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40 years of FDE: An Introductory Overview

Abstract. In this introduction to the special issue "40 years of **FDE**", we offer an overview of the field and put the papers included in the special issue into perspective. More specifically, we first present various semantics and proof systems for **FDE**, and then survey some expansions of **FDE** by adding various operators starting with constants. We then turn to unary and binary connectives, which are classified in a systematic manner (affirmative/negative, extensional/intensional). First-order **FDE** is also briefly revisited, and we conclude by listing some open problems for future research.

Keywords: First-degree entailment logic, Expansions of **FDE**, Paraconsistent logic, Paracomplete logic, Modal logic.

1. Introduction

1.1. A Brief History

Logicians explore various logical consequence relations for a wide range of formal languages, not only the one known as classical logic, but also other consequence relations referred to as *nonclassical* logics.[1] There is a continuum of nonclassical logics, but some systems have emerged as particularly interesting and useful. Among these distinguished nonclassical logics is a system of propositional logic that has become well-known as Belnap and Dunn's *useful four-valued logic* or *first-degree entailment logic*, **FDE**. In its now standard presentation as an extensional four-valued logic, **FDE** first appeared in print around 1977, i.e., approximately 40 years ago, in three seminal papers by Nuel D. Belnap and J. Michael Dunn [6,7,10]. In a now less-standard presentation, however, **FDE** was already introduced in the late 1950s in Belnap's unpublished doctoral dissertation and in [1] as a fragment of the system **E** of entailment, namely as a set of certain first-degree entailments, i.e., implications $A \rightarrow B$, where A and B are formulas containing at most conjunction, disjunction, and negation. A first-degree entailment $A \rightarrow B$

[1] We highly recommend [34] for an excellent overview as well as an introduction to nonclassical logics.

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is in normal form if A is in disjunctive and B is in conjunctive normal form. Moreover, $A \rightarrow B$ is a *tautological entailment* iff it can be put into a provably equivalent normal form $A_1 \vee \ldots \vee A_m \rightarrow B_1 \wedge \ldots \wedge B_m$ and for all $A_j \rightarrow B_k$, the conjunction A_j and the disjunction B_k share a propositional variable (so that $A_j \rightarrow B_k$ is tautologically valid in this sense). In [1] it is shown that a first-degree entailment $A \rightarrow B$ is provable in system **E** iff $A \rightarrow B$ is a tautological entailment.

In this introduction, we offer an overview of **FDE** and various extensions (in the same vocabulary) and expansions of it. Moreover, we point to the main contributions of the papers included in this special issue and collect some open problems that will hopefully be a helpful guide to continuing research in this field.

1.2. Preliminaries

Our propositional language consists of a finite set C of propositional connectives and a countable set Prop of propositional variables which we refer to as $\mathcal{L}_{\mathsf{C}}$. Furthermore, we denote by $\mathsf{Form}_{\mathsf{C}}$ the set of formulas defined as usual in $\mathcal{L}_{\mathsf{C}}$. In this paper, we always assume that $\{\sim, \wedge, \vee\} \subseteq \mathsf{C}$ and just include the propositional connective(s) not from $\{\sim, \wedge, \vee\}$ in the subscript of $\mathcal{L}_{\mathsf{C}}$. For example, we write $\mathcal{L}_{\{\circ\}}$ and $\mathsf{Form}_{\{\circ\}}$ to mean $\mathcal{L}_{\{\sim,\wedge,\vee,\circ\}}$ and $\mathsf{Form}_{\{\sim,\wedge,\vee,\circ\}}$ respectively. Moreover, we denote a formula of $\mathcal{L}_{\mathsf{C}}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}_{\mathsf{C}}$ by Γ, Δ, Σ, etc.

2. FDE: Some Basics

2.1. Semantics for FDE

We first present four representative semantics for **FDE**.

Four-valued semantics Although it is not the only way to present the system, **FDE** is probably best known as a system of four-valued logic characterized through the following truth tables due to Timothy Smiley (cf. [7, p. 16]):[2]

x	$\sim x$
t	**f**
b	**b**
n	**n**
f	**t**

$x \wedge y$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$x \vee y$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

[2] The logic **FDE** is also characterized by a certain eight-valued matrix, M_0, from [1], see [3, p. 205, Theorem 3].

