1. Introduction

A differentiable manifold $M$ is said to be formal if the algebra $\wedge^*(M)$ of the differential forms on $M$ is quasi isomorphic to its DeRham cohomology. (We recall that a morphism between Differential Graded Algebras is said to be a quasi isomorphism if it induces an isomorphism in cohomology and that two DGA’s are said to be quasi isomorphic if they are equivalent with respect to the equivalence relation generated by quasi isomorphisms (cf. also [2]).)

It is well known that (cf. [6])

$$\wedge^*(M), \wedge^*\to \wedge^{k+1}(M)$$

is formal $\Rightarrow$ from $H^*(M, \mathbb{R})$ we can reconstruct (via its minimal model, Postnikov towers etc...) the whole rational (i.e. all the cofinite) homotopy theory of $M$.

One (actually, almost the only effective) way to get formality is to be able to produce a suitable derivation $\delta$ on $\wedge^*(M)$, $\delta : \wedge^k \to \wedge^{k+1}$ (for $k = 0, ..., n$), satisfying $\delta^2 = 0$ and such that $d\delta$-lemma holds, i.e. $(\text{Ker} \ d \cap \text{Ker} \ \delta) \cap (\text{Im} \ d + \text{Im} \ \delta) = \text{Im} \ d\delta$.

More precisely, the following general statements holds:

**Theorem 1.1** (cf. [6]). Let $M$ be a smooth manifold with a derivation $\delta : \wedge^k \to \wedge^{k+1}$ (for $k = 0, ..., n$), satisfying $\delta^2 = 0$ such that $d\delta$-lemma holds. Then

$$H(\wedge^*(M), d) = (\text{Ker} \ d \cap \text{Ker} \ \delta)/\text{Im} \ d\delta$$

$$H(\wedge^*(M), \delta) = (\text{Ker} \ d \cap \text{Ker} \ \delta)/\text{Im} \ d\delta$$

and so $(\wedge^*(M), d)$ and $(\wedge^*(M), \delta)$ are formal.

An example of such a situation is provided by Kähler manifolds: in this case, $\delta = d^c := J^{-1}dJ$, where $J$ is the complex structure (cf. again [6]).

We first show (Lemma 2.1, Remark 2.2) that the derivation $\delta$ satisfying properties above must be of the form $\delta = d_R := RdR^{-1}$, with $R \in \text{End}(TM)$ (i.e., $R$ is a field of non degenerate linear transformations of the tangent spaces).

Then, we prove (Lemma 2.4) that the supercommutation of $d$ and $\delta = d_R$ (which is a natural, essentially necessary condition to get a $d\delta$-lemma) amounts to $N_R \equiv 0$, $N_R$ being the Nijenhuis tensor of $R$. Then, we are looking for sufficient conditions that ensure the $dd_R$-lemma holds. For $R$ self adjoint with respect to a Riemannian metric, it is done in Section 3. For $R$ compatible with an almost symplectic structure this is done in Section 4. Finally, we show that, if $'R = -R$ and $\det R \equiv 1$, then

$$N_R \equiv 0 \implies N_J \equiv 0$$

where $J$ is the orthogonal component of $R$, in its polar decomposition and this also provides a new characterization of Kähler structures.
2. Preliminary remarks: on the space of derivations

We begin with the following Lemma 2.1.

**Lemma 2.1.** Let $M$ be a smooth compact manifold of dimension $n$ and let $\delta \in \text{End}(\wedge^*(M))$ such that:

a. $\delta(\alpha \wedge \beta) = \delta \alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \delta \beta$ and $|\delta| = 1$, i.e. $\delta : \wedge^p(M) \rightarrow \wedge^{p+1}(M)$

b. $\text{Ker} \delta \cap \wedge^0(M) = \mathbb{R}$

c. $\delta^2 = 0$;

then $R : X \mapsto \delta X$ belongs to $\text{End}(TM)$ and $\delta = d_R := R^*dR^{-1}$ (where, clearly, $\delta X : f \mapsto \delta f(X)$).

