TWO REMARKS ON $PQ^\epsilon$-PROJECTIVITY OF RIEMANNIAN METRICS

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Abstract. We show that $PQ^\epsilon$-projectivity of two Riemannian metrics introduced in [15] implies affine equivalence of the metrics unless $\epsilon \in \{0, -1, -3, -5, -7, \ldots \}$. Moreover, we show that for $\epsilon = 0$, $PQ^\epsilon$-projectivity implies projective equivalence.

1. Introduction

1.1. $PQ^\epsilon$-projectivity of Riemannian metrics. Let $g, \bar{g}$ be two Riemannian metrics on an $m$-dimensional manifold $M$. Consider $(1,1)$-tensors $P, Q$ which satisfy

\[
g(P.,.) = -g(.,P.), \quad g(Q.,.) = -g(.,Q.)
\]

\[
\bar{g}(P.,.) = -\bar{g}(.,P.), \quad \bar{g}(Q.,.) = -\bar{g}(.,Q.)
\]

\[
PQ = \epsilon Id,
\]

where $Id$ is the identity on $TM$ and $\epsilon$ is a real number, $\epsilon \neq 1, m + 1$. The following definition was introduced in [15].

Definition 1. The metrics $g, \bar{g}$ are called $PQ^\epsilon$-projective if for a certain 1-form $\Phi$ the Levi-Civita connections $\nabla$ and $\bar{\nabla}$ of $g$ and $\bar{g}$ satisfy

\[

\bar{\nabla}X Y - \nabla X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX
\]

for all vector fields $X, Y$.

Example 1. If the two metrics $g$ and $\bar{g}$ are affinely equivalent, i.e. $\nabla = \bar{\nabla}$, then they are $PQ^\epsilon$-projective with $P, Q, \epsilon$ arbitrary and $\Phi \equiv 0$.

Example 2. Suppose that $\Phi(P.) = 0$ or $Q = 0$ and $\epsilon = 0$. It follows that equation (2) becomes

\[

\nabla X Y - \bar{\nabla} X Y = \Phi(X)Y + \Phi(Y)X.
\]

By Levi-Civita [4], equation (3) is equivalent to the condition that $g$ and $\bar{g}$ have the same geodesics considered as unparametrized curves, i.e., that $g$ and $\bar{g}$ are projectively equivalent. The theory of projectively equivalent metrics has a very long tradition in differential geometry, see for example [13, 10, 7, 5, 6] and the references therein.

Example 3. Suppose that $P = Q = J$ and $\epsilon = -1$. It follows that $J$ is an almost complex structure, i.e., $J^2 = -Id$, and by (1) the metrics $g$ and $\bar{g}$ are required to be hermitian with respect to $J$. Equation (2) now reads

\[

\nabla X Y - \bar{\nabla} X Y = \Phi(X)Y + \Phi(Y)X - \Phi(JX)JY - \Phi(JY)JX.
\]

This equation defines the $h$-projective equivalence of the hermitian metrics $g$ and $\bar{g}$ and was introduced for the first time by Otsuki and Tashiro in [12, 14] for Kählerian metrics. The theory of $h$-projectively equivalent metrics was introduced as an analog of projective geometry in the Kählerian situation and has been studied actively over the years, see for example [11, 3, 1, 2, 8] and the references therein.

Remark 1. $PQ^\epsilon$-projectivity of Riemannian metrics is a special case of so-called $F$-planar mappings introduced and investigated in [9], whose defining equation [9, (1)] clearly generalises equation (2) above.
1.2. Results. The aim of our paper is to give a proof of the following two theorems:

**Theorem 1.** Let Riemannian metrics $g$ and $\bar{g}$ be $PQ^\epsilon$-projective. If $g$ and $\bar{g}$ are not affinely equivalent, the number $\epsilon$ is either zero or an odd negative integer, i.e., $\epsilon \in \{0, -1, -3, -5, -7, \ldots\}$.

**Theorem 2.** Let Riemannian metrics $g$ and $\bar{g}$ be $PQ^\epsilon$-projective. If $\epsilon = 0$ then $g$ and $\bar{g}$ are projectively equivalent.

