Local normal forms for geodesically equivalent pseudo-Riemannian metrics

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Abstract

Two pseudo-Riemannian metrics \( g \) and \( \bar{g} \) are geodesically equivalent, if they share the same (unparameterized) geodesics. We give a complete local description of such metrics which solves the natural generalisation of Beltrami problem for pseudo-Riemannian metrics.

1 Introduction

1.1 Definition and history

Two pseudo-Riemannian metrics \( g \) and \( \bar{g} \) on one manifold \( M^n \) are geodesically equivalent, if every \( g \)-geodesic, after an appropriate reparameterisation, is a \( \bar{g} \)-geodesic. Theory of geodesically equivalent metrics has a long and rich history, the first examples were constructed already by Lagrange [23], and many important results about geodesically equivalent metrics were obtained by classics such as Beltrami [3, 4, 5], Levi-Civita [25], Painlevé [35], Lie [26], Liouville [27], Fubini [18], Eisenhart [16, 17], Weyl [43], Thomas und Veblen [40, 41, 42]. In the 50th-90th, the theory of geodesically equivalent metrics was one of the main research areas of the Soviet and Japanese differential geometry schools, see the surveys [2, 34]. In the recent time, the theory of geodesically equivalent metrics has a revival because of new mathematical methods that came here, in particular, from the theory of integrable systems [28] and from parabolic Cartan geometry [12, 15]. Using these methods led, in the last ten years, to the solution of many problems explicitly stated by classics including Sophus Lie problems [11, 33], Lichnerowicz conjecture [31], and Weyl-Ehlers problems [20, 32].

In the present paper we solve the natural generalization of one more problem explicitly stated by a classic to the pseudo-Riemannian metrics, namely

**Beltrami Problem**. Describe all pairs of geodesically equivalent metrics.

From the context it is clear that Beltrami actually considered this problem locally and in a neighborhood of almost every point, so do we. From the context it is also clear that Beltrami thought about two dimensional Riemannian surfaces; in our answer we do not have this restriction: the dimension of the manifold, and the signatures of the metrics are arbitrary.

Special cases of the Beltrami problem were known before. The two-dimensional Riemannian case was solved already by Dini [14]: he has shown that two Riemannian geodesically equivalent metrics on a surface in a neighborhood of almost every point are given in a certain coordinate system by
the following formulas

\[ g = (Y(y) - X(x))(dx^2 + dy^2) \text{ and } \left( \frac{1}{X(x)} - \frac{1}{Y(y)} \right) \left( \frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)} \right). \]  

(1)

Here \( X \) and \( Y \) are functions of the indicated variables. For every (smooth) functions \( X, Y \) such that the formulas (1) correspond to Riemannian metrics (i.e., \( 0 < X(x) < Y(y) \) for all \( (x, y) \)) the metrics \( g \) and \( \bar{g} \) are geodesically equivalent.

For an arbitrary dimension, the Beltrami problem in the Riemannian case was solved by Levi-Civita [25]. We will recall the Levi-Civita’s 3-dimensional analog of the formulas (1) below, in Example 2.

The methods of Levi-Civita and Dini can not be directly generalized to the pseudo-Riemannian case. Levi-Civita and Dini consider the tensor \( G \) defined by the condition \( g(G,.) = \bar{g}(..) \). Levi-Civita has shown, that the eigenspaces of this tensor are simultaneously integrable which implies that (in a neighborhood of almost every point) there exists a local coordinate system

\[(x^1, ..., x^n) = (x_1^1, ..., x_1^{m_1}, ..., x_k^1, ..., x_k^{m_k})\]

such that in these coordinates the matrix of \( g \) is blockdiagonal with \( k \) blocks of dimension \( m_1, m_2, ..., m_k \) and the matrix of \( G \) is diagonal \( \text{diag}(\rho_1, ..., \rho_k, ..., \rho_k) \). In the two-dimensional Riemannian case considered by Dini, the existence of such coordinate system is obvious. Now, in this coordinate system, the partial differential equations on the entries of \( g \) and on \( \rho_i \) expressing the geodesic equivalence condition for \( g \) and \( \bar{g} \) are relatively easy (though they are still coupled) and, after some nontrivial work, can be solved.

The methods of Levi-Civita also work in the pseudo-Riemannian case under the additional assumption that \( G \) is diagonalizable. Unfortunately, in the pseudo-Riemannian case the tensor \( G \) may have complex eigenvalues and Jordan blocks. From the point of view of partial differential equation, the case of many Jordan blocks poses the main difficulties: unlike the case when \( G \) is diagonalizable, there is no ‘best’ coordinate system, and the equation corresponding to the entries of the metrics coming from different blocks are coupled in a very nasty manner.

This difficulty was overcome in [9]. In §1.2 we recall the main result of [9] and explain that the description of geodesically equivalent metrics \( g \) and \( \bar{g} \) in a neighborhood of almost every point can be reduced to the case when the tensor \( G \) has one real eigenvalue only, or two complex-conjugated eigenvalues. The biggest part of our paper is devoted to the local description of geodesically equivalent metrics under this assumption.

Special cases of the local description of geodesically equivalent pseudo-Riemannian metrics were known before. The 2-dimensional case was described essentially by Darboux [13, §§593, 594], see also [7, 8]. Three dimensional case was solved by Petrov [36], it is one of the results Petrov obtained in 1972 the Lenin price for, the most important scientific award of the Soviet Union. According to [2], under the additional assumptions that the metrics \( g \) and \( \bar{g} \) have Lorentz signature, the Beltrami problem was solved by Golikov [19] in dimension 4, and by Kruchkovich [22] in all dimensions; unfortunately, we were not able to find and to verify these references.

It was generally believed that the Beltrami problem was solved in full generality in [1]. Unfortunately, this result of Aminova seems to be wrong. More precisely, in view of [1, Theorem 1.1] and the formulas [1, (1.17),(1.18)] for \( k = 1, n = 4 \) and all \( \varepsilon \)'s equal to +1, the following two metrics \( g \) and \( \bar{g} \) given by the matrices (where \( \omega \) is an arbitrary function of the variable \( x_4 \))

\[
\begin{bmatrix}
0 & 0 & 0 & 3x_3 + 3\omega(x_4) \\
0 & 0 & 1 & 2x_2 \\
0 & 1 & 0 & x_1 \\
3x_3 + 3\omega(x_4) & 2x_2 & x_1 & 4x_1x_2
\end{bmatrix},
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 3x_3 + 3\omega(x_4) \\
0 & 0 & 1 & 2x_2 \\
0 & 1 & 0 & x_1 \\
3x_3 + 3\omega(x_4) & 2x_2 & x_1 & 4x_1x_2
\end{bmatrix},
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 3x_3 + 3\omega(x_4) \\
0 & 0 & 1 & 2x_2 \\
0 & 1 & 0 & x_1 \\
3x_3 + 3\omega(x_4) & 2x_2 & x_1 & 4x_1x_2
\end{bmatrix}
\]
Given two metrics $g$ and $\tilde{g}$ on the same manifold, instead of considering the $(1,1)$-tensor $G^i_j = g^{ik}\tilde{g}_{kj}$, we consider the $(1,1)$-tensor $L = L(g, \tilde{g})$ defined by

$$L^i_j := \left( \frac{\det(\tilde{g})}{\det(g)} \right)^{\frac{1}{n+1}} \tilde{g}^{ik}g_{kj},$$

(2)

where $\tilde{g}^{ik}$ is the contravariant inverse of $\tilde{g}_{kj}$. The tensors $G$ and $L$ are related by a simple relation

$$L = \det(G)^{\frac{1}{n+1}}G^{-1}, \quad G = \frac{1}{\det(L)}L^{-1},$$

(3)

so in particular they have the same structure of Jordan blocks (though the eigenvalues are, in general, different). Since the metric $\tilde{g}$ can be uniquely reconstructed from $g$ and $L$, namely:

$$\tilde{g}(\cdot, \cdot) = \frac{1}{\det(L)}g(L^{-1}\cdot, \cdot),$$

(4)

the condition that $\tilde{g}$ is geodesically equivalent to $g$ can be written as a PDE-system on the components of $L$. From the point of view of partial differential equations, the tensor $L$ is more convenient than $G$: the corresponding system of partial differential equations on $L$ turns out to be linear. In the index-free form, it can be written as the condition (where “*$” means $g$-adjoint)

$$\nabla_u L = \frac{1}{2}(u \otimes d\text{tr} L + (u \otimes d\text{tr} L)^*),$$

(5)

which should be fulfilled at every point and for every vector field $u$.

In tensor notation, the condition (5) reads

$$L_{ij,k} = \lambda_{i}g_{jk} + \lambda_{j}g_{ik},$$

(6)

where $L_{ij} := L^{k}_{i}g_{kj}$ and $\lambda := \frac{1}{2}L_{i}^{i} = \frac{1}{4}\text{tr}(L)$. The tensor $L^{i}_{j}$ defined in (2) is essentially the same as the tensor introduced by Sinjukov (see equations (32, 34) on the page 134 of the book [37], and also Theorem 4 on page 135); the equation (6) is also due to him; see also [6, Theorem 2].

Remark 1. If $n$ is even, the tensor $L$ is always well defined. If $n$ is odd, the ratio $\det(\tilde{g})/\det(g)$ may be negative, and then formula (2) has no sense. There is the following way to avoid this (rather, formal) difficulty: we can replace $\tilde{g}$ by $-\tilde{g}$ and make the ratio $\det(\tilde{g})/\det(g)$ positive and $L$ well defined. Moreover, since the equations (5) are linear, we can assume that $L$ is close to $1$ in the neighborhood we are working in, implying that $g$ and $\tilde{g}$ have the same signatures, and the problem with the sign does not appear at all.

We say that the $(1,1)$-tensor $L$ is compatible with $g$, if it is $g$-selfadjoint, nondegenerate at every point, and satisfies (5) at any point and for all tangent vectors $u$. As we explain above, $L$ is compatible with $g$ if and only if $\tilde{g}(\cdot, \cdot) = \frac{1}{\det(L)}g(L^{-1}\cdot, \cdot)$ is a pseudo-Riemannian metric geodesically equivalent to $g$. 

\[\begin{array}{cccc}
0 & 0 & 0 & \frac{3}{4} - \frac{3}{4}x_{4}x_{4} \\
0 & 0 & 2x_{4}^{-5} & -\frac{3}{4}x_{4}x_{4} \\
0 & 2x_{4}^{-5} & -x_{4}^{-6} & \frac{3}{4}x_{4} \times x_{4} \\
\frac{3}{4}x_{4}^{-3}x_{4} + 3x_{4} - 2x_{4} & 3x_{4} - 3x_{4}x_{4} - 2x_{4} - x_{4} & \frac{3}{4}x_{4} \times x_{4} & \frac{3}{4}x_{4} \times x_{4}
\end{array}\]
Let us now recall the gluing construction from [9]. The construction, and also the splitting construction to be recalled below, is due to [9]; in the Riemannian case it appeared slightly earlier, see [29, §4], [31, Lemma 2] and [30, §2.2, 2.3].

Consider two pseudo-Riemannian manifolds \((M_1, h_1)\) and \((M_2, h_2)\). Assume that \(L_1\) on \(M_1\) is compatible with \(h_1\), and that \(L_2\) on \(M_2\) is compatible with \(h_2\). Assume in addition that \(L_1\) and \(L_2\) have no common eigenvalues in the sense that for any two points \(x \in M_1, y \in M_2\) we have

\[
\text{Spectrum } L_1(x) \cap \text{Spectrum } L_2(y) = \emptyset. \tag{7}
\]

Then one can naturally and canonically construct a pseudo-Riemannian metric \(g\) and a tensor \(L\) compatible with \(g\) on the direct product \(M = M_1 \times M_2\). The new metrics \(g\) differs from the direct product metrics \(h_1 + h_2\) on \(M_1 \times M_2\) and is given by the following formula involving \(L_1\) and \(L_2\); we denote by \(\chi_i, i = 1, 2\), the characteristic polynomial of \(L_i\): \(\chi_i = \det(t \cdot 1_i - L_i)\) (where \(1_i\) is the identical operator \(1_i : TM_i \to TM_i\)). We treat the \((1,1)\)-tensors \(L_i\) as linear operators acting on \(TM_i\). For a polynomial \(f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m\) and an \((1,1)\)-tensor \(A\) we put \(f(A)\) to be the \((1,1)\)-tensor

\[
f(A) = a_0 \cdot 1 + a_1 A + a_2 A \circ A + \cdots + a_m A \circ \cdots \circ A. 
\]

If no eigenvalue of \(A\) is a root of \(f, f(A)\) is nondegenerate; if \(A\) is \(g\)-selfadjoint, \(f(A)\) is \(g\)-selfadjoint as well.