**Proof.** The linearity of $R$ is evident. By (b), $R$ is nondegenerate. In order to prove $\delta = d_R$, let us note that for any $f \in \wedge^0(M)$, $X \in TM$, we have:

$$d_R f(X) = df(RX) = RX f = \delta_X f = (\delta f)(X)$$

i.e. $\delta$ coincides with $d_R$ on $\wedge^0(M)$; this, together with (a), (c), is sufficient to insure $\delta \equiv d_R$. □

**Remark 2.2.** Assume $\delta$ satisfies (a), (c) of lemma 2.1. Suppose

- the $d\delta$-lemma holds, i.e.:

$$\text{Ker} d \cap \text{Ker} \delta \cap (\text{Im} d + \text{Im} \delta) = \text{Im} d\delta$$

- $d\delta + \delta d = 0$.

Then also (b) of lemma (2.1) is fulfilled.

**Proof.** Indeed, if $f \in \text{Ker} \delta \cap \wedge^0(M)$, $f \neq \text{const}$, then

$$0 \neq df \in (\text{Ker} d \cap \text{Ker} \delta) \cap (\text{Im} d + \text{Im} \delta),$$

contradicting $df \notin \text{Im} d\delta$. □

For any $S \in \text{End}(TM)$, we define the Nijenhuis tensor of $S$ as the element $N_S \in \wedge^2(M) \otimes TM$ given by

$$N_S(X, Y) := [SX, SY] + S^2[X, Y] - S[SX, Y] - S[X, SY];$$

It is known (and follows direct from definitions), that

- $N_{I+S} = N_S$ (where $I : TM \rightarrow TM$ is the identity)
- for any $\lambda \in C^\infty(M, \mathbb{R})$, $N_{\lambda I} = 0$
- if $R \in \text{End}(TM)$ then

$$N_{R^{-1}}(X, Y) = R^{-2}N_R(R^{-1}X, R^{-1}Y).$$

Let $V$ be a vector space. For any $L \in \text{End}(V)$ we consider

$$\tau(L) \in \text{End}(\wedge^*V^*)$$

defined as follows:

$$(\tau(L)(\alpha))(v_1, ..., v_p) := \sum_{h=1}^p \alpha(v_1, ..., L(v_h), ..., v_p).$$

We recall that a Differential Graded Lie Algebra (DGLA) is a graded vector space $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$
together with a bilinear map $[ , ] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ and a degree one graded derivation $d$ on $\mathfrak{g}$ in such a way that:

- $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$
Lemma 2.3. The gradation is obvious:

\[
[a, b] = -(-1)^{|a||b|}[b, a]
\]

\[
[a, [b, c]] = [[[a, b], c] + (-1)^{|a||b|}[b, [a, c]]]
\]

Jacobi identity

\[d[a, b] = [da, b] + (-1)^{|a|}[a, db].\]

\[d^2 = 0\]

For example, there is a natural structure of DGLA on \(\text{End}(\wedge^*(M))\):

\[\text{End}(\wedge^*(M))\]

and the bracket \([\ , \ ]\) and the derivation (we use the letter \(\gamma\) for it) are given by

\[
[P, Q] := P Q - (-1)^{|P||Q|} Q P,
\]

\[
\gamma P := [d, P].
\]

Let us recall the lemma (cf. e.g. [4]),

Lemma 2.3. Let \(R \in \text{Aut}(TM)\). Then,

\[d_R = d + [\tau(S), d] - r(R),\]

where \(\tau\) is defined by (1), \(S := R - \mathbf{1}\), and \(r(R)\) is a zero order differential operator quadratic in \(S\) defined as follows:

- \(r(R) \equiv 0\) on \(\wedge^0(M)\)
- for \(\alpha \in \wedge^1(M)\) \(r(R)(\alpha)(X, Y) := \alpha(R^{-1}N_R(X, Y))\)
- extend to general \(\alpha\) as skew-symmetric derivation.

