S_{F + \tau}, \quad f(y) = \frac{1}{1 + \tau(y)} y.$$ 

Consider $\xi \in K^*_F$. Then, for every element $y' = f(y)$ of $S_{F + \tau}$ (for $y \in S_F$) we have

$$(\xi + \tau)(y') = (\xi + \tau) \left( \frac{1}{1 + \tau(y)} y \right) = \frac{1}{1 + \tau(y)} \xi(y) + \frac{1}{1 + \tau(y)} \tau(y) \leq \frac{1 + \tau(y)}{1 + \tau(y)} = 1.$$ 

We see that for every 1-form $\xi \in K^*_F$ the 1-form $\xi + \tau \in K^*_{F + \tau}$. Analogous we prove that for every $\xi \in K^*_{F + \tau}$ the 1-form $\xi - \tau$ lies in $K^*_F$. Thus, $K^*_{F + \tau}$ is the $\tau$-parallel translation of $K^*_F$. Then, the barycenter $b_{F + \tau}$ corresponding to $K^*_{F + \tau}$ is $b_F + \tau$, and the bodies $K_{\text{better}}$ corresponding to $F + \tau$ and $F$ coincide, so the transformation $F \mapsto F + \tau$ does not change the metric $F_{\text{better}}$.

Since by Fact 2 every almost isometry sends $F$ to a $F + df$, and since the addition of $\tau = df$ does not change the metric $F_{\text{better}}$, every almost isometry of $F$ is an isometry of $F_{\text{better}}$.

By assumptions, the initial Finsler metric $F$ has at least $n^2 - n + 2$-dimensional space of almost Killing vector fields. Thus, the Finsler metric $F_{\text{better}}$ has at least $n^2 - n + 2$-dimensional space of Killing vector field. By [6, Theorem 3.1], it is essentially a Riemannian metric, i.e., $F(x, y) = \sqrt{g_x(y, y)}$ for a certain Riemannian metric $g$. Then, as we explain in Example 10, the metric $F$ is a Randers metric as we claimed. Theorem 4 is proved.

**Proof of Theorem 5.** We assume $n = \text{dim}(M) \neq 2, 4$ and consider a Finsler metric $F$ such that the vector space of its almost Killing vector fields is at least $n^2 - n + 2$-dimensional. By Theorem 4, the Finsler metric is a Randers one, $F(x, y) = \sqrt{g_x(y, y)} + \tau(y)$.

Take a point $x \in M$ and consider the Lie subgroup of $SO(T_x M; g_x)$ (=the group of $g_x$-isometries of $T_x M$ preserving the orientation) generated by almost Killing vector fields that vanish at 0. It is at least $\left( \frac{n^2 - n}{2} + 2 - n \right)$-dimensional. Take it closure; it is a closed Lie subgroup of $SO(T_x M; g_x)$ of dimension at least $\frac{n^2 - n}{2} + 2 - n = \frac{(n-1)(n-2)}{2} + 1$. 
By the classical result of [7], for \( n \neq 4 \), every closed subgroup of \( SO_n \) of dimension greater than \( \frac{(n-1)(n-2)}{2} \) coincides with the whole \( SO_n \). Evidently, the flow of every almost Killing vector field preserves \( d\tau \) (since it preserves the metric \( g \) and in view of Fact 2 sends \( \tau \) to \( \tau + df \) which have the same differential as \( \tau \)). Then, the differential of \( \tau \), which is a 2-form, is preserved by the whole group \( SO(T_x M; g_x) \) implying it is vanishes at the (arbitrary) point \( x \).

Since \( SO(T_x M; g_x) \) acts transitively on 2-planes in \( T_x M \), the Riemannian metric \( g \) has constant sectional curvature. Theorem 2 is proved.

**Proof of Theorem 8.** Consider a Randers metric \( F(x, y) = \sqrt{g_x(y, y)} + \tau \); we assume \( n = \dim(M) = 2 \) and the existence of 3 linearly independent almost Killing vector fields. Then, the Riemannian metric \( g \) has 3 Killing vector fields implying it has constant sectional curvature. The Killing vector fields preserve the differential of \( \tau \) implying \( d\tau \) is proportional to the volume form. Theorem 8 is proved.

**Proof of Theorem 9.** Consider a Randers metric \( F(x, y) = \sqrt{g_x(y, y)} + \tau \); we assume \( n = \dim(M) = 4 \) and the existence of at least 8 linearly independent almost Killing vector fields. Since almost Killing vector fields for \( F \) are Killing for \( g \), the metric \( g \) has at least 8 linearly independent Killing vector fields. 4-dimensional Riemannian metrics with at least 8 linearly independent Killing vector fields are all known (see for example [3]): they are of constant sectional curvature, or of constant holomorphic sectional curvature. In both cases the local pseudogroup of almost isometries generated by the almost Killing vector fields acts transitively.

Suppose now \( d\tau \neq 0 \) at a certain point. Since the local pseudogroup of almost isometries generated by the almost Killing vector fields acts transitively, \( d\tau \neq 0 \) at all points. Consider an arbitrary point \( x \) and again, as in the proof of Theorem 5, consider the almost Killing vector fields that vanish at \( x \), the Lie subgroup of \( SO(T_x M; g_x) \) generated by these vector fields, and its closure which we denote \( H_x \subseteq SO(T_x M; g_x) \approx SO_4 \). It is at least 4 dimensional. Now, the closed subgroups of \( SO(T_x M; g_x) \approx SO_4 \) of dimensions 4 and larger are well-understood: any 4-dimensional closed (connected) subgroup is essentially \( U_2 \) (in the sense that in a certain \( g_x \)-orthonormal basis of \( T_x M \) the group \( SO(T_x M; g_x) \) and its subgroup \( H \) are precisely the standard \( SO_4 \) and the standard \( U_2 \) standardly embedded in \( SO_4 \)). Any closed (connected) subgroup of dimension \( \geq 5 \) is the whole \( SO_4 \).

In all cases the element \(-id : T_x M \to T_x M\) is an element of \( H_x \) which implies that every geodesic reflection (i.e., a local isometry \( I \) of \( g \) such that it takes the point \( x \) to itself and whose differential at \( x \) is \(-id\)
can be realized by an almost isometry of $F$ generated by almost Killing vector fields.

By our assumptions, the differential $d\tau$ of the form $\tau$ is not zero. Since it is preserved by $H_x$, it also preserved by the holonomy group of $g$, since the holonomy group of a symmetric space is the subgroup of the group generated by geodesic reflections. Then, $d\tau$ is covariantly constant. Since the group $H_x$ acts transitively on $T_x M$, all the eigenvalues of the endomorphism $J : TM \to TM$ defined by the condition $d\tau(y,.) = g(J(y),.)$ are $\pm c \cdot i$, so the endomorphism $\frac{1}{c} \cdot J$ is an almost complex structure. It is covariantly constant (because $g$ and $d\tau$ are covariantly constant), so $c$ is a constant. Then, it is a complex structure and $(M, g, \omega = \frac{1}{c}d\tau)$ is a Kähler manifold. Then, the metric has constant holomorphic sectional curvature and $d\tau$ is proportional to $\omega$ with the constant coefficient as we claimed. Finally, if there exists more than 8 almost Killing vector fields, then the group $H_x$ is more than 4-dimensional, which, as we explained above, implies that it is the whole $SO(T_x M; g_x)$ so the metric has constant sectional curvature and the form $\tau$ is closed.

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