For two tangent vectors \(u = (u_1, u_2)\), \(v = (v_1, v_2)\) \(\in TM\) we put

\[
g(u, v) = h_1(\chi_2(L_1)(u_1), v_1) + h_2(\chi_1(L_2)(u_2), v_2), \tag{8}
\]

\[
L(v) = (L_1(v_1), L_2(v_2)) \tag{9}.
\]

We see that the \((1,1)\)-tensor \(L\) is the direct sum of \(L_1\) and \(L_2\) in the natural sense.

It might be convenient to understand the formulas (8, 9) in the matrix notation: we consider the coordinate system \((x^1, \ldots, x^r, y^{s+1}, \ldots, y^n)\) on \(M\) such that \(x^i\)’s are coordinates on \(M_1\) and \(y^i\)’s are coordinates on \(M_2\). Then, in this coordinate system, the matrices of \(g\) and \(L\) have the block diagonal form

\[
g = \begin{pmatrix} h_1 \chi_2(L_1) & 0 \\ 0 & h_2 \chi_1(L_2) \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}. \tag{10}
\]

If (7) is fulfilled, then \(g\) is a pseudo-Riemannian metric (i.e., symmetric and nondegenerate) and \(L\) is nondegenerate and \(g\)-selfadjoint.

**Theorem 1** (Gluing Lemma from [9]). If \(L_1\) is compatible with \(h_1\) on \(M_1\) and \(L_2\) is compatible with \(h_2\) on \(M_2\), and if the condition (7) is fulfilled, then \(L\) given by (9) is compatible with \(g\) given by (8).

**Example 1** (Gluing construction and Dini’s Theorem). As the manifolds \(M_1, M_2\) we take two intervals \(I_1\) with the coordinate \(x\) and \(I_2\) with the coordinate \(y\). Next, take the metrics \(h_1 = dx^2\) on \(I_1\) and \(h_2 = -dy^2\) on \(I_2\). Consider the \((1,1)\)-tensors \(L_1 = X dx \otimes \frac{\partial}{\partial x}\) on \(I_1\) and \(L_2 = Y dy \otimes \frac{\partial}{\partial y}\) on \(I_2\). We assume that \(0 < X(x) < Y(y)\) for all \(x \in I_1\) and \(y \in I_2\). The tensors \(L_1\) and \(L_2\) are compatible with \(h_1\) resp. \(h_2\) (which can be checked by direct calculation and which is trivial in view of the obvious fact that in dimension 1 all metrics are geodesically equivalent). Then, \(\chi_1 = (t - X(x))\), \(\chi_2 = (t - Y(y))\), so the formulas (10) read

\[
g = \begin{pmatrix} Y(y) - X(x) & X(x) - Y(x) \\ X(x) - Y(x) & Y(y) - X(x) \end{pmatrix}, \quad L = \begin{pmatrix} X(x) & Y(y) \\ Y(y) & X(x) \end{pmatrix}. 
\]
Now, combining this with (4), we obtain that this $g$ is geodesically equivalent to the metric
\[
\bar{g} = \frac{Y(y) - X(x)}{X(x)Y(y)} \left( \frac{1}{x(x)} \right) = \left( \frac{1}{X(x)} - \frac{1}{Y(y)} \right) \left( \frac{1}{x(x)} \right) .
\]
Comparing the above formulas with (1), we see that gluing construction applied to two intervals proves Dini’s local description of geodesically equivalent metrics in one direction.

One can iterate this construction: having three pseudo-Riemannian manifolds $(M_1, h_1)$, $(M_2, h_2)$, $(M_3, h_3)$ carrying $g$-compatible $(1, 1)$-tensors $L_i$ such that the condition (7) is mutually fulfilled, one can glue $M_1$ and $M_2$ and then glue the result with $M_3$. Actually the gluing construction is associative. Indeed, one obtains the same metric $g$ and the same $g$-compatible $(1, 1)$-tensor $L$ on $M_1 + M_2 + M_3$ if one first glues $(M_1, h_1, L_1)$ and $(M_2, h_2, L_2)$ and then glue the result with $(M_3, h_3, L_3)$, or if one first glues $(M_2, h_2 L_2)$ and $(M_3, h_3, L_3)$ and then glues $(M_1, h_1, L_1)$ with the result:
\[
((M_1, h_1, L_1) \overleftarrow{g} + (M_2, h_2, L_2)) \overleftarrow{g} + (M_3, h_3, L_3) = (M_1, h_1, L_1) + ((M_2, h_2, L_2) \overleftarrow{g} + (M_3, h_3, L_3)) .
\]

The gluing construction is commutative as well:
\[
(M_1, h_1, L_1) \overleftarrow{g} + (M_2, h_2, L_2) \overrightarrow{g} = (M_2, h_2, L_2) \overleftarrow{g} + (M_1, h_1, L_1) ,
\]
where $\overleftarrow{g}$ means the existence of a diffeomorphism that preserves the metric and $L$. Actually, this diffeomorphism is given by the natural formula
\[
M_1 \times M_2 \ni (x, y) \mapsto (y, x) \in M_2 \times M_1 .
\]
In the case we “glue” $k$ manifolds $(M_i, h_i)$ ($i = 1, \ldots, k$) such that each manifold is equipped with $h_i$-compatible $L_i$, we obtain a metric $g$ on $M = M_1 \times \cdots \times M_k$ and $g$-compatible $L$ on $M$ such that in the matrix notation in the natural coordinate system they have the form
\[
g = \begin{pmatrix}
    h_1 \chi_2(L_1) \cdots \chi_k(L_1) \\
    h_2 \chi_1(L_2) \chi_3(L_2) \cdots \chi_k(L_2) \\
    \cdots \\
    h_k \chi_1(L_k) \cdots \chi_{k-1}(L_k)
\end{pmatrix} ,
\]
\[
L = \begin{pmatrix}
    L_1 \\
    L_2 \\
    \cdots \\
    L_k
\end{pmatrix} .
\]

**Example 2** (Gluing construction and Levi-Civita’s Theorem in dim 3). We now take three intervals $I_1, I_2, I_3$ with the coordinates $x$, resp. $y$, $z$, the metrics $h_1 = dx^2$, $h_2 = -dy^2$, $h_3 = dz^2$, and the $h_i$-compatible $(1, 1)$-tensors $L_1 = X(x)dx \otimes \frac{\partial}{\partial x}$, $L_2 = Y(y)dy \otimes \frac{\partial}{\partial y}$, and $L_3 = Z(z)dz \otimes \frac{\partial}{\partial z}$. We again assume that the spectra of $L_i$ are mutually disjoint at every point, without loss of generality we assume
\[
0 < X(x) < Y(y) < Z(z) \quad \forall x \in I_1, \forall y \in I_2 \forall z \in I_3 .
\]
Applying the gluing construction two times, we obtain that, on $M^3 = I_1 \times I_2 \times I_3$ with the natural coordinate system, the $(1, 1)$-tensor $L$ below is compatible with the metric $g$ below.
\[
g = \begin{pmatrix}
    (Y(y) - X(x))(Z(z) - X(x)) \\
    (Y(y) - X(x))(Z(z) - Y(y)) \\
    (Z(z) - Y(y))(Z(z) - X(x))
\end{pmatrix} .
\]
\[ L = \begin{pmatrix} X(x) & Y(y) \\ Z(z) \end{pmatrix}. \]

Combining this with (4), we obtain a special case of Levi-Civita’s local description of geodesically equivalent metrics in dimension 3 (when the tensor \( G \) has three different eigenvalues).

The **splitting construction** is the inverse operation. We will describe its local version only since it is sufficient for our goals.

Suppose \( g \) is a pseudo-Riemannian metric on \( M^n \) and \( L \) is compatible with \( g \). We consider an arbitrary point \( p \in M \).

We take a point \( p \) of the manifold such that in the neighborhood \( U(p) \) of this point the eigenvalues of \( L \) do not bifurcate (i.e., the number of different eigenvalues is constant in the neighborhood). Then, the eigenvalues are smooth possibly complex-values functions: we denote them by

\[ \lambda_1, \lambda, \ldots, \lambda_r : U(p) \to \mathbb{C}, \quad \lambda_{r+1}, \ldots, \lambda_k : U(p) \to \mathbb{R}. \]

We assume that for \( i \leq r \) the eigenvalue \( \lambda_i \) is complex-conjugate to \( \lambda_i \). We think that the eigenvalue \( \lambda_i \) has algebraic multiplicity \( m_i, 2m_1 + \cdots + 2m_r + m_{r+1} + \cdots + m_k = n \).

Next, let us consider the polynomial functions \( \chi_i : \mathbb{R} \times U(p) \to \mathbb{R} \):

\[ \chi_i = (t - \lambda_i)^{m_i} (t - \bar{\lambda}_i)^{m_i} \text{ for } i = 1, \ldots, r \text{ and } \chi_i = (t - \lambda_i)^{m_i} \text{ for } i = r + 1, \ldots, k, \]

and the polynomial function \( \chi := \chi_1 + \cdots + \chi_k \), where \( \chi_i = \frac{\chi}{\chi_i} \) \text{ and } \chi = \det(t \cdot 1 - L) \text{ is the characteristic polynomial of } L. \text{ It is easy to see that the } (1,1)-\text{tensor } \chi(L) \text{ is } g\text{-selfadjoint and nondegenerate. Then we can introduce a new pseudo-Riemannian metric } h \text{ on } U(p) \text{ by}

\[ h(u,v) := g(\chi(L)^{-1}u,v), \quad u,v \in T_q M, \quad q \in U(p). \]

\textbf{Theorem 2} (Follows from the splitting Lemma, see §2.1 of [9]). \textit{In a neighborhood of } \( p \text{ there exists a coordinate system}

\[ (\bar{x}_1, \ldots, \bar{x}_k) = (x_1^1, \ldots, x_1^{m_1}, \ldots, x_r^1, \ldots, x_r^{m_r}, x_{r+1}^1, \ldots, x_{r+1}^{m_{r+1}}, \ldots, x_k^1, \ldots, x_k^{m_k}) \]

\textit{such that in this coordinate system the matrices of } \( h \) \text{ and of } \( L \) \text{ are given by}

\[ h = \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & \vdots \\ & & & h_k \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & & \\ & L_2 & \\ & & \vdots \\ & & & L_k \end{pmatrix}. \]

\textbf{Moreover,}

\begin{itemize}
  \item the entries of the blocks } h_i \text{ and } L_i \text{ depend on the coordinates } \bar{x}_i \text{ only;}
  \item for } i = 1, \ldots, r \text{ the eigenvalues of } L_i \text{ are } \lambda_i \text{ and } \bar{\lambda}_i, \text{ and for } i = r+1, \ldots, k \text{ the only eigenvalue of } L_i \text{ is } \lambda_i; \quad \text{\textbf{Example 3} (Splitting construction and Dini’s Theorem). We consider a two dimensional manifold } M^2 \text{ and geodesically equivalent Riemannian metrics } g \text{ and } \bar{g} \text{ on it. We take a point where the metrics are not proportional; then, } L(g, \bar{g}) \text{ has two (real) eigenvalues in every point of a small}
\end{itemize}
neighborhood $U(p)$. Then, in the notation of Theorem 2, $k = 2$, $r = 0$, and $m_1 = m_2 = 1$. Then, $\chi_1 = t - \lambda_1$, $\chi_2 = t - \lambda_2$ and $\hat{\chi} = (t - \lambda_2) + (t - \lambda_1)$. Then, there exists a coordinate system $x, y$ such that $h$, $L$, and $\hat{\chi}(L)$ are given by the matrices

$$
\begin{align*}
  h &= \begin{pmatrix}
    \tilde{X}(x) \\
    \tilde{Y}(y)
  \end{pmatrix}, \\
  L &= \begin{pmatrix}
    X(x) & Y(y)
  \end{pmatrix}, \\
  \hat{\chi}(L) &= \begin{pmatrix}
    (Y(y) - X(x)) & X(x) - Y(y)
  \end{pmatrix}.
\end{align*}
$$

Combining this with (4), (13), we see that the metrics $g$ and $\tilde{g}$ are given by

$$
g = (Y(y) - X(x))(\tilde{X}(x)dx^2 + \tilde{Y}(y)dy^2) \text{ and } \begin{pmatrix}
  1 \\
  \tilde{X}(x)
\end{pmatrix} - \begin{pmatrix}
  1 \\
  \tilde{Y}(y)
\end{pmatrix} = \begin{pmatrix}
  \tilde{X}(x)dx^2 \\
  \tilde{Y}(y)dy^2
\end{pmatrix}.
$$

By a coordinate change of the form $x_{new} = x_{new}(x), y_{new} = y_{new}(x)$ one can ‘hide’ $\tilde{X}$ in $dx^2$ and $\tilde{Y}$ in $dy^2$ and obtain the metrics of the form (1).