1.3. Motivation and open questions. As it was shown in [15], $PQ^\epsilon$-projectivity of the metrics $g, \bar{g}$ allows us to construct a family of commuting integrals for the geodesic flow of $g$ (see Fact 2 and equation (9) below). The existence of these integrals is an interesting phenomenon on its own. Besides, it appeared to be a powerful tool in the study of projectively equivalent and $h$-projectively equivalent metrics (Examples 2, 3), see [3, 7, 5, 6, 8]. Moreover, in [15] it was shown that given one pair of $PQ^\epsilon$-projective metrics, one can construct an infinite family of $PQ^\epsilon$-projective metrics. Under some non-degeneracy condition, this gives rise to an infinite family of integrable flows.

From the other side, the theories of projectively equivalent and $h$-projectively equivalent metrics appeared to be very useful mathematical theories of deep interest.

The results in our paper suggest to look for other examples in the case when $\epsilon = -1, -3, -5, \ldots$ If $\epsilon = -1$ but $P^2 \neq -Id$, a lot of examples can be constructed using the "hierarchy construction" from [15]. It is interesting to ask whether every pair of $PQ^{-1}$-projective metrics is in the hierarchy of some $h$-projectively equivalent metrics.

Another attractive problem is to find interesting examples for $\epsilon = -3, -5, \ldots$. Besides the relation to integrable systems provided by [15], one could find other branches of differential geometry of similar interest as projective or $h$-projective geometry.

1.4. PDE for $PQ^\epsilon$-projectivity. Given a pair of Riemannian metrics $g, \bar{g}$ and tensors $P, Q$ satisfying (1), we introduce the $(1,1)$-tensor $A = A(g, \bar{g})$ defined by

$$A = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{\epsilon+1}} \bar{g}^{-1} g. \tag{5}$$

Here we view the metrics as vector bundle isomorphisms $g : TM \to T^* M$ and $\bar{g}^{-1} : T^* M \to TM$.

We see that $A$ is non-degenerate and self-adjoint with respect to $g$ and $\bar{g}$. Moreover $A$ commutes with $P$ and $Q$.

**Fact 1** (Lemma 2 in [15], see also Theorems 5,6 in [9]). Two metrics $g$ and $\bar{g}$ are $PQ^\epsilon$-projective if for a certain vector field $A$, the $(1,1)$-tensor $A$ defined in (5) is a solution of

$$\left(\nabla_X A\right) Y = g(Y, X)A + g(Y, A)X + g(Y, QX)PA + g(Y, PA)QX \text{ for all } X, Y \in TM. \tag{6}$$

Conversely, if $A$ is a $g$-self-adjoint positive solution of (6) which commutes with $P$ and $Q$, the Riemannian metric

$$\bar{g} = (\det A)^{-\frac{1}{\epsilon+1}} gA^{-1}$$

is $PQ^\epsilon$-projective to $g$.

**Remark 2.** Taking the trace of the $(1,1)$-tensors in equation (6) acting on the vector field $Y$, we obtain

$$\Lambda = \frac{1}{2(1-\epsilon)} \text{grad trace } A, \tag{7}$$

hence, (6) is a linear first order PDE on the $(1,1)$-tensor $A$.

**Remark 3.** From Fact 1 it follows that the metrics $g, \bar{g}$ are affinely equivalent if and only if $\Lambda \equiv 0$ on the whole $M$.

**Remark 4.** The relation between the 1-form $\Phi$ in (2) and the vector field $A$ in (6) is given by $\Lambda = -Ag^{-1}\Phi$ (again $g^{-1} : T^* M \to TM$ is considered as a bundle isomorphism), see [15]. Recall from Example 2 that projective equivalence is a special case of $PQ^\epsilon$-projectivity with $\Phi(P) = 0$ or $Q = 0$ and $\epsilon = 0$. In view of Fact 1, we now have that $g$ and $\bar{g}$ are projectively equivalent if and only if $A = A(g, \bar{g})$ given by (5) (with $\epsilon = 0$), satisfies (6) with $P\Lambda = 0$ or $Q = 0$, i.e.,

$$\left(\nabla_X A\right) Y = g(Y, X)A + g(Y, A)X \text{ for all } X, Y \in TM. \tag{8}$$
2. Proof of the results

2.1. Topalov’s integrals. We first recall

\textbf{Fact 2} (Proposition 3 in [15]). Let $g$ and $\tilde{g}$ be $PQ^*\text{-projective metrics}$ and let $A$ be defined by (5). We identify $TM$ with $T^*M$ by $g$, and consider the canonical symplectic structure on $TM \cong T^*M$. Then the functions $F_t : TM \to \mathbb{R}$,

$$F_t(X) = |\det (A - tId)|^{\frac{1}{2}}g((A - tId)^{-1}X, X), \quad X \in TM$$

are commuting quadratic integrals for the geodesic flow of $g$.

