Belnap's Four-Valued Logic and De Morgan Lattices

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Abstract

This paper contains some contributions to the study of Belnap's four-valued logic from an algebraic point of view. We introduce a finite Hilbert-style axiomatization of this logic, along with its well-known semantical presentation, and a Gentzen calculus that slightly differs from the usual one in that it is closer to Anderson and Belnap's formalization of their "logic of first-degree entailments". We prove several Completeness Theorems and reduce every formula to an equivalent normal form. The Hilbert-style presentation allows us to characterize the Leibniz congruence of the matrix models of the logic, and to find that the class of algebraic reducts of its reduced matrices is strictly smaller than the variety of De Morgan lattices. This means that the links between the logic and this class of algebras cannot be fully explained in terms of matrices, as in more classical logics. It is through the use of abstract logics as models that we are able to confirm that De Morgan lattices are indeed the algebraic counterpart of Belnap's logic, in the sense of Font and Jansana's recent theory of full models for sentential logics. Among other characterizations, we prove that its full models are those abstract logics that are finitary, do not have theorems, and satisfy the metalogical properties of Conjunction, Disjunction, Double Negation, and Weak Contraposition. As a consequence, we find that the Gentzen calculus presented at the beginning is strongly adequate for Belnap's logic and is algebraizable in the sense of Rebagliato and Verdú, having the variety of De Morgan lattices as its equivalent algebraic semantics.

Keywords: Many valued logic, De Morgan lattice, abstract logic, full model, non-protoalgebraic logic, algebraizable Gentzen system.

1 Introduction

Belnap's four-valued logic is widely known, specially to applied logicians and theoretical computer scientists. Although its early appearance was in the investigations of the notions of relevance and entailment (see [1, 30] for instance), its popularity grew out of the "epistemic" interpretation of the four values given by Belnap in [6, 7] (see also §81 of [3], where both papers are reproduced with some changes): Besides known-only-true (**t**) and known-only-false (**f**), there are two intermediate (but unrelated) values, namely unknown (**n**) and known-both-true-and-false (**b**). Belnap's seminal ideas have lead to interesting developments in several fields, such as the study of the logic of deductive data-bases and of distributed logic programs (in general, and specially of those dealing with information that can contain conflicts or gaps), in some extensions of Kripke's theory of truth where the truth predicate is partial, and in some works connected with situation semantics or, more generally, partial logic, to mention but a few; the presence of two ordering relations on it (the truth order and the knowledge order) has given rise to the interesting notion of *bilattice* [16, 28], which has found applications in many areas of Theoretical Computer Science. Note that, from the two orderings, it is the *truth ordering* the one used in Belnap's original definition of his logic; cf. Epstein and Dunn's introduction to [7], pp. 5–7.

The connection between this logic and the class of De Morgan lattices (or algebras) was also recognized from the very beginning, as [30] elegantly shows (there the logic is called "De Morgan implication"), but only completeness and some consequences were considered, as in our Proposition 2.5 and Corollary 2.7. The purpose of this paper is to explore further this connection, with the ultimate goal of obtaining a sounder, metalogical basis for the statement that *the class of De Morgan lattices is the algebraic counterpart of Belnap's logic*. Since this logic is not protoalgebraic (Theorem 2.11) and this class of algebraic structures is not the equivalent algebraic semantics of any algebraizable logic (Proposition 2.12), such a statement must be based on a more general theory than the standard, matrix-based ones: This will be *the theory of full models for sentential logics*, announced in [18] and developed in [21] (a summary also appears in [20]). The mathematical objects taken in this theory to act as models of a logic, the *abstract logics* of [13], are strongly related to Scott's *information systems* [39]; see [14, chapter 3] for an exposition and discussion of this relationship.

In Section 2, besides presenting the class of De Morgan lattices $\mathbf{DM}$ and some general terminology and notations, we introduce Belnap's logic $\mathcal{B}$ both semantically (its original birth dress) and through a Gentzen-style calculus $G_{\mathcal{B}}$, inspired in the Hilbert-style calculus for first-degree entailments appearing in [2, §15]. It is worth noting that the calculus we present follows it more closely than other, better-known ones, in that it has a rule of contraposition for negation, while other incorporate rules corresponding to De Morgan Laws, which do not appear as primitive rules in [2]; a more detailed comparison of this difference and its consequences appears in the last part of Section 4. After Completeness (Theorem 2.9), we prove that $\mathcal{B}$ is characterized by the following set of so-called "Tarski-style conditions" [41]: Conjunction, Disjunction, Double Negation, and Weak Contraposition (Proposition 2.10). The section ends with the classification of $\mathcal{B}$ as a non-protoalgebraic, selfextensional, and non-Fregean logic.

Section 3 begins with the presentation of a Hilbert-style calculus $\vdash_H$ and the proof that it is indeed a presentation of Belnap's logic $\mathcal{B}$; the proof of this fact is based on a Normal Form Theorem (3.10) which proof shows precisely why each one of the rules of the calculus has been introduced (we do not claim, however, that the axiomatics is an independent one). The fact that $\mathcal{B}$ is a purely inferential logic, i.e., it has no theorems, does not prevent such a Hilbert-style presentation from existing, contrary to what is claimed in [4, p. 37]; but of course this axiomatization has no axioms and just rules; its interest, rather than proof-theoretic, is that it allows us to find a finite characterization of the Leibniz congruence in matrices of $\mathcal{B}$, and from it we obtain a description of the class of reduced matrices for $\mathcal{B}$. As a consequence we check that the class of their algebraic reducts forms a proper subclass of $\mathbf{DM}$; this fact, together with Theorem 2.11 and Proposition 2.12, confirms our thesis that the algebraization of $\mathcal{B}$ does not fit in the framework of the ordinary theory of logical matrices [9, 10, 36, 41].

In Section 4 it is shown how the algebraization of $\mathcal{B}$ satisfactorily results from the application of the general theory of [21]. After presenting an indispensable minimum of this theory, we prove (Theorem 4.1) that $\mathbf{DM}$ is the algebraic counterpart of $\mathcal{B}$ and give several characterizations of the class of *full models* of $\mathcal{B}$, seing that they are associated in a natural way with De Morgan lattices (Proposition 4.2), that they inherit from $\mathcal{B}$ the property of being semantically generated from the four-element

De Morgan lattice (Theorem 4.4), and that they are characterized by the same set of Tarski-style conditions found in Section 2 to characterize $\mathcal{B}$ (Theorem 4.6). We go on to remark that putting this last result in parallel with some general results of [21] shows that the Gentzen calculus $G_{\mathcal{B}}$ introduced in Section 2 is uniquely determined by $\mathcal{B}$ through a special relationship between their respective models: The full models of $\mathcal{B}$ are exactly the finitary models of $G_{\mathcal{B}}$ without theorems (using a natural, precise sense of an abstract logic being a model of a Gentzen calculus). We confirm this by comparing $G_{\mathcal{B}}$ to $G_{\mathcal{BL}}$, a well-known Gentzen calculus for $\mathcal{B}$ having rules for the De Morgan Laws instead of the contraposition rule: In Proposition 4.10 we prove that both are adequate for $\mathcal{B}$ (i.e., both have the same derivable sequents, which correspond to the Hilbert-style rules of $\mathcal{B}$) but they have different derivable rules (namely, $G_{\mathcal{BL}}$ is a *proper subsystem* of $G_{\mathcal{B}}$), that is, they differ at the *metalogical* level. Moreover, in Theorems 4.11 and 4.12 we show that $G_{\mathcal{B}}$ is *algebraizable* as a Gentzen system, in the sense of the recent theory of Rebagliato and Verdú [27, 37, 38], while $G_{\mathcal{BL}}$ is not. Thus our characterizations of $\mathcal{B}$ through its full models have a distinct metalogical significance, and in this sense one can say that we are using Gentzen calculi for one of the purposes they were originally designed for: calculi of metalogical properties, that is, calculi of (Hilbert-style) rules.

Finally in Section 5 we indicate how to modify some points of our work in order to perform a similar analysis of several related logics: Kleene's three-valued logic, the versions of Belnap's and Kleene's logics with truth or falsity constants, and classical logic.

2 Semantical and Gentzen-style presentations.

Let us begin with a common introduction to the concrete many-valued structure that defines our logic and to the abstract algebraic structures that will be used as its models.

A **De Morgan lattice** is an algebra $\mathfrak{A} = \langle A, \wedge, \vee, \neg \rangle$ of type (2,2,1) such that:

(DM1) The reduct $\langle A, \wedge, \vee \rangle$ is a distributive lattice; we denote its order by $\leq$.

(DM2) The unary operation $\neg$ satisfies the following equations:

$$x \approx \neg\neg x \quad , \quad \neg(x \vee y) \approx (\neg x \wedge \neg y) \quad , \quad \neg(x \wedge y) \approx (\neg x \vee \neg y)$$

(this is not a minimal presentation). De Morgan lattices were introduced in 1935 by Moisil [31, 32], and also independently in 1958 by Kalman [29] under the name of *distributive i-lattices*; see also [36, pp. 44ff]. The theory of De Morgan lattices is very similar, though not so well-known, to that of De Morgan algebras (the *quasi-Boolean algebras* of [8, 36]): the results in [5, 33, 34] can be easily adapted; here we summarize the ones we will need in the paper.

We will denote by **DM** the **variety of De Morgan lattices**. This variety is generated by the four-element De Morgan lattice $\mathfrak{M}_4$ with universe $M_4 = \{\mathsf{f}, \mathsf{n}, \mathsf{b}, \mathsf{t}\}$ and with the algebraic structure specified by the Hasse diagram and negation table shown in Figure 1. This set of four values is sometimes called $\mathcal{FOUR}$ in the literature. In that diagram, the ordering relation $\leq$ goes upwards, thus $\mathsf{f} = \min M_4$ and $\mathsf{t} = \max M_4$, that is, we are considering the so-called *truth-ordering* relation, which was the one originally considered by Belnap to define his entailment in the way we do in Definition

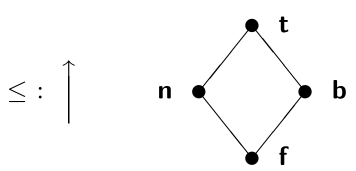
2.1. It is this ordering relation the one giving that set the structure of a De Morgan lattice. Note that another ordering relation is also possible (the *knowledge ordering*, going from left to right, with $\mathbf{n}$ and $\mathbf{b}$ as bounds) but then negation has quite a different behaviour; the notion of *bilattice* has been introduced to organize the coexistence of both orders, and of their respective lattice operations (those for the knowledge order are usually denoted by $\oplus, \otimes$), but we do not deal with this extended structure; see [4] for a study (not in the algebraic logic framework) of the logic of bilattices.

The algebra $\mathfrak{M}_4$ has two non-isomorphic proper subalgebras: the two-element Boolean algebra $\mathfrak{M}_2$ with $M_2 = \{\mathbf{f}, \mathbf{t}\}$ and the three-element chain $\mathfrak{M}_3$ with $M_3 = \{\mathbf{f}, \mathbf{n}, \mathbf{t}\}$; these algebras generate the two proper subvarieties of $\mathsf{\mathbf{DM}}$, namely the variety of Boolean algebras (or, rather, of their $(\wedge, \vee, \neg)$-reducts) and the variety of Kleene lattices (see Section 5).

A **lattice filter** of a De Morgan lattice is a *non-empty* $F \subseteq A$ such that for all $x, y \in A$, $x \wedge y \in F \Leftrightarrow x \in F$ and $y \in F$; a **prime filter** is a *proper* (i.e., $F \neq A$) lattice filter F such that for all $x, y \in A$, $x \vee y \in F \Leftrightarrow x \in F$ or $y \in F$. Usually we will denote by $\mathcal{F}(\mathfrak{A})$ the family of all lattice filters of a De Morgan lattice $\mathfrak{A}$, and for a non-empty $X \subseteq A$ we denote by $\mathbf{F}(X)$ the least lattice filter containing X. If we want to extend this operator to the case $X = \emptyset$ we cannot, however, take the same definition, for if $\mathfrak{A}$ is unbounded then there is no least filter in $\mathfrak{A}$. But if we generally define $\mathbf{F}(\emptyset) = \emptyset$, then $\mathbf{F}$ becomes a closure operator, whose associated closure system will always be $\mathcal{F}(\mathfrak{A}) \cup \{\emptyset\}$. Recall that, as in every distributive lattice, the set of prime filters plus $\emptyset$ generates this closure system. In the case of $\mathfrak{M}_4$ we put $\mathbf{F}_4(X)$. The lattice $\mathfrak{M}_4$ has two prime filters $F_{\mathbf{n}} = \{\mathbf{t}, \mathbf{n}\}$ and $F_{\mathbf{b}} = \{\mathbf{t}, \mathbf{b}\}$.

A **De Morgan algebra** is a bounded De Morgan lattice; of course it is enough to add only one of the bounds. De Morgan algebras form a variety in the extended similarity type with one or two *truth constants*, and have been extensively studied in the literature; see for instance [5, chapter XI].

We denote by $\mathfrak{Fm} = \langle Fm, \wedge, \vee, \neg \rangle$ the absolutely free algebra of similarity type (2,2,1) generated by a denumerable set Var. The elements of Var are called **variables** or **atomic formulas** and those of Fm **formulas**; lowercase greek letters $\varphi, \psi, \delta, \xi \ldots$ represent formulas, and uppercase ones $\Gamma, \ldots$ represent arbitrary sets of formulas. By a **sentential logic** we understand in general a pair $\mathcal{S} = \langle \mathfrak{Fm}, \mathbf{C}_{\mathcal{S}} \rangle$ where $\mathfrak{Fm}$ is the formula algebra of some similarity type and $\mathbf{C}_{\mathcal{S}}$ is a finitary and structural closure operator over Fm; it is customary to use also the symbol $\vdash_{\mathcal{S}}$ for the **relation of**



x	$\neg x$
$\mathbf{f}$	$\mathbf{t}$
$\mathbf{n}$	$\mathbf{n}$
$\mathbf{b}$	$\mathbf{b}$
$\mathbf{t}$	$\mathbf{f}$

FIG. 1: The generating De Morgan lattice $\mathfrak{M}_4$ described by its lattice structure and its negation operation.

consequence: $\Gamma \vdash_{\mathcal{S}} \varphi$ iff $\varphi \in \mathbf{C}_{\mathcal{S}}(\Gamma)$; however, when the logic has been defined semantically, it is also customary to use something similar to $\vDash_{\mathcal{S}}$ instead of $\vdash_{\mathcal{S}}$. In this paper, with the exception of Section 5, we only consider sentential logics over the language $(\wedge, \vee, \neg)$, and also all algebras we deal with have this similarity type. If $\mathfrak{A}$ is any such algebra, an equivalence relation $\theta \subseteq A \times A$ is said to be a **congruence of** $\mathfrak{A}$ if $a\theta a'$ and $b\theta b'$ jointly imply $\neg a\theta\neg a'$, $(a \wedge b)\theta(a' \wedge b')$ and $(a \vee b)\theta(a' \vee b')$. $\mathrm{Co}\mathfrak{A}$ denotes the set of all congruences of $\mathfrak{A}$.