The truth values are written as **t**, **b**, **n** and **f**, taken from **t**rue only, **b**oth true and false, **n**either true nor false and **f**alse only respectively. We fix the set of designated values $\mathcal{D}$ to be $\{\mathbf{t}, \mathbf{b}\}$ in this introduction unless stated otherwise.[3] We refer to the algebra $\langle\{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}, \{\sim, \wedge, \vee\}\rangle$, the algebraic counterpart of **FDE**, as $\mathcal{BD}$. The formal definition of the four-valued semantic consequence relation will be given as follows.

Definition 1. A *four-valued valuation for* Form is a homomorphism from the free algebra of formulas in $\mathcal{L}$ to $\mathcal{BD}$.

Definition 2. A formula A is a *four-valued semantic consequence* of Γ ($\Gamma \models_4 A$) iff for all four-valued valuations v, if $v(B) \in \mathcal{D}$ for all $B \in \Gamma$ then $v(A) \in \mathcal{D}$.

Two-valued Dunn semantics We now turn to the two-valued relational semantics due to Dunn (cf. [10]). This semantics justifies the intuitive reading of the four truth values in the four-valued semantics.

Definition 3. A *Dunn-interpretation* is a relation, r, between propositional variables and the values 1 and 0, namely $r \subseteq \mathsf{Prop} \times \{1, 0\}$. Given an interpretation, r, this is extended to a relation between all formulas and truth values by the following clauses:

- $\sim Ar1$ iff $Ar0$,
- $\sim Ar0$ iff $Ar1$,
- $A \wedge Br1$ iff $Ar1$ and $Br1$,
- $A \wedge Br0$ iff $Ar0$ or $Br0$,
- $A \vee Br1$ iff $Ar1$ or $Br1$,
- $A \vee Br0$ iff $Ar0$ and $Br0$.

Definition 4. A formula A is a *two-valued semantic consequence* of Γ ($\Gamma \models_2 A$) iff for all Dunn-interpretations r, if $Br1$ for all $B \in \Gamma$ then $Ar1$.

Remark 5. As one can easily observe, there is a close connection between the four-valued semantics and two-valued Dunn semantics. More specifically, there is a *mechanical* procedure to turn truth tables into pairs of truth and falsity conditions and vice versa. For the details, see [30].

[3] For a variant of **FDE** with **t** being the only designated value, see [20,31,51], and for a variant of **FDE** with **f** being the only non-designated value, see [20].

Star semantics Yet another semantics was devised by Richard Routley (later Sylvan) and Val Routley (later Plumwood).

Definition 6. A *Routley interpretation* is a structure $\langle W, *, v\rangle$ where

- W is a set of worlds,
- $* : W \longrightarrow W$ is a function with $w^{**} = w$,
- $v : W \times \mathsf{Prop} \longrightarrow \{0, 1\}$.

The function v is extended to an assignment I of truth values for all pairs of worlds and formulas by the conditions:

- $I(w, p) = v(w, p)$,
- $I(w, \sim A) = 1$ iff $I(w^*, A) \neq 1$,
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$,
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$.

Definition 7. A formula A is a *star semantic consequence* of Γ ($\Gamma \models_* A$) iff for all Routley interpretations $\langle W, *, v\rangle$ and for all $w \in W$, if $I(w, B) = 1$ for all $B \in \Gamma$ then $I(w, A) = 1$.

Remark 8. As is well-known, there are two approaches to the semantics of negation in relevance logics. The one with star semantics is known as the *Australian* plan, and the other with four-valued (or equivalently two-valued à la Dunn) semantics is known as the *American* plan. One of the notable differences lies in the validity or non-validity of contraposition, once a strict or intensional conditional has been added to the language, see [34, Chapter 9] and §5.1. Indeed, in that case, $A \models_* B$ implies $\sim B \models_* \sim A$, but $A \models_4 B$ does *not* imply $\sim B \models_4 \sim A$.

Algebraic semantics Finally, there is also an algebraic semantics, defined in the following manner.