We call a point $p \in M$ regular; if in some neighborhood $U(p)$ of $p$ the Jordan type of $L$ is constant (that is, the number of eigenvalues and Jordan blocks is the same at all points $x \in U(p)$; the sizes of Jordan blocks are assumed to be fixed too, whereas the eigenvalues can, of course, depend on the point). It is easy to see that almost every point of $M$ is regular, that is, the set of regular points is open and everywhere dense on the manifold.

Now, applying the Splitting Lemma in the neighborhood of a regular point, we obtain the metrics $h_i$ on $2m_i$- or $m_i$-dimensional discs, and $(1,1)$-tensors $L_i$ compatible with $h_i$. Moreover, each $L_i$ has one real or two complex eigenvalues, and the Jordan type of $L_i$ is the same at all points.

If we describe all possible $h_i$ and $L_i$ satisfying these conditions, we will describe then all possible geodesically equivalent metrics $g$ and $\tilde{g}$ near regular points: all possible $g$ and $g$-compatible $L$ can be obtained by the gluing construction (which is given by explicit formulas (11,12), and the metric $\tilde{g}$ is constructed from $g$ and $L$ by the formula $\tilde{g}(\cdot, \cdot) = \frac{1}{\det(L)} g(L^{-1}\cdot, \cdot)$.

Finally, in order to describe the metric and $L$ in the neighborhood of almost any point, it is sufficient to describe the metrics $h_i$ and the $h_i$-compatible $L_i$ such that $L_i$ has one real eigenvalue, or two complex-conjugate eigenvalues, and the type of the Jordan block is the same in the whole neighborhood. We will formulate the result in § 1.3, see Theorems 3, 4, 5 there, the proof of these theorems will be given in Sections 2, 3 and 4.

### 1.3 Canonical forms for basic blocks

Throughout this section we assume that $p \in M$ is a regular point, i.e. the Jordan type of $L$ remains unchanged in some neighborhood of $p$. Our goal is to find local normal forms for compatible $L$ and $g$ nearby $p$.

According to the previous section (Theorem 2), it is sufficient to describe the structure of compatible pairs $(g, L)$ in the case when $L$ either has a single real eigenvalue $\lambda$, or has a pair of complex eigenvalues $\lambda, \bar{\lambda}$. However even in these cases, the situation depends essentially of the algebraic structure of $L$, more precisely of the geometric multiplicity of $\lambda$, i.e., the number of linearly independent eigenvectors. There are three essentially different possibilities: 1) the geometric multiplicity of $\lambda$ is at least two, 2) $L$ is conjugate to a real Jordan block (i.e. $L$ has a single real eigenvalue $\lambda$ of geometric multiplicity one) and 3) $L$ is conjugate to a pair of complex conjugate Jordan blocks (i.e., $L$ has a pair of complex conjugate eigenvalues each of geometric multiplicity one). These cases are described by Theorems 3, 4 and 5 below.

We start with the case of multiplicity $\geq 2$. This situation turns out to be very special. Namely, the following statement holds.

**Theorem 3.** Let $g$ and $L$ be compatible, i.e., satisfy (5) and in a neighborhood $U$ of a point $p \in M$ the operator $L$ has either a unique real eigenvalue $\lambda = \lambda(x)$ or a unique pair of complex conjugate
eigenvalues $\lambda(x), \bar{\lambda}(x)$. Suppose that the geometric multiplicity of $\lambda$ is at least two at each point $x \in U$. Then the function $\lambda(x)$ is constant and $L$ is covariantly constant in $U$, i.e., $\nabla L = 0$. In particular the metrics $g$ and $\bar{g}$ given by (4) are affinely equivalent.

Thus, in the case of geometric multiplicity $\geq 2$, our problem is reduced to another rather non-trivial problem of local classification of pairs $g, L$ satisfying $\nabla L = 0$, which has been recently completely solved by Charles Boubel and we refer to his work [10] for further details.

We now give the answer for $L$ being conjugate to a single real Jordan block, in other words we assume that the eigenvalue $\lambda$ is real and $L$ possesses a unique (up to proportionality) eigenvector.

**Theorem 4.** Let $g$ and $L$ be compatible, i.e., satisfy (5), and $L$ be conjugate to a single Jordan block with a real eigenvalue $\lambda$. Then there exists a local coordinate system $x^1, \ldots, x^n$ such that

$$
g = \begin{pmatrix}
1 & & & & & \\
& a_{n-1} & & & & \\
& & 1 & & & \\
& & & \ddots & & & \\
& & & & a_1 & & \\
& & & & & \cdots & \\
a_{n-1} & a_{n-2} & \cdots & a_2 & \sum_{i=1}^{n-2} a_i a_{n-i-1} & \\
\end{pmatrix}
$$

and

$$
L = \begin{pmatrix}
\lambda(x_n) & 1 & & & & \\
& \lambda(x_n) & 1 & & & \\
& & \ddots & & & \\
& & & \lambda(x_n) & a_{n-1} & \\
& & & & \lambda(x_n) & \\
\end{pmatrix}
$$

where

$$
a_1 = \lambda'_{x_n} x_1,
$$

$$
a_2 = 2 \lambda'_{x_n} x_2,
$$

$$
\ldots
$$

$$
a_{n-2} = (n-2) \lambda'_{x_n} x_{n-2},
$$

$$
a_{n-1} = 1 + (n-1) \lambda'_{x_n} x_{n-1}.
$$

and $\lambda = \lambda(x_n)$ is an arbitrary function. Conversely, if $\lambda = \lambda(x_n)$ is an arbitrary smooth function such that $\lambda(x_n) \neq 0$ for all $x_n$, then $g$ and $L$ given by the above formulas are compatible (in the domain where $g$ is non-degenerate, i.e., $1 + (n-1) \lambda'_{x_n} x_{n-1} \neq 0$).

**Remark 2.** Equivalently, one can write $g$ as the symmetric 2-form

$$
\sum_{k=1}^{n} (dx_k + (k-1) \lambda'_{x_n} x_{k-1} dx_n) (dx_{n-k+1} + (n-k) \lambda'_{x_n} x_{n-k} dx_n).
$$

The (1,1)-tensor $L$, in this notation, takes the following form:

$$
L = \lambda(x_n) \cdot 1 + \sum_{k=1}^{n-1} \partial x_k \otimes dx_k + \lambda'_{x_n} \left( \sum_{k=1}^{n-1} k x_k \partial x_k \right) \otimes dx_n.
$$

**Remark 3.** In the case when $\lambda'_{x_n} \neq 0$ at the point $p$, we can simplify these formulas even further taking $\lambda(x_n)$ as a new coordinate. After the change $x_{n}^{\text{new}} = \lambda(x_n)$ we obtain the following normal forms for $L$ and $g$ (we keep the “old” notation $x_n$ for the “new” coordinate).

Let $g$ and $L$ be compatible, i.e., satisfy (5), and $L$ be conjugate to a Jordan block with a real eigenvalue $\lambda$. If $d\lambda(p) \neq 0$, then in a neighborhood of $p \in M$ there exists a local coordinate system
The transformation that preserves the structure of (15) and (16) has the form

\[ g = \begin{pmatrix} h(x_n) + (n-1)x_{n-1} & \cdots & \cdots & h(x_n) + (n-1)x_{n-2} \\ 1 & \cdots & \cdots & 1 \\ \vdots & \cdots & \cdots & \vdots \\ \end{pmatrix} \]  

and

\[ L = \begin{pmatrix} x_n & 1 & \cdots & 1 \\ x_n & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \vdots \\ x_n & h(x_n) + (n-1)x_{n-1} \\ \end{pmatrix} \]  

where \( \sum = \sum_{i=1}^{n-2} i(n-i+1)x_{i}x_{n-i-1} \) and \( h(x_n) \) is an arbitrary function such that \( h(0) \neq 0 \). Conversely, \( g \) and \( L \) given by these formulas are compatible for every \( h(x_n) \) (in the domain where \( h(x_n) + (n-1)x_{n-1} \neq 0 \)).

**Remark 4.** It follows immediately from the proof (see Section 3) that the canonical coordinate system (and hence canonical forms) for \( g \) and \( L \) from Theorem 4 is uniquely defined (up to a finite group) if we fix the position of the initial point \( p \in M \) by saying that \( p \) is the origin of the canonical coordinate system. If we do not fix \( p \), i.e., move the origin to another point \( p' \in U(p) \), then the function \( h(x_n) \) playing the role of the parameter for canonical forms (15) and (16) changes. It is not difficult to check that the transformation that preserves the structure of (15) and (16) has the following form:

\[
\begin{align*}
\tilde{x}_n &= x_n, \\
\tilde{x}_{n-1} &= x_{n-1} + p(x_n), \\
\tilde{x}_{n-2} &= x_{n-2} - \frac{1}{n-2}p'(x_n), \\
&\vdots \\
\tilde{x}_{n-k} &= x_{n-k} + (-1)^{k-1} \frac{1}{(n-2)(n-3)\cdots(n-k)}p^{(k-1)}(\tilde{x}_n), \\
&\vdots \\
\tilde{x}_1 &= x_1 + (-1)^{n-2} \frac{1}{(n-2)}p^{(n-2)}(x_n).
\end{align*}
\]

Here \( p(x_n) \) is an arbitrary polynomial of degree \( n-2 \) and \( p^{(k)} \) denotes its \( k \)th derivative. The function \( h(x_n) \) after this change of variables takes the form \( h(\tilde{x}_n) - (n-1)p(\tilde{x}_n) \). Thus we see that the function \( h \), the parameter of our canonical form, is defined modulo a polynomial of degree \( n-2 \).

The next case is a complex Jordan block, i.e., we assume that the only eigenvalues of \( L \) are a pair of complex conjugate numbers \( \lambda \) and \( \bar{\lambda} \) for each of which there is a single (up to proportionality) eigenvector over \( \mathbb{C} \). Equivalently, this means that the corank of the real operator \((L - \lambda \cdot \mathbf{1})(L - \bar{\lambda} \cdot \mathbf{1})\) is two. In this case, the normal form for \( g \) and \( L \) can be described in the following way.

**Theorem 5.** Let \( g \) and \( L \) be compatible, i.e., satisfy (5), and \( L \) be conjugate to a complex Jordan block with complex conjugate eigenvalues \( \lambda \) and \( \bar{\lambda} \) (\( \text{Im} \lambda \neq 0 \)). Then there exists a complex structure \( J \) and a local complex coordinate system \((z_1, \ldots, z_n)\) such that

1. the eigenvalue \( \lambda \) is a holomorphic function of \( z_n \),
2. \( L \) is a complex linear operator on \((T_p M, J)\) given in this coordinate system by the matrix:

\[
L^C = \begin{pmatrix} 
\lambda(z_n) & 1 & \cdots & \cdots & 1 \\
\lambda(z_n) & 1 & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\lambda(z_n) & 1 & \cdots & \cdots & 1 \\
\lambda(z_n) & \lambda(z_n) & \cdots & \cdots & \lambda(z_n) \\
\end{pmatrix}
\]
3. the metric $g$ is the real part of the complex bilinear form on $(T_P M, J)$ given in this coordinate system by the matrix:

$$g^C = -i \begin{pmatrix} 1 & a_{n-1} & \cdots & a_1 \\ a_{n-1} & 1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_{n-2} & \cdots & 1 \end{pmatrix} (L^C - \bar{\lambda} \cdot 1)^n,$$

where

$$a_1 = \lambda'_n z_1,$$
$$a_2 = 2 \lambda'_n z_2,$$
$$\cdots$$
$$a_{n-2} = (n-2) \lambda'_n z_{n-2},$$
$$a_{n-1} = 1 + (n-1) \lambda'_n z_{n-1}.$$ 

In the real coordinate system $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ (where $z_k = x_k + iy_k$), the operator $L$ and metric $g$ are defined by the $2n \times 2n$ real matrices which can be obtained from $L^C$ and $g^C$ by following the standard rule:

— each complex entry $a + ib$ of $L^C$ is replaced by the $2 \times 2$ block $egin{pmatrix} a & -b \\ b & a \end{pmatrix}$

— each complex entry $a + ib$ of $g^C$ is replaced by the $2 \times 2$ block $egin{pmatrix} a & -b \\ -b & -a \end{pmatrix}$.

Conversely, $g$ and $L$ defined by the above formulas are compatible for every holomorphic function $\lambda(z_n)$ (in the domain where $\det g \neq 0$, i.e. $1 + (n-1) \lambda'_n z_{n-1}$).