A **(logical) matrix** is a pair $\langle \mathfrak{A}, D\rangle$ where $\mathfrak{A}$ is an algebra and $D \subseteq A$. The **Leibniz congruence** $\Omega_{\mathfrak{A}}D$ of the matrix $\langle \mathfrak{A}, D\rangle$ is the largest $\theta \in \mathrm{Co}\mathfrak{A}$ such that $x\theta y$ and $x \in D$ imply $y \in D$. A matrix is **reduced** if $\Omega_{\mathfrak{A}}D$ is the identity relation; from every matrix $\langle \mathfrak{A}, D\rangle$ we can obtain a reduced matrix by factoring it modulo $\Omega_{\mathfrak{A}}D : \langle \mathfrak{A}/\Omega_{\mathfrak{A}}D,\ D/\Omega_{\mathfrak{A}}D\rangle$ is always reduced.

The notion of logical matrix is a generalization of the primitive idea of *truth-table* with a subset of *designated elements*, and accordingly any matrix $\langle \mathfrak{A}, D\rangle$ defines a sentential logic by putting

$$\Gamma \vDash_{\langle \mathfrak{A}, D\rangle} \varphi \Longleftrightarrow \forall h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A}), \text{ if } h[\Gamma] \subseteq D \text{ then } h(\varphi) \in D\ .$$

The matrix $\langle \mathfrak{A}, D\rangle$ is said to be a **matrix model of a sentential logic** $\mathcal{S}$, or simply an $\mathcal{S}$**-matrix**, if $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \vDash_{\langle \mathfrak{A}, D\rangle} \varphi$. In such a case the set D is called an $\mathcal{S}$**-filter** or a **filter for** $\mathcal{S}$. We denote by $\mathcal{F}i_{\mathcal{S}}\mathfrak{A}$ the family of all $\mathcal{S}$-filters on the algebra $\mathfrak{A}$. For more on the classical theory of matrices, see [41]. In this theory *the algebraic counterpart of a sentential logic* is usually taken to be the class

$$\mathsf{Alg}^{*}\mathcal{S} = \{\, \mathfrak{A} : \mathfrak{A} \text{ is the algebraic reduct of a reduced matrix for } \mathcal{S}\,\}\ .$$

This has been known to work for the so-called *protoalgebraic logics*, but will not work here as we see in Theorem 2.11 and in Theorem 3.14.

Belnap's four-valued logic $\mathcal{B} = \langle \mathfrak{Fm}, \mathbf{C}_{\mathcal{B}}\rangle$ appears in [7] as a semantically defined entailment relation between sentences, and is naturally extended to a relation $\vDash_{\mathcal{B}}$ between arbitrary sets of sentences and a sentence as in the following definition:

Definition 2.1 *For any $\Gamma \subseteq Fm$, $\varphi \in Fm$, we say that $\Gamma \vDash_{\mathcal{B}} \varphi$ if and only if there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that for every $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{M}_4)$, $h(\varphi_1) \wedge \cdots \wedge h(\varphi_n) \leq h(\varphi)$, that is, $h(\varphi) \in \mathbf{F}_4\big(h(\varphi_1), \ldots, h(\varphi_n)\big)$. **Belnap's four-valued logic** is the sentential logic $\mathcal{B} = \langle \mathfrak{Fm}, \mathbf{C}_{\mathcal{B}}\rangle$ associated with the consequence relation $\vDash_{\mathcal{B}}$.*

That this actually defines a sentential logic can be directly checked; but the following result contains this and more information.

Proposition 2.2 *$\mathcal{B}$ is the logic defined by the family of two matrices $\{\langle \mathfrak{M}_4, F_{\mathsf{n}}\rangle\,,\ \langle \mathfrak{M}_4, F_{\mathsf{b}}\rangle\}$; that is, $\vDash_{\mathcal{B}} = \vDash_{\langle \mathfrak{M}_4, F_{\mathsf{n}}\rangle} \cap \vDash_{\langle \mathfrak{M}_4, F_{\mathsf{b}}\rangle}$.*

PROOF. If we put $\vDash' = \vDash_{\langle \mathfrak{M}_4, F_{\mathsf{n}}\rangle} \cap \vDash_{\langle \mathfrak{M}_4, F_{\mathsf{b}}\rangle}$, then this is a sentential logic defined by a finite family of finite matrices, i.e., it is a "strongly finite" logic. It is well-known [41, p. 261] that such logics are finitary. Therefore $\Gamma \vDash' \varphi$ iff $\exists \varphi_1 \ldots \varphi_n \in \Gamma$ such that $\{\varphi_1, \ldots, \varphi_n\} \vDash' \varphi$. By definition, this happens iff $\forall h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{M}_4)$, if $h(\varphi_i) \in F_x\ \forall i = 1, \ldots, n$ then $h(\varphi) \in F_x$, for $x = \mathsf{n}, \mathsf{b}$. Since $F_{\mathsf{n}}, F_{\mathsf{b}}$ are the prime filters of $\mathfrak{M}_4$, this says that $h(\varphi)$ belongs to every prime filter to which all the $h(\varphi_i)$ belong; and since the family $\{F_{\mathsf{n}}, F_{\mathsf{b}}\}$ constitutes a basis of the closure system of all filters of $\mathfrak{M}_4$, we actually have that $h(\varphi) \in \mathbf{F}_4\big(h(\varphi_1), \dots, h(\varphi_n)\big)$, and this is equivalent to $h(\varphi_1) \wedge \cdots \wedge h(\varphi_n) \leq h(\varphi)$. By Definition 2.1, thus, $\vDash'$ and $\vDash_{\mathcal{B}}$ agree on finite sets; since, also by Definition 2.1, $\vDash_{\mathcal{B}}$ is finitary, we conclude that $\vDash' = \vDash_{\mathcal{B}}$. ∎

Proposition 2.3 *$\mathcal{B}$ is the logic defined by the matrix $\langle \mathfrak{M}_4, F_{\mathsf{n}} \rangle$ alone, and also by $\langle \mathfrak{M}_4, F_{\mathsf{b}} \rangle$ alone.*

Proof. This is because the two matrices $\langle \mathfrak{M}_4, F_{\mathsf{n}} \rangle$ and $\langle \mathfrak{M}_4, F_{\mathsf{b}} \rangle$ are isomorphic: the mapping $s : M_4 \to M_4$ given by $s(\mathsf{f}) = \mathsf{f}\,, s(\mathsf{t}) = \mathsf{t}\,, s(\mathsf{n}) = \mathsf{b}\,, s(\mathsf{b}) = \mathsf{n}$ is an automorphism of $\mathfrak{M}_4$ such that $s^{-1}[F_{\mathsf{n}}] = F_{\mathsf{b}}$ and conversely. As a consequence, the logics $\vDash_{\langle \mathfrak{M}_4, F_{\mathsf{n}} \rangle}$ and $\vDash_{\langle \mathfrak{M}_4, F_{\mathsf{b}} \rangle}$ are equal; therefore, after Proposition 2.2, each one of them equals $\vDash_{\mathcal{B}}$. ∎

Generally speaking, it is useful to know that some logic is characterized by a single, finite matrix; but in our case Proposition 2.2 will often make things easier.

Observe that, since the two mappings constantly equal to n, and to b, are homomorphisms, by Proposition 2.3 it results that $\mathcal{B}$ does not have theorems, hence it is a *purely inferential* logic, to use the term of [41, p. 41]; this can be hardly surprising, given its origin in the *tautological entailments* surroundings.

One gains some insight into $\mathcal{B}$ by introducing a notation for formal order: Let us denote by $\varphi \preccurlyeq \psi$ a formal ordering relation between φ and ψ (in the same sense that $\varphi \approx \psi$ is a notation for a formal equation; both things are just pairs of formulas, but we interpret them in two different ways). This formalism has been used since some time in several applications of Universal Algebra to Computer Science (see for instance [40, §4.2.1], but note that in this book only ordered algebras with *monotonic* operations are considered, so it does not cover DM).

Definition 2.4 *Let $\mathfrak{A}$ be any De Morgan lattice (or more generally, any algebraic structure possessing an order relation $\leq$). Then we say that $\mathfrak{A}$ **satisfies** $\varphi \preccurlyeq \psi$, in symbols $\mathfrak{A} \vDash \varphi \preccurlyeq \psi$, iff $\forall h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A})$, $h(\varphi) \leq h(\psi)$; and we say that **a class** K **of algebras satisfies** $\varphi \preccurlyeq \psi$ $(\mathsf{K} \vDash \varphi \preccurlyeq \psi)$ iff every $\mathfrak{A} \in \mathsf{K}$ satisfies it.*

Proposition 2.5 *For any $\varphi_1, \dots, \varphi_n, \varphi \in Fm$, the following are equivalent:*

(i) $\{\varphi_1, \dots, \varphi_n\} \vDash_{\mathcal{B}} \varphi$.
(ii) $\mathfrak{M}_4 \vDash \varphi_1 \wedge \cdots \wedge \varphi_n \preccurlyeq \varphi$.
(iii) $\mathsf{DM} \vDash \varphi_1 \wedge \cdots \wedge \varphi_n \preccurlyeq \varphi$.

Proof. Proposition 2.2 proves (i)⇔(ii). Since in any (semi)lattice $\mathfrak{A}$ it is straightforward that $\mathfrak{A} \vDash \varphi \preccurlyeq \psi$ iff $\mathfrak{A} \vDash \varphi \wedge \psi \approx \varphi$, and $\mathfrak{M}_4$ generates the variety DM, it results that (ii)⇔(iii). ∎

Corollary 2.6 *The class of matrices $\{\langle \mathfrak{A}, F \rangle : \mathfrak{A} \in \mathsf{DM}\,,\ F \in \mathcal{F}(\mathfrak{A})\}$ is complete for $\mathcal{B}$. Thus in particular, every lattice filter of a De Morgan lattice is a filter for $\mathcal{B}$.* ∎

In Section 3 we will prove that, conversely, on every $\mathfrak{A} \in \mathsf{DM}$, every non-empty $\mathcal{B}$-filter is a lattice filter of $\mathfrak{A}$. Note that $\emptyset$ is a $\mathcal{B}$-filter on any $\mathfrak{A}$, due to its lack of theorems. If we consider the **interderivability relation** $\varphi \dashv\vDash_{\mathcal{B}} \psi \iff \varphi \vDash_{\mathcal{B}} \psi$ and $\psi \vDash_{\mathcal{B}} \varphi$, then we have:

Corollary 2.7 *For every $\varphi, \psi \in Fm\,,\ \varphi \dashv\vDash_{\mathcal{B}} \psi$ if and only if $\mathsf{DM} \vDash \varphi \approx \psi$.* ∎

$$\frac{\Gamma \to \varphi}{\Gamma, \psi \to \varphi}\ (\mathrm{W}) \qquad \frac{\Gamma \to \varphi \qquad \Gamma, \varphi \to \psi}{\Gamma \to \psi}\ (\mathrm{Cut})$$

$$\frac{\Gamma, \varphi, \psi \to \xi}{\Gamma, \varphi \wedge \psi \to \xi}\ (\wedge \to) \qquad \frac{\Gamma \to \varphi \qquad \Gamma \to \psi}{\Gamma \to \varphi \wedge \psi}\ (\to \wedge)$$

$$\frac{\Gamma, \varphi \to \xi \qquad \Gamma, \psi \to \xi}{\Gamma, \varphi \vee \psi \to \xi}\ (\vee \to) \qquad \frac{\Gamma \to \varphi}{\Gamma \to \varphi \vee \psi}\ ,\ \frac{\Gamma \to \psi}{\Gamma \to \varphi \vee \psi}\ (\to \vee)$$

$$\frac{\varphi \to \psi}{\neg\psi \to \neg\varphi}\ (\neg)$$

$$\frac{\Gamma, \varphi \to \psi}{\Gamma, \neg\neg\varphi \to \psi}\ (\neg\neg \to) \qquad \frac{\Gamma \to \varphi}{\Gamma \to \neg\neg\varphi}\ (\to \neg\neg)$$

FIG. 2. The rules of the sequent calculus $G_{\mathcal{B}}$.

Some metalogical properties of $\mathcal{B}$ can be expressed in the form of a Gentzen calculus. The one we are going to introduce is inspired in that presented in [2, § 15], where it appears disguised as a Hilbert-style formalism for the "calculus of tautological entailments".

Definition 2.8 *Let us consider* ***sequents*** *of the form* $\Gamma \to \varphi$, *where* $\Gamma \subseteq Fm$ *is finite and non-empty, and* $\varphi \in Fm$. *We will call* $G_{\mathcal{B}}$ *the Gentzen calculus whose only* ***axiom*** *is*

$$\varphi \to \varphi$$

and whose ***rules*** *are the ones appearing in Figure 2.*

The fact that $\mathcal{B}$ has no theorems motivates our choice of sequents: there is no point in having sequents of the form $\emptyset \to \varphi$ in a sequent calculus whose derivable sequents will determine such a logic. Note also that we have explicitly included only Weakening (W) and Cut as structural rules in our presentation of $G_{\mathcal{B}}$; this is because Exchange and Contraction are implicit in the fact that in our sequents $\Gamma \to \varphi$, the left-hand component Γ is a non-empty and finite *set* of formulas, rather than a *multiset* or a *sequence.*

Theorem 2.9 (First Completeness) *The logic* $\mathcal{B}$ *is the finitary logic defined by the derivable sequents of* $G_{\mathcal{B}}$*; that is,* $\Gamma \vDash_{\mathcal{B}} \varphi$ *if and only if there are* $\varphi_1, \dots, \varphi_n \in \Gamma$ *such that the sequent* $\varphi_1, \dots, \varphi_n \to \varphi$ *is derivable in* $G_{\mathcal{B}}$.

PROOF. By using (W), Cut and $(\to \wedge)$ it is straightforward to see that the rule $(\wedge \to)$ can be reversed, that is, that the rule

$$\frac{\Gamma, \varphi \wedge \psi \to \xi}{\Gamma, \varphi, \psi \to \xi}$$

is derivable in $G_{\mathcal{B}}$. As a consequence, an arbitrary sequent $\varphi_1, \ldots, \varphi_n \to \varphi$ is derivable iff the sequent $\varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi$ is derivable. Let us call in this proof *first-degree entailments (fde's)* the sequents of the form $\xi \to \gamma$ with $\xi, \gamma \in Fm$. It is straightforward to check that the calculus for first-degree entailments presented in [2, § 15.2], where fde's are treated as formulas and $\to$ as a connective, is equal to the restriction of $G_{\mathcal{B}}$ to them plus the distribution axiom $\varphi \wedge (\psi \vee \xi) \to (\varphi \wedge \psi) \vee \xi$; since this sequent is derivable in $G_{\mathcal{B}}$, it follows that both calculi actually yield the same derivable fde's. As a consequence, a sequent $\varphi_1, \ldots, \varphi_n \to \varphi$ is derivable in $G_{\mathcal{B}}$ iff the fde $\varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi$ is derivable in the calculus of [2]. A four-element matrix, corresponding to our $\mathfrak{M}_4$ plus an interpretation for the operation $\to$, is presented in § 15.3 of this book, and it is proved that it is characteristic for fde's in the sense that an fde $\xi \to \gamma$ is derivable iff for all assignments v in $\mathfrak{M}_4$, $v(\xi \to \gamma) = 1$; inspection of the table for $\to$ in $\mathfrak{M}_4$ shows that this happens iff $v(\xi) \leq v(\gamma)$, therefore by our Definition 2.1 this amounts to $\xi \vDash_{\mathcal{B}} \gamma$. ∎

As a consequence it is straightforward to prove that $\mathcal{B}$ is characterized by the following set of metalogical properties, of the kind sometimes called *Tarski-style conditions*:

Proposition 2.10 *The logic $\mathcal{B} = \langle \mathfrak{Fm}, \mathbf{C}_{\mathcal{B}} \rangle$ has the following properties, for every $\Gamma \subseteq Fm$ and every $\varphi, \psi \in Fm$:*

(PC) *Property of Conjunction:* $\mathbf{C}_{\mathcal{B}}(\varphi \wedge \psi) = \mathbf{C}_{\mathcal{B}}(\varphi, \psi)$.