Proof. It is enough to prove the lemma for \(f \in \wedge^0(M)\) and for \(\alpha \in \wedge^1(M)\). If \(f \in \wedge^0(M)\), we have:

\[d_R f(X) = df((I + S)X) = df(X) + ([\tau(S), d])f(X);\]

if \(\alpha \in \wedge^1(M)\) we have first:

\[([\tau(S), d] \alpha)(X, Y) = SX\alpha(Y) - SX\alpha(X) + \alpha(S[X, Y] - [SX, Y] - [X, SY]);\]

then:

\[(d_R \alpha)(X, Y) = (dR^{-1}\alpha)(I + S)X, (I + S)Y) = \ldots\]

We will need the following

Lemma 2.4.

\[d_R = 0 \iff N_R = 0 \iff d_R = d + [\tau(S), d].\]

Proof. Let us first show that \(d\) commutes with \(d + [\tau(S), d]\). Since \(d^2 = 0\), \(d\) commutes with itself.

In order to show that \(d\) commutes with \([\tau(S), d]\) = 0, we use the Jacobi identity:

\[[d, [\tau(S), d]] = 0 = [d, [\tau(S), d]] = [d, [\tau(S), d]] = 0.\]

(In particular, the above observation shows the “\(\iff\)” direction of the lemma)
In order to show that \( d \) commutes with \( r(R) \) if and only if \( N_R = 0 \), we use that, for every \( f \in \Lambda^0(M) \), \( X, Y \in TM \), we have,
\[
[d, d_R]f(X, Y) = (r(R)df)(X, Y) = df(R^{-1}N_R(X, Y)).
\]
Clearly, the right hand side vanishes for all \( X, Y \) if and only if \( N_R \equiv 0 \). \( \square \)

**Remark 2.5.** The previous lemma says that, in \( (\text{End}(\Lambda^*(M)), [, ], \nabla) \),
\[
\nabla d_R = 0 \iff N_R = 0 \iff d - d_R = \nabla r(S) \iff <d> = <d_R>.
\]
Note also that:
\[
[d, d_R] = 0 \iff d_R \text{ satisfies the Maurer–Cartan equation } \nabla d + \frac{1}{2}[d_R, d_R] = 0.
\]

3. **\( dd_R \)-lemma in the presence of a Riemannian metric**

Let \( g \) be a Riemannian metric on \( M \). We denote by \( * \) the Hodge-star operation. The next two lemmas says that when certain (natural) conditions on \( R \) are fulfilled, then the \( dd_R \)-lemma holds.

**Lemma 3.1.** Let \( R \in \text{Aut}(TM) \) such that:
\[
\begin{align*}
(1) \quad N_R &= 0 \ (\text{Lem. 2.4}) \quad d_R = d + [\tau(S), d] \\
& \quad \text{[Lem. 2.4]} \quad [d, d_R] = 0
\end{align*}
\]
\[
(2) \quad \text{there exists a Riemannian metric } g \text{ on } M \text{ such that}
\]
\[
a. \quad [d_R, dd^*] = 0 \\
b. \quad [d_R, d^*d] = 0.
\]
Then,
\[
\text{Ker } d \cap \text{Im } d_R = \text{Im } dd_R
\]

**Proof.** Set
\[
\Delta_R := [d_R, d^*_R].
\]
Clearly,
\[
[\Delta_R, d] = 0 = [\Delta, d_R].
\]
Note that \( \Delta_R = R^{-1} \tilde{\Delta} R \), where \( \tilde{\Delta} \) is the Laplacian operator with respect to \( \tilde{g} = g(R, \cdot, R \cdot) \), consider the Hodge decomposition with respect to \( \Delta \) and \( \Delta_R :\)
\[
I = H + \Delta G
\]
\[
I = H_R + \Delta_R G_R.
\]
Given \( \alpha \in \text{Ker } d \cap \text{Im } d_R \) we have:
\[
\alpha = H(\alpha) + dd^*G(\alpha)
\]
\[
\text{and so}
\]
\[
\gamma = H(\gamma) + dd^*G(\gamma) + d^*G(\gamma)
\]
and so
\[
d_R \gamma = d_R H(\gamma) + dd^*G(d_R \gamma),
\]
i.e.,
\[
H(d_R \gamma) = d_R H(\gamma) = (d + [\tau(S), d])H(\gamma) = -d\tau(\gamma)H(\gamma) = 0
\]
and so:
\[
\alpha = dd^*G(\alpha) = d_R d^*_R G_R(\alpha).
\]
and finally
\[
\alpha = dd^*G(\alpha) = d_R d^*_R G_R(dd^*G(\alpha)) = dd_R G_R(d^*_R d^*G(\alpha)). \quad \square
\]