Remark 5. Note that the function $F_t$ in equation (9) is not defined in the points $x \in M$ such that $t \in \text{spec} A_x$. From the proof of Theorem 1 it will be clear that in the non-trivial case one can extend the functions $F_t$ to these points as well.

2.2 Proof of Theorem 1. Suppose that $g$ and $\tilde{g}$ are $PQ^*\text{-projective Riemannian metrics}$ and let $A = A(g, \tilde{g})$ be the corresponding solution of (6) defined by (5). Since $A$ is self-adjoint with respect to the positively-definite metric $g$, the eigenvalues of $A$ in every point $x \in M$ are real numbers. We denote them by $\mu_1(x) \leq ... \leq \mu_m(x)$; depending on the multiplicity, some of the eigenvalues might coincide. The functions $\mu_i$ are continuous on $M$. Denote by $M^0 \subseteq M$ the set of points where the number of different eigenvalues of $A$ is maximal on $M$. Since the functions $\mu_i$ are continuous, $M^0$ is open in $M$. Moreover, it was shown in [15] that $M^0$ is dense in $M$ as well. The implicit function theorem now implies that $\mu_i$ are differentiable functions on $M^0$.

From Remark 3 and equation (7) we immediately obtain that $g$ and $\tilde{g}$ are affine equivalent, if and only if all eigenvalues of $A$ are constant. Suppose that $g$ and $\tilde{g}$ are not affine equivalent, that is, there is a non-constant eigenvalue $\rho$ of $A$ with multiplicity $k \geq 1$. Let us choose a point $x_0 \in M^0$ such that $d\rho_{x_0}(0) \neq 0$, define $c := \rho(x_0)$ and consider the hypersurface $H = \{x \in U : \rho(x) = c\}$, where $U \subseteq M^0$ is a geodesically convex neighborhood of $x_0$. We think that $U$ is sufficiently small such that $\mu(x) \neq c$ for all eigenvalues $\mu$ of $A$ different from $\rho$ and all $x \in U$.

\textbf{Lemma 1.} There is a smooth nowhere vanishing $(0, 2)$-tensor $T$ on $U$ such that on $U \setminus H$, $T$ coincides with

$$sgn(\rho - c) |\det (A - cId)|^{\frac{1}{2}}g((A - cId)^{-1}..).$$

\textbf{Proof.} Let us denote by $\rho = \rho_1, \rho_2, ..., \rho_r$ the different eigenvalues of $A$ on $M^0$ with multiplicities $k = k_1, k_2, ..., k_r$ respectively. Since the eigenspace distributions of $A$ are differentiable on $M^0$, we can choose a local frame $\{U_1, ..., U_m\}$ on $U$, such that $g$ and $A$ are given by the matrices

$$g = \text{diag}(1, ..., 1) \quad \text{and} \quad A = \text{diag}(\rho_1, ..., \rho_r),$$

with respect to this frame. The tensor (10) can now be written as

$$sgn(\rho - c) |\det (A - cId)|^{\frac{1}{2}}g(A - cId)^{-1} =$$

$$= (\rho - c) \prod_{i=2}^{r} |\rho_i - c|^\frac{k_i}{2} \text{diag}\left(\frac{1}{\rho - c}, ..., \frac{1}{\rho - c}, ..., \frac{1}{\rho_r - c}, ..., \frac{1}{\rho_r - c}\right) =$$

$$= \prod_{i=2}^{r} |\rho_i - c|^\frac{k_i}{2} \text{diag}\left(\frac{1}{\rho - c}, ..., \frac{\rho - c}{\rho_r - c}, ..., \frac{\rho - c}{\rho_r - c}\right).$$

Since $\rho_i \neq c$ on $U \subseteq M^0$ for $i = 2, ..., r$, we see that (11) is a smooth nowhere vanishing $(0, 2)$-tensor on $U$. $\square$

\textbf{Lemma 2.} The multiplicity of the non-constant eigenvalues of $A$ is equal to $1 - c$. 

