Appendix: Dini theorem for pseudo-Riemannian metrics

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1 Introduction

Consider a Riemannian or a pseudo-Riemannian metric $g = (g_{ij})$ on a surface $M^2$. We say that a metric $\bar{g}$ on the same surface is projectively equivalent to $g$, if every geodesic of $\bar{g}$ is a reparametrized geodesic of $g$. In 1865 Beltrami [2] asked1 to describe all pairs of projectively equivalent Riemannian metrics on surfaces. From the context it is clear that he considered this problem locally, in a neighbourhood of almost every point.

Theorem A below, which is the main result of this note, gives an answer to the following generalization of the question of Beltrami: we allow the metrics $g$ and $\bar{g}$ to be pseudo-Riemannian.

Theorem A. Let $g, \bar{g}$ be projectively equivalent metrics on $M^2$, and $\bar{g} \neq \text{const} \cdot g$ for every const $\in \mathbb{R}$. Then, in the neighbourhood of almost every point there exist coordinates $(x, y)$ such that the metrics are as in the following table.

<table>
<thead>
<tr>
<th>Liouville case</th>
<th>Complex-Liouville case</th>
<th>Jordan-block case</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>$(\bar{X}(x) - \bar{Y}(y))(dx^2 + dy^2)$</td>
<td>$2\Im(h)dx dy$</td>
</tr>
<tr>
<td>$\bar{g}$</td>
<td>$\left(\frac{1}{\bar{Y}(y)} - \frac{1}{\bar{X}(z)}\right)\left(\frac{dx}{\bar{X}(z)} \pm \frac{dy}{\bar{Y}(y)}\right)$</td>
<td>$-\left(\frac{\Im(h)}{(\Re(h))^2 + (\Im(h))^2}\right)^2 dx^2$ + $2\frac{(\Re(h)\Im(h))}{(\Re(h))^2 + (\Im(h))^2} dx dy$ + $\left(\frac{\Im(h)}{(\Re(h))^2 + (\Im(h))^2}\right)^2 dy^2$</td>
</tr>
</tbody>
</table>

where $h := \Re(h) + i \cdot \Im(h)$ is a holomorphic function of the variable $z := x + i \cdot y$.

Remark 1. It is natural to consider the metrics from the Complex-Liouville case as the complexification of the metrics from the Liouville case: indeed, in the complex coordinates $z = x + i \cdot y$, $\bar{z} = x - i \cdot y$, the metrics have the form

$$\begin{align*}
\bar{d}s_\bar{g}^2 &= -\frac{1}{4}\left(h(\bar{z}) - h(z)\right)(dz^2 - d\bar{z}^2), \\
\bar{d}s_\bar{g}^2 &= -\frac{1}{2}\left(\frac{1}{h(z)} - \frac{1}{h(\bar{z})}\right)(d\bar{z}^2 - dz^2).
\end{align*}$$

Remark 2. In the Jordan-block case, if $dY \neq 0$ (which is always the case at almost every point, if the restriction of $g$ to any neighborhood does not admit a Killing vector field), after a local coordinate change, the metrics $g$ and $\bar{g}$ have the form

$$\begin{align*}
\bar{d}s_\bar{g}^2 &= \left(\bar{Y}(y) + x\right)dx dy, \\
\bar{d}s_\bar{g}^2 &= -2\frac{(\bar{Y}(y) + x)}{y^3}dx dy + \frac{(\bar{Y}(y) + x)^2}{y^4}dy^2.
\end{align*}$$

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1Italian original from [2]: La seconda ... generalizzazione ... del nostro problema, vale a dire: riportare i punti di una superficie sopra un’altra superficie in modo che alle linee geodetiche della prima corrispondano linee geodetiche della seconda.
We see that the metrics from Complex-Liouville and Jordan-block cases always have signature $(+,−)$, and the metric $g$ from the Liouville case has signature $(+,+)$ or $−−$, only if the sign “±” is “+”. In this case, the formulas from Theorem A are precisely the formulas obtained by Dini in [5].

We do not insist that we are the first to find these normal forms of projectively equivalent pseudo-Riemannian metrics. According to [1], a description of projectively equivalent metrics was obtained by P. Shiroyok in [13]. Unfortunately, we were not able to find the reference [13] to check it. The result of Theorem A could be even more classical, see Remark 4.