As we see, the case of a complex Jordan block is very similar to the real one. However, there is one very essential difference: the additional factor $(L^C - \bar{\lambda} \cdot 1)^n$ in the formula for $g$. Notice, by the way, that unlike $L^C$ the components of $g^C$ are not holomorphic functions on $M$ because of $\bar{\lambda}$ involved in this formula.

**Remark 5.** If the differential of $\lambda$ does not vanish at the point $p$, then just in the same way as in the case of a real Jordan block, we can take $\lambda$ as the coordinate $z_n$ and obtain the following version of Theorem 5.

Let $g$ and $L$ be compatible, i.e., satisfy (5), and $L$ be conjugate to a complex Jordan block with complex conjugate eigenvalues $\lambda$ and $\bar{\lambda}$ ($\Im \lambda \neq 0$). If $d\lambda(p) \neq 0$, then in a neighborhood of $p$ there exists a complex structure $J$ and a local complex coordinate system $(z_1, \ldots, z_n)$ on $M$ such that

1. $L$ is a complex linear operator on $(T_P M, J)$ given in this coordinate system by the matrix:

$$L^C = \begin{pmatrix} z_n & 1 & \cdots & z_1 \\ z_n & 1 & \cdots & 2z_2 \\ \vdots & \vdots & \ddots & \vdots \\ z_n & h(z_n) + (n-1)z_{n-1} & \cdots & z_n \end{pmatrix}$$

2. the $g$ is the real part of the complex bilinear form on $(T_P M, J)$ given in this coordinate
there exists a local coordinate system \( x \) given as follows

\[
g^C = -i \begin{pmatrix}
1 & h(z_n) + (n-1)z_{n-1} \\
& 1 & \ddots & \vdots \\
& & \ddots & 1 \\
h(z_n) + (n-1)z_{n-1} & \cdots & \cdots & z_1 \\
\end{pmatrix}
\]

where \( \sum = \sum_{i=1}^{n-2} i(n-i+1)z_i z_{n-i-1} \) and \( h(z_n) \) is a holomorphic function such that \( h(0) \neq 0 \).

The passage from the complex coordinates \( z_i \) to real coordinates \( x_k, y_k \) \((z_k = x_k + iy_k)\) follow the same rules as explained in Theorem 5.

**Remark 6.** We can easily rewrite this result in real coordinates. Namely, if the differential of the complex eigenvalue \( \lambda \) does not vanish at the point \( p \in M \), then in a neighbourhood of this point there exists a local coordinate system \( x_1, y_1, \ldots, x_n, y_n \) such that the metric \( g \) and operator \( L \) are given as follows

\[
L = C^{-1}L_0C, \quad g = C^T g_0 \tilde{L}_0^T C
\]

where

\[
L_0 = \begin{pmatrix}
Z_n & 1_2 \\
\vdots & \ddots & \ddots \\
Z_n & \vdots & 1_2 \\
1_2 & \vdots & \vdots \\
1_2 & \cdots & 1_2 \\
\end{pmatrix}, \quad \tilde{L}_0 = \begin{pmatrix}
2iY_n & 1_2 \\
\vdots & \ddots & \ddots \\
2iY_n & \vdots & 1_2 \\
1_2 & \cdots & 0_2 \\
1_2 & \cdots & (n-2)Z_n-2 \\
\end{pmatrix}
\]

Each of the indicated entries represents a \( 2 \times 2 \)-matrix of the following form:

\[
1_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{1}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad 0_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Z_i = \begin{pmatrix} x_i & -y_i \\ y_i & x_i \end{pmatrix}, \quad 2iY_n = \begin{pmatrix} 0 & -2y_n \\ 2y_n & 0 \end{pmatrix}
\]

and

\[
H = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}
\]

with \( u = u(x_n, y_n), v = v(x_n, y_n) \) being functions satisfying the Cauchy–Riemann conditions (i.e., \( h = u + iv \) is a holomorphic function of \( z_n = x_n + iy_n \)).

### 1.4 Perspectives and first global results

It is hard to overestimate the role of the Levi-Civita theorem in the local and global theory of geodesically equivalent Riemannian metrics. Almost all local results are based on it, or can be easily reproved using it. Though Levi-Civita theorem is local, most global (= when the manifold is compact) results on geodesically equivalent Riemannian metrics also use Levi-Civita theorem as an important tool. Roughly speaking, using the Levi-Civita description one can reduce any problem that can be stated using geometric partial geometric equations (for example, any problem involving the curvature) to solving or analysing a system of ODEs.

We expect that our result will play the same role in the pseudo-Riemannian case. We suggest to use it to prove the natural generalization of the projective Lichnerowicz-Obata conjecture and the Sophus Lie problem for the pseudo-Riemannian case, see [9, §2.2] for the description of the problems. Note that the Lichnerowicz-Obata conjecture was solved in the Riemannian case, in [38]...
under additional assumptions and in [31] in full generality, and the solution essentially used the Levi-Civita theorem. The Sophus Lie problem was solved in the Riemannian case for dimensions $n > 2$ in [38]; the solution again used the Levi-Civita theorem, and in the 2-dimensional case for all signatures of the metrics in [33], and the solution essentially used the description of two dimensional Riemannian and pseudo-Riemannian metrics obtained for example in [8].

We also hope that our description will be helpful in understanding of the global structure of the manifolds carrying geodesically equivalent pseudo-Riemannian metrics. One of the ultimate goals could be to understand the “possible topology” of such manifolds. Though our main theorem is local, it can be effectively used (as it was the case with the Levi-Civita theorem) in the global setting as well, we will demonstrate this to prove the following results:

**Theorem 6.** Let $M^n$ be a closed connected manifold. Suppose $g$ and $\bar{g}$ are geodesically equivalent metrics on it. Suppose $L$ given by (2) has a complex (= not real) eigenvalue $\lambda$ in one point. Then, at every point of $M$ this number $\lambda$ is an eigenvalue of $L$.

**Corollary 1.** Let $M^3$ be a closed connected 3-dimensional manifold. Suppose $g$ and $\bar{g}$ are geodesically equivalent metrics on it. Suppose $L$ given by (2) has a complex (= not real) eigenvalue $\alpha + i\beta$ at least at one point. Then, the manifold can be finitely covered by the 3-torus.

2 Proof of Theorem 3: case of geometric multiplicity $\geq 2$

We assume that $(M, g)$ is connected and that (a selfadjoint (1,1)-tensor) $L$ is compatible with $g$. Our first goal is to prove

**Proposition 1.** Assume that in a neighborhood $U \subseteq M$ there exists a continuous function $\lambda : U \to \mathbb{R}$ or $\lambda : U \to \mathbb{C}$ such that for every $x \in U$ the number $\lambda(x)$ is an eigenvalue of $L$ at $x$ of geometric multiplicity at least two. Then, the function $\lambda$ is constant; moreover, for every point $x \in M$ the number $\lambda$ is an eigenvalue of $L$ at $x$ of geometric multiplicity at least two.

**Proof.** Our proof will use the following theorem due to [6, 28, 39]. For any $(1,1)$-tensor $A$ on $M$, let us denote by $\text{co}(A)^T$ the $(1,1)$-tensor whose matrix in a local coordinate system is the comatrix of (= adjoint matrix to) $A$ transposed. It is indeed a well-defined tensor field: smoothness follows from the fact that the components of $\text{co}(A)^T$ are algebraic expressions in the components of $A$. The correct transformation law w.r.t. to the coordinate change is trivial if $A$ is nondegenerate, since in this case $\text{co}(A)^T = \text{det}(A)A^{-1}$. Since nondegenerate matrices are dense in the set of all (quadratic) matrices, the correct transformation law w.r.t. to the coordinate change holds for all $A$.

**Theorem 7.** Let $L$ be compatible with $g$. Then for any $t \in \mathbb{R}$, the function

$$I_t : TM \to \mathbb{R}, \ g(\text{co}(L - t \cdot 1)^T \xi, \xi)$$

is an integral of the geodesic flow of the metric $g$.

Recall that a function $I$ is an integral, if for every geodesic $\gamma$ parameterized by a natural parameter $s$ (such that $\nabla_s \dot{\gamma} = 0$), the function $s \mapsto I(\dot{\gamma}(s))$ is constant.

We will first consider the case when the function $\lambda$ is real. We assume without loss of generality that the (algebraic) multiplicity of the eigenvalue $\lambda(x)$ is the same at all points $x \in U$. Then, $\lambda$ is a smooth function.

First we prove that $\lambda$ is constant. By contradiction, assume that there exists $p \in U$ where the derivative of $\lambda$ is not zero. Then, in a small neighborhood of $p$, the set

$$M_{\lambda(p)} := \{ q \in M \mid \lambda(q) = \lambda(p) \}$$
is a smooth submanifold of $M$ of codimension 1. At every point of $M_{\lambda(p)}$, the matrix of the tensor $(L - \lambda(p) \cdot 1)$ has rank at most $n - 2$ so $\det(L - \lambda(p) \cdot 1)^T = 0$. Consequently, for every point $q \in M_{\lambda(p)}$ and for every $\xi \in T_q M$ we have $I_{\lambda(p)}(\xi) = 0$. Now, take a point $x \in U$, $x \not\in M_{\lambda(p)}$, and consider all geodesics $\gamma_{q,x}$ connecting the points $q \in M_{\lambda(p)}$ with $x$. We assume that the vector $s$ on the geodesic is natural and $\gamma_{q,x}(0) = q \in M_{\lambda(p)}$, $\gamma_{q,x}(1) = x$. Since $M_{\lambda(p)}$ has codimension one, for all $x$ that are sufficiently close to $p$, the vectors that are proportional to the velocity vectors $\gamma_{q,x}(1)$ of such geodesics contains an open nonempty subset. Then, $\det(L - \lambda(p) \cdot 1)^T$ is zero at the point $x$. It follows immediately that $\lambda(p)$ is an eigenvalue of $L$ at the point $x$. Then, $\lambda$ is constant in a neighborhood of $p$ which contradicts our assumption that $d\lambda_{|p} \neq 0$. The contradiction shows that $\lambda$ is a constant on $U$.

Let us now consider the case when $L$ has two complex-conjugate eigenvalues $\lambda, \bar{\lambda} : U \subseteq M \rightarrow \mathbb{C}$. We again assume without loss of generality that the algebraic multiplicity of the eigenvalue $\lambda(x)$ is the same at all points $x \in U$, which in particular implies that $\lambda$ is a smooth function. We first note that, for every $(1,1)$-tensor $A$, the $(1,1)$-tensor $\det(A - t \cdot 1)^T$ is a polynomial in $t$ of degree $n - 1$ whose coefficients are $(1,1)$-tensors. Then, for every complex number $\tau$, the real and imaginary parts of the complex-valued function

$$I_\tau : TM \rightarrow \mathbb{C}, \quad I_\tau(\xi) := \det(\tau(L - \lambda \cdot 1)^T \xi, \xi)$$

are also integrals. Since rank $(L(q) - \lambda(q) \cdot 1)\leq n - 2$, for every $q$ such that $\lambda(q) = \tau$ we have that $I_\tau(\xi) = 0$ for every $\xi \in T_q M$.

Suppose $\lambda$ is not constant in $U$; then for a certain point $p$ of $U$ its differential is not zero. Suppose first that the differential of the real part of $\lambda$ is proportional to the differential of the imaginary part in all points of a certain neighborhood of $p$. Then, in a sufficiently small neighborhood $U'(p) \subseteq U$ of $p$ the set $M_{\lambda(p)} := \{q \in M \mid \lambda(q) = \lambda(p)\}$ is a submanifold of dimension $n - 1$, as it was in the case of a real eigenvalue $\lambda$. Then, repeating the same arguments as above we conclude that $\lambda(x) = \lambda(p)$ for all $x$ from a small neighborhood of $p$, which gives us a contradiction with the assumption that the differential of $\lambda$ does not vanish at $p$. The contradiction shows that $\lambda$ is a constant provided the differential of the real part of $\lambda$ is proportional to the differential of the imaginary part in some $U' \subseteq U$.