Our goal is to show that \( \Lambda \) is proven.

Using the tensor \( T \) from Lemma 1, we can write

\[
F_c(X) = \text{sgn}(\rho - c) |\det (A - cI)|^{\frac{1}{1-c} - \frac{1}{\kappa}} T(X, X), \quad X \in TM.
\]

Let us consider the integral \( F_c : TM \to \mathbb{R} \) defined in equation (9). Using the tensor \( T \) from Lemma 1, we can write \( F_c \) as

\[
F_c(X) = \text{sgn}(\rho - c) |\det (A - cI)|^{\frac{1}{1-c} - \frac{1}{\kappa}} T(X, X), \quad X \in TM.
\]

Our goal is to show that \( \frac{1}{1-c} - \frac{1}{\kappa} = 0 \).

First suppose that \( \frac{1}{1-c} - \frac{1}{\kappa} > 0 \) and let be \( y \in U \setminus H \). We choose a geodesic \( \gamma : [0,1] \to U \) such that \( y = \gamma(0) \) and \( \gamma(1) \in H \), see figure 1. Since \( \rho(\gamma(t)) \overset{t \to 1}{\to} c \), we see from equation (12) that \( f_c(\gamma(t)) \overset{t \to 1}{\to} 0 \). It follows that \( F_c(\gamma(t)) \overset{t \to 1}{\to} 0 \). On the other hand, since \( F_c \) is an integral for the geodesic flow of \( g \), \( F_c(\gamma(t)) \) is independent of \( t \) and, hence, \( F_c(\gamma(0)) = 0 \).

We have shown that \( F_c(\gamma(0)) = 0 \) for all initial velocities \( \gamma(0) \in T_yM \) of geodesics connecting \( y \) with points of \( H \). Since \( H \) is a hypersurface, it follows that the quadric \( \{ X \in T_yM : F_c(X) = 0 \} \) contains an open subset which implies that \( F_c \equiv 0 \) on \( T_yM \). This is a contradiction to Lemma 1, since \( T \) is non-vanishing in \( y \). We obtain that \( \frac{1}{1-c} - \frac{1}{\kappa} \leq 0 \).

Let us now treat the case when \( \frac{1}{1-c} - \frac{1}{\kappa} < 0 \). We choose a vector \( X \in T_{x_0}M \) which is not tangent to \( H \) and satisfies \( T(X, X) \neq 0 \). Such a vector exists, since \( T_{x_0}M \setminus T_{x_0}H \) is open in \( T_{x_0}M \) and \( H \) is not identically zero on \( T_{x_0}M \) by Lemma 1. Let us consider the geodesic \( \gamma \) with \( \gamma(0) = x_0 \) and \( \gamma(1) = X \), see figure 2. Since \( X \notin T_{x_0}H \), the geodesic \( \gamma \) has to leave \( H \) for \( t > 0 \). In a point \( \gamma(t) \in U \setminus H \) the value \( F_c(\gamma(t)) \) will be finite. On the other hand, since \( f_c(\gamma(t)) \overset{t \to 0}{\to} \infty \) and \( T(\gamma(0), \gamma(0)) \neq 0 \), we have \( F_c(\gamma(t)) \overset{t \to 0}{\to} \infty \). Again this contradicts the fact that the value of \( F_c \) must remain constant along \( \gamma \) by Fact 2. We have shown that \( \frac{1}{1-c} - \frac{1}{\kappa} = 0 \) and finally, Lemma 2 is proven. \( \square \)

As a consequence of Lemma 2, if the metrics \( g, \bar{g} \) are not affinely equivalent (i.e., at least one eigenvalue of \( A \) is non-constant), \( \epsilon \) is an integer less or equal to zero. If \( \epsilon \neq 0 \), the condition \( PQ = \epsilon I \mathrm{Id} \) in (1) implies that \( P \) is non-degenerate and by the first condition in (1), \( g(P, \cdot) \) is a non-degenerate 2-form on each eigenspace of \( A \) (note that \( A \) and \( P \) commute). This implies that

\[
\frac{1}{1-c} - \frac{1}{\kappa} = 0.
\]
for $\epsilon \neq 0$ the eigenspaces of $A$ have even dimension, in particular, $1 - \epsilon \in \{2, 4, 6, 8, \ldots\}$. Theorem 1 is proven.