Given two projectively equivalent metrics, it is easy to understand what case they belong to. Indeed, the $(1,1)$-tensor $G^i_j := \sum_{\alpha=1}^2 \bar{g}_{i\alpha}g^{\alpha j}$, where $g^{\alpha j}$ is inverse to $g_{i\alpha}$, has two different real eigenvalues in the Liouville case, two complex-conjugated eigenvalues in the Complex-Liouville case, and is (conjugate to) a Jordan-block in the Jordan-block case.

There exists an interesting and useful connection of projectively equivalent metrics with integrable systems.

Recall that a function $F : T^*M^2 \to \mathbb{R}$ is called an integral of the geodesic flow of $g$, if $\{H,F\} = 0$, where $H := \frac{1}{2}g^{ij}p_ip_j : T^*M^2 \to \mathbb{R}$ is the kinetic energy corresponding to the metric, and $\{ , \}$ is the standard Poisson bracket on $T^*M^2$. Geometrically, this condition means that the function is constant on the orbits of the Hamiltonian system with the Hamiltonian $H$. We say the integral $F$ is quadratic in momenta, if for every local coordinate system $(x,y)$ on $M^2$ it has the form

$$F(x,y,p_x, p_y) = a(x,y)p_x^2 + b(x,y)p_xp_y + c(x,y)p_y^2$$

in the canonical coordinates $(x,y,p_x,p_y)$ on $T^*M^2$. Geometrically, the formula (1) means that the restriction of the integral to every cotangent space $T^*_y M^2 \equiv \mathbb{R}^2$ is a homogeneous quadratic function. Of course, $H$ itself is an integral quadratic in the momenta for $g$. We will say that the integral $F$ is nontrivial, if $F \neq \text{const} \cdot H$ for all const $\in \mathbb{R}$.

**Theorem B.** Suppose the metric $g$ on $M^2$ admits a nontrivial integral quadratic in momenta. Then, in a neighbourhood of almost every point there exist coordinates $(x,y)$ such that the metric and the integral are as in the following table

<table>
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<td>$F$</td>
<td>$(X(x)−Y(y))(dx^2 + dy^2)$</td>
<td>$\Re(h) dxdy$</td>
<td>$(1 + xY'(y))dxdy$</td>
</tr>
<tr>
<td></td>
<td>$\frac{X(x)p_x^2 + Y(y)p_y^2}{X(x)−Y(y)}$</td>
<td>$p_x^2 − p_y^2 + 2 \Re(h)\frac{X(x)−Y(y)}{2X(x)−Y(y)}p_xp_y$</td>
<td>$p_x^2 − 2 \frac{Y(y)}{1+ xY'(y)}p_xp_y$</td>
</tr>
</tbody>
</table>

where $h := \Re(h) + i \cdot \Im(h)$ is a holomorphic function of the variable $z := x + iy$.

Indeed, as it was shown in [7, 8], and as it was essentially known to Darboux [4, §§600–608], if two metrics $g$ and $\bar{g}$ are projectively equivalent, then

$$I : TM^2 \to \mathbb{R}, \quad I(\xi) := \bar{g}(\xi, \xi) \left( \frac{\det(g)}{\det(\bar{g})} \right)^{2/3}$$

is an integral of the geodesic flow of $g$. Moreover, it was shown in [3], see Section 2.4 there, see also [10], the above statement is proven to be true² in the other direction: if the function (1) is an integral for the geodesic flow of $g$, then the metrics $g$ and $\bar{g}$ are projectively equivalent. Thus, Theorem A and Theorem B are equivalent. In this paper, we will actually prove Theorem B obtaining Theorem A as its consequence.

**Remark 3.** The corresponding natural Hamiltonian problem on the hyperbolic plane has been recently treated in [11] following the approach used by Rosquist and Uggla [12].

**Remark 4.** The formulas that will appear in the proof are very close to that in §593 of [4]. Darboux worked over complex numbers and therefore did not care about whether the metrics are Riemannian or pseudo-Riemannian. For example, Liouville and Complex-Liouville case are the same for him. Moreover, in §594, Darboux gets the formulas that are very close to that of Jordan-block case, though he was interested in the Riemannian case only, and, hence, treated this “imaginary” case as not interesting.