**Corollary 3.2.** Let \( R \in \text{End}(TM) \) such that:
\[
(1) \quad N_R = 0 \ (\text{and so } d_R = d + [\tau(S), d] \text{ and } [d, d_R] = 0)
\]
\[
(2) \quad \text{there exists a Riemannian metric } g \text{ on } M \text{ such that}
\]
a. \([d_R, dd^*] = 0\)
b. \([d_R, d^*d] = 0\)
c. \([d, d_Rd_R^*] = 0\)
d. \([d, d_R^*d_R] = 0\);
then the \(dd_R\)-lemma holds, i.e.

\[(Ker \, d \cap \text{Ker} \, d_R) \cap (\text{Im} \, d + \text{Im} \, d_R) = \text{Im} \, dd_R\]

4. \(dd_R\)-Lemma in the Almost Symplectic Setting

Let \((M, \kappa)\) be an almost symplectic, \(2n\)-dimensional compact manifold. We consider

\[\mathcal{M}_\kappa(M) := \left\{ g \in \text{Riem}(M) \mid d\mu(g) = \frac{\kappa^n}{n!} \right\}.\]

Recall (cf. [2]) that we can define the symplectic analog of the Hodge-star \(\star\) by means of the relation

\[\alpha \wedge \star \beta = \kappa(\alpha, \beta) \frac{\kappa^n}{n!}\]

for \(\alpha, \beta \in \wedge^r(M)\).

Analog to the Riemannian case, we consider on \(\wedge^r(M)\):

\[d^* := (-1)^r \star d \star.\]

For any

\[g \in \mathcal{M}_\kappa(M)\]

there exists \(R \in \text{End}(TM)\) such that

\[g(X, Y) = \kappa(R^{-1}X, Y).\]

Clearly, \(R\) and \(R^{-1}\) are \(g\)-antisymmetric and \(det \, R \equiv 1\);

on \(\wedge^r(M)\) we have (cf. [3]):

\[\star R^{-1} = (-1)^r \star \text{ i.e. } \star R = (-1)^r \star \text{ and } R^{-1} \star = \star;\]

consequently:

\[d^*_R = -R^{-1} \star d \star = -d^*.\]

We have the following

**Lemma 4.1.** Assume

- \(N_R \equiv 0\)
- \(d_R[d, d^*] = 0 = [d, d^*]d_R\)
- \(d_{R^{-1}}[d, d^*] = 0 = [d, d^*]d_{R^{-1}}\)

then, the \(dd_R\)-lemma holds.

**Proof.** We have:

\([d, d_Rd_R^*] = dd_Rd_R^* - d_Rd_R^*d = -d_R[d, d_R^*] = d_R[d, d^*]\]

and, similarly:

\([d, d^*_Rd_R] = -[d, d^*_R]d_R;\]

finally, we have:

\([d_R, dd^*] = -d^*[d^*, d_R]\]

and

\[R^{-1}d^*[d^*, d_R]R = \pm d_{R^{-1}}[d^*, d].\]

Repeating this procedure with the other relation and applying Lemma 3.2 we obtain that \(dd_R\)-lemma holds. \(\square\)

**Remark 4.2.** If \(\kappa\) defines a symplectic structure, i.e. \(d\kappa = 0\), then \([d, d^*] = 0\) (cf. e.g. [3]), and so we only need \(N_R \equiv 0\):
5. Relaxed condition $J^2 = -I$ in the definition of Kähler manifold.

One of the equivalent definitions of the Kähler manifold is the following one: A Kähler manifold is a symplectic manifold $(M, \kappa)$ equipped with $J \in \text{End}(TM)$ such that the bilinear form $g$ defined by the equality $g(X, Y) := \kappa(X, JY)$ is a Riemannian metric and such that

$I \quad J^2 = -I$

$II \quad N_J = 0.$

A lot of papers study the consequences of relaxing the second condition $N_J = 0$. In this case, the structure $J$ is called an almost complex structure, and many papers are dedicated to almost complex structures satisfying additional conditions, see for example [8].

What about relaxing the first condition?