(PDI) *Property of Disjunction:* $\mathbf{C}_{\mathcal{B}}(\Gamma, \varphi \vee \psi) = \mathbf{C}_{\mathcal{B}}(\Gamma, \varphi) \cap \mathbf{C}_{\mathcal{B}}(\Gamma, \psi)$.

(PDN) *Property of Double Negation:* $\mathbf{C}_{\mathcal{B}}(\varphi) = \mathbf{C}_{\mathcal{B}}(\neg\neg\varphi)$.

(PWC) *Property of Weak Contraposition: If* $\varphi \in \mathbf{C}_{\mathcal{B}}(\psi)$ *then* $\neg\psi \in \mathbf{C}_{\mathcal{B}}(\neg\varphi)$.

Moreover, $\mathcal{B}$ is the weakest logic satisfying them.

Proof. The properties are obtained from the facts that $\mathcal{B}$ is finitary, that the set of derivable sequents of $G_{\mathcal{B}}$ is closed under its own rules, and by using the (Cut) rule extensively. If $\mathcal{S} = \langle \mathfrak{Fm}, \vdash_{\mathcal{S}} \rangle$ is another logic with these properties, then it results to be closed under the rules of $G_{\mathcal{B}}$, and therefore every derivation in $G_{\mathcal{B}}$ starting from $\varphi \to \varphi$ (which is obviously derivable in any $\mathcal{S}$) produces only sequents derivable in $\mathcal{S}$. Therefore, by Theorem 2.9, if $\Gamma \vDash_{\mathcal{B}} \varphi$ then $\Gamma \vdash_{\mathcal{S}} \varphi$. ∎

In Section 4 we will show that, in a certain strong sense, the calculus $G_{\mathcal{B}}$ is canonically associated with $\mathcal{B}$. Now we are going to classify $\mathcal{B}$ according to several criteria appearing in [9], [21] and [41]:

Theorem 2.11 *Belnap's four valued logic $\mathcal{B}$ is non-protoalgebraic, selfextensional, and non-Fregean.*

Proof. In $\mathfrak{M}_4$ the sets $\emptyset$ and F_{n} are $\mathcal{B}$-filters. We have that $\Omega_{\mathfrak{M}_4}\emptyset = M_4 \times M_4$ by a general argument, and it is easy to check that $\Omega_{\mathfrak{M}_4} F_{\mathsf{n}} = \Delta_{M_4}$; so $\emptyset \subseteq F_{\mathsf{n}}$ while $\Omega_{\mathfrak{M}_4}\emptyset \not\subseteq \Omega_{\mathfrak{M}_4} F_{\mathsf{n}}$, that is, the Leibniz operator is not monotone on $\mathcal{B}$-filters, which implies that $\mathcal{B}$ is not protoalgebraic (see [9]).

That $\mathcal{B}$ is selfextensional means, see [41], that the relation of interderivability $=\!\mid\vDash_{\mathcal{B}}$ is a congruence of the formula algebra $\mathfrak{Fm}$. This is true because, by Corollary 2.7, $\varphi =\!\mid\vDash_{\mathcal{B}} \psi$ iff $\mathsf{DM} \vDash \varphi \approx \psi$, and the replacement property of equational logic implies that $=\!\mid\vDash_{\mathcal{B}}$ is a congruence.

Finally, that $\mathcal{B}$ be Fregean would mean that for any $\Gamma \subseteq Fm$, the interderivability relation modulo Γ (defined as $\xi \equiv_\Gamma \eta$ iff $\Gamma, \xi \vDash_{\mathcal{B}} \eta$ and $\Gamma, \eta \vDash_{\mathcal{B}} \xi$) is also a congruence of $\mathfrak{Fm}$. In particular, since it is trivially true that $\varphi \wedge \neg\psi \equiv_{\{\varphi\}} \neg\psi$, it should also be true that $\neg(\varphi \wedge \neg\psi) \equiv_{\{\varphi\}} \neg\neg\psi$, which implies $\varphi \wedge \neg(\varphi \wedge \neg\psi) \vDash_{\mathcal{B}} \neg\neg\psi$. But at the same time we have $\varphi \wedge \neg\psi \vDash_{\mathcal{B}} \varphi$, therefore by contraposition $\neg\varphi \vDash_{\mathcal{B}} \neg(\varphi \wedge \neg\psi)$, hence also $\varphi \wedge \neg\varphi \vDash_{\mathcal{B}} \varphi \wedge \neg(\varphi \wedge \neg\psi)$. Now using that $\psi \vDash_{\mathcal{B}} \neg\neg\psi$ we conclude that $\varphi \wedge \neg\varphi \vDash_{\mathcal{B}} \psi$. But this would imply that $\forall \mathfrak{A} \in \mathbf{DM}$ and $\forall h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A})$, $h(\varphi \wedge \neg\varphi)$ would be a lower bound of $\mathfrak{A}$, which is impossible since not all De Morgan lattices are bounded, an example being the set of rational numbers with the usual order and $\neg x = -x$. ∎

Therefore, the algebraic study of Belnap's logic needs a more general framework than that of logical matrices for protoalgebraic logics [9] (or for general logics [41], as we shall confirm in Section 3). This will be done in Section 4. On the other hand, there is no possibility that **DM** is the class of algebras associated with some logic by the more restricted approach of [10], as the next proposition shows.

Proposition 2.12 DM *is not the equivalent algebraic semantics of any algebraizable logic.*

PROOF. One of the key results of [10] is that, if $\mathcal{S}$ is an algebraizable sentential logic with equivalent algebraic semantics a variety **K**, then $\forall \mathfrak{A} \in \mathbf{K}$, $\Omega_{\mathfrak{A}}$ is an isomorphism

$\theta \in \mathrm{Co}\mathfrak{A}$	Blocks of θ	$F \subseteq A$ such that $\Omega_{\mathfrak{A}} F = \theta$
$A \times A$	A	$\emptyset$ A
θ_1	$\{0\}$ $\{a, b, \neg b, \neg a\}$ $\{1\}$	$\{0\}$ $\{1\}$ $\{0,1\}$ $\{a, b, \neg b, \neg a\}$ $\{0, a, b, \neg b, \neg a\}$ $\{a, b, \neg b, \neg a, 1\}$
θ_2	$\{0, a\}$ $\{b, \neg b\}$ $\{\neg a, 1\}$	$\{0, a\}$ $\{b, \neg b\}$ $\{\neg a, 1\}$ $\{0, a, b, \neg b\}$ $\{0, a, \neg a, 1\}$ $\{b, \neg b, \neg a, 1\}$
θ_3	$\{0, a, b\}$ $\{\neg b, \neg a, 1\}$	$\{0, a, b\}$ $\{\neg b, \neg a, 1\}$
θ_{12} $(= \theta_1 \cap \theta_2)$	$\{0\}$ $\{a\}$ $\{b, \neg b\}$ $\{\neg a\}$ $\{1\}$	$\{a, \neg a\}$ $\{a, \neg a, 0\}$ $\{a, \neg a, 1\}$ $\{b, \neg b, 0\}$ $\{b, \neg b, 1\}$ $\{b, \neg b, 0, 1\}$ $\{a\}$ $\{\neg a\}$ $\{b, \neg b, a\}$ $\{b, \neg b, \neg a\}$
θ_{13} $(= \theta_1 \cap \theta_3)$	$\{0\}$ $\{a, b\}$ $\{\neg b, \neg a\}$ $\{1\}$	$\{\neg b, \neg a\}$ $\{a, b, 1\}$ $\{\neg a, \neg b, 0\}$ $\{a, b\}$ $\{a, b, 0, 1\}$ $\{\neg a, \neg b, 0, 1\}$
θ_{23} $(= \theta_2 \cap \theta_3)$	$\{0, a\}$ $\{b\}$ $\{\neg b\}$ $\{\neg a, 1\}$	$\{b\}$ $\{\neg b\}$ $\{0, a, \neg b\}$ $\{b, \neg a, 1\}$ $\{0, a, \neg a, b, 1\}$ $\{0, a, \neg a, \neg b, 1\}$
Δ_A	$\{0\}$ $\{a\}$ $\{b\}$ $\{\neg b\}$ $\{\neg a\}$ $\{1\}$	The remaining 26 subsets of A

FIG. 3. The congruences of the six-element chain as a De Morgan lattice.

between the lattices $\mathcal{F}i_{\mathcal{S}}\mathfrak{A}$ and $\mathrm{Co}\mathfrak{A}$. If such an $\mathcal{S}$ exists for $\mathbf{DM}$, then $\forall\mathfrak{A} \in \mathbf{DM}$ there must be some lattice of subsets $\mathcal{C} = \mathcal{F}i_{\mathcal{S}}\mathfrak{A} \subseteq P(A)$ such that $\Omega_{\mathfrak{A}} : \mathcal{C} \cong \mathrm{Co}\mathfrak{A}$. But consider the following De Morgan lattice $A = \{0, a, b, \neg b, \neg a, 1\}$ totally ordered as $0 < a < b < \neg b < \neg a < 1$, with negation obviously given ($\neg 0 = 1$, etc). $\mathfrak{A} \in \mathbf{DM}$. Its congruences are $\mathrm{Co}\mathfrak{A} = \{\Delta_A, \theta_{12}, \theta_{13}, \theta_{23}, \theta_1, \theta_2, \theta_3, A \times A\}$; they are described in Figure 3 by giving their equivalence classes (blocks). The set $\mathrm{Co}\mathfrak{A}$ ordered under $\subseteq$ is a 8-element Boolean algebra. The table in Figure 3 also lists the subsets of A whose Leibniz congruence is a given congruence. Inspection of the table shows that it is impossible to find a family $\mathcal{C} \subseteq P(A)$ that is isomorphic to $\mathrm{Co}\mathfrak{A}$ through $\Omega_{\mathfrak{A}}$ (and hence has the structure of a 8-element Boolean algebra when ordered under $\subseteq$): For the only $F \subseteq A$ such that $\Omega_{\mathfrak{A}}F = \theta_3$ are three-element subsets; since in $\mathrm{Co}\mathfrak{A}$, $\Delta_A \subsetneq \theta_{13} \subsetneq \theta_3$, this implies that in $\mathcal{C}$ one should have $F_0 \subsetneq F_1 \subsetneq F$ with $\Omega_{\mathfrak{A}}F_0 = \Delta_A$ and $\Omega_{\mathfrak{A}}F_1 = \theta_{13}$; this implies that F_0 should be a one-element subset (since $\Omega_{\mathfrak{A}}\emptyset = A \times A$; moreover any algebraizable logic has theorems, so the empty subset would not be a filter), but we see that no singleton has the identity as its Leibniz congruence. Therefore such $\mathcal{S}$ cannot exist. ∎

3 Hilbert-style presentation and matrices

In this section we present a finite Hilbert-style axiomatization[1] for $\mathcal{B}$, prove completeness, and find some facts about its reduced matrices.

Definition 3.1 *Denote by $\langle \mathfrak{Fm}, \mathbf{C}_H \rangle$ or simply by $\vdash_H$ the sentential logic defined through the following set of rules (and no axioms), where $p, q, r \in Var$.*

$$\text{(R1)}\ \frac{p \wedge q}{p} \qquad \text{(R2)}\ \frac{p \wedge q}{q} \qquad \text{(R3)}\ \frac{p \quad q}{p \wedge q}$$

$$\text{(R4)}\ \frac{p}{p \vee q} \qquad \text{(R5)}\ \frac{p \vee q}{q \vee p} \qquad \text{(R6)}\ \frac{p \vee p}{p}$$

$$\text{(R7)}\ \frac{p \vee (q \vee r)}{(p \vee q) \vee r} \qquad \text{(R8)}\ \frac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)} \qquad \text{(R9)}\ \frac{(p \vee q) \wedge (p \vee r)}{p \vee (q \wedge r)}$$

$$\text{(R10)}\ \frac{p \vee r}{\neg\neg p \vee r} \qquad \text{(R12)}\ \frac{\neg(p \vee q) \vee r}{(\neg p \wedge \neg q) \vee r} \qquad \text{(R14)}\ \frac{\neg(p \wedge q) \vee r}{(\neg p \vee \neg q) \vee r}$$

$$\text{(R11)}\ \frac{\neg\neg p \vee r}{p \vee r} \qquad \text{(R13)}\ \frac{(\neg p \wedge \neg q) \vee r}{\neg(p \vee q) \vee r} \qquad \text{(R15)}\ \frac{(\neg p \vee \neg q) \vee r}{\neg(p \wedge q) \vee r}$$

This system of rules is an extension of that given in [15] for the $\{\wedge, \vee\}$-fragment of classical logic. Note that in [19] it was shown that the following rules of [15], not reproduced above, can actually be derived from rules (R1) to (R9):

$$\frac{p \vee (p \vee q)}{p \vee q} \qquad \frac{(p \vee q) \vee r}{p \vee (q \vee r)} \qquad \frac{p \wedge (q \vee r)}{(p \wedge q) \vee (p \wedge r)}$$

[1] The same calculus has been found independently by Pynko and presented in [35]; his completeness proof follows a different technique. Cf. footnote 3 on page 443 of [35].

By the completeness of [15], all rules corresponding to order relations $\varphi \preccurlyeq \psi$ true in all distributive lattices follow from rules (R1) to (R9). We also have:

Proposition 3.2 *The following rules follow from rules* (R1) *to* (R15)*:*

(a) *The rule* (Ri^+) $\dfrac{\varphi}{\psi}$ *, for each one of the rules* (Ri) $\dfrac{\varphi \vee r}{\psi \vee r}$ $(i = 10, \dots, 15)$.

(b) *The rule* $\dfrac{\varphi \wedge r}{\psi \wedge r}$ *in the same cases.*

Proof. (a) $$\dfrac{\dfrac{\dfrac{\varphi}{\varphi \vee \psi}\,(\mathrm{R4})}{\psi \vee \psi}\,(\mathrm{R}i)}{\psi}\,(\mathrm{R6})$$ (b) $$\dfrac{\dfrac{\dfrac{\varphi \wedge r}{\varphi}\,(\mathrm{R1})}{\psi}\,(\mathrm{a}) \qquad \dfrac{\varphi \wedge r}{r}\,(\mathrm{R2})}{\psi \wedge r}\,(\mathrm{R3})$$ ∎

Note that rules (R10$^+$) to (R15$^+$) are the usual rules corresponding to the Law of Double Negation and to the De Morgan Laws.