²with a good will, one also can attribute this result to Darboux
2 Proof of Theorem B (and, hence, of Theorem A)

If the metric $g$ has signature $(+, +)$ or $(-, -)$, Theorem A and, hence, Theorem B, were obtained by Dini in [5]. Below we assume that the metric $g$ has signature $(+, -)$.

2.1 Admissible coordinate systems and Birkhoff-Kolokoltsov forms

Let $g$ be a pseudo-Riemannian metric on $M^2$ of signature $(+, -)$. Consider (and fix) two linear independent vector fields $V_1, V_2$ on $M^2$ such that

- $g(V_1, V_1) = g(V_2, V_2) = 0$
- $g(V_1, V_2) > 0$.

Such vector fields always exist locally (and, since our result is local, this is sufficient for our proof).

We will say that a local coordinate system $(x, y)$ is admissible, if the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are proportional to $V_1, V_2$ with positive coefficient of proportionality:

$$V_1(x, y) = \lambda_1(x, y) \frac{\partial}{\partial x}, \quad V_2(x, y) = \lambda_2(x, y) \frac{\partial}{\partial y},$$

where $\lambda_i > 0$, $i = 1, 2$.

Obviously,

- admissible coordinates exist in a sufficiently small neighborhood of every point,
- the metric $g$ in admissible coordinates has the form
  $$ds^2 = f(x, y) dx dy,$$
  where $f > 0$,
- two admissible coordinate systems in one neighbourhood are connected by
  $$(x_{new}, y_{new}) = \left(x_{new}(x_{old}), y_{new}(y_{old})\right),$$
  where $\frac{dx_{new}}{dx_{old}}, \frac{dy_{new}}{dy_{old}} > 0$.

Lemma. Let $(x, y)$ be an admissible coordinate system for $g$. Let $F$ given by (1) be an integral for $g$. Then, $B_1 := \frac{1}{\sqrt{|a(x, y)|}} dx$ ($B_2 := \frac{1}{\sqrt{|c(x, y)|}} dy$, respectively) is a 1-form, which is defined at points such that $a \neq 0$ ($c \neq 0$, respectively). Moreover, the coefficient $a$ ($c$, respectively) depends only on $x$ ($y$, respectively), which in particular imply that the forms $B_1, B_2$ are closed.

Remark 5. The forms $B_1, B_2$ are not the direct analog of the “Birkhoff” 2-form introduced by Kolokoltsov in [6]. In a certain sense, they are the real analog of the different branches of the square root of the Birkhoff form.

Proof of the Lemma. The first part of the statement, namely that the $\frac{1}{\sqrt{|a|}} dx$ ($\frac{1}{\sqrt{|c|}} dy$, respectively) transforms as a 1-form under admissible coordinate changes is evident: indeed, after the coordinate change (4), the momenta transform as follows: $p_{x_{old}} = p_{x_{new}} \frac{dx_{new}}{dx_{old}}$, $p_{x_{old}} = p_{x_{new}} \frac{dx_{new}}{dx_{old}}$.

Then, the integral $F$ in the new coordinates has the form

$$\left( a \frac{dx_{new}}{dx_{old}} \right)^2 p_{x_{new}}^2 + b \frac{dx_{new}}{dx_{old}} \frac{dy_{new}}{dy_{old}} p_{x_{new}} p_{y_{new}} + c \frac{dy_{new}}{dy_{old}}^2 p_{y_{new}}^2.$$

Then, the formal expression $\frac{1}{\sqrt{|a|}} dx_{old}$ ($\frac{1}{\sqrt{|c|}} dy_{old}$, respectively) transforms in

$$\frac{1}{\sqrt{|a|}} dx_{old} d x_{new},$$

$$\frac{1}{\sqrt{|c|}} dy_{old} d y_{new},$$

respectively.
which is precisely the transformation law of 1-forms.