Let us now suppose that the differential of the real part of $\lambda$ at the point $p$ is not proportional to the differential of the imaginary part. Then, the set $M_{\lambda(p)} := \{q \in M \mid \lambda(q) = \lambda(p)\}$ is (in a sufficiently small neighborhood $U'(p) \subseteq U$) a submanifold of dimension $n - 2$. We again take an arbitrary point $x$ that is sufficiently close to $p$ and consider all geodesics $\gamma_{q,x}$ connecting the points $q \in M_{\lambda(p)}$ with $x$ assuming as above that $\gamma_{q,x}(0) \in M_{\lambda(p)}$ and $\gamma_{q,x}(1) = x$. The set of the tangent vectors at $x$ that are proportional to the velocity vectors to such geodesics at the point $x$ contains a submanifold of codimension 1 implying that the real part of $I_{\lambda(p)}(\xi)$ is proportional to the imaginary part of $I_{\lambda(p)}(\xi)$ for all $\xi \in T_x M$ (the coefficient of the proportionality is a constant on each $T_x M$ but may a priori depend on $x$). Now, since both functions, the real and the imaginary parts of $I_{\lambda(p)}$, are integrals, the coefficient of the proportionality of these functions is an integral as well implying it is constant. Then, for a certain complex constant $a + ib \neq 0$, for every $x \in U'(p)$ and every $\xi \in T_x M$ we have $(a + ib)I_{\lambda(p)}(\xi) = (a - ib)I_{\lambda(p)}(\xi)$ so

$$(a + ib)\det(L - \lambda \cdot 1)^T = (a - ib)\det(L - \bar{\lambda} \cdot 1)^T.$$  \hspace{1cm} (18)$$

For points $x$ such that $\lambda(x) \neq \lambda(p)$ the matrix $L - \lambda(p) \cdot 1$ is nondegenerate and (18) implies that $L$ is proportional to $1$ which contradicts the assumption that $\lambda$ is not real. The contradiction shows that at all points of the neighborhood $U$ (such that $\lambda(x)$ is an eigenvalue of $L$ of geometric multiplicity at least two at every point $x$) the function $\lambda$ is a constant.

Let us now show that this (real or complex) constant $\lambda$ is an eigenvalue of $L$ of geometric multiplicity at least two at every point of the whole $M$. We first consider a point $p \in M \setminus U$ such that one can connect it with a point of $U$ by a geodesic $\gamma$, where $U$ is a neighborhood such that at each its point $L$ has (constant) eigenvalue $\lambda$ of multiplicity at least 2; we think that $\gamma(0) = p$.
and \( \gamma(1) \in U \). We consider a small open neighborhood \( V \subseteq T_p M \) of \( \xi = \dot{\gamma}(0) \). If \( V \) is sufficiently small, for every \( \eta \in V \) the point \( \gamma_{p,\eta}(1) \) of the geodesic \( \gamma_{p,\eta} \) such that \( \gamma(0) = p \) and \( \dot{\gamma}(0) = \eta \) lies in \( U \). Since at each point of \( U \) the constant \( \lambda \) is an eigenvalue of \( L \) of multiplicity at least 2, \( I_\lambda(\dot{\gamma}_{p,\eta}(1)) = 0 \) implying \( I_\lambda(\dot{\gamma}_{p,\eta}(0)) = 0 \). Hence, \( I_\lambda \equiv 0 \) on a nonempty open subset of \( T_p M \) implying \( I_\lambda \equiv 0 \) on the whole \( T_p M \) so \( \lambda \) is an eigenvalue of \( L \) at \( p \) of multiplicity at least two. Now, if \( p \) can be connected by a geodesic with a point of \( U \), then any point from a sufficiently small neighborhood of \( p \) can also be connected by a geodesic with a point of \( U \) so \( \lambda \) is an eigenvalue of \( L \) of multiplicity at least two at every point of a small neighborhood of \( p \). To come to the same conclusion on the whole \( M \), it suffices to notice that every point \( x \in M \) can be joined with \( U \) by a piecewise smooth curve such that its each smooth segment is a geodesic. Proposition 1 is proved.

**Corollary 2.** In the hypotheses of Proposition 1, assume in addition that \( \lambda \) is the unique eigenvalue of \( L \) (or \( \lambda, \bar{\lambda} \) are unique eigenvalues of \( L \), if \( \lambda \in \mathbb{C} \)), then \( L \) is covariantly constant, i.e., \( \nabla L = 0 \) or, equivalently, \( g \) and \( \bar{g} \) are affinely equivalent.

The proof is obvious: since \( \lambda \) is constant, so is \( \text{tr } L \). Hence, the right hand side of (5) vanishes, and we get \( \nabla L = 0 \).

Thus, if the eigenspace of \( L \) has dimension \( \geq 2 \), then our problem is reduced to the classification of pairs of affinely equivalent pseudo-Riemannian metrics, which has been recently obtained by Boubel in [10].

### 3 Proof of Theorem 4: case of a real Jordan block

#### 3.1 Canonical frames and uniqueness lemma

Let \( L \) be \( g \)-selfadjoint operator on a real vector space \( V \). It is a natural question to ask to which canonical form we can reduce (the matrices of) \( L \) and \( g \) simultaneously by an appropriate change of a basis. The answer is well known (see, for example, [24]) and is given by the following

**Proposition 2.** There exists a canonical basis \( e_1, \ldots, e_n \in V \) in which \( L \) and \( g \) can be simultaneously reduced to the following block diagonal canonical forms:

\[
L_{\text{can}} = \begin{pmatrix} L_1 & & \\ & L_2 & \\ & & \ddots \end{pmatrix}, \quad g_{\text{can}} = \begin{pmatrix} g_1 & & \\ & g_2 & \\ & & \ddots \end{pmatrix},
\]

where

\[
g_j = \begin{pmatrix} \begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{array} \end{pmatrix},
\]

and

\[
L_j = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix}.
\]
in the case of a real eigenvalue $\lambda \in \mathbb{R}$ (real Jordan block), or

$$L_j = \begin{pmatrix}
a & -b & 1 & 0 \\
b & a & 0 & 1 \\
a & -b & \ddots & \ddots \\
\ddots & \ddots & 1 & 0 \\
0 & 1 & a & -b \\
b & a & \end{pmatrix} \quad (21)$$

in the case of complex conjugate eigenvalues $\lambda_{1,2} = a \pm ib$ (complex Jordan block). It is assumed that for each $j$ the blocks $g_j$ and $L_j$ are of the same size, and that the corresponding eigenvalues depend on $j$.

Notice that the canonical forms $g_{\text{can}}$ and $L_{\text{can}}$ can be chosen in many different ways. For example, in the complex case we can replace $g_{\text{can}}$ by $g_{\text{can}}p(L_{\text{can}})$ where $p(t)$ is an arbitrary polynomial such that $p(L_{\text{can}})$ is invertible. For our purposes, by a canonical form of $L$ and $g$ it is convenient to understand any forms where the entries of $L_{\text{can}}$ and $g_{\text{can}}$ depend on the eigenvalues of $L$ only.

Now let $g$ be a pseudo-Riemannian metric on a smooth manifold $M$ and $L$ be a $g$-self-adjoint $(1,1)$ tensor field. Assume that $L$ is regular at each point of a small neighborhood $U(p_0)$ of a point $p_0 \in M$. Recall that the regularity of $L$ means that each eigenvalue of $L$ is of geometric multiplicity one or, equivalently, the Jordan normal form contains exactly one Jordan block for each eigenvalue. This condition implies that the eigenvalues of $L$ are smooth functions on $U(p_0)$ and the Jordan type of $L$ does not change in $U(p_0)$, in particular, $p_0$ is a regular point. In such a situation we can choose smooth linearly independent vector fields $e_1, \ldots, e_n \in T_p M$ in which $L$ and $g$ both take canonical forms. For a regular $L$, these vectors are uniquely defined (up to a discrete group), and we will say that $e_1, \ldots, e_n$ is a canonical moving frame for $L$ and $g$.

In general, the vectors $e_1, \ldots, e_n$ do not commute. To reconstruct a canonical coordinate system on $U(p_0)$ we need to analyse the commutation relations between them. It turns out that these relations can be obtained from the compatibility equation (5).

**Lemma 1.** Let $e_1, \ldots, e_n$ be a canonical moving frame for $L$ and $g$ in a neighborhood of a generic point $p_0 \in M$. If $L$ and $g$ are compatible and $L$ is regular, then the covariant derivatives $\nabla_{e_i} e_j$ and hence the commutators $[e_i, e_j]$ can be uniquely expressed as certain linear combinations of $\xi_l$ with coefficients being functions of $e_s(\lambda_r)$ and $\lambda_r$, where $\lambda_r$ are eigenvalues of $L$.

**Proof.** For the frame $e_1, \ldots, e_n$ we introduce $B_u$ to be $(1,1)$-tensor field defined by

$$B_u v = \nabla_u v \quad (22)$$

where $u$ and $v$ are vector fields with constant coordinates w.r.t. the frame.

Clearly, $B_u$ defines the Levi-Civita connection in the frame $e_1, \ldots, e_n$ and our goal is to reconstruct it from the compatibility equation (5). The covariant derivative of $L$ in terms of $B_u$ can be written as

$$\nabla_u L = D_u(L) + [B_u, L]$$

where $D_u L$ denotes the operator obtained by differentiating $L$ componentwise along $u = u^k e_k$, i.e., for $L = L^i_j e_i \otimes e_j$, we have $D_u L = u^k e_k (L^i_j) e_i \otimes e_j$.

To find $B_u$, it is convenient to rewrite the compatibility equation in the form

$$[B_u, L] = \frac{1}{2} (u \otimes dtr L + (u \otimes dtr L)^*) - D_u L. \quad (23)$$

In addition to that we have

$$D_u (g(e_i, e_j)) = \nabla_u g(e_i, e_j) = g(B_u e_i, e_j) + g(e_i, B_u e_j)$$
or, equivalently

\[ Bu + B_u^* = g^{-1}D_u g \]

Thus, \( B_u \) satisfies two equations of the form

\[
\begin{align*}
[B_u, L] &= C \\
B_u + B_u^* &= D
\end{align*}
\]  

(24)

where \( C \) and \( D \) are certain operators (whose components w.r.t. the moving frame are functions of the eigenvalues \( \lambda_r \) and their derivatives \( e_s(\lambda_r) \)).

The uniqueness of the solution (if it exists!) is a purely algebraic fact. Indeed, consider the corresponding homogeneous system

\[
\begin{align*}
[B_u, L] &= 0 \\
B_u + B_u^* &= 0
\end{align*}
\]  

(25)

The first equation means that \( B_u \) commutes with \( L \), i.e., belongs to the centralizer of \( L \). Since \( L \) is regular, its centralizer is generated by powers of \( L \), i.e., \( 1, L, L^2, \ldots, L^{n-1} \). It follows from this that \( B_u \) is \( g \)-selfadjoint. But then the second equation can be rewritten simply as \( 2B_u = 0 \). Thus, the homogeneous system has the only trivial solution which proves the statement.

The commutators \([e_i, e_j]\) can now be uniquely reconstructed by means of the standard formula:

\[
[e_i, e_j] = \nabla e_i e_j - \nabla e_j e_i = B_{e_i} e_j - B_{e_j} e_i.
\]

In the next section we show how the commutation relations between the elements of the canonical moving frame can be found in practice.

Notice that Lemma 1 does not say that (24) is always consistent for every \( C \) and \( D \). In fact, these matrices have to satisfy some additional relations (for example, \( \text{tr} CL^k = 0 \)). These equations, in particular, imply vanishing of the Nijenhuis torsion of \( L \) and, therefore, the fact that \( e_s(\lambda_r) = 0 \) for those \( e_s \) which “do not belong” to the \( \lambda_r \)-block. Another condition of this kind is discussed below in Lemma 2.

### 3.2 Canonical frame and canonical coordinate system for a real Jordan block

Let \( L \) be conjugate to a Jordan block with a real eigenvalue. Then we can chose a moving frame \( e_1, \ldots, e_n \) in which \( L \) and \( g \) take the following canonical forms:

\[
L_{\text{can}} = \begin{pmatrix}
\lambda(x) & 1 \\
& \ddots & \ddots \\
& & \lambda(x) & 1 \\
& & & \ddots & \ddots \\
& & & & \lambda(x)
\end{pmatrix}, \quad g_{\text{can}} = \pm \begin{pmatrix}
1 \\
& \ddots \\
& & 1 \\
& & & \ddots \\
& & & & 1
\end{pmatrix}
\]  

(26)

Here we apply the ideas from § 3.1 to describe the commutation relations between \( \xi_1, \ldots, \xi_n \) and then to solve them in order to construct a canonical coordinate system for the pair \( g \) and \( L \).