Definition 9. A *De Morgan lattice* is an algebra $\mathfrak{A} = \langle A, \cap, \cup, -\rangle$ of type (2,2,1) such that

- The reduct $\langle A, \cap, \cup\rangle$ is a distributive lattice; we denote its order by $\leq$.
- The unary operation satisfies the following equations:
  - $\star$ $x \approx --x$,
  - $\star$ $-(x \cup y) \approx (-x \cap -y)$,
  - $\star$ $-(x \cap y) \approx (-x \cup -y)$.

Definition 10. Let $\mathfrak{Fm} = \langle \mathsf{Form}, \wedge, \vee, \sim \rangle$ be the absolutely free algebra of similarity type (2,2,1) generated by a denumerable set Prop, and **DM** the variety of De Morgan lattices. Then, A is an *algebraic semantic consequence* of $\Gamma = \{B_1, \ldots, B_n\}$ ($\Gamma \models_a A$) iff for all homomorphisms $h : \mathfrak{Fm} \longrightarrow \mathfrak{A}$, $h(B_1) \cap \cdots \cap h(B_n) \leq h(A)$ holds for all $\mathfrak{A} \in \mathbf{DM}$.

2.2 Proof Systems for FDE

In this section we present, by way of example, four proof systems for **FDE**.[4]

Hilbert-style system The first proof system we introduce is a Hilbert-style system, due to Josep Maria Font [12]. Since **FDE** has no theorems, the calculus has no axioms.

Definition 11. Let $\mathcal{H}$**FDE** be the Hilbert-style system with the following rules.

$$\frac{A \wedge B}{A}\ (\text{R1}) \qquad \frac{A \vee A}{A}\ (\text{R6}) \qquad \frac{\sim\sim A \vee C}{A \vee C}\ (\text{R11})$$

$$\frac{A \wedge B}{B}\ (\text{R2}) \qquad \frac{A \vee (B \vee C)}{(A \vee B) \vee C}\ (\text{R7}) \qquad \frac{\sim(A \vee B) \vee C}{(\sim A \wedge \sim B) \vee C}\ (\text{R12})$$

$$\frac{A \quad B}{A \wedge B}\ (\text{R3}) \qquad \frac{A \vee (B \wedge C)}{(A \vee B) \wedge (A \vee C)}\ (\text{R8}) \qquad \frac{(\sim A \wedge \sim B) \vee C}{\sim(A \vee B) \vee C}\ (\text{R13})$$

$$\frac{A}{A \vee B}\ (\text{R4}) \qquad \frac{(A \vee B) \wedge (A \vee C)}{A \vee (B \wedge C)}\ (\text{R9}) \qquad \frac{\sim(A \wedge B) \vee C}{(\sim A \vee \sim B) \vee C}\ (\text{R14})$$

$$\frac{A \vee B}{B \vee A}\ (\text{R5}) \qquad \frac{A \vee C}{\sim\sim A \vee C}\ (\text{R10}) \qquad \frac{(\sim A \vee \sim B) \vee C}{\sim(A \wedge B) \vee C}\ (\text{R15})$$

We write $\Gamma \vdash_{\mathbf{h}} A$ iff there is a sequence of formulas $B_1, \ldots, B_n, A$ ($n \geq 0$) such that every formula in the sequence either (i) belongs to Γ or (ii) is obtained by one of the rules from formulas preceding it in the sequence.

Theorem 1. (Font). *For any* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{\mathbf{h}} A$ *iff* $\Gamma \models_4 A$.

Gentzen-style system The second system, also due to Font, is a Gentzen-style sequent calculus.

Definition 12. Let $\mathcal{G}$**FDE** be the Gentzen-style system with the axiom $A \rightarrow A$ and the following rules.

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[4] David Nelson's four-valued constructive propositional logic with strong negation **N4** is a conservative expansion of **FDE** by intuitionistic implication, and the survey of proof systems for **N4** in [18] contains various other proof-theoretic characterizations of **FDE**.