Let us prove that the forms are closed. If $g$ is given by (3), its Hamiltonian $H$ is given by $\frac{\partial H}{\partial y}$, and the condition $0 = \{H, F\}$ reads

$$0 = \left\{ \frac{\partial H}{\partial y}, a \frac{\partial x}{\partial y} + b \frac{\partial y}{\partial y} + c \frac{\partial z}{\partial y} \right\}$$

$$= \gamma \left( p_2^x(f_{ax} + f_{by} + 2f_x a + f_y b) + p_y^2(f_{bx} + f_{cy} + f_x b + 2f_y) + p_0^y(c_x f) \right),$$

i.e., is equivalent to the following system of PDE:

$$\begin{cases}
    a_y = 0 \\
    b_x + f_k y + 2f_x a + f_y b = 0 \\
    c_x = 0
\end{cases} \quad (5)$$

Thus, $a = a(x)$, $c = c(y)$, which is equivalent to $B_1 := \frac{1}{\sqrt{|x|}} dx$ and $B_2 := \frac{1}{\sqrt{|y|}} dy$ are closed forms (assuming $a \neq 0$ and $c \neq 0$).

**Remark 6.** For further use let us formulate one more consequence of the equations (5): if $a \equiv c \equiv 0$ in a neighborhood of a point, then $bf = \text{const } \implies H = \text{const }$ in the neighborhood.

Assume $a \neq 0$ (or $c \neq 0$, respectively) at a point $P_0$. For every point $P_1$ in a small neighborhood $U$ of $P_0$ consider

$$x_{\text{new}} := \int_{x_0}^x B_1, \quad y_{\text{new}} := \int_{y_0}^y B_2,$$

and locally, in the admissible coordinates, the functions $x_{\text{new}}$ and $y_{\text{new}}$ are given by

$$x_{\text{new}}(x) = \int_{x_0}^x \frac{1}{\sqrt{|a(t)|}} dt, \quad y_{\text{new}}(y) = \int_{y_0}^y \frac{1}{\sqrt{|c(t)|}} dt.$$

The new coordinates $(x_{\text{new}}, y_{\text{new}})$ (or $(x_{\text{old}}, y_{\text{old}})$ if $c_{\text{old}} \equiv 0$, or $(x_{\text{old}}, y_{\text{new}})$ if $a_{\text{old}} \equiv 0$) are admissible. In these coordinates, the forms $B_1$ and $B_2$ are given by $\text{sign}(a_{\text{old}}) dx_{\text{new}}$, $\text{sign}(c_{\text{old}}) dy_{\text{new}}$ (we assume $\text{sign}(0) = 0$).

### 2.2 Proof of Theorem B

We assume that $g$ of signature $(+, -)$ on $M^2$ admits a nontrivial quadratic integral $F$ given by (1).

Consider the matrix $F^{ij} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It can be viewed as a $(2,0)$-tensor: if we change the coordinate system and rewrite the function $F$ in the new coordinates, the matrix changes according to the tensor rule. Then,

$$\sum_{\alpha = 1}^2 g_{j\alpha} F^{i\alpha}$$

is a $(1,1)$-tensor. In a neighborhood $U$ of almost every point the Jordan normal form of this $(1,1)$-tensor is one of the following matrices:

Case 1 $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$,  \hspace{1cm} Case 2 $\begin{pmatrix} \lambda + i\mu & 0 \\ 0 & \lambda - i\mu \end{pmatrix}$,  \hspace{1cm} Case 3 $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$,

where $\lambda, \mu : U \to \mathbb{R}$. Moreover, in view of Remark 6, there exists a neighborhood of almost every point such that $\lambda \neq \mu$ in Case 1 and $\mu \neq 0$ in Case 2. In the admissible coordinates, up to multiplication of $F$ by $-1$, and renaming $V_1 \leftrightarrow V_2$, Case 1 is equivalent to the condition $a > 0$, $c > 0$, Case 2 is equivalent to the condition $a > 0$, $c < 0$, and Case 3 is equivalent to the condition $c \equiv 0$.

We now consider all three cases.
2.2.1 Case 1: \( a > 0, \ c > 0 \).