As usual, it is convenient to decompose \( L \) canonically into the semisimple and nilpotent parts:

\[ L = \lambda(x) \cdot 1 + N. \]

Obviously, \( N \) is self-adjoint with respect to \( g \). The compatibility equation can naturally be rewritten in terms of \( N \):

\[
\nabla_u(L) = u(\lambda) \cdot 1 + \nabla_u N = \frac{n}{2} (u \otimes d\lambda + (u \otimes d\lambda)^*)
\]

or

\[
[B_u, N] = \frac{n}{2} (u \otimes d\lambda + (u \otimes d\lambda)^*) - u(\lambda) \cdot 1,
\]

(27)
where $B_u$, as before, is defined by (22), and we use the fact that the components of $N$ in the frame are all constants so that $\nabla_u N = [B_u, N]$. This equation implies

**Lemma 2.** We have $e_i(\lambda) = 0$, for $i = 1, \ldots, n - 1$.

**Proof.** This property is well known for $L$ with zero Nijenhuis torsion (for $L$ this condition is fulfilled, see e.g. [6, Theorem 1]). However, we can easily derive this fact from (27). Indeed, multiply the both sides of this equation by $N$

Thus, for any vector $v = N\lambda \in \text{Im} N$, we have $v(\lambda) = 0$. It remains to notice that $\text{Im} N = \text{span}(\xi_1, \ldots, \xi_{n-1})$. Thus, in our basis $\text{d}\lambda = (0, \ldots, 0, e_n(\lambda))$. This allows us to get the following explicit form for the right hand side of (27):

$$\nabla_u N = [B_u, N] = e_n(\lambda) \begin{pmatrix} \frac{n-2}{2}u_n & \frac{n}{2}u_{n-1} & \cdots & \frac{n}{2}u_2 & \frac{n}{2}u_1 \\ -u_n & \frac{n}{2}u_2 & \cdots & \frac{n}{2}u_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -u_n & \frac{n}{2}u_2 & \cdots & \frac{n}{2}u_2 & \frac{n}{2}u_n \\ -u_n & \frac{n}{2}u_{n-1} & \cdots & \frac{n}{2}u_2 & \frac{n}{2}u_n \end{pmatrix}$$  \hspace{1cm} (28)

According to Lemma 1, the solution of this equation is unique. We just give the final answer (reader can check this result by substitution into (28)).

$$B_u = e_n(\lambda) \begin{pmatrix} \frac{n}{2}u_{n-1} & \frac{n}{2}u_{n-2} & \cdots & \frac{n}{2}u_2 & \frac{n}{2}u_1 & 0 \\ (1 - \frac{n}{2})u_n & 2 & \cdots & \frac{n}{2}u_2 & \frac{n}{2}u_2 & -\frac{n}{2}u_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (2 - \frac{n}{2})u_n & \cdots & \frac{n}{2}u_2 & \frac{n}{2}u_2 & \frac{n}{2}u_n & -\frac{n}{2}u_1 \\ (\frac{n}{2} - 2)u_n & \cdots & \frac{n}{2}u_2 & \frac{n}{2}u_2 & \frac{n}{2}u_n & -\frac{n}{2}u_1 \\ (\frac{n}{2} - 1)u_n & \cdots & \frac{n}{2}u_2 & \frac{n}{2}u_2 & \frac{n}{2}u_n & -\frac{n}{2}u_1 \\ \end{pmatrix}$$  \hspace{1cm} (29)

The next step is to find pairwise commutators $[e_i, e_j]$.

**Lemma 3.** The vector fields $e_1, \ldots, e_{n-1}$ commute.

**Proof.** Let $u = u_1 e_1 + \cdots + u_n e_n$. It follows from (29) that

$$\nabla_u e_j = e_n(\lambda) \left( \frac{n}{2}u_{n-j} e_1 + (j - \frac{n}{2})u_n e_{j+1} \right), \quad j < n.$$

Hence $\nabla e_i e_j = e_n(\lambda) \frac{n}{2} e_1$ if and only if $i + j = n$, otherwise $\nabla e_i e_j = 0$. In any case

$$[e_i, e_j] = \nabla e_i e_j - \nabla e_j e_i = 0$$

for $i, j < n$.

It remains to find the commutators $[e_i, e_n]$.

**Lemma 4.** For $i = 1, \ldots, n - 1$, we have

$$[e_i, e_n] = -i e_n(\lambda) \cdot e_{i+1}.$$
Proof. From (29) we have
\[ \nabla_{e_r} e_n = -\frac{n}{2} e_n(\lambda) \cdot e_{i+1} \quad \text{and} \quad \nabla_{e_r} e_i = -(i-\frac{n}{2}) e_n(\lambda) \cdot e_{i+1}. \]
Thus, \[ [e_i, e_n] = -e_n(\lambda) \left( \frac{n}{2} e_{i+1} + -(i-\frac{n}{2}) e_{i+1} \right) = -i e_n(\lambda) \cdot e_{i+1}, \]
as stated. \[ \square \]
Our goal now is to find a coordinate system with respect to which \( N \) and \( g \) have the simplest form. Since the vector fields \( e_1, \ldots, e_{n-1} \) commute we can choose a coordinate system \( x_1, \ldots, x_n \) in such a way that \( e_1 = \partial_{x_1}, \ldots, e_{n-1} = \partial_{x_{n-1}} \). To make our choice unambiguous, we assume that our point \( p_0 \in M \) has all coordinates zero and, in addition,
\[ e_n = \partial_{x_n} \quad \text{on the } x_n\text{-axes}, \quad (30) \]
i.e. on the curve \( x_1 = x_2 = \cdots = x_n = 0 \). Notice that the foliation generated by \( \text{Im} \, N \) is given by \( x_n = \text{const} \) and the eigenvalue \( \lambda \) depends on \( x_n \) only.
To rewrite \( L \) and \( g \) in this coordinate system we just need to find the transition matrix between \( e_1, \ldots, e_n \) and \( \partial_{x_1}, \ldots, \partial_{x_n} \). Since \( \partial_{x_i} = e_i, i = 1, \ldots, n-1 \), it remains to determine the coefficients (yet unknown) of the linear combination
\[ \partial_{x_n} = a_0 e_1 + \cdots + a_{n-1} e_n \]
First we use the fact that \( \lambda \) does not depend on \( x_1, \ldots, x_{n-1} \). Therefore
\[ \lambda'_{x_n} = \partial_{x_n}(\lambda) = (a_0 e_1 + \cdots + a_{n-1} e_n)(\lambda) = a_{n-1} e_n(\lambda). \]
Since \( \partial_{x_n} \) must commute with each \( e_i = \partial_{x_i} \) (\( i < n \)), we obtain a system of differential equations on \( a_j \):
\[ 0 = [e_i, a_0 e_1 + \cdots + a_{n-1} e_n] = \sum_{i=1}^n \frac{\partial a_{i-1}}{\partial x_i} \cdot e_i - a_{n-1} i e_n(\lambda) \cdot e_{i+1} = \sum_{i=1}^n \frac{\partial a_{i-1}}{\partial x_i} \cdot e_i - i \lambda'_{x_n} \cdot e_{i+1}, \]
or, equivalently,
\[ \frac{\partial a_{i-1}}{\partial x_i} = 0, \quad \text{if } l \neq i + 1 \quad \text{and} \quad \frac{\partial a_{i-1}}{\partial x_i} = i \lambda'_{x_n}, \quad i = 1, \ldots, n - 1. \]
In other words, \( a_0 = a_0(x_n) \), whereas \( a_{i-1} \) depends on \( x_{i-1} \) and \( x_n \) and satisfies the equation
\[ \frac{\partial a_{i-1}}{\partial x_{i-1}} = (i-1) \lambda'_{x_n}, \quad i \neq 1, \]
which can be easily solved. Its general solution is
\[ a_{i-1}(x_{i-1}, x_n) = (i-1) \lambda'_{x_n} x_{i-1} + f_i(x_n), \]
where \( f_i(x_n) \) is an arbitrary function. But we have a kind of initial condition (30) that requires
\[ a_{i-1}(0, \ldots, 0, x_n) = 0 \quad \text{for } i \neq n, \quad \text{and} \quad a_n(0, \ldots, 0, x_n) = 1. \]
It follows immediately from this that
\[ a_{i-1} = (i-1) \lambda'_{x_n} x_{i-1}, \quad i \neq n, \quad (31) \]
and
\[ a_{n-1} = (n-1) \lambda'_{x_n} x_{n-1} + 1. \quad (32) \]
In this section we assume that $L$ has two complex conjugate eigenvalues $\lambda = a + ib$ and $\bar{\lambda} = a - ib$, $b \neq 0$ (each of geometric multiplicity one), so that $L$ and $g$ can be simultaneously reduced to the following canonical forms

$$L_{\text{can}} = \begin{pmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ a & -b & \ddots & \ddots \\ b & a & \ddots & \ddots & \ddots \\ & 1 & 0 & \ddots & \ddots \\ & & 0 & 1 & \ddots & \ddots \\ & & & a & -b & \ddots \\ & & & & b & a \end{pmatrix}$$

and $g_{\text{can}} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix}$ (33)

By using the “moving frame” machinery as above, we can find the commutation relations between the elements of the canonical frame (associated with the canonical forms (33) of $L$ and $g$) and describe the corresponding canonical coordinate system. However, this approach leads to serious technical difficulties because the commutation relations turn out to be quite complicated. To simplify them we will change the canonical forms $L$ and $g$ in a certain way which is, in fact, motivated by the splitting construction from [9] which we recalled in §1.2. Namely, we set:

$$L_{\text{can}} = L_{\text{can}}^{\text{old}}, \quad g_{\text{can}} = g_{\text{can}}^{\text{old}}((L_{\text{can}}^{\text{old}} - \lambda \cdot 1)^n + (L_{\text{can}}^{\text{old}} - \bar{\lambda} \cdot 1)^n),$$

where $L_{\text{can}}^{\text{old}}$ and $g_{\text{can}}^{\text{old}}$ are as in (33), and $n = \frac{1}{2} \dim M$. Notice that the operator $((L - \lambda \cdot 1)^n + (L - \bar{\lambda} \cdot 1)^n)$ is real, so $g_{\text{can}}$ is a real symmetric matrix.

Let $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$ be the canonical frame associated with these (real) canonical forms. To simplify the commutation relations between them, we need one more modification. Namely, we pass from $e_i, f_i$ ($i = 1, \ldots, n$) to the natural complex frame $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ by putting

$$\xi_k = \frac{1}{2}(e_k - if_k), \quad \eta_k = \frac{1}{2}(e_k + if_k) = \bar{\xi}_k,$$ (35)

Thus, from now on we allow ourselves to use formal complex combinations of tangent vectors, i.e., we pass from the real tangent bundle $TM$ to its complexification $T^cM$. In particular, we consider...
the complex vector fields \( \xi = e + if, e, f \in \Gamma(TM) \), and treat them as differential operators on the space of complex-valued smooth functions \( w(x) = u(x) + iv(x) \) on \( M \):

\[
\xi(w) = (e(u) - f(v)) + i(e(v) + f(u)).
\]

The commutators of complex-valued vector fields and other objects of this kind are defined in the natural way.

According to Lemma 1, we now can uniquely reconstruct the commutation relations between the elements of the frame \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \) and information about derivatives of \( \lambda \) and \( \bar{\lambda} \) along these elements. Here is the result

**Proposition 3.** Let \( e_1, f_1, e_2, f_2, \ldots, e_n, f_n \) be the canonical frame associated with canonical forms (34). Then the complex frame

\[
\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n
\]

defined by (35) satisfies the following properties:

1. \( \xi_k \) and \( \eta_m \) commute for all \( k, m \);
2. \( \xi_1, \ldots, \xi_{n-1} \) commute and \( \eta_1, \ldots, \eta_{n-1} \) commute (in particular, all real vector fields \( e_k \) and \( f_m \) commute for all \( k, m \leq n - 1 \));
3. the only non-trivial derivatives are \( \xi_n(\lambda) \) and \( \eta_n(\bar{\lambda}) \);
4. non-trivial commutation relations are:

\[
\begin{align*}
[\xi_1, \xi_n] & = -\xi_n(\lambda) \cdot \xi_2, \\
[\xi_2, \xi_n] & = -2\xi_n(\lambda) \cdot \xi_3, \\
[\xi_3, \xi_n] & = -3\xi_n(\lambda) \cdot \xi_4, \\
& \vdots \\
[\xi_{n-1}, \xi_n] & = -(n-1)\xi_n(\lambda) \cdot \xi_n, \\
[\eta_{n-1}, \eta_n] & = -(n-1)\eta_n(\bar{\lambda}) \cdot \eta_n.
\end{align*}
\]

Proof. One can find these relations by straightforward (linear-algebraic) computation, but we shall give another proof based on the splitting construction (see §1.2) and the uniqueness lemma (Lemma 1).

Before discussing the case of a complex Jordan block (more precisely, of two complex conjugate blocks), consider the case of two real Jordan blocks with distinct eigenvalues as an illustrating example. Take two compatible pairs \((g_1, L_1)\) and \((g_2, L_2)\), each of which represents a single Jordan block with eigenvalue \( \lambda_i \in \mathbb{R} \) (see the previous section for complete description). Let \( \xi_1, \ldots, \xi_n \) be the canonical frame for the first pair \((g_1, L_1)\) and \( \eta_1, \ldots, \eta_k \) for the second one \((g_2, L_2)\). The gluing lemma (Theorem 1) allows us to construct a new compatible pair \( L, g \) by putting:

\[
L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \left( \chi_1(L) + \chi_2(L) \right) (36)
\]

where \( \chi_i(t) \) is the characteristic polynomial of \( L_i \). The compatibility of \( L_i \) and \( g_i, i = 1, 2 \), guarantees the compatibility of \( L \) and \( g \).