**Theorem 5.1.** Let $(M, \kappa)$ be an almost symplectic, $2n$-dimensional connected manifold; let again

$$(2) \quad \mathcal{M}_\kappa(M) := \left\{ g \in \mathcal{Riem}(M) \mid d\mu(g) = n! \kappa^n \right\}.$$  

Assume there exists $g \in \mathcal{M}_\kappa(M)$ such that, representing $g$ via $\kappa$ by $R \in \text{End}(TM)$, i.e. for $R$ satisfying $g(X, Y) = \kappa(RX, Y)$,

we have

$N_R \equiv 0.$

Then, the orthogonal component $J$ of $R$ in its $g$-polar decomposition is $g$-scw-symmetric and satisfies

$N_J \equiv 0.$

Moreover, if $d\kappa = 0$, then $(M, g, J)$ is a Kähler manifold.

**Proof.** The proof is organized as follows: we will first show that the orthogonal component $J$ of $R$ in its $g$-polar decomposition is actually a polynomial of $R$ (we will also see that the polynomial is real and odd). The property $N_J = 0$ will then follow from $N_R = 0$ by [5]. The closedness of the form $g(J\lambda, \cdot)$ will require certain additional work.

We consider $-R^2 := -R \circ R$. It is clearly self-adjoint and positively definite with respect to $g$; by (2) we have $\det(R^2) = \text{const}$. Then, it is semi-simple, and all its eigenvalues are positive by linear algebra.

We denote by $m(x)$ the number of different eigenvalues of $-R^2$ at $x \in M$ and by $\lambda_1(x)^2 > ... > \lambda_{m(x)}(x)^2$ ($\lambda_j > 0$, $1 \leq j \leq m(x)$) the eigenvalues of $-R^2$ at $x \in M$.

We say that a point $x \in M$ is stable if $m(x)$ is constant in a neighborhood of $x$. By [9, Lemma 4], the set of stable points is open and everywhere dense on $M$. Later, we will even show that all points are stable. We shall first work near a stable point $x$.

By [5, Lemma 6], the Nijenhuis tensor $N_{-R^2} = 0$. By [7], in the neighborhood of $x$ there exists a coordinate system $\bar{x} = (\bar{x}_1 = (x_1^1, ..., x_{2^{k_1}}^1), ..., \bar{x}_m = (x_1^m, ..., x_{2^{k_m}}^m))$ such that in this coordinate system the matrix of $-R^2$ is block diagonal, the dimensions of the blocks are $2k_1, ..., 2k_m$, and such that the $j$th block is $\lambda_j^2$ times the identity $2k_j \times 2k_j$-matrix:

$$(3) \quad -R^2 = \begin{pmatrix} \lambda_1^2 \cdot I_{2k_1} & & & \\ & \ddots & & \\ & & \lambda_m^2 \cdot I_{2k_m} & \end{pmatrix}.$$ 

Moreover, the function $\lambda_j$ does not depend on the variables $x_i^j$ for $j \neq i$.

This in particular implies that all eigenvalues $\lambda_i$ are actually constant: indeed, from (3) we know that the determinant of $-R^2$ is the product $(\lambda_1)^{2k_1} \cdot \ldots \cdot (\lambda_m)^{2k_m}$. By assumption, the determinant is constant. Since the functions $\lambda_i$ depend on its own variables, all functions $\lambda_i$ must be constant. Then, all points must be stable as we claimed before.
Remark 5.2. For further use let us note that, since the eigenspaces of $R$ corresponding to different eigenvalues are orthogonal, in these coordinates the matrix of $g$ is also block-diagonal with the same as in (3) dimensions of the blocks; by construction, the components of $R$ are also orthogonal with the same dimensions of the blocks

\[(4) \quad g = \begin{pmatrix} g_1 & \cdots & \cdots & g_m \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & \cdots & \cdots & R_m \end{pmatrix}.\]

Let us now cook with the help of $R$ the field of endomorphisms $J$ such that it is the orthogonal component $R$ in its $g$–polar decomposition. We take the polynomial $P(X) = a_2mX^{2m−1} + \ldots + a_0$ of degree at most $2m − 1$ such that its value at the points $X = iλ_1, \ldots, iλ_m$ is equal to $i$ and such that its value at the points $X = −iλ_1, \ldots, −iλ_m$ is equal to $−i$. From general theory it follows that such polynomial is unique (since the values in $2m$ points determine a unique polynomial of degree $2m − 1$, see [1, §2 Ch. 1]). Since $P(\bar{X}) = \bar{P}(X)$ for $2m$ points $X = ±iλ_1, \ldots, ±iλ_m$, the coefficients of the polynomial are real. Since $P(−X) = −P(X)$ for $2m$ points $X = ±iλ_1, \ldots, ±iλ_m$, the polynomial is odd (i.e., all terms of even degree are zero).