Proposition 3.3 *The interderivability relation* $\dashv\vdash_H$ *(defined as* $\varphi \dashv\vdash_H \psi \iff \varphi \vdash_H \psi$ *and* $\psi \vdash_H \varphi$*) is a congruence with respect to the operations* $\wedge$ *and* $\vee$*.*

Proof. It is enough to show that for each rule of Definition 3.1 of the form $\dfrac{\varphi}{\psi}$, the rules $\dfrac{\varphi \vee \xi}{\psi \vee \xi}$ and $\dfrac{\varphi \wedge \xi}{\psi \wedge \xi}$ hold in $\vdash_H$, and that the rules $\dfrac{p \wedge r \quad q \wedge r}{(p \wedge q) \wedge r}$ and $\dfrac{p \vee r \quad q \vee r}{(p \wedge q) \vee r}$ also hold. The ones not involving negation are known to follow just from rules (R1) to (R9); Proposition 3.2 shows the conjunction case for the remaining ones, and the disjunction case for these is easily shown by using associativity of $\vee$. ∎

This will enable us to *replace* interderivable formulas inside *formulas built up with only* $\wedge$ *and* $\vee$, as we will soon do.

Lemma 3.4 *If* $\mathfrak{A} \in \mathbf{DM}$ *and* $F \in \mathcal{F}(\mathfrak{A})$ *then* $\langle \mathfrak{A}, F \rangle$ *is a matrix for* $\vdash_H$*.*

Proof. It is straightforward to check that, if $\mathfrak{A} \in \mathbf{DM}$, then $\mathfrak{A} \vDash \varphi \preccurlyeq \psi$ for any rule of the form $\dfrac{\varphi}{\psi}$ of Definition 3.1, that is, for all the rules except (R3). Therefore every lattice filter of $\mathfrak{A}$ is closed under these rules; and it is also closed under (R3) by definition. ∎

Taking Corollary 2.6 into account, this implies:

Corollary 3.5 $\vdash_H \ \le\ \vDash_{\mathcal{B}}$*, that is, if* $\Gamma \vdash_H \varphi$ *then* $\Gamma \vDash_{\mathcal{B}} \varphi$*.* ∎

Note that, since $\vdash_H$ has no axioms, it has no theorems, and again $\emptyset$ is also a filter for it on every algebra $\mathfrak{A}$. To prove the converse of the implication in Corollary 3.5, we introduce some (well-known) special types of formulas.

Definition 3.6 $\mathcal{L}it = Var \cup \{\neg p : p \in Var\}$ *is the set of* ***literals****.* $\mathcal{C}l$*, the set of* ***clauses****, is the least set of formulas containing* $\mathcal{L}it$ *and closed under* $\vee$*. For any* $\varphi \in Fm$*, the set of* ***variables of*** φ*,* $var(\varphi)$*, is defined in the usual way; and for* $\Gamma \subseteq Fm$*,* $var(\Gamma) = \bigcup_{\varphi \in \Gamma} var(\varphi)$*. For any* $\varphi \in \mathcal{C}l$*, the set of* ***literals of*** φ*,* $lit(\varphi)$*, is defined inductively by:* $lit(\varphi) = \{\varphi\}$ *if* $\varphi \in \mathcal{L}it$*, and* $lit(\varphi \vee \psi) = lit(\varphi) \cup lit(\psi)$*; for* $\Gamma \subseteq \mathcal{C}l$*,* $lit(\Gamma) = \bigcup_{\varphi \in \Gamma} lit(\varphi)$*.*

Thus clauses are just disjunctions of literals (in [2, p. 154] they are called *primitive disjunctions*); actually we can recognize that any clause φ is a disjunction of literals in $lit(\varphi)$, modulo some associations, permutations, repetitions, etc. Note also that a variable $p \in Var$, although it is a literal, will not be a literal of a clause φ if p appears in φ only under the form $\neg p$: for while $p, \neg p \in \mathcal{L}it$, the definition of the set of literals of a clause is such that $lit(\neg p) = \{\neg p\}$.

Lemma 3.7 *For all $\varphi \in Fm$ there is a finite $\Gamma \subseteq \mathcal{C}l$ such that $var(\varphi) = var(\Gamma)$ and for every $\psi \in Fm$, $\mathbf{C}_H(\varphi \vee \psi) = \mathbf{C}_H\big(\{\gamma \vee \psi : \gamma \in \Gamma\}\big)$.*

Proof. By induction on the length of φ.

If $\varphi = p \in Var$, then $\Gamma = \{p\}$.

If $\varphi = \varphi_1 \wedge \varphi_2$ and Γ_1, Γ_2 correspond to φ_1, φ_2 resp. by inductive hypothesis, then $\Gamma = \Gamma_1 \cup \Gamma_2$ satisfies $var(\Gamma) = var(\varphi)$ and $\mathbf{C}_H(\varphi \vee \psi) = \mathbf{C}_H\big((\varphi_1 \wedge \varphi_2) \vee \psi\big) = \mathbf{C}_H\big((\varphi_1 \vee \psi) \wedge (\varphi_2 \vee \psi)\big) =$, by R1, R2, R3, $= \mathbf{C}_H\big(\varphi_1 \vee \psi,\ \varphi_2 \vee \psi\big) = \mathbf{C}_H\big(\mathbf{C}_H(\varphi_1 \vee \psi) \cup \mathbf{C}_H(\varphi_2 \vee \psi)\big) = \mathbf{C}_H\big(\mathbf{C}_H(\{\gamma_1 \vee \psi : \gamma_1 \in \Gamma_1\}) \cup \mathbf{C}_H(\{\gamma_2 \vee \psi : \gamma_2 \in \Gamma_2\})\big) = \mathbf{C}_H\big(\{\gamma \vee \psi : \gamma \in \Gamma\}\big)$.

If $\varphi = \varphi_1 \vee \varphi_2$ and Γ_1, Γ_2 correspond to φ_1, φ_2, then $\Gamma = \{\gamma_1 \vee \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$ satisfies $var(\Gamma) = var(\varphi)$ and $\mathbf{C}_H(\varphi \vee \psi) = \mathbf{C}_H\big((\varphi_1 \vee \varphi_2) \vee \psi\big) = \mathbf{C}_H\big(\varphi_1 \vee (\varphi_2 \vee \psi)\big) =$ by inductive hypothesis $= \mathbf{C}_H\big(\{\gamma_1 \vee (\varphi_2 \vee \psi) : \gamma_1 \in \Gamma_1\}\big) = \mathbf{C}_H\big(\{\varphi_2 \vee (\gamma_1 \vee \psi) : \gamma_1 \in \Gamma_1\}\big) = (\dots) = \mathbf{C}_H\big(\{\gamma_2 \vee (\gamma_1 \vee \psi) : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}\big) = \mathbf{C}_H\big(\{(\gamma_1 \vee \gamma_2) \vee \psi : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}\big)$.

If $\varphi = \neg\varphi'$ then we distinguish cases on φ' :

If $\varphi' = p \in Var$ then $\varphi \in \mathcal{L}it \subseteq \mathcal{C}l$ and $\Gamma = \{\varphi\}$ works.

If $\varphi' = \neg\varphi''$ then $\varphi = \neg\neg\varphi''$ and by (R10) and (R11) we have $\mathbf{C}_H(\varphi \vee \psi) = \mathbf{C}_H(\varphi'' \vee \psi)$; φ'' is shorter than φ and its set Γ also works for φ.

If $\varphi' = \varphi_1 \wedge \varphi_2$ then $\varphi = \neg(\varphi_1 \wedge \varphi_2)$ and by (R14) and (R15) we have $\mathbf{C}_H(\varphi \vee \psi) = \mathbf{C}_H\big((\neg\varphi_1 \vee \neg\varphi_2) \vee \psi\big)$; both $\neg\varphi_1$ and $\neg\varphi_2$ are shorter than $\neg(\varphi_1 \wedge \varphi_2)$, and the same procedure followed in the case $\varphi = \varphi_1 \vee \varphi_2$ works.

If $\varphi' = \varphi_1 \vee \varphi_2$ then $\varphi = \neg(\varphi_1 \vee \varphi_2)$ and by (R12) and (R13) we have $\mathbf{C}_H(\varphi \vee \psi) = \mathbf{C}_H\big((\neg\varphi_1 \wedge \neg\varphi_2) \vee \psi\big)$ and as before, the procedure given in the case $\varphi = \varphi \wedge \varphi_2$ also works. ∎

Proposition 3.8 *For all $\varphi \in Fm$ there is a finite $\Gamma_\varphi \subseteq \mathcal{C}l$ such that $var(\varphi) = var(\Gamma_\varphi)$ and $\mathbf{C}_H(\varphi) = \mathbf{C}_H(\Gamma_\varphi)$.*

Proof. By induction on the length of φ.

If $\varphi = p \in Var$, take $\Gamma_\varphi = \{\varphi\}$.

If $\varphi = \varphi_1 \wedge \varphi_2$, since by (R1), (R2) and (R3) we have $\mathbf{C}_H(\varphi) = \mathbf{C}_H(\varphi_1, \varphi_2)$, taking $\Gamma_\varphi = \Gamma_{\varphi_1} \cup \Gamma_{\varphi_2}$ we are done.

If $\varphi = \varphi_1 \vee \varphi_2$ then directly by Lemma 3.7 and (R5): $\mathbf{C}_H(\varphi) = \mathbf{C}_H\big(\{\gamma_1 \vee \varphi_2 : \gamma_1 \in \Gamma_1\}\big) = \mathbf{C}_H\big(\{\varphi_2 \vee \gamma_1 : \gamma_1 \in \Gamma_1\}\big) = \mathbf{C}_H\big(\{\gamma_2 \vee \gamma_1 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}\big)$ and since $\Gamma_1, \Gamma_2 \subseteq \mathcal{C}l$ are finite, also $\Gamma_\varphi = \{\gamma_1 \vee \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\} \subseteq \mathcal{C}l$ works and is finite.

If $\varphi = \neg\psi$ then we distinguish cases on ψ :

If $\varphi = \neg p,\ p \in Var$, then $\varphi \in \mathcal{C}l$ and $\Gamma_\varphi = \{\varphi\}$ works.

If $\varphi = \neg\neg\varphi'$ then by (R10$^+$), (R11$^+$) $\mathbf{C}_H(\varphi) = \mathbf{C}_H(\varphi')$, and since φ' is shorter than φ, we are done.

If $\varphi = \neg(\varphi_1 \wedge \varphi_2)$ then by (R14$^+$), (R15$^+$) $\mathbf{C}_H(\varphi) = \mathbf{C}_H\big(\neg\varphi_1 \vee \neg\varphi_2)$ and the same procedure for $\varphi = \varphi_1 \vee \varphi_2$ works.

If $\varphi = \neg(\varphi_1 \vee \varphi_2)$ then by (R12$^+$), (R13$^+$) $\mathbf{C}_H(\varphi) = \mathbf{C}_H(\neg\varphi_1, \neg\varphi_2)$ and the inductive hypothesis ends the proof. ∎

Theorem 3.9 (Normal Form) *Every formula is equivalent, both through $\dashv\vdash_H$ and through $\dashv\models_{\mathcal{B}}$, to a conjunction of clauses with the same variables.*

Proof. ¿From Proposition 3.8 it results at once that $\mathbf{C}_H(\varphi) = \mathbf{C}_H(\bigwedge\Gamma_\varphi)$, where $\bigwedge\Gamma_\varphi$ is any conjunction (with any order and association of parentheses) of all clauses in Γ_φ. By Corollary 3.5 this implies that also $\mathbf{C}_{\mathcal{B}}(\varphi) = \mathbf{C}_{\mathcal{B}}(\bigwedge\Gamma_\varphi)$. ∎

Now we analyse closely the behaviour of both logics on clauses:

Proposition 3.10 *Let $\Gamma \subseteq \mathcal{C}l$, $\varphi \in \mathcal{C}l$. Then the following are equivalent:*

(i) $\Gamma \vdash_H \varphi$.
(ii) $\Gamma \models_{\mathcal{B}} \varphi$.
(iii) $\exists\, \gamma \in \Gamma$ *such that* $lit(\gamma) \subseteq lit(\varphi)$.
(iv) $\exists\, \gamma \in \Gamma$ *such that* $\gamma \vdash_H \varphi$.

Proof. (i)⇒(ii) follows from Corollary 3.5.

(ii)⇒(iii): For a fixed $\varphi \in \mathcal{C}l$ define $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{M}_4)$ by putting, for every $p \in Var$:

$$h(p) \;=\; \begin{cases} \mathsf{t} & \text{if} \quad p \notin lit(\varphi) \;\text{ and }\; \neg p \in lit(\varphi) \\ \mathsf{n} & \text{if} \quad p, \neg p \notin lit(\varphi) \\ \mathsf{b} & \text{if} \quad p, \neg p \in lit(\varphi) \\ \mathsf{f} & \text{if} \quad p \in lit(\varphi) \;\text{ and }\; \neg p \notin lit(\varphi) \end{cases}$$

Thus $h(p) \in \{\mathsf{f}, \mathsf{b}\}$ when $p \in lit(\varphi)$, but also $h(\neg p) \in \{\mathsf{f}, \mathsf{b}\}$ when $\neg p \in lit(\varphi)$; therefore $h(\varphi) \in \{\mathsf{f}, \mathsf{b}\}$ because this set is a filter. If (iii) were to fail, then for each $\gamma \in \Gamma$ there would be a $\psi_\gamma \in lit(\gamma)$ such that $\psi_\gamma \notin lit(\varphi)$; we can check that then we would have $h(\psi_\gamma) \in \{\mathsf{n}, \mathsf{t}\}$ and as a consequence $h(\gamma) \in \{\mathsf{n}, \mathsf{t}\}$. Thus $h[\Gamma] \subseteq F_{\mathsf{n}}$ while $h(\varphi) \notin F_{\mathsf{n}}$, against (ii).

(iii)⇒(iv): If $lit(\gamma) \subseteq lit(\varphi)$ and both are clauses, then φ is a disjunction of the same literals appearing in γ plus other ones, modulo some associations, etc … Therefore by (R4), (R7), (R5) and (R6) and making heavy use of Proposition 3.3 we get $\gamma \vdash_H \varphi$.