Now ask ourselves the converse question. Let \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k \) be the canonical frame for a compatible pair \( g, L \) having the (non-standard) canonical form (36) with \( L_i, g_i \) being the standard canonical forms as (26). What are commutation relations between the elements of the frame and conditions on the derivatives of the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) along the frame? We mean, of course, those relations which can be derived from the compatibility equation for \( g \) and \( L \).

Using the uniqueness result (Lemma 1), we immediately conclude that these relations will be exactly the same as for two separate Jordan blocks, namely, \( \xi_i \)'s commute with \( \eta_j \)'s and the relations within each of these two groups will be those given in Lemmas 3 and 4.
We now notice that in this construction noting changes if we allow $\lambda_1$ and $\lambda_2$ to be complex conjugate, i.e., $\lambda_1 = \lambda$ and $\lambda_2 = \bar{\lambda}$, $\text{Im} \lambda \neq 0$. The elements $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ of the canonical frame will be, of course, vectors of the complexified tangent space $(T_p M)^C$. The point is that Lemmas 1, 2, 3 and 4 are of purely algebraic nature and therefore can be applied for complexified objects without any change.

If $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$ is the canonical frame associated with the (real) canonical forms (34), then in the complex frame $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ defined by (35), $L_{\text{can}}$ and $g_{\text{can}}$ take the form:

$$L_{\text{can}} \mapsto L'_{\text{can}} = \begin{pmatrix} \lambda & 1 & \cdots & \cdots & \cdots \\
\lambda & 1 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \lambda \end{pmatrix} = \begin{pmatrix} L_\lambda & 0 \\
0 & L_\bar{\lambda} \end{pmatrix},$$

$$g_{\text{can}} \mapsto g'_{\text{can}} = \frac{1}{2} \begin{pmatrix} -i & \cdots & \cdots & \cdots & i \\
\cdots & \ddots & \ddots & \ddots & \cdots \\
\cdots & \ddots & \ddots & \ddots & \cdots \\
\cdots & \ddots & \ddots & \ddots & \cdots \\
i & \cdots & \cdots & \cdots & -i \end{pmatrix} \cdot (\chi_1(L'_{\text{can}}) + \chi_2(L'_{\text{can}})), $$

where $\chi_1(t) = (t - \lambda)^n$ and $\chi_2(t) = (t - \bar{\lambda})^n$ are the characteristic polynomials of $L_\lambda$ and $L_{\bar{\lambda}}$ respectively.

According to Lemma 1, we now can uniquely reconstruct the commutation relations between the elements of the frame $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ and information about derivatives of $\lambda$ and $\bar{\lambda}$ along these elements. This reconstruction can be done by straightforward computation, but we do not need to repeat it once again. Instead it suffices to notice that we are now exactly in the same situation as in the case of two real Jordan blocks, see formula (36) and discussion around. So we can repeat the above argument to get the conclusion of Proposition 3.

Therefore, the Nijenhuis torsion $N(L)$ of $L$ vanishes is well known (see e.g. [6]), the implication $N(L) = 0$ if $N(J) = 0$ can be verified directly. Alternatively, one can use [9, Lemma 6]. Indeed, $J$ can be represented as $J = f(L)$ where $f$ is the analytic (locally constant!) function defined on $\mathbb{C} \setminus \mathbb{R}$ in the following way: $f(a + ib) = i$ if $b > 0$ and $f(a + ib) = -i$ if $b < 0$.

For the basis vectors $\xi_k$ and $\eta_k$ we have $J_f \xi_k = i \xi_k$ and $J_f \eta_k = -i \eta_k$. This means that for any holomorphic coordinate system $z_1, \ldots, z_n$, the vectors $\xi_k$'s are linear combinations of $\partial_{z_k}$ and $\eta_k$'s are linear combinations of $\partial_{\bar{z}_k}$. Moreover, item 1 of Proposition 3 says that the vector fields $\xi_1, \ldots, \xi_n$ are holomorphic.

Remark 7. The fact that the Nijenhuis torsion $N(L)$ of $L$ vanishes is well known (see e.g. [6]), the implication $N(L) = 0$ if $N(J) = 0$ can be verified directly. Alternatively, one can use [9, Lemma 6]. Indeed, $J$ can be represented as $J = f(L)$ where $f$ is the analytic (locally constant!) function defined on $\mathbb{C} \setminus \mathbb{R}$ in the following way: $f(a + ib) = i$ if $b > 0$ and $f(a + ib) = -i$ if $b < 0$.

For the basis vectors $\xi_k$ and $\eta_k$ we have $J_f \xi_k = i \xi_k$ and $J_f \eta_k = -i \eta_k$. This means that for any holomorphic coordinate system $z_1, \ldots, z_n$, the vectors $\xi_k$'s are linear combinations of $\partial_{z_k}$ and $\eta_k$'s are linear combinations of $\partial_{\bar{z}_k}$. Moreover, item 1 of Proposition 3 says that the vector fields $\xi_1, \ldots, \xi_n$ are holomorphic.

From now on we can forget about $\eta_k$'s and work with $\xi_k$'s only. The following repeats the arguments for the real Jordan blocks, but in the complex (holomorphic) setting. Since the holomorphic vector

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fields $\xi_1, \ldots, \xi_{n-1}$ pairwise commute we can find a local complex coordinate system $z_1, \ldots, z_n$ such that

$$\xi_k = \partial_{z_k} \quad k = 1, \ldots, n - 1$$

Moreover, this coordinate system can be chosen in such a way that on the two-dimensional surface $z_1 = z_2 = \cdots = z_{n-1} = 0$, we have

$$\xi_n = \partial_{z_n}. \quad (37)$$

The eigenvalue $\lambda$ is a holomorphic function (since $\xi_k$'s are holomorphic and $[\xi_{n-1}, \xi_n] = -(n-1)\xi_n(\lambda) \cdot \xi_n$). Moreover, $\lambda$ depends of $z_n$ only because $\xi_k(\lambda) = \partial_{z_k}(\lambda) = 0$, $k = 1, \ldots, n - 1$.

Now our goal is to determine the transition matrix between two bases $\xi_1, \ldots, \xi_n$ and $\partial_{z_1}, \ldots, \partial_{z_n}$. As before, we consider the relation

$$\partial_{z_n} = \sum_{k=1}^{n} a_{k-1} \xi_k = \sum_{k=1}^{n-1} a_{k-1} \partial_{z_k} + a_{n-1} \xi_n.$$  

Applying this differential operator to $\lambda$ we obtain:

$$\lambda'_{z_n} = \frac{\partial \lambda}{\partial z_n} = a_{n-1} \xi_n(\lambda)$$

Next applying it to $z_k$, $k = 1, \ldots, n - 1$, we see that $a_k$ depends on $z_k$ and $z_n$ only and satisfies:

$$\frac{\partial a_k}{\partial z_k} = k a_{n-1} \lambda' = k \frac{\partial \lambda}{\partial z_n}.$$ 

These equations can be easily solved:

$$a_k = k z_k \frac{\partial \lambda}{\partial z_n} + h_k(z_n, \bar{z}_n)$$

And taking into account the initial conditions (37), we conclude that

$$a_0 = 0,$$
$$a_1 = \lambda'_{z_n} z_1,$$
$$a_2 = 2 \lambda'_{z_n} z_2, \ldots$$
$$a_{n-2} = (n-2) \lambda'_{z_n} z_{n-2},$$
$$a_{n-1} = 1 + (n-1) \lambda'_{z_n} z_{n-1}. \quad (38)$$

These formulas are, of course, identical to those for the real case. The only difference is that now we work with complex variables.

Now we have all the information to rewrite the formulas for $L$ and $g$ in the basis $\partial_{z_i}$. Notice that $L$ commutes with the complex structure $J$ and therefore $L$ can be treated as a complex operator. The form $g$ is also compatible with $J$ in the sense that $g(Ju, v) = g(u, Jv)$ so that $g$ can be understood as the real part of the complex form on $T_P M$ treated as an $n$-dimensional complex space (w.r.t. $J$). This allows us to represent $L$ and $g$ by $n \times n$ complex matrices. In the canonical frame $\xi_1, \ldots, \xi_n$ these matrices are:

$$L_{can}^c = \begin{pmatrix} \lambda & 1 \\ \lambda & \ddots \\ \vdots & \ddots & 1 \\ \lambda \end{pmatrix}.$$ 

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To determine $L$ and $g$ (more precisely their complex representations $L^C$ and $g^C$) in the coordinates $z_1, \ldots, z_n$, we use the standard transformation:

\[
L^C_{\text{can}} \mapsto L^C = C^{-1} L^C_{\text{can}} C, \quad g^C_{\text{can}} \mapsto g^C = C^\top g^C_{\text{can}} C
\]

with the transition matrix $C$

\[
\begin{pmatrix}
\partial_{z_1}, & \ldots, & \partial_{z_n} \\
(\xi_1, \ldots, \xi_n)C,
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
1 & a_0 \\
1 & a_1 \\
& \ddots \\
1 & a_n-2 \\
& a_n-1
\end{pmatrix}
\]

where $a_k$ are defined by (38).

Now a straightforward computation of $g^C$ and $L^C$ immediately leads to the conclusion of Theorem 5.

5 Applications: some global results

Here we give the proofs of Theorem 6 and Corollary 1.

5.1 Proof of Theorem 6

Consider two projectively equivalent pseudo-Riemannian metrics $g$ and $\bar{g}$ on $M$ and the $(1,1)$-tensor fields $L = L(g, \bar{g})$ defined by

\[
L^i_j = \left( \frac{\det \bar{g}}{\det g} \right)^{1/2} \bar{g}^{ik} g_{kj}.
\]

Theorem 6 can be reformulated as follows:

If $M$ is compact, then $L$ has no non-constant complex eigenvalues. In other words, complex eigenvalues of $L$ are all constant.

The idea of the proof is very natural. As we know from § 4, the complex eigenvalue is a holomorphic function in an appropriate coordinate system. Roughly speaking, our proof is somehow equivalent to saying that “a holomorphic function on a compact manifold has to be constant”. However, to make sense out of this principle we have to deal with two issues:

- the complex structure $J$ (see Theorem 5 and Section 4) is not globally defined;
- the complex eigenvalues may collide and near the collision points the complex eigenvalues are not well defined functions (these points should be considered as branching points for eigenvalues).

To avoid these difficulties, we use the following two observations:

- a natural complex structure $J$ is well defined as soon as we have complex eigenvalues even at collision points,
monic polynomials of degree 2 in such a way that the roots of $\chi_b$ in a neighborhood of $p$. Proof. We first notice (exactly in the same way as we did in our splitting construction [9]) that in a neighborhood of $p$ the characteristic polynomial of $L$ admits the following factorization:

$$\chi_L(t) = p_z(t) \cdot \bar{p}_z(t) \cdot q_v(t)$$

(39)

where

$$p_z(t) = t^k + a_{k-1}(z)t^{k-1} + \cdots + a_1(z)t + a_0(z)$$

with coefficients $a_m(z)$ being holomorphic functions of the complex variables $z = x + iy$, 
$$\bar{p}_z(t) = t^k + \bar{a}_{k-1}(z)t^{k-1} + \cdots + \bar{a}_1(z)t + \bar{a}_0(z),$$

and

$$q_v(t) = t^l + b_{l-1}(v)t^{l-1} + \cdots + b_1(v)t + b_0(v)$$

where $b_m(v)$ are smooth real valued functions of $v$, and the polynomial $p_z(t)$ at the point $p_0$ takes the form $(t - \mu_0)^k$.

Proof. We first notice (exactly in the same way as we did in our splitting construction [9]) that in a neighborhood of $p_0$ the characteristic polynomial $\chi_L(t)$ can be uniquely factorized into two monic polynomials of degree 2$k$ and 1 respectively with smooth real coefficients

$$\chi_L(t) = \chi_1(t) \chi_2(t)$$

in such a way that the roots of $\chi_1(t)$ at point $p_0$ are $\mu_0$ and $\bar{\mu}_0$, both with multiplicity $k$. Locally, in a neighborhood of $p_0$ these polynomials do not have common roots. This factorization immediately lead (see [9, Theorem 2]) to the existence of a coordinate system $u^1, \ldots, u^k, v^1, \ldots, v^l$ in which $L$ splits into blocks each of which depend on its own group of coordinates:

$$L = \begin{pmatrix} L_1(u) & 0 \\ 0 & L_2(v) \end{pmatrix},$$

so that $\chi_1(t)$ and $\chi_2(t)$ are the characteristic polynomials of $L_1$ and $L_2$. In other words, locally we may think of $M$ with $L$ as a direct product $(M_1, L_1) \times (M_2, L_2)$ of two “independent” manifolds with (1,1)-tensor fields on them. We put $q_v(t) = \chi_2(t)$ and continue working, from now on, with the first factor $(M_1, L_1)$ only.