We would like to point out that, since $λ_i$ are constant, the coefficients of the polynomial are constant.

We now consider $J := P(R) = a_2m−1R^{2m−1} + \ldots + a_1 \cdot R$ (we understand $R^r$ as $R \circ R \circ \ldots \circ R$). Let us show that $J$ is indeed the orthogonal component of $R$ in its $g$–polar decomposition.

Evidently, the eigenvalues of $J$ are $P(±iλ_1) = ±i$, and the algebraic multiplicity of each eigenvalue coincides with its geometric multiplicity. Then, $J^2 = −I$.

Now, since the polynomial $P$ is even, the bilinear form $g(J\cdot, \cdot)$ is skew-symmetric. Indeed, all terms of the polynomial of even degree are zero, and for every term of odd degree we have $g(a_{2\ell−1}R^{2\ell−1}(U), V) = −g(a_{2\ell−1}R^{2\ell−2}(U), R(V)) = g(a_{2\ell−1}R^{2\ell−3}(U), R^2(V)) = \ldots = −g(U, a_{2\ell−1}R^{2\ell−1}(V))$ (each time we transport one $R$ to the right hand side we change the sign; all together we make odd number the sign change). Then, each term $g(a_{2\ell−1}R^{2\ell−1}, \cdot)$ is skew-symmetric implying $g(J\cdot, \cdot)$ is skew-symmetric as well.

Then, $J$ is a $g$–orthogonal operator. Indeed,

\[g(JV, JU') = −g(J(U'), V) = g(U, V) = g(V, U).\]

Now, the operator $R \cdot J = R \cdot P(R)$ is $g$–symmetric (implying $R = SJ$ for a certain $g$–symmetric operator $S$). Indeed, arguing as above, we have $g(a_{2\ell−1}R^{2\ell}(U), V) = −g(a_{2\ell−1}R^{2\ell−1}(U), R(V)) = g(a_{2\ell−1}R^{2\ell−3}(U), R^2(V)) = \ldots = g(U, a_{2\ell−1}R^{2\ell}(V))$ (this time we transport $2\ell$ $R$’s from left to right, so we change the sign even number of times). Finally, $J = P(R)$ satisfies the following properties:

- It is $g$–orthogonal,
- $R = SJ$ for a certain $g$–symmetric operator.

Thus, $J$ is the orthogonal component of $R$ in its $g$–polar decomposition.

Our goal is to show that $(g, J)$ is a Kähler structure on $M$ provided $κ$ is closed. We already have seen that $J$ is $g$–skew-symmetric. The property $N_J \equiv 0$ follows from [5, Lemma 6].

Let us now prove prove that the form $g(J\cdot, \cdot)$ is also closed. We will work locally, in a coordinate system $\bar{x}$ constructed above. Combining these with the form (3) of $−R^2$, we obtain that the matrix of $J$ is given by

\[(5) \quad J = \begin{pmatrix} \frac{1}{λ_1} R_1 & \cdots & \cdots & \frac{1}{λ_m} R_m \end{pmatrix}.\]

Combining (3) and (4) we see that the matrix of $κ(\cdot, \cdot) := g(R\cdot, \cdot)$ (in the coordinate system $\bar{x}$ above) is given by the matrix
\[(6) \quad \kappa = \begin{pmatrix} \frac{1}{\kappa} & \cdots & \frac{1}{m\kappa} \\ \vdots & \ddots & \vdots \\ \frac{1}{\kappa} & \cdots & \frac{1}{m\kappa} \end{pmatrix} = \begin{pmatrix} -R_1 g_1 \\ \vdots \\ -R_m g_m \end{pmatrix}.\]