(iv)⇒(i) is obvious. ∎

Thus we see that $\vdash_H$ and $\models_{\mathcal{B}}$ agree on clauses. Putting this together with Theorem 3.9 we immediately get:

Theorem 3.11 (Second Completeness) $\vdash_H \;=\; \models_{\mathcal{B}}$*, that is, Belnap's logic can be axiomatized by the finite list of Hilbert-style rules given in Definition 3.1.* ∎

One of the applications of this result is a characterization of the $\mathcal{B}$-filters on De Morgan lattices:

Proposition 3.12 *If $\mathfrak{A} \in \mathbf{DM}$ then $\forall F \subseteq A$, F is a $\mathcal{B}$-filter iff F is a lattice filter of $\mathfrak{A}$ or $F = \emptyset$; that is, $\mathcal{F}i_{\mathcal{B}}\mathfrak{A} = \mathcal{F}(\mathfrak{A}) \cup \{\emptyset\}$.*

Proof. That $\emptyset$ and every lattice filter is a filter for $\vDash_{\mathcal{B}}$ was seen in Corollary 2.6. By Theorem 3.11, a set $F \subseteq A$ is a filter for $\mathcal{B}$ iff it is closed under all rules of $\vdash_H$; if $\mathfrak{A} \in \mathbf{DM}$ then rules (R5) to (R15) become equalities, and being closed under rules (R1) to (R4) amounts to being a lattice filter or $= \emptyset$. ∎

Another application is a characterization of the Leibniz congruence on arbitrary matrices for $\mathcal{B}$; as a by-product, we will obtain a description of its reduced matrices, and some information on the class $\mathbf{Alg}^*\mathcal{B}$:

Proposition 3.13 *If $\langle \mathfrak{A}, D\rangle$ is a matrix for $\mathcal{B}$ then $\forall\, a, b \in A$, $\langle a, b\rangle \in \Omega_{\mathfrak{A}} D$ if and only if for any $c \in A$ the following hold:*

$$\begin{array}{lrcl} \text{(a)} & a \vee c \in D & \iff & b \vee c \in D \\ \text{(b)} & \neg a \vee c \in D & \iff & \neg b \vee c \in D \end{array}$$

Proof. ($\Rightarrow$): The following characterization [9] of the Leibniz congruence will be useful: $\langle a, b\rangle \in \Omega_A D$ iff $\forall \varphi(p_0, p_1, \ldots, p_n) \in Fm$, $\forall c_1, \ldots, c_n \in A$, $\varphi^{\mathfrak{A}}(a, c_1, \ldots, c_n) \in D$ iff $\varphi^{\mathfrak{A}}(b, c_1, \ldots, c_n) \in D$. In particular, taking $\varphi = p_0 \vee p_1$ we obtain (a), and taking $\varphi = \neg p_0 \vee p_1$ we obtain (b).

($\Leftarrow$): If $\forall c \in A$, (a) and (b) hold, then using that $\langle \mathfrak{A}, D\rangle$ is a matrix for $\mathcal{B}$ we can show that $\forall c \in A$ the following also hold:

$$\begin{array}{lrcl} \text{(c)} & a \in D & \iff & b \in D \\ \text{(d)} & \neg a \in D & \iff & \neg b \in D \\ \text{(e)} & a \vee \neg a \vee c \in D & \iff & b \vee \neg b \vee c \in D \\ \text{(f)} & a \vee \neg a \in D & \iff & b \vee \neg b \in D \end{array}$$

(c) holds because $a \in D \Leftrightarrow a \vee a \in D \Leftrightarrow b \vee a \in D \Leftrightarrow a \vee b \in D \Leftrightarrow b \vee b \in D \Leftrightarrow b \in D$; (d) is similar, using (b) instead of (a); and (e) and (f) use both (a) and (b).

Now suppose that $\langle a, b\rangle \notin \Omega_{\mathfrak{A}} D$: There is some $\varphi(p_0, p_1, \ldots, p_n) \in Fm$ and some $c_1, \ldots, c_n \in A$ with $\varphi^{\mathfrak{A}}(a, c_1, \ldots c_n) \in D$ but $\varphi^{\mathfrak{A}}(b, c_1, \ldots, c_n) \notin D$. By Theorem 3.9 we can assume without loss of generality that $\varphi \in \mathcal{C}l$. The variable p_0 must obligatorily appear in φ; by deleting repeated appearances of p_0 and of $\neg p_0$ (this uses several rules of $\vdash_H$, Proposition 3.3, and the assumption that $\langle \mathfrak{A}, D\rangle$ is a matrix for it) we see that φ can be assumed to have one of the six following forms (where $\psi \in \mathcal{C}l$):

$$\begin{array}{lll} p_0 & \neg p_0 & p_0 \vee \neg p_0 \\ p_0 \vee \psi(p_1, \ldots, p_n) & \neg p_0 \vee \psi(p_1, \ldots, p_n) & p_0 \vee \neg p_0 \vee \psi(p_1, \ldots, p_n) \end{array}$$

But from (a) , ... , (f) we know that neither of these can satisfy the requirement. Thus $\langle a, b\rangle \in \Omega_{\mathfrak{A}} D$ must be true. ∎

We already know that for any $\mathfrak{A}$, $\langle \mathfrak{A}, \emptyset\rangle$ is a matrix for $\mathcal{B}$; and since $\Omega_{\mathfrak{A}}\emptyset = A \times A$, the only case where this matrix is reduced is when $\mathfrak{A}$ is a one-element algebra, that is, a trivial one; in this case the matrix $\langle \mathfrak{A}, A\rangle$ is also reduced. The non-trivial reduced matrices can be described as follows:

Theorem 3.14 *Let $\mathfrak{A}$ be a non-trivial algebra. Then $\langle \mathfrak{A}, D\rangle$ is a reduced matrix for $\mathcal{B}$ iff $\mathfrak{A} \in \mathbf{DM}$ and D is a lattice filter of $\mathfrak{A}$ such that $\forall a, b \in A$, if $a < b$ then there is a $c \in A$ such that [$a \vee c \notin D$ and $b \vee c \in D$] or [$\neg b \vee c \notin D$ and $\neg a \vee c \in D$].*

Proof. Let $\langle \mathfrak{A}, D\rangle$ be a reduced matrix for $\mathcal{B}$ on a non-trivial algebra $\mathfrak{A}$. We first show that $\mathfrak{A} \in \mathbf{DM}$: Since $\mathbf{DM}$ is a variety, in order to prove this we just show that $\mathfrak{A}$ satisfies all equations that hold in $\mathbf{DM}$. Thus, assume that $\mathbf{DM} \vDash \varphi \approx \psi$, and pick any $p_0 \in \mathit{Var}$ not appearing in $\varphi \approx \psi$. By replacement, also $\mathbf{DM} \vDash \varphi \vee p_0 \approx \psi \vee p_0$ and $\mathbf{DM} \vDash \neg\varphi \vee p_0 \approx \neg\psi \vee p_0$. By Corollary 2.7, this implies $\varphi \vee p_0 \dashv\vDash_{\mathcal{B}} \psi \vee p_0$ and $\neg\varphi \vee p_0 \dashv\vDash_{\mathcal{B}} \neg\psi \vee p_0$. Since $\langle \mathfrak{A}, D\rangle$ is a matrix for $\mathcal{B}$, this implies that $\forall h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A})$, $h(\varphi) \vee h(p_0) \in D$ iff $h(\psi) \vee h(p_0) \in D$, and $\neg h(\varphi) \vee h(p_0) \in D$ iff $\neg h(\psi) \vee h(p_0) \in D$. By the choice of p_0, $h(p_0) \in A$ is arbitrary, so we have proved that $\forall c \in A$, $h(\varphi) \vee c \in D$ iff $h(\psi) \vee c \in D$, and $\neg h(\varphi) \vee c \in D$ iff $\neg h(\psi) \vee c \in D$. By the characterization in Proposition 3.13 we have that $\langle h(\varphi), h(\psi)\rangle \in \Omega_{\mathfrak{A}} D$; but the assumption that the matrix is reduced implies $h(\varphi) = h(\psi)$. Since the homomorphism h is arbitrary, we have shown that $\mathfrak{A} \vDash \varphi \approx \psi$. That is, $\mathfrak{A} \in \mathbf{DM}$. Since the assumption that $\mathfrak{A}$ is non-trivial excludes the case $D = \emptyset$, Proposition 3.12 implies that D is a lattice filter of $\mathfrak{A}$. Finally, if $a < b$ and the matrix is reduced, this means that $\langle a, b\rangle \notin \Omega_{\mathfrak{A}} D$, and by Proposition 3.13 the remaining condition must be true.

For the converse, Proposition 3.12 tells us that for any non-trivial $\mathfrak{A} \in \mathbf{DM}$ and any lattice filter D of $\mathfrak{A}$, $\langle \mathfrak{A}, D\rangle$ is a matrix for $\mathcal{B}$, and the additional condition plus Proposition 3.13 implies it is reduced: If $a \neq b$ then $a \wedge b < a \vee b$, and we can apply the assumption to $a \wedge b$ and $a \vee b$; but what we find actually means that $\langle a \wedge b, a \vee b\rangle \notin \Omega_{\mathfrak{A}} D$, which is equivalent to $\langle a, b\rangle \notin \Omega_{\mathfrak{A}} D$. ∎

We can now obtain some more information about the class of reduced matrices for $\mathcal{B}$ and the class $\mathbf{Alg}^*\mathcal{B}$ of their algebraic reducts. Since $\mathfrak{M}_4$ is a simple algebra, it results that $\langle \mathfrak{M}_4, F_{\mathsf{n}}\rangle$, $\langle \mathfrak{M}_4, F_{\mathsf{b}}\rangle$, and $\langle \mathfrak{M}_4, \{\mathsf{t}\}\rangle$ are all reduced matrices for $\mathcal{B}$. Thus, on one hand, we see that not all non-trivial reduced matrices for $\mathcal{B}$ have the form $\langle \mathfrak{A}, \{1\}\rangle$, as one could naïvely expect (as it happens with $\mathcal{CPC}_{\wedge\vee}$, the $\{\wedge, \vee\}$-fragment of CPC, see [19]). On the other hand, we also see that a single $\mathfrak{A} \in \mathbf{Alg}^*\mathcal{B}$ can support several reduced matrices, a thing that cannot happen in $\mathcal{CPC}_{\wedge\vee}$. One can also ask a reverse question: If an $\mathfrak{A} \in \mathbf{DM}$ is bounded above, say by 1, is then the matrix $\langle \mathfrak{A}, \{1\}\rangle$ reduced? The answer is no, a counterexample being the six-element De Morgan lattice $\mathfrak{M}_6$ on the set $M_6 = \{0, a, b, c, d, 1\}$ with Hasse diagram and negation given in Figure 4; besides Δ_{M_6} and $M_6 \times M_6$, it has two more congruences, θ_1 and θ_2, whose blocks are shown in the table in Figure 4; this table also shows all the $\mathcal{B}$-filters on $\mathfrak{M}_6$, and their Leibniz congruences: None of them is the identity, therefore $\mathfrak{M}_6 \notin \mathbf{Alg}^*\mathcal{B}$. In particular, we see that $\mathbf{Alg}^*\mathcal{B}$ is a proper subclass of $\mathbf{DM}$. More precisely, since it contains the generators $\mathfrak{M}_2$ and $\mathfrak{M}_3$ of the only proper subvarieties of $\mathbf{DM}$, it results that $\mathbf{Alg}^*\mathcal{B}$ is not a variety, and that it properly contains the classes of Boolean algebras and Kleene algebras, see Section 5. Other miscellaneous facts that one can prove about $\mathbf{Alg}^*\mathcal{B}$, using basically Theorem 3.14, are that the only chains (either finite or infinite) in $\mathbf{Alg}^*\mathcal{B}$ are $\mathfrak{M}_2$ and $\mathfrak{M}_3$, and that the only algebras supporting reduced matrices of $\mathcal{B}$ where the filter is prime are $\mathfrak{M}_2$, $\mathfrak{M}_3$ and $\mathfrak{M}_4$. We leave the proofs of these facts as an exercise for the reader.

We end this section with the characterization of a kind of matrices that individually give completeness for $\mathcal{B}$, as $\langle \mathfrak{M}_4, F_{\mathsf{n}}\rangle$ or $\langle \mathfrak{M}_4, F_{\mathsf{b}}\rangle$ do. To this end we consider on any De Morgan lattice $\mathfrak{A}$ the mapping

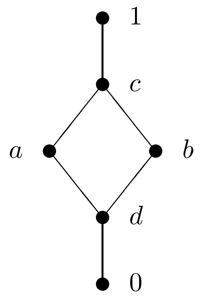
$$F \subseteq A \quad \longmapsto \quad \Phi(F) = A \smallsetminus \{\neg x : x \in F\}\ , \tag{3.1}$$

informally called by A. Monteiro "the Birula-Rasiowa transformation", probably because it comes from the representation of De Morgan algebras given in [8]. It is known that if F is a prime filter of $\mathfrak{A}$ then $\Phi(F)$ is also a prime filter and $\Phi\big(\Phi(F)\big) = F$, and that congruences of $\mathfrak{A}$ are determined by families of prime filters closed under Φ; these results were worked out in detail, for De Morgan algebras, by Monteiro in his unpublished course [33]; parts of it can also be found in [5, 34, 36].

Proposition 3.15 *Let $\mathfrak{A} \in \mathbf{DM}$ and let F be a prime filter of $\mathfrak{A}$ such that F and $\Phi(F)$ are not comparable by $\subseteq$. Then the matrix $\langle \mathfrak{A}, F\rangle$ is complete for $\mathcal{B}$.*

PROOF. Consider the congruence $\theta \in \mathrm{Co}\mathfrak{A}$ associated with the family of prime filters $\{F, \Phi(F)\}$; it is defined as : $a\,\theta\, b$ if and only if both $\big[a \in F \Leftrightarrow b \in F\big]$ and $\big[a \in \Phi(F) \Leftrightarrow b \in \Phi(F)\big]$. As a consequence of this definition, the quotient $\mathfrak{A}/\theta \in \mathbf{DM}$ and has four elements, corresponding to the equivalence classes $F \cap \Phi(F)$, $F \smallsetminus \Phi(F)$, $\Phi(F) \smallsetminus F$ and $A \smallsetminus \big(F \cup \Phi(F)\big)$; they are all non-empty by the assumptions that F and $\Phi(F)$ are non-empty, proper prime filters and not comparable. By taking into account that both F and $\Phi(F)$ are prime filters, and the definition of Φ, one can check that $\mathfrak{A}/\theta \cong \mathfrak{M}_4$, and the projection $\pi : \mathfrak{A} \to \mathfrak{A}/\theta = \mathfrak{M}_4$ is an epimorphism such that $F = \pi^{-1}[F_{\mathsf{n}}]$ (if we put $\pi(x) = \mathsf{n}\ \ \forall x \in F \smallsetminus \Phi(F)\ldots$; the symmetric one with b is equally possible). Using the terminology of the classical theory of matrices [41], we have that π is a strict epimorphism from the matrix $\langle \mathfrak{A}, F\rangle$ onto the matrix $\langle \mathfrak{M}_4, F_{\mathsf{n}}\rangle$, and it is well-known that in such situation the two matrices are semantically equivalent. Then Proposition 2.3 ends the proof. ∎

An example of this situation is the six-element De Morgan lattice $\mathfrak{M}_6$ described in Figure 4, if we take $F = \{1, c, a\}$ or $F = \{1, c, b\}$. It is interesting to note that this lattice was used in [22] to formalize a six-valued logic that had been introduced in [26] as a common extension of Łukasiewicz and Kleene's strong three-valued logics in connection with an analysis of the changes in truth-values that a sentence can



x	$\neg x$
0	1
a	a
b	b
c	d
d	c
1	0

$F \in \mathcal{F}i_{\mathcal{B}}\mathfrak{M}_6$	$\theta = \Omega_{\mathfrak{M}_6} F$	blocks of θ
$\emptyset$	$M_6 \times M_6$	M_6
M_6		
$\{1\}$	θ_1	$\{1\}\quad\{0\}$
$\{1, a, b, c, d\}$		$\{a, b, c, d\}$
$\{1, c\}$	θ_2	$\{1, c\}$
$\{1, c, a\}$		$\{a\}\quad\{b\}$
$\{1, c, b\}$		$\{d, 0\}$

FIG. 4: The De Morgan lattice $\mathfrak{M}_6$ described by its lattice structure and its negation operation. The table lists the $\mathcal{B}$-filters on $\mathfrak{M}_6$ with their Leibniz congruences described by their blocks.

experience during a given interval of time; in [22] it was proved, using algebraic techniques, that the resulting sentential logic (which was defined from $\mathfrak{M}_6$ as we do from $\mathfrak{M}_4$ in our Definition 2.1) was in fact Belnap's logic, that is, it was not really six-valued, but simply four-valued. This construction was further generalized in [17], where the connections with bilattices are also studied.