2.3. Proof of Theorem 2. Let $g, \bar{g}$ be two $PQ'$-projective metrics and let $A$ be the corresponding solution of equation (6) defined by (5). As it was already stated in the proof of Theorem 1, the eigenspace distributions of $A$ are differentiable in a neighborhood of almost every point of $M$.

First let us prove

Lemma 3. Let $X$ be an eigenvector of $A$ corresponding to the eigenvalue $\rho$. If $\mu$ is another eigenvalue of $A$ and $\rho \neq \mu$, then $X(\mu) = 0$. In particular, $\text{grad} \mu$ is an eigenvector of $A$ corresponding to the eigenvalue $\mu$.

Remark 6. Lemma 3 is known for projectively equivalent (Example 2) and $h$-projectively equivalent (Example 3) metrics. For projectively equivalent metrics it is a classical result which was already known to Levi-Civita [4]. For $h$-projectively equivalent metrics, it follows from [1, 8].

Proof. Let $Y$ be an eigenvector field of $A$ corresponding to the eigenvalue $\mu$. For arbitrary $X \in TM$, we obtain $\nabla_X (AY) = \nabla_X (\mu Y) = X(\mu)Y + \mu \nabla_X Y$ and $\nabla_X (AY) = (\nabla_X A)Y + A \nabla_X Y$.

Combining these equations and replacing the expression $(\nabla_X A)Y$ by (6) we obtain

\begin{equation}
(A - \mu \text{Id}) \nabla_X Y = X(\mu)Y - g(Y, X)A - g(Y, A)X - g(Y, QX)PA - g(Y, PA)QX.
\end{equation}

Now let $X$ be an eigenvector of $A$ corresponding to the eigenvalue $\rho$ and suppose that $\rho \neq \mu$. Since $A$ is $g$-self-adjoint, the eigenspaces of $A$ corresponding to different eigenvalues are orthogonal to each other. Moreover, since $A$ and $Q$ commute, $Q$ leaves the eigenspaces of $A$ invariant. Using (13) we obtain

\begin{equation}
(A - \mu \text{Id}) \nabla_X Y + g(Y, A)X + g(Y, PA)QX = X(\mu)Y.
\end{equation}

Since the left-hand side is orthogonal to the $\mu$-eigenspace of $A$, we necessarily have $X(\mu) = 0$.

We have shown that $g(\text{grad} \mu, X) = X(\mu) = 0$ for any eigenvalue $\mu$ and any eigenvector field $X$ corresponding to an eigenvalue different form $\mu$. This forces $\text{grad} \mu$ to be contained in the eigenspace of $A$ corresponding to $\mu$. \hfill \square

Now suppose that $\epsilon = 0$. Let us denote the non-constant eigenvalues of $A$ by $\rho_1, \ldots, \rho_l$. Using Lemma 2, the corresponding eigenspaces are 1-dimensional and Lemma 3 implies that they are spanned by the gradients $\text{grad} \rho_1, \ldots, \text{grad} \rho_l$ respectively. Since $P$ and $A$ commute, $P$ leaves the eigenspaces of $A$ invariant, hence, $P \text{grad} \rho_i = p_i \text{grad} \rho_i$ for some real number $p_i$. Now $P$ is skew with respect to $g$ and we obtain $0 = g(\text{grad} \rho_i, P \text{grad} \rho_i) = p_i g(\text{grad} \rho_i, \text{grad} \rho_i)$ which implies that

\[ P \text{grad} \rho_i = 0. \]

On the other hand, by equation (7)

\[ \Lambda = \frac{1}{2} \text{grad} \text{trace } A = \frac{1}{2} (\text{grad} \rho_1 + \ldots + \text{grad} \rho_l). \]

Combining the last two equations, we obtain $PA = 0$. It follows from Remark 4 that $g$ and $\bar{g}$ are projectively equivalent and, hence, Theorem 2 is proven.

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References