Consider the coordinates (6). In this coordinates, \( a = 1, \ c = 1 \), and equations (5) are:

\[
\begin{align*}
(f b)_y + 2f_x &= 0, \\
(f b)_x + 2f_y &= 0.
\end{align*}
\]

This system can be solved. Indeed, it is equivalent to

\[
\begin{align*}
(f b + 2f)_x + (f b + 2f)_y &= 0, \\
(f b - 2f)_x - (f b - 2f)_y &= 0,
\end{align*}
\]

which, after the change of coordinates \( x_{new} = x + y, \ y_{new} = x - y \), has the form

\[
\begin{align*}
(f b + 2f)_x &= 0, \\
(f b - 2f)_y &= 0,
\end{align*}
\]

implying \( f b + 2f = Y(y), \ f b - 2f = X(x) \). Thus, \( f = \frac{Y(y) - X(x)}{4 Y(y) - X(x)} \), \( b = 2 \frac{X(x) + Y(y)}{Y(y) - X(x)} \).

Finally, in the new coordinates, the metric and the integral have (up to a possible multiplication by a constant) the form

\[
(X - Y)(dx^2 - dy^2)
\]

\[
2 \left( p_x^2 - \frac{X(x) + Y(y)}{X(x) - Y(y)} (p_x^2 - p_y^2) + p_y^2 \right) = 4 \frac{p_x^2 X(x) - p_y^2 Y(y)}{X(x) - Y(y)}.
\]

Theorem B is proved under the assumptions of Case 1.

2.2.2 Case 2: \( a > 0, \ c < 0 \).

Consider the coordinates (6). In this coordinates, \( a = 1, \ c = -1 \), and the equations (5) are:

\[
\begin{align*}
(f b)_y + 2f_x &= 0, \\
(f b)_x - 2f_y &= 0.
\end{align*}
\]

We see that these conditions are the Cauchy-Riemann conditions for the complex-valued function \( f b + 2i \cdot f \). Thus, for an appropriate holomorphic function \( h = h(x + i \cdot y) \), we have \( f b = \Re(h), \ 2f = \Im(h) \).

Finally, in a certain coordinate system the metric and the integral are (up to multiplication by constants):

\[
2 \Im(h) dx dy \quad \text{and} \quad p_x^2 - p_y^2 + 2 \Re(h) p_x p_y.
\]

Theorem B is proved under the assumptions of Case 2.

2.2.3 Case 3: \( a > 0, \ c = 0 \).

Consider admissible coordinates \( x, y \), such that \( x \) is the coordinate from (6). In these coordinates, \( a = 1, \ c = 0 \), and the equations (5) are:

\[
\begin{align*}
(f b)_y + 2f_x &= 0, \\
(f b)_x &= 0.
\end{align*}
\]

This system can be solved. Indeed, the second equation implies \( f b = -Y(y) \). Substituting this in the first equation we obtain \( Y' = 2f_x \) implying

\[
f = \frac{x}{2} Y'(y) + \tilde{Y}(y) \quad \text{and} \quad b = -\frac{Y(y)}{\frac{x}{2} Y'(y) + \tilde{Y}(y)}.
\]

Finally, the metric and the integral are

\[
\left( \tilde{Y}(y) + \frac{x}{2} Y'(y) \right) dx dy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{\tilde{Y}(y) + \frac{x}{2} Y'(y)} p_x p_y
\]

(7)
Moreover, by the change $y_{\text{new}} = \beta(y_{\text{old}})$ the metric and the integral (7) will be transformed to:

\[
\left( Y(y)\beta' + \frac{x}{2} Y'(y) \right) dx dy \quad \text{and} \quad p_x^2 + \frac{Y(y)}{Y(y)\beta' + \frac{x}{2} Y'(y)} p_x p_y
\]

Thus, by putting $\beta(y) = \int_{y_0}^y \frac{1}{Y(t)} dt$, we can make the metric and the integral to be

\[
\left( 1 + \frac{x}{2} Y'(y) \right) dx dy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{1 + \frac{x}{2} Y'(y)} p_x p_y.
\]

Moreover, after the coordinate change $x_{\text{new}} = \frac{x}{2\\sqrt{4}}$ and dividing/multiplication of the metric/integral by 2, the metric and the integral have the form from Theorem B

\[
(1 + x Y'(y)) dx dy \quad \text{and} \quad p_x^2 - 2 - \frac{Y(y)}{1 + x Y'(y)} p_x p_y
\]

Theorem B is proved.

References