4 Full models and algebraic counterpart

In this section we are going to show that the class **DM** is the algebraic counterpart of $\mathcal{B}$ in the sense of the general theory of [18, 21], and will find the canonical class of its full models. We begin by summarizing some terms and notations from [21]; for more on the general theory of abstract logics see also [13].

The models are now not matrices, but families of matrices having the same underlying algebra; that is, abstract logics. An **abstract logic** $\mathbb{L} = \langle \mathfrak{A}, \mathbf{C} \rangle$ or $\mathbb{L} = \langle \mathfrak{A}, \mathcal{C} \rangle$ is constituted by an algebra $\mathfrak{A}$ and a closure operator $\mathbf{C}$ on A, or, equivalently, a closure system $\mathcal{C}$ on A. Because of the one-to-one dual correspondence between closure operators and closure systems on a given set, sometimes we define an abstract logic by giving its closure operator, and sometimes by its closure system; the link between them is the usual one: $\mathcal{C} = \{X \subseteq A : \mathbf{C}(X) = X\}$, and $\mathbf{C}(X) = \bigcap\{T \in \mathcal{C} : T \supseteq X\}$. We adopt the customary abbreviations $\mathbf{C}(a)$ for $\mathbf{C}(\{a\})$ (if $a \in A$), $\mathbf{C}(X, Y)$ for $\mathbf{C}(X \cup Y)$ (if $X, Y \subseteq A$), and so on.

There are two groups of natural examples of abstract logics related to our subject: Every sentential logic presented in the form $\mathcal{S} = \langle \mathfrak{Fm}, \mathbf{C}_{\mathcal{S}} \rangle$ can actually be defined as an abstract logic on the formula algebra satisfying the property of structurality; and with every De Morgan lattice $\mathfrak{A}$ we can associate the abstract logic $\langle \mathfrak{A}\,, \mathbf{F} \rangle$ or $\langle \mathfrak{A}\,, \mathcal{F}(\mathfrak{A}) \cup \{\emptyset\} \rangle$. One of the purposes of this section is to see to what extent does this second group of abstract logics mirror the metalogical properties of the former, in our case the sentential logic $\mathcal{B}$ presented as $\langle \mathfrak{Fm}\,, \mathbf{C}_{\mathcal{B}} \rangle$.

With every abstract logic we associate its **Frege relation** $\Lambda\mathbb{L} = \{\langle a, b \rangle \in A \times A : \mathbf{C}(a) = \mathbf{C}(b)\}$, which is always an equivalence relation, and its **Tarski congruence** $\widetilde{\Omega}\mathbb{L} = \max\{\theta \in \mathrm{Co}\mathfrak{A} : \theta \subseteq \Lambda\mathbb{L}\}$, the greatest congruence below the Frege relation. The **reduction** $\mathbb{L}^*$ of an abstract logic $\mathbb{L} = \langle \mathfrak{A}, \mathcal{C} \rangle$ is the abstract logic $\mathbb{L}^* = \langle \mathfrak{A}^*, \mathcal{C}^* \rangle$ where $\mathfrak{A}^* = \mathfrak{A}/\widetilde{\Omega}\mathbb{L}$ and $\mathcal{C}^* = \{S \subseteq A^* : \pi^{-1}[S] \in \mathcal{C}\}$, where $\pi : A \to A^*$ $(= A/\widetilde{\Omega}\mathbb{L})$ is the canonical projection. An abstract logic is called **reduced** when $\widetilde{\Omega}\mathbb{L} = \Delta_A$. Every reduction is reduced.

One says that $\mathbb{L}$ is a **model of a sentential logic** $\mathcal{S}$ iff $\Gamma \vdash_{\mathcal{S}} \varphi$ implies that $\forall h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A})$, $h(\varphi) \in \mathbf{C}(h[\Gamma])$; since $\mathbf{C}(h[\Gamma]) = \bigcap\{F \in \mathcal{C} : h[\Gamma] \subseteq F\}$, it is straightforward to see that $\mathbb{L}$ is a model of $\mathcal{S}$ if and only if $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\mathfrak{A}$. An abstract logic $\mathbb{L}$ is a **full model of** $\mathcal{S}$ iff its reduction satisfies $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}}(\mathfrak{A}^*)$. An arbitrary family of matrices makes a model, and ordinary models are just models of the entailment relations of $\vdash_{\mathcal{S}}$. By contrast, full models in some sense (which is discussed in [21]) inherit, in abstract form, many metalogical properties of $\mathcal{S}$, like the Deduction Theorem or, in our case, the Properties of Disjunction PDI and of Weak Contraposition PWC appearing in Proposition 2.10, as we prove in Theorem 4.6; that these properties are not inherited by arbitrary models is also shown in this section. In this framework the **algebraic counterpart of a sentential logic** is the so-called class of $\mathcal{S}$-**algebras**:

$$\begin{aligned}\mathbf{Alg}\mathcal{S} &= \{\,\mathfrak{A} : \mathfrak{A} \text{ is the algebra reduct of a reduced full model of } \mathcal{S}\,\} \\ &= \{\,\mathfrak{A} : \text{ the abstract logic } \langle \mathfrak{A}, \mathcal{F}i_{\mathcal{S}}\mathfrak{A}\rangle \text{ is reduced}\,\}\end{aligned}$$

For protoalgebraic logics it results that $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}^*\mathcal{S}$; hence this definition agrees with the one commonly used in the classical matrix approach in the class of logics where matrices are well-behaved, as shown in [9, 11].

In the case of Belnap's logic, although $\mathcal{B}$ is not protoalgebraic (Theorem 2.11), it is well-behaved enough for making the determination of $\mathbf{Alg}\mathcal{B}$ rather straightforward:

Theorem 4.1 $\mathbf{Alg}\mathcal{B} = \mathbf{DM}$*, that is, the class of De Morgan lattices is the algebraic counterpart of Belnap's logic, according to the criteria of* [21].

Proof. We have already seen, after Theorem 3.14, that $\mathbf{Alg}^*\mathcal{B} \subseteq \mathbf{DM}$ and that $\mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_4 \in \mathbf{Alg}^*\mathcal{B}$; since $\mathbf{DM}$ is a variety and $\mathfrak{M}_2$ and $\mathfrak{M}_3$ are the generators of its only proper subvarieties, it follows that the variety generated by $\mathbf{Alg}^*\mathcal{B}$ is exactly $\mathbf{DM}$. On the other hand, it is not difficult to show, using just the definitions, that $\mathbf{Alg}\mathcal{B}$ is the class of all subdirect products of algebras in $\mathbf{Alg}^*\mathcal{B}$ (this is a general fact, see [21, Theorem 2.23]), therefore $\mathbf{Alg}\mathcal{B} \subseteq \mathbf{DM}$. For the converse inclusion, let $\mathfrak{A} \in \mathbf{DM}$; as in any lattice, different elements of A generate different principal filters, that is, $\mathbf{F}(a) = \mathbf{F}(b)$ implies $a = b$; since by Proposition 3.12 $\mathcal{F}i_{\mathcal{B}}\mathfrak{A} = \mathcal{F}(\mathfrak{A}) \cup \{\emptyset\}$, we see that the Tarski congruence of the abstract logic $\langle \mathfrak{A}, \mathcal{F}i_{\mathcal{B}}\mathfrak{A}\rangle$ must be the identity relation, that is, this abstract logic is reduced. According to the definition we have just given, this means that $\mathfrak{A} \in \mathbf{Alg}\mathcal{B}$. ∎

The main tool used in [13] and [21] to handle abstract logics, and more precisely to express equivalence between them, is the notion of **bilogical morphism**: If $\mathbb{L}_1 = \langle \mathfrak{A}_1, \mathbf{C}_1\rangle$ and $\mathbb{L}_2 = \langle \mathfrak{A}_2, \mathbf{C}_2\rangle$ are two abstract logics, a bilogical morphism between them (or from $\mathbb{L}_1$ onto $\mathbb{L}_2$) is an epimorphism $h : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $a \in \mathbf{C}_1(X) \iff h(a) \in \mathbf{C}_2(h[X])$ for all $a \in A_1$ and all $X \subseteq A_1$; a convenient equivalent definition in terms of closure systems is to say that exactly $\mathcal{C}_1 = \{h^{-1}[S] : S \in \mathcal{C}_2\}$. Then the mapping induced by h establishes that $\mathcal{C}_1 \cong \mathcal{C}_2$ as complete lattices, and moreover $\mathbf{C}_1 = h^{-1} \circ \mathbf{C}_2 \circ h$ and $\mathbf{C}_2 = h \circ \mathbf{C}_1 \circ h^{-1}$; therefore the relation between the two logics is a very close one. For instance, the projection $\pi : \mathbb{L} \to \mathbb{L}^*$ corresponding to the reduction process is a bilogical morphism. It is easy to prove (Proposition 2.21 of [21]) that $\mathbb{L}$ is a full model of $\mathcal{S}$ iff there is a bilogical morphism from $\mathbb{L}$ onto a logic of the form $\langle \mathfrak{B}, \mathcal{F}i_{\mathcal{S}}\mathfrak{B}\rangle$ for some $\mathfrak{B} \in \mathbf{Alg}\mathcal{S}$. Then we have, in our case:

Proposition 4.2 *For every abstract logic* $\mathbb{L} = \langle \mathfrak{A}, \mathcal{C}\rangle$ *the following are equivalent:*

(i) $\mathbb{L}$ *is a full model of* $\mathcal{B}$.

(ii) $\mathbb{L}^* = \langle \mathfrak{A}^*, \mathcal{C}^*\rangle$ *is such that* $\mathfrak{A}^* \in \mathbf{DM}$ *and* $\mathcal{C}^* = \mathcal{F}(\mathfrak{A}^*) \cup \{\emptyset\}$.

(iii) *There is a bilogical morphism between* $\mathbb{L}$ *and a logic* $\langle \mathfrak{A}', \mathcal{C}'\rangle$ *with* $\mathfrak{A}' \in \mathbf{DM}$ *and* $\mathcal{C}' = \mathcal{F}(\mathfrak{A}') \cup \{\emptyset\}$.

Proof. Given Theorem 4.1 and Proposition 3.12, the equivalence (i)⇔(ii) follows from the first definition of full model we have given. And the equivalence (i)⇔(iii) follows from the equivalent formulation given just above. ∎

Therefore, each full model of $\mathcal{B}$ is essentially equivalent to the natural, intrinsic abstract logic associated with a certain De Morgan lattice. This is the first new way we find to express the relationship between $\mathcal{B}$ and $\mathbf{DM}$. Starting from it, and by adapting previous purely algebraic results obtained in collaboration with V. Verdú [23, 24, 25] we will be able to find several characterizations of those full models. These papers work on algebras with a different similarity type, and thus their results cannot be directly applied here; moreover, some of them are not easily accessible, therefore we will give fairly complete proofs[2]. A family $\mathcal{E}$ of subsets of A is said to be a **basis** of a closure system $\mathcal{C}$ when every $T \in \mathcal{C}$ is the intersection of a subfamily of $\mathcal{E}$ (the whole A being considered as the intersection of the empty subfamily). The first result uses the following transformation, an abstract and more convenient version of the one used at the end of Section 3:

$$X \subseteq A \quad \longmapsto \quad \Phi(X) = \{a \in A : \neg a \notin X\} \tag{4.1}$$

It is straightforward that if $\mathfrak{A} \in \mathbf{DM}$ then (4.1) equals (3.1); but for arbitrary algebras we will take Φ as defined by (4.1).

Theorem 4.3 *An abstract logic $\mathbb{L}$ is a full model of $\mathcal{B}$ if and only if $\mathbb{L}$ is finitary and its closure system has a basis $\mathcal{E}$ such that $\emptyset \in \mathcal{E}$ and every nonempty $T \in \mathcal{E}$ satisfies:*

(1) *T is an $\wedge$-filter, i.e., $a \wedge b \in T \iff \big[\, a \in T$ and $b \in T \,\big]$.*

(2) *T is $\vee$-prime, i.e., $a \vee b \in T \iff \big[\, a \in T$ or $b \in T \,\big]$.*

(3) *$\Phi(T) \in \mathcal{E}$ and $\Phi\big(\Phi(T)\big) = T$.*

Proof. We will actually prove that an abstract logic $\mathbb{L}$ satisfies condition (iii) of Proposition 4.2 if and only if it satisfies the condition in the statement.

($\Rightarrow$) Let $\mathbb{L}' = \mathfrak{A}', \mathcal{C}'$ be the abstract logic of 4.2(iii). From the properties of general lattices it follows that $\mathbb{L}$ is finitary; from the distributivity of the lattice it follows that $\mathcal{C}' = \mathcal{F}(\mathfrak{A}') \cup \{\emptyset\}$ has a basis made of all prime filters of $\mathfrak{A}'$ plus $\emptyset$, thus this family satisfies (1) and (2). Finally, from the properties of De Morgan lattices it follows that condition (3) is also satisfied. Bilogical morphisms transform bases into bases, preserve finitarity and also the conditions (1) to (3). Therefore, $\mathbb{L}$ satisfies them for the basis $\mathcal{E}$ made of the converse images of $\mathcal{F}(\mathfrak{A}') \cup \{\emptyset\}$.

($\Leftarrow$) Let us consider the Frege relation $\Lambda\mathbb{L}$: Since $\mathcal{E}$ is a basis of $\mathcal{C}$, we have that $a \Lambda\mathbb{L} b$ iff for every $T \in \mathcal{E}$, $[a \in T \iff b \in T]$. Moreover, by (1) and (2) $\Lambda\mathbb{L}$ a congruence with respect to $\wedge$ and $\vee$ [25, Theorem 3.5], and by (3) it is a congruence with respect to $\neg$: If $a \Lambda\mathbb{L} b$ then for any $T \in \mathcal{E}$, $\neg a \in T \iff a \notin \Phi(T) \iff b \notin \Phi(T) \iff \neg b \in T$, therefore $\neg a \Lambda\mathbb{L} \neg b$. Hence $\Lambda\mathbb{L} = \widetilde{\Omega}\mathbb{L}$ and the reduction is actually the quotient by $\Lambda\mathbb{L}$; moreover we can use Theorem 3.5 of [25] again and we know that $\mathfrak{A}^*$ is a distributive lattice and that $\mathcal{F}^* = \mathcal{F}(\mathfrak{A}^*) \cup \{\emptyset\}$. Thus it only remains to prove DM2: For any $a \in A$ and any $T \in \mathcal{E}$, $\neg\neg a \in T \iff \neg a \notin \Phi(T) \iff a \in \Phi(\Phi(T)) = T$, so $\mathbf{C}(a) = \mathbf{C}(\neg\neg a)$ and the quotient $\mathfrak{A}^* = \mathfrak{A}/\Lambda\mathbb{L}$ satisfies the equation $x \approx \neg\neg x$. By using the definition of Φ and conditions (2) and (3) one can check that $\mathbf{C}(\neg(a \vee b)) = \mathbf{C}(\neg a \wedge \neg b)$, so $\mathfrak{A}^*$ satisfies $\neg(x \vee y) \approx \neg x \wedge \neg y$, and, similarly, from (1) and (3) we prove that $\mathfrak{A}^*$ satisfies $\neg(x \wedge y) \approx \neg x \vee \neg y$. Therefore $\mathfrak{A}^* \in \mathbf{DM}$. ∎

[2] Following the suggestion of one referee.