$$\frac{\Gamma \to A}{\Gamma, B \to A}\ (\mathrm{W}) \qquad \frac{\Gamma \to A \qquad \Gamma, A \to B}{\Gamma \to B}\ (\mathrm{Cut})$$

$$\frac{\Gamma, A, B \to C}{\Gamma, A \wedge B \to C}\ (\wedge\!\to) \qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \wedge B}\ (\to\!\wedge)$$

$$\frac{\Gamma, A \to C \qquad \Gamma, B \to C}{\Gamma, A \vee B \to C}\ (\vee\!\to) \qquad \frac{\Gamma \to A}{\Gamma \to A \vee B} \quad \frac{\Gamma \to B}{\Gamma \to A \vee B}\ (\to\!\vee)$$

$$\frac{\Gamma, A \to C}{\Gamma, {\sim}{\sim} A \to C}\ ({\sim}{\sim}\!\to) \qquad \frac{\Gamma, {\sim}{\sim} A \to C}{\Gamma, A \to C}\ (\to\!{\sim}{\sim})$$

$$\frac{A \to B}{{\sim} B \to {\sim} A}\ ({\sim})$$

We write $\Gamma \vdash_{\mathbf{g}} A$ iff there is a sequence of formulas $B_1, \ldots, B_n \in \Gamma$ $(n \geq 0)$ such that the sequent $B_1, \ldots, B_n \to A$ is derivable.

Theorem 2. (Font). *For any* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{\mathbf{g}} A$ *iff* $\Gamma \models_4 A$.

Remark 13. Note that a cut-free sequent calculus for **FDE** can be found in [38].

Natural deduction system The third system is a natural deduction calculus in the style of Gentzen and Prawitz. The following system is due to Priest and can be found in [33, p. 309].

Definition 14. Let $\mathcal{N}\mathbf{FDE}$ be the natural deduction system with the following rules.

$$\frac{A \quad B}{A \wedge B}\ (\wedge I) \qquad \frac{A \wedge B}{A}\ (\wedge E) \qquad \frac{A \wedge B}{B}\ (\wedge E)$$

$$\frac{A}{A \vee B}\ (\vee I) \qquad \frac{B}{A \vee B}\ (\vee I) \qquad \frac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\ (\vee E)$$

$$\frac{{\sim}{\sim} A}{A}\ ({\sim}{\sim}) \qquad \frac{{\sim}(A \wedge B)}{{\sim} A \vee {\sim} B}\ ({\sim}\wedge) \qquad \frac{{\sim}(A \vee B)}{{\sim} A \wedge {\sim} B}\ ({\sim}\vee)$$

Note here that the double underlining indicates a two-way rule of inference.

Finally, $\Gamma \vdash_{\mathbf{n}} A$ iff for some finite $\Gamma' \subseteq \Gamma$, there is a derivation of A from Γ' with the above rules.

Theorem 3. (Priest). *For any* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{\mathbf{n}} A$ *iff* $\Gamma \models_2 A$.

A note on the papers in the special issue. The joint contribution by Yaroslav Shramko, Dmitry Zaitsev and Alexander Belikov focuses on various less studied semantic consequence relations that are closely related to $\models_2$ and introduces proof systems that combine elements of natural deduction and of Hilbert-style proof systems.

Tableaux system Finally, we present a tableau system, again due to Priest. The details can be found in [34, Chap. 8]

DEFINITION 15. Let $\mathcal{T}\mathbf{FDE}$ be the tableau system consisting of the following rules.

$$
\begin{array}{cccc}
\begin{array}{c} A \wedge B, + \\ \downarrow \\ A, + \\ B, + \end{array} &
\begin{array}{c} A \wedge B, - \\ \swarrow \quad \searrow \\ A, - \qquad B, - \\ \end{array} &
\begin{array}{c} A \vee B, + \\ \swarrow \quad \searrow \\ A, + \qquad B, + \\ \end{array} &
\begin{array}{c} A \vee B, - \\ \downarrow \\ A, - \\ B, - \end{array}
\end{array}
$$

$$
\begin{array}{cccc}
\begin{array}{c} {\sim}(A \wedge B), + \\ \swarrow \quad \searrow \\ {\sim} A, + \qquad {\sim} B, + \\ \end{array} &
\begin{array}{c} {\sim}(A \wedge B), - \\ \downarrow \\ {\sim} A, - \\ {\sim} B, - \end{array} &
\begin{array}{c} {\sim}(A \vee B), + \\ \downarrow \\ {\sim} A, + \\ {\sim} B, + \end{array} &
\begin{array}{c} {\sim}(A \vee B), - \\ \swarrow \quad \searrow \\ {\sim} A, - \qquad {\sim} B, - \\ \end{array}
\end{array}
$$

$$
\begin{array}{cc}
\begin{array}{c} {\sim}{\sim} A, + \\ \downarrow \\ A, + \end{array} &
\begin{array}{c} {\sim}{\sim} A, - \\ \downarrow \\ A, - \end{array}
\end{array}
$$