In a neighborhood of $p_0$, the characteristic polynomial $\chi_1(t)$ of $L_1$ admits a further factorization:

$$\chi_1(t) = p_u(t) \bar{p}_u(t)$$

into two complex conjugate polynomials with smooth complex valued coefficients satisfying the required property: at the point $p_0$ we have $p_u(t) = (t - \mu_0)^k$. So far this construction is purely algebraic. But now we need to pass from real coordinates $u$ to complex coordinates $z = x + iy$ in such a way that the coefficients of $p(t)$ become holomorphic functions of $z$.

First we construct a natural complex structure $J$ on $M_1$. The spectrum of $L_1$ in the neighborhood of $p_0$ consists of complex eigenvalues only located nearby either $\mu_0$ or $\bar{\mu}_0$. In this case $J$ can be defined as an analytic function of $L_1$, i.e.,

$$J = f(L_1)$$
Here $f : \mathbb{C} \to \mathbb{C}$ is a locally constant function defined on the disjoint union of two small discs $B(\mu_0)$ and $B(\mu_0)$ centered at $\mu_0$ and $\mu_0$:

$$f(z) = \begin{cases} i, & \text{if } z \in B(\mu_0), \\ -i, & \text{if } z \in B(\mu_0) \end{cases}$$

For definiteness, we assume that $\text{Im} \mu_0 > 0$, and $\text{Im} \mu_0 < 0$.

An equivalent definition of $J$ is as follows. Let $L_1 : V^{2n} \to V^{2n}$ be a linear operator whose eigenvalues are all complex (with non-zero imaginary parts!). We consider the decomposition of $V^C$ into two $L_1$-invariant subspaces $U^+ \oplus U^-$ corresponding to eigenvalues of $L_1$ with positive and negative imaginary parts respectively. Such a decomposition is obviously unique. Now we define $J$ to be the multiplication by $i$ on $U^+$ and multiplication by $-i$ on $U^-$. It is easy to see that $V$ as a subspace of $V^C$ is $J$-invariant, i.e., $J$ gives a well-defined operator on $V$, satisfying $J^2 = -1$ and commuting with $L_1$.

Thus, $J = f(L_1)$ is a well-defined almost complex structure in a neighborhood of $p_0$. By Lemma 6 from [9], the vanishing of the Nijenhuis torsion of $L_1$ implies the same property for $J$. Therefore, $J$ is integrable and is indeed a complex structure in a neighborhood of $p_0$.

We now need to show that the coefficients of $p(t)$ are holomorphic with respect to $J$. Though it can be done independently, we shall easily derive this property from §4.

Indeed, in §4 we have shown that in a neighborhood of every generic point each complex eigenvalue $\lambda_i$ of $L$ is a holomorphic function (in the “singular” case studied in §2, $\lambda_i$ is constant, so this property holds automatically). On the other hand the coefficients $a_j(z)$ of $p(t)$ are symmetric polynomials in $\lambda_i$, so they are holomorphic at each generic point too. Now it remains to notice that generic points form open everywhere dense subset and $a_j(z)$ are smooth everywhere. This obviously implies that $a_j(z)$ are holomorphic on the whole neighborhood of $p_0$. This completes the proof.

**Remark 8.** In the final part of the proof we have used the fact that the complex structure $J$ is in natural sense compatible with the complex structures defined in §4 (cf. Remark 7). More precisely, in a neighborhood of a generic point, $J$ from Lemma 5 is the direct sum of the complex structures constructed for each individual $(\lambda_i, \bar{\lambda}_i)$-block by the method explained in §4. Therefore, “holomorphic on a single complex Jordan block” implies “holomorphic on the sum of all complex Jordan blocks”.

We also shall use the following almost obvious statement.

**Lemma 6.** Let $p_z(t) = t^k + a_{k-1}(z) t^{k-1} + \cdots + a_1(z) t + a_0(z)$ be a polynomial in $t$ whose coefficients are holomorphic functions of $z = (z^1, \ldots, z^k) \in U$, where $U \subset \mathbb{C}^k$ is an open connected domain. Assume that at some point $z_0 \in U$ the polynomial takes the form $p_{z_0}(t) = (t - \mu_0)^k$ and at any other point $z \in U$ all the roots $\lambda_i(z)$ of $p_z(t)$ satisfy the condition

$$\text{Im} \lambda_i(z) \leq c = \text{Im} \mu_0, \quad i = 1, \ldots, k.$$

Then $\lambda_i(z) \equiv \mu_0$ for all $z \in U$, i.e., the roots of $p_z(t)$ are all constant and equal to $\mu_0$. In particular, $p_z(t) \equiv (t - \mu_0)^k$ on $U$.

**Proof.** Consider the sum $\sum_{i=1}^k \lambda_i(z)$ of the roots of $p_z(t)$. Since $\sum_{i=1}^k \lambda_i(z) = -a_{k-1}(z)$, this sum is a holomorphic function on $U$. On the other hand, we see that $\text{Im}(-a_{k-1}(z)) = \text{Im} \left( \sum_{i=1}^k \lambda_i(z) \right) \leq k \cdot c$ and $\text{Im}(-a_{k-1}(z_0)) = k \cdot c$, i.e., the imaginary part of the holomorphic function $-a_{k-1}(z)$ attains maximum at a certain point $z_0 \in U$. This implies (by maximum principle) that $a_{k-1}(z)$ is constant on $U$. From this, in turn, it is easy to derive that the imaginary part of each $\lambda_i(z)$ and, therefore, $\lambda_i(z)$ itself is constant.

We are now ready to complete the proof of Theorem 6.
Consider the roots $\lambda_1(P), \ldots, \lambda_n(P)$, $n = \text{dim } M$ of the characteristic polynomial $\chi_L(t)$ at $P \in M$ and let
$$c = \max_{P \in M, i=1, \ldots, n} \text{Im} \lambda_i(P)$$
We assume that some complex eigenvalues exist, so $c > 0$.

Since the roots $\lambda_i(P)$ depend on $P$ continuously (in natural sense) and $M$ is compact, then $c$ is attained, i.e., there is a point $Q \in M$ at which $\chi_L(t)$ has a complex root $\mu_0$ such that $\text{Im} \mu_0 = c$.

In general, $\mu_0$ may have different multiplicities at different points. Let $k$ be maximal multiplicity of $\mu_0$ on $M$.

Consider the following subset $A \subset M$:
$$A = \{Q \in M \mid \mu_0 \text{ is a root of } \chi_L(t) \text{ of multiplicity } k \text{ at the point } Q\}.$$ 

By our assumption $A$ is not empty, and the standard continuity argument shows that $A$ is closed. Let us show that $A$ is open. Indeed, let $p_0 \in A$. We first apply Lemma 5 at this point to get the factorization (39) in some neighborhood $U(p_0)$, and then apply Lemma 6 to see that $\chi_L(t) = (t - \mu_0)^k(t - \mu_0)^k q_0(t)$ on $U(Q)$/ In other words, $\mu_0$ is a root of $\chi_L(t)$ of multiplicity $k$ for all points $P \in U(Q)$, i.e. $U(Q) \subset A$ and therefore $A$ is open.

Thus, $A$ is open, closed and non-empty. Hence, $A = M$ and we see that $\chi_L(t) = (t - \mu_0)^k(t - \mu_0)^k q(t)$ everywhere on $M$. In other words, $\mu_0$ is a constant complex eigenvalue of $L$ of multiplicity $k$ on the whole manifold $M$.

If $q(t)$ has some complex roots at some points of $M$, we simply repeat the same argument to show that these roots have to be constant. This completes the proof of Theorem 6.

**Remark 9.** The proof is based on the fact that the Nijenhuis torsion of $L$ vanishes identically on $M$. This condition itself implies the conclusion of Theorem 6. In other words, one can prove the following more general fact: if the Nijenhuis torsion of $L$ vanishes identically on a compact manifold $M$, then the complex eigenvalues of $L$ have all to be constant.

### 5.2 Proof of Corollary 1

Thus, we need to prove the following result:

*Let $M^3$ be a closed connected 3-dimensional manifold. Suppose $g$ and $\bar{g}$ are geodesically equivalent metrics on it. Suppose $L$ given by (2) has a complex (≠ not real) eigenvalue $\alpha + i\beta$ at least at one point. Then, the manifold can be infinitely covered by the 3-torus.*

Without loss of generality we think that the metric has signature $(-, +, +)$. Then, by Theorem 6, the complex eigenvalue is a constant; we denote it by $\alpha + i\beta$. The complex conjugate number $\alpha - i\beta$ is also an eigenvalue of $L$; the remaining third eigenvalue will be denoted by $\lambda$; it is a (smooth) real valued function on the manifold.

At every point $p \in M$, let us consider a basis $\{v_1, v_2, v_3\}$ in $T_p M$ such that in this basis the matrices of the metric and of $L$ are given by
$$g = \begin{pmatrix} (\lambda - \alpha)^2 + \beta^2 & -\beta & \alpha - \lambda \\ -\beta & \alpha - \beta & \beta \\ \alpha - \lambda & \beta & \lambda \end{pmatrix}, \quad L = \begin{pmatrix} \lambda & \alpha & \beta \\ -\beta & \alpha & \beta \\ \alpha - \lambda & \beta & \lambda \end{pmatrix}.$$ 

The existence of such basis follows from [24, Theorem 12.2]; it is an easy exercise to show that the basis is unique up to the transformations $v_1 \mapsto -v_1; \ (v_2, v_3) \mapsto (v_2, -v_3)$.

Now, consider the positive-definite Euclidean structure at $T_p M$ such that this basis is orthonormal. This Euclidean structure does not depend on the freedom in the choice of the basis and is therefore well-defined; as it will be clear later, it smoothly depend on the point $p$ (in fact, one can easily
show the smoothness using the implicit function theorem) and therefore generates a Riemannian metric on $M$ which we denote by $g_0$. Let us show that the metric $g_0$ is flat.

In order to do it, we will use our description of the compatible pair $(g, L)$. As we explained in §1.2, in a neighborhood of every point the metric $g$ could be obtained by gluing $(I, h_1, L_1)$ and $(U^2, h_2, L_2)$ such that

- $I$ is one-dimensional, the metric $g_1$ is positively definite, and the eigenvalue of $L_1$ is $\lambda$.
- $U^2$ is two-dimensional, $g_2$ has signature $(-, +)$ and the eigenvalues of $L_2$ are $\alpha + i\beta, \alpha - i\beta$.

Then, for the certain choice of the coordinate $x_1$ on $I$ the metric $g_1$ is $(dx_1)^2$ and the only component of the $(1 \times 1)$-matrix of $L$ is $\lambda(x_1)$. Now, since $L_2$ is compatible with $g_2$ and since the trace of $L_2$ is a constant, $L_2$ is covariantly constant with respect to $h_2$. Then, in a certain (local) coordinate system $(x_2, x_3)$ on $U^2$ the metric $h_2$ and the $(1,1)$-tensor $L$ are given by the matrices

$$h_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Applying the gluing construction to $(I, h_1, L_1)$ and $(U^2, h_2, L_2)$, we obtain that the metric $g$ and the tensor $L$ are given by

$$g = \begin{pmatrix} (\lambda(x_1) - \alpha)^2 + \beta^2 & -\beta & \alpha - \lambda(x_1) \\ -\beta & \alpha - \lambda(x_1) & \beta \end{pmatrix}, \quad L = \begin{pmatrix} \lambda(x_1) & \alpha & \beta \\ \alpha & \beta & -\beta \alpha \end{pmatrix}.$$

We see that the vector fields $v_1 = \frac{\partial}{\partial x_1}, v_2 = \frac{\partial}{\partial x_2}, v_3 = \frac{\partial}{\partial x_3}$ form a basic such that $g$ and $L$ are as in (40) implying that the metric $g_0$ is given by $g_0 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$ and therefore is flat.

Thus, there exists a flat Riemannian metric on $M^3$. Then, the manifold is a 3-dimensional Bieberbach manifold (i.e., is a quotient of $\mathbb{R}^3$ modulo a freely acting crystallographic group) and can be finitely covered by the torus $T^3$ as we claimed.

References