This result has an undoubtedly semantical flavour; it will become more explicit in the following result, where we use a generalization of the construction of a sentential logic defined from a logical matrix: We say that an abstract logic $\mathbb{L} = \langle \mathfrak{A}, \mathcal{C} \rangle$ is **generated from** a matrix $\langle \mathfrak{A}', F \rangle$ by a family of homomorphisms $\mathcal{H} \subseteq \mathrm{Hom}(\mathfrak{A}, \mathfrak{A}')$ when the family $\{h^{-1}[F] : h \in \mathcal{H}\}$ is a basis of $\mathcal{C}$. Then:

Theorem 4.4 *An abstract logic $\mathbb{L}$ is a full model of $\mathcal{B}$ iff $\mathbb{L}$ is finitary and is generated from the matrix $\langle \mathfrak{M}_4, F_{\mathsf{n}} \rangle$ by some family $\mathcal{H} \subseteq \mathrm{Hom}(\mathfrak{A}, \mathfrak{M}_4)$ such that the mapping constantly equal to b is in $\mathcal{H}$ and such that if $h \in \mathcal{H}$ then also $s \circ h \in \mathcal{H}$, where s is the non-trivial automorphism of $\mathfrak{M}_4$ given in Proposition 2.3.*

Proof. Let $\mathbb{L}$ be a full model of $\mathcal{B}$, and let $\mathcal{E}$ be the basis of $\mathcal{C}$ that satisfies the condition in Theorem 4.3. Since $\{\mathsf{b}\}$ is a subalgebra of $\mathfrak{M}_4$, the mapping h_{b} constantly equal to b satisfies $h_{\mathsf{b}} \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{M}_4)$ and $\emptyset = h_{\mathsf{b}}^{-1}[F_{\mathsf{n}}]$. Now let $T \in \mathcal{E}\,, T \neq \emptyset$. Then also $\Phi(T) \in \mathcal{E}$ and the closure system $\mathcal{C}_T = \{\emptyset\,, T \cap \Phi(T)\,, T\,, \Phi(T)\,, A\}$ also satisfies the conditions in Theorem 4.3 with $\mathcal{E}_T = \{\emptyset\,, T\,, \Phi(T)\}$ as a basis. Therefore it defines a full model $\mathbb{L}_T = \langle \mathfrak{A}\,, \mathcal{C}_T \rangle$ of $\mathcal{B}$. AS in the proof of Theorem 4.3, we know that the associated equivalence relation $\Lambda\mathbb{L}_T \in \mathrm{Co}\mathfrak{A}$ and that the quotient $\mathfrak{A}/\Lambda\mathbb{L}_T \in \mathsf{DM}$. But $\langle a\,, b \rangle \in \Lambda\mathbb{L}_T$ iff both $(a \in T \iff b \in T)$ and $(a \in \Phi(T) \iff b \in \Phi(T))$, thus the relative positions of T and $\Phi(T)$ determine the structure of the quotient $\mathfrak{A}/\Lambda\mathbb{L}_T$: If $T = \Phi(T)$ then $\mathfrak{A}/\Lambda\mathbb{L}_T \cong \mathfrak{M}_2$, the two-element De Morgan lattice, and $T = \pi^{-1}[\{\mathsf{t}\}]$. If $\Phi(T) \subsetneq T$ or $T \subsetneq \Phi(T)$ then $\mathfrak{A}/\Lambda\mathbb{L}_T \cong \mathfrak{M}_3$, the three-element De Morgan chain, with $T = \pi^{-1}[\{\mathsf{t}\}]$ or $T = \pi^{-1}[\{\mathsf{n}\,, \mathsf{t}\}]$. Finally if T and $\Phi(T)$ are incomparable, then $T \cap \Phi(T) \neq \emptyset$ because for any $a \in T$ and any $b \in \Phi(T)$ by 4.3(2) we have that $a \vee b \in T \cap \Phi(T)$; in this case $\mathfrak{A}/\Lambda\mathbb{L}_T \cong \mathfrak{M}_4$ and this can be done in such a way that $T = \pi^{-1}[F_{\mathsf{n}}]$. Using a convenient embedding of $\mathfrak{M}_2$ or of $\mathfrak{M}_3$ into $\mathfrak{M}_4$ we can, in all cases, obtain an homomorphism $h \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{M}_4)$ such that $T = h^{-1}[F_{\mathsf{n}}]$. We conclude that $\mathbb{L}$ is generated from the matrix $\langle \mathfrak{M}_4\,, F_{\mathsf{n}} \rangle$ by the family of homomorphisms thus obtained; this family contains h_{b} by construction, and is closed under composition with s because if $T = h^{-1}[F_{\mathsf{n}}]$ then $\Phi(T) = (s \circ h)^{-1}[F_{\mathsf{n}}]$, and $T \in \mathcal{E}$ implies $\Phi(T) \in \mathcal{E}$ by 4.3(3).

To prove the converse it is enough to check that the family $\mathcal{E} = \{h^{-1}[F_{\mathsf{n}}] : h \in \mathcal{H}\}$ satisfies the conditions of Theorem 4.3, and this is straightforward from the assumptions on $\mathcal{H}$ and the structure of $\mathfrak{M}_4$. ∎

Comparing this with Proposition 2.3, we conclude that full models of $\mathcal{B}$ inherit from $\mathcal{B}$, to some extent, the property of being semantically determined from the matrix $\langle \mathfrak{M}_4, F_{\mathsf{n}} \rangle$ by a family of homomorphisms satisfying some conditions, called "admissible family of valuations" in some contexts. With slight modifications in the above proof one obtains the following:

Proposition 4.5 *An abstract logic $\mathbb{L}$ is a full model of $\mathcal{B}$ iff $\mathbb{L}$ is finitary and is generated from the set of two matrices $\{\, \langle \mathfrak{M}_4\,, F_{\mathsf{n}} \rangle\,, \langle \mathfrak{M}_4\,, F_{\mathsf{b}} \rangle \,\}$ by some family $\mathcal{H} \subseteq \mathrm{Hom}(\mathfrak{A}, \mathfrak{M}_4)$ such that the mapping constantly equal to b is in $\mathcal{H}$.* ∎

The next characterization of full models of $\mathcal{B}$ has, in turn, a clear proof-theoretic flavour.

Theorem 4.6 *Let $\mathbb{L}$ be an arbitrary abstract logic. Then $\mathbb{L}$ is a full model of $\mathcal{B}$ iff it is finitary and satisfies, for all $a, b \in A$ and all $X \subseteq A$, the following properties:*

(E) $\mathbf{C}(\emptyset) = \emptyset$.

(PC) $\mathbf{C}(a \wedge b) = \mathbf{C}(a, b)$.

(PDI) $\mathbf{C}(X, a \vee b) = \mathbf{C}(X, a) \cap \mathbf{C}(X, b)$.

(PDN) $\mathbf{C}(a) = \mathbf{C}(\neg\neg a)$.

(PWC) $b \in \mathbf{C}(a) \;\Rightarrow\; \neg a \in \mathbf{C}(\neg b)$.

Proof. ($\Rightarrow$) Let $\mathcal{E}$ be a basis of $\mathcal{C}$ satisfying properties (1) to (3) of Theorem 4.3. That $\mathcal{E}$ is a basis of $\mathcal{C}$ means that for any $Y \subseteq A$, $\mathbf{C}(Y) = \bigcap\{T \in \mathcal{E} : Y \subseteq T\}$. Then the condition that $\emptyset \in \mathcal{E}$ implies that $\mathbf{C}(\emptyset) = \emptyset$. It is straightforward to see that conditions (1) and (2) of Theorem 4.3 imply (PC) and (PDI), respectively. Finally from 4.3(3) one obtains (PDN) and (PWC): For any $T \in \mathcal{E}$, $a \in T = \Phi(\Phi(T)) \iff \neg a \notin \Phi(T) \iff \neg\neg a \in T$; if $b \in \mathbf{C}(a)$ and $\neg b \in T$ then $b \notin \Phi(T)$ which implies $a \notin \Phi(T)$, that is, $\neg a \in T$, thus proving that $\neg a \in \mathbf{C}(\neg b)$.

($\Leftarrow$) Let $\mathbb{L} = \langle \mathfrak{A}, \mathbf{C} \rangle$ be finitary and satisfy conditions (E) to (PWC). Then properties (PC), (PDI) and (PWC) imply that the Frege relation $\Lambda\mathbb{L}$ is a congruence with respect to the operations $\wedge, \vee, \neg$, respectively, that is, it is a congruence of $\mathfrak{A}$, which implies that $\Lambda\mathbb{L} = \widetilde{\Omega}\mathbb{L}$. This makes working with the reduction $\mathfrak{A}^*$ very easy, and one can check directly (i.e., using an equational presentation of De Morgan lattices like the one implicit in the definition given at the beginning of Section 2) that $\mathfrak{A}^* \in \mathsf{DM}$. Since the reduction mapping is a bilogical morphism, the reduced logic $\langle \mathfrak{A}^*, \mathbf{C}^* \rangle$ is also finitary and satisfies (E) to (PWC). In particular (E) plus (PC) imply that $\mathcal{C}^* \subseteq \mathcal{F}(\mathfrak{A}^*) \cup \{\emptyset\}$, and finitarity and (PC) together imply that actually $\mathcal{C}^* = \mathcal{F}(\mathfrak{A}^*) \cup \{\emptyset\}$, as in [25, Theorem 4.2]. So we can apply Proposition 4.2 once more and conclude that $\mathbb{L}$ is a full model of $\mathcal{B}$. ∎

The reader may have noticed that two properties that have a prominent rôle in several of the proofs in this and the previous section are PC and the fact that the relation $\Lambda\mathbb{L}$ is a congruence. A general study of full models of logics with these two properties is performed in Section 4.2 of [21]. Here we can see that, at the abstract level, congruence is the key property that an arbitrary model of $\mathcal{B}$ needs to satisfy in order to be a full model:

Proposition 4.7 *An abstract logic $\mathbb{L}$ is a full model of $\mathcal{B}$ if and only if $\mathbb{L}$ is a finitary model of $\mathcal{B}$, without theorems, such that the Frege relation $\Lambda\mathbb{L}$ is a congruence of $\mathfrak{A}$.*

Proof. ($\Rightarrow$) By definition all full models are finitary models, and by 4.6(E) they do not have theorems. Moreover, in the proof of Theorem 4.3 we have already seen that the relation $\Lambda\mathbb{L}$ is a congruence.

($\Leftarrow$) Let $\varphi \approx \psi$ be any equation true in De Morgan lattices, that is, $\mathsf{DM} \models \varphi \approx \psi$. By Corollary 2.7 $\varphi \dashv\models_{\mathcal{B}} \psi$. If $\mathbb{L}$ is a model of $\mathcal{B}$ satisfying the stated conditions, then $\Lambda\mathbb{L} = \widetilde{\Omega}\mathbb{L}$ and the reduction $\mathbb{L}^*$ is also a finitary model of $\mathcal{B}$ without theorems and such that $\Lambda\mathbb{L}^* = \widetilde{\Omega}\mathbb{L}^* = \Delta_{A^*}$. From it being a model we have that for any $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A})$, $\mathbf{C}^*(h(\varphi)) = \mathbf{C}^*(h(\psi))$, therefore $h(\varphi) = h(\psi)$, which means that $\mathfrak{A}^* \models \varphi \approx \psi$. That is, $\mathfrak{A}^* \in \mathsf{DM}$. Now we can prove that $\mathcal{C}^* = \mathcal{F}(\mathfrak{A}^*) \cup \{\emptyset\}$ in the same way as in the proof of Theorem 4.6, since that proof uses only finitarity, that we have by assumption, and PC, that holds in every model of $\mathcal{B}$. Finally, using Proposition 4.2 we obtain that $\mathbb{L}$ is a full model of $\mathcal{B}$. ∎

The significance of Theorem 4.6 lies in that full models of $\mathcal{B}$ are characterized by the abstract counterparts of exactly the same "Tarski-style conditions" shown in Proposition 2.10 to characterize the sentential logic $\mathcal{B}$, plus condition (E). In particular, this tells us that full models of $\mathcal{B}$ inherit these key metalogical properties of the logic; that these properties are not inherited by arbitrary models is seen by a simple counterexample: Take on $\mathfrak{M}_4$ the closure system $\mathcal{C} = \{\{\mathbf{t}\}, M_4\}$. Since the two closed sets we have chosen are filters of the logic, the abstract logic $\mathbb{L} = \langle \mathfrak{M}_4, \mathcal{C} \rangle$ is a model of $\mathcal{B}$, but it does not satisfy (PDI), because $\mathbf{C}(\mathbf{n}) \cap \mathbf{C}(\mathbf{b}) = M_4 \cap M_4 = M_4$ while $\mathbf{C}(\mathbf{n} \vee \mathbf{b}) = \mathbf{C}(\mathbf{t}) = \{\mathbf{t}\}$. We already saw that properties (E) to (PWC) of Theorem 4.6 appear in Proposition 2.10 due to the fact that $\mathcal{B}$ can be defined from the sequent calculus $G_{\mathcal{B}}$ presented in Definition 2.8, because its rules are precisely the expression of those abstract properties. For a finitary logic, to satisfy any of these properties is the same as to be a model of these rules, in the following sense:

Definition 4.8 *We say that an abstract logic* $\mathbb{L}$ ***is a model of the Gentzen-style rule***

$$\frac{\{\, \Gamma_i \rightarrow \varphi_i \ : \ i < k \,\}}{\Gamma \rightarrow \varphi} \tag{4.2}$$

if and only if every $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A})$ *satisfies the property that if for all* $i < k$ *,* $h(\varphi_i) \in \mathbf{C}(h[\Gamma_i])$*, then also* $h(\varphi) \in \mathbf{C}(h[\Gamma])$*.*

$\mathbb{L}$ ***is a model of a Gentzen system*** *when it is a model of all its rules.*

Note that every abstract logic, by its own definition, is a model of the structural rules, like (W) and (Cut). Moreover, in order to check that an abstract logic is a model of a certain Gentzen system it is enough to check that it is a model of the non-structural rules of any calculus that defines the system. After comparing Definition 2.8 with Theorem 4.6 we immediately obtain:

Proposition 4.9 *An abstract logic* $\mathbb{L}$ *is a full model of* $\mathcal{B}$ *if and only if* $\mathbb{L}$ *is a finitary model of* $G_{\mathcal{B}}$ *without theorems.* ∎

According to the general theory of [21], in this situation $G_{\mathcal{B}}$ is said to be **strongly adequate for** $\mathcal{S}$. From the definitions and discussions therein, it follows that an strongly adequate Gentzen system for a given sentential logic, if it exists, is unique, and is so-to-speak a kind of Gentzen system *canonically associated* with the sentential logic. Here when we speak of uniqueness of Gentzen systems, we understand these as sets of derived rules; that is, two distinct sets of primitive rules may lead to the same derived rules, and in this case one speaks of different *calculi* or of different *presentations* of the same Gentzen system.