Then, by (5), the matrix of \(g(J\cdot, \cdot)\) is
\[(7) \quad \begin{pmatrix} -J g_1 \\ \vdots \\ -J g_m \end{pmatrix} = \begin{pmatrix} -\frac{1}{\lambda_1} R_1 g_1 \\ \vdots \\ -\frac{1}{\lambda_m} R_m g_m \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda_1} \kappa \\ \vdots \\ \frac{1}{\lambda_m} \kappa \end{pmatrix}.\]

In what follows we will use the convention
\[\tilde{x} = (\tilde{x}_1 = (x_1^1, \ldots, x_1^{2k_1}), \ldots, \tilde{x}_m = (x_m^1, \ldots, x_m^{2k_m})) = (y_1^1, \ldots, y_m^{2n}),\]
i.e., \(y_1^1 := x_1^1, \ldots, y_{2k_1}^1 := x_1^{2k_1}, y_{2k_1+1}^2 := x_2^1, \ldots, y_n^2 := x_m^{2k_m} \).

Now we use that the differential of the form is given by
\[d\left(\sum_{p,q=1}^{2n} \kappa_{pq} dy^p \wedge dy^q \right) = \sum_{p,q,s=1}^{2n} \left(\frac{\partial}{\partial y^s} \kappa_{pq}\right) dy^p \wedge dy^q \wedge dy^s.\]

If the matrix of the form \(\kappa\) is as in (6), i.e., if
\[\kappa = \sum_{\alpha,\beta=1}^{2k_1} \frac{1}{\kappa} \kappa_{\alpha\beta} dx_1^\alpha \wedge dx_1^\beta + \sum_{\alpha,\beta=1}^{2k_m} \kappa_{\alpha\beta} dx_m^\alpha \wedge dx_m^\beta,\]
then, the differential of \(\kappa\) is
\[d\kappa = d\frac{1}{\kappa} + \cdots + d\frac{m}{\kappa} = \sum_{i=1}^{m} \left(\sum_{p=1}^{2n} \sum_{\alpha,\beta=1}^{2k_i} \left(\frac{\partial}{\partial y^p} \kappa_{\alpha\beta}\right) dy^p \wedge dx_i^\alpha \wedge dx_i^\beta \right).\]

We see that the components of the differentials of \(d\frac{i}{\kappa}\) and \(d\frac{k}{\kappa}\) do not combine for \(i \neq j\). Indeed, every component of \(d\frac{i}{\kappa}\) is proportional to a certain \(dy^p \wedge dx_i^\alpha \wedge dx_i^\beta\), and every component of \(d\frac{j}{\kappa}\) is proportional to a certain \(dy^p \wedge dx_j^\alpha \wedge dx_j^\beta\). Then, \(d\kappa = 0\) implies \(d\frac{i}{\kappa} = 0\) for all \(i\).

Now, by (7), the form \(g(J\cdot, \cdot)\) is given by
\[(8) \quad \frac{1}{\lambda_1} \kappa + \cdots + \frac{1}{\lambda_m} \kappa.\]

Since \(\lambda_i\) are constants as we explained above, and \(d\frac{k}{\kappa} = 0\), then the differential of (8) vanishes. Thus, \(g(J\cdot, \cdot)\) is closed as we claim. Theorem 5.1 is proved. \(\square\)

**Definition 5.3.** Let \((M, \kappa)\) be a \(2n\)-dimensional (compact) symplectic manifold; \(R \in \text{End}(TM)\) is said to be \(\kappa\)-calibrated if
\[g := \kappa(R\cdot, \cdot)\]
is a Riemannian metric such that \(d\mu(g) = \frac{\kappa^n}{n!}.\)

From Theorem 5.1 we immediately obtain

**Corollary 5.4.** Let \((M, \kappa)\) be a \(2n\)-dimensional connected symplectic manifold. Then the following statements are equivalent:
- \((M, \kappa)\) admits a Kähler structure \(g, J\) such that \(\kappa(\cdot, \cdot) = g(J\cdot, \cdot)\).
there exists $R \in \text{End}(TM)$ such that it is $\kappa$-calibrated and such that $N_R \equiv 0$.

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