Based on these rules, we write $A_1, \ldots, A_n \vdash_{\mathbf{t}} B$ iff there exists a completed and closed tableau for the following initial list:

$$
\begin{array}{c} A_1, + \\ A_2, + \\ \cdots \\ A_n, + \\ B, - \end{array}
$$

THEOREM 4. (Priest). *For any finite set* $\Gamma \cup \{A\} \subset \mathsf{Form}$*,* $\Gamma \vdash_{\mathbf{t}} A$ *iff* $\Gamma \models_2 A$.

2.3 Remarks on Extensions of FDE

Before turning to additional connectives, let us briefly make a comment on three extensions of **FDE**. The most familiar one, of course, is the two-valued classical logic. This is obtained by eliminating the intermediate values **b** and **n** in the four-valued semantics, assuming that the relation r is actually a function in the two-valued Dunn semantics, and assuming that the star world is identical to the starred world in the Star semantics.

Moreover, there are two famous three-valued extensions. One is a rather old logic known as the strong Kleene logic. The other is the logic known as the Logic of Paradox which has been studied in detail by Graham Priest since [32].

A note on the papers in the special issue. The joint contribution by Hugo Albuquerque, Adam Přenosil and Umberto Rivieccio, as well as Adam Přenosil's contribution, focus on extensions of **FDE**, building on the results of Rivieccio and Přenosil in [39] and [37] respectively. Moreover, the joint contribution by Stefan Wintein and Reinhard Muskens also explores extensions of **FDE** with special attention to the interpolation property.

2.4 Preliminaries for Expansions of FDE

Before discussing various expansions, we present some semantic frameworks that will be useful later. The first one will be used in discussing various modal operators.

Definition 16. A *Kripke frame* is a structure $\langle W, R\rangle$ where

- W a non-empty set,
- R is a binary relation on W.

Definition 17. A *Kripke model* is a structure $\langle W, R, v^+, v^-\rangle$ where

- $\langle W, R\rangle$ is a Kripke frame,
- v^i: $\mathsf{Prop} \longrightarrow 2^W$ $(i \in \{+, -\})$.

Remark 18. Given a Kripke model $\mathcal{M} = \langle W, R, v^+, v^-\rangle$, we will write $\mathcal{M}, w \models^+ p$ and $\mathcal{M}, w \models^- p$ for $w \in v^+(p)$ and $w \in v^-(p)$ respectively. The same notation will be also deployed for other models introduced in this subsection.

Second, we introduce a special kind of a Kripke models for the purpose of discussing various binary connectives.

Definition 19. A *constructive frame* is a structure $\langle W, \leq\rangle$ where

- W a non-empty set,
- $\leq$ is a reflexive and transitive binary relation on W.

Definition 20. A *intuitionistic Kripke model* is a structure $\langle W, \leq, v\rangle$ where

- $\langle W, \leq\rangle$ is a constructive frame,

- v: $\mathsf{Prop} \longrightarrow 2^W$ such that for all $w, w' \in W$ and for all $p \in \mathsf{Prop}$:

$$\text{if } w \leq w',\ \text{then } w \in v(p) \text{ implies } w' \in v(p).$$

DEFINITION 21. A *Nelson model* is a structure $\langle W, \leq, v^+, v^- \rangle$ where

- $\langle W, \leq \rangle$ is a constructive frame,
- v^i: $\mathsf{Prop} \longrightarrow 2^W$ such that for both $i \in \{+, -\}$, for all $w, w' \in W$ and for all $p \in \mathsf{Prop}$:

$$\text{if } w \leq w',\ \text{then } w \in v^i(p) \text{ implies } w' \in v^i(p)$$

Finally, we introduce yet another world semantics with a *ternary* relation.

DEFINITION 22. A *Routley-Meyer frame* is a structure $\langle W, R \rangle$ where

- W a non-empty set,
- R is a ternary relation on W.

DEFINITION 23. A *Routley-Meyer model* is a structure $\langle W, R, v^+, v^- \rangle$ where

- $\langle W, R \rangle$ is a