A different Gentzen system for the sentential logic $\mathcal{B}$ can be obtained by having the **De Morgan Laws**,

$$\frac{\Gamma, \neg\varphi \rightarrow \xi \qquad \Gamma, \neg\psi \rightarrow \xi}{\Gamma, \neg(\varphi \wedge \psi) \rightarrow \xi} \ (\neg\wedge\rightarrow) \qquad \frac{\Gamma \rightarrow \neg\varphi}{\Gamma \rightarrow \neg(\varphi \wedge \psi)} \, , \, \frac{\Gamma \rightarrow \neg\psi}{\Gamma \rightarrow \neg(\varphi \wedge \psi)} \ (\rightarrow\neg\wedge)$$

$$\frac{\Gamma, \neg\varphi, \neg\psi \rightarrow \xi}{\Gamma, \neg(\varphi \vee \psi) \rightarrow \xi} \ (\neg\vee\rightarrow) \qquad \frac{\Gamma \rightarrow \neg\varphi \qquad \Gamma \rightarrow \neg\psi}{\Gamma \rightarrow \neg(\varphi \vee \psi)} \ (\rightarrow\neg\vee)$$

as primitive rules *instead of* rule $(\neg)$ corresponding to contraposition; let us call $G_{\mathcal{BL}}$ both this well-known Gentzen calculus and the Gentzen system it defines. The precise relationship between $G_{\mathcal{B}}$ and $G_{\mathcal{BL}}$ is described in the following result:

Proposition 4.10 *As a Gentzen system, $G_{\mathcal{BL}}$ is a proper subsystem of $G_{\mathcal{B}}$ with the same derivable sequents (and hence with the same admissible rules).*

PROOF. In order to complete the proof we will show the following three facts:

(1) Every rule derivable in $G_{\mathcal{BL}}$ is also derivable in $G_{\mathcal{B}}$.
(2) Every sequent derivable in $G_{\mathcal{B}}$ is also derivable in $G_{\mathcal{BL}}$.
(3) There is a rule, namely the Weak Contraposition rule $(\neg)$, which is derivable in $G_{\mathcal{B}}$ but not in $G_{\mathcal{BL}}$; actually it is only admissible in $G_{\mathcal{BL}}$.

Part (1) says that $G_{\mathcal{BL}}$ is a subsystem of $G_{\mathcal{B}}$, and part (3) says that it is a proper one, while parts (1) and (2) together imply that the two systems have the same derivable sequents. In order to prove part (1) it suffices to show that the De Morgan rules just given are derivable in $G_{\mathcal{B}}$, which is an easy exercise. Part (2) follows from the facts that derivable sequents of each of the systems correspond to the entailments of Belnap's logic $\mathcal{B}$; for $G_{\mathcal{B}}$ this was proved in Theorem 2.9, and for $G_{\mathcal{BL}}$ this is well-known, see for instance [4, Theorem 3.9]. Finally, to show (3) it is enough to present a model of $G_{\mathcal{BL}}$ not being a model of the rule $(\neg)$, for instance the abstract logic $\langle \mathfrak{M}_4 , \{F_{\mathsf{n}}, M_4\}\rangle$. ∎

Therefore we see that $G_{\mathcal{B}}$ and $G_{\mathcal{BL}}$, although defining the same derivable sequents, are not the same Gentzen system (i.e., they are not two presentations, or calculi, of the same system). Another consequence is that, since $G_{\mathcal{BL}}$ also defines $\mathcal{B}$, this logic can also be characterized as the least sentential logic satisfying the corresponding properties (i.e., those of Proposition 2.10 save PWC which has to be replaced by the abstract version of the De Morgan Laws); this result has already been obtained in [35]. However, such characterization is not a 'best' one in the sense that abstract logics satisfying this set of properties might not be full models of $\mathcal{B}$; the example given in the proof of Proposition 4.10 is one such case. This fact is also related to the uniqueness of the strongly adequate Gentzen system for $\mathcal{B}$, which is $G_{\mathcal{B}}$ and not $G_{\mathcal{BL}}$.

We will also see the difference between these two Gentzen systems at the metalogical level by examining their *algebraizability* in the context of the theory of Rebagliato and Verdú [27, 37, 38]. This is important since, by Theorem 2.11, the sentential logic $\mathcal{B}$ is not protoalgebraic, hence it is not algebraizable either, and then one of the ways of studying its "degree of algebraizability" is indirectly through that of some Gentzen system defining it. When using $G_{\mathcal{B}}$ we obtain a satisfactory result:

Theorem 4.11 *The Gentzen system $G_{\mathcal{B}}$ is algebraizable, and its equivalent algebraic semantics is the variety* DM *of De Morgan lattices.*

PROOF. The translations from sequents into equations and conversely we are going to use are the following:

$$\begin{aligned} \boldsymbol{\tau}(\Gamma \to \varphi) &= \bigwedge \Gamma \preccurlyeq \varphi \\ \boldsymbol{\rho}(\varphi \approx \psi) &= \{\, \varphi \to \psi \,, \psi \to \varphi \,\} \end{aligned}$$

where here we take $\bigwedge\Gamma \preccurlyeq \varphi$ as shorthand for the equation $(\bigwedge\Gamma) \wedge \varphi \approx \bigwedge\Gamma$; this translation is inspired by the (much weaker) result in Proposition 2.5. Note that in this way every Gentzen-style rule like (4.2) gets translated into the quasi-equation

$$\mathop{\&}_{i<k} \boldsymbol{\tau}(\Gamma_i \to \varphi_i) \Longrightarrow \boldsymbol{\tau}(\Gamma \to \varphi) \tag{4.3}$$

The first condition for algebraizability, in this case with respect to **DM**, is:

(A1) A Gentzen-style rule (4.2) is a derivable rule of $G_{\mathcal{B}}$ *if and only if* the quasi-equation (4.3) holds in the variety **DM**.

Now to check the "only if" part of (A1) it is enough to check it for the primitive rules of $G_{\mathcal{B}}$, and this is straightforward by the properties of De Morgan lattices; for instance rule ($\neg$) is translated by $\boldsymbol{\tau}$ into the quasi-equation $\varphi \preccurlyeq \psi \Rightarrow \neg\psi \preccurlyeq \neg\varphi$, which is true in **DM**. To prove the "if" part we reason by contraposition: Assume that for some rule (4.2) the quasi-equation (4.3) does not hold in **DM**. This means there is some algebra $\mathfrak{A} \in \mathbf{DM}$ and some assignment $h \in \mathrm{Hom}(\mathfrak{Fm}, \mathfrak{A})$ such that $\bigwedge h[\Gamma_i] \leq h(\varphi_i)$ for all $i < k$ but $\bigwedge h[\Gamma] \not\leq h(\varphi)$; that is, using the operator $\mathbf{F}$ of lattice-filter-generation, $h(\varphi_i) \in \mathbf{F}(h[\Gamma_i])$ for all $i < k$ but $h(\varphi) \notin \mathbf{F}(h[\Gamma])$. According to Definition 4.8, this says that the abstract logic $\langle \mathfrak{A}, \mathbf{F} \rangle$ is not a model of (4.2). Since by Proposition 4.2 this abstract logic is a full model of $\mathcal{B}$, then by Proposition 4.7 (strong adequacy) it is a model of $G_{\mathcal{B}}$, therefore the questioned rule cannot be a derivable rule of $G_{\mathcal{B}}$.

The second condition for algebraizability says, roughly speaking, that the composition of both translations should be equivalent to identity modulo the class of algebras; this means precisely:

(A2) The quasi-equations $\varphi \approx \psi \Longrightarrow \delta \approx \varepsilon$ for every $\delta \approx \varepsilon \in \boldsymbol{\tau}[\boldsymbol{\rho}(\varphi \approx \psi)]$, and the quasi-equation $\& \boldsymbol{\tau}[\boldsymbol{\rho}(\varphi \approx \psi)] \Longrightarrow \varphi \approx \psi$ hold in the class **DM**.

In our case this amounts to saying that the three quasi-equations $\varphi \approx \psi \Longrightarrow \varphi \preccurlyeq \psi$, $\varphi \approx \psi \Longrightarrow \psi \preccurlyeq \varphi$ and $[\varphi \preccurlyeq \psi \;\&\; \psi \preccurlyeq \varphi] \Longrightarrow \varphi \approx \psi$ hold in **DM**, which is trivial. ∎

A Gentzen system algebraizable with a given equivalent algebraic semantics and through given translations is unique; therefore, taking Proposition 4.10 into account, we conclude that $G_{\mathcal{BL}}$ is not algebraizable in the same sense as $G_{\mathcal{B}}$ is, that is, with respect to the same class of algebras and with the same translations. However, the power of the general theory of algebraizability shows itself as we can also obtain a much stronger and absolute result:

Theorem 4.12 *The Gentzen system $G_{\mathcal{BL}}$ is not algebraizable.*

PROOF. Assume it is so, with some quasi-variety **K** of algebras as equivalent algebraic semantics, and some translation we do not need to specify. Then (by the general theory of algebraizability) there is a dual order-isomorphism between the family of all extensions of $G_{\mathcal{BL}}$ and the family of all sub-quasi-varieties of **K**. Since by Proposition 4.10 the system $G_{\mathcal{B}}$ is one such extension, we obtain that $\mathbf{DM} \subseteq \mathbf{K}$. But on the other hand (again by the general theory) the variety generated by **K** is determined intrinsically (i.e., by the Leibniz operator, independently of the translation used) by the derivable sequents of $G_{\mathcal{BL}}$; since, also by Proposition 4.10, these are the same for $G_{\mathcal{BL}}$ and for $G_{\mathcal{B}}$, we obtain that $\mathbf{V}(\mathbf{K}) = \mathbf{V}(\mathbf{DM}) = \mathbf{DM}$, thus completing the proof that $\mathbf{DM} = \mathbf{K}$. But then the above-mentioned isomorphism would imply that $G_{\mathcal{BL}} = G_{\mathcal{B}}$, which is certainly not the case. Therefore $G_{\mathcal{BL}}$ cannot be algebraizable. ∎

The system $G_{\mathcal{BL}}$ has been regarded as more convenient for proof-theoretical purposes. In contrast we conclude, from several of the above results, that $G_{\mathcal{B}}$ seems to be more convenient from the algebraic point of view.

5 Some related logics

We mention briefly the most relevant results concerning some other sentential logics whose treatment can be made closely parallel to that of $\mathcal{B}$. The interested reader will find no difficulty in supplying details.

5.1 Kleene's three-valued logic

Among several three-valued logics, the one we have in mind is the implication-free fragment of Kleene's "strong" three-valued logic; let us call it $\mathcal{K}$. It is linked to the variety of **Kleene lattices** in the same way as $\mathcal{B}$ is linked to De Morgan lattices, of which they form a proper subvariety (the other only one is Boolean algebras); this variety is generated by the three-element chain $\mathfrak{M}_3$, with $\mathsf{f} < \mathsf{n} < \mathsf{t}$. All these facts can be shown by adapting the results of [5, §XI.3] on Kleene algebras. The matrix-style definition of $\mathcal{K}$ uses the two matrices $\langle \mathfrak{M}_3, F_{\mathsf{n}} \rangle$, where $F_{\mathsf{n}} = \{\mathsf{n}, \mathsf{t}\}$, and $\langle \mathfrak{M}_3, \{\mathsf{t}\} \rangle$; this time none of them can be eliminated, that is, no analogue of Proposition 2.3 holds. It is well-known that a Kleene lattice is a De Morgan lattice satisfying the formal ordering relation $\varphi \wedge \neg\varphi \preccurlyeq \psi \vee \neg\psi$; and accordingly the Gentzen-style calculus defining $\mathcal{K}$ has the same rules as $G_{\mathcal{B}}$ and the same axioms plus:

$$\varphi \wedge \neg\varphi \ \rightarrow \ \psi \vee \neg\psi \ .$$

The logic $\mathcal{K}$ is non-protoalgebraic, selfextensional, and non-Fregean, and the variety of Kleene lattices is not the equivalent algebraic semantics of any algebraizable logic. The Hilbert-style presentation of $\mathcal{K}$ results from that of $\mathcal{B}$ after the addition of the rule $\varphi \wedge \neg\varphi \vdash \psi \vee \neg\psi$; the Leibniz relation on its matrices is characterized as in Proposition 3.13, and its reduced matrices are also described as in Theorem 3.14, *mutatis mutandis.* The $\mathcal{K}$-algebras are, of course, Kleene lattices, and its full models are bilogical inverse images of the abstract logics constituted by Kleene lattices with all their lattice filters plus $\emptyset$; they can also be characterized by adding the condition "$\forall T \in \mathcal{E}$, T and $\Phi(T)$ are comparable by $\subseteq$" to Theorem 4.3, and also by adding "$b \vee \neg b \in \mathbf{C}(a \wedge \neg a)$" to Theorem 4.6.

5.2 Classical Logic

This corresponds to the smallest subvariety of DM, which is the class of Boolean algebras; it is an extension of $\mathcal{K}$, but can be obtained directly from $\mathcal{B}$ by strengthening the Contraposition Rule in $G_{\mathcal{B}}$ to the form:

$$\frac{\Gamma, \varphi \rightarrow \psi}{\Gamma, \neg\psi \rightarrow \neg\varphi} \ .$$

In order to obtain classical logic from the resulting sequent calculus one has to refine Theorem 2.9: Use the same procedure for $\Gamma \neq \emptyset$, while for $\Gamma = \emptyset$ one says that $\emptyset \vdash \varphi$ holds if and only if for all $\psi \in Fm$ the sequent $\psi \rightarrow \varphi$ is derivable.

The full models arising by the restriction of previous results are those of $\mathcal{B}$ satisfying the condition "$\forall T \in \mathcal{E}\,,\ \Phi(T) = T$", or the property "$a \in \mathbf{C}(X, b)$ implies $\neg b \in \mathbf{C}(X, \neg a)$"; it is not hard to check that the reduced ones are Boolean algebras with all their filters, that is, the "Boolean abstract logics" of [12], shown there to be (in our present terminology) the reduced full models of Classical Propositional Calculus.

5.3 Bounded lattices and logics with "falsum".

The lack of theorems of the logics $\mathcal{B}$ and $\mathcal{K}$ can be remedied, if one wants, by adding them artificially; enlarge the language with a propositional constant $\top$ ("truth") or $\perp$ ("falsum"), and repeat the whole process in the new algebraic type (2,2,1,0). Homomorphisms always map $\top$ to 1, or $\perp$ to 0. The resulting algebraic structures are **De Morgan algebras** and **Kleene algebras**, respectively: see [5, Chapter XI]. One needs to add $\top$ (or $\neg\perp$) as an axiom for the logic. The full models are conveniently characterized; small adjustments have to be made here and there. Although this extension is very common in applications, when one wants to have a means of expressing "truth" and "falsity" in the language, note that in the case of $\mathcal{B}$ and $\mathcal{K}$ this is to a certain extent vacuous, since these extensions are conservative in theorems: Every theorem of the extensions must have an appearance of $\top$ (or of $\perp$).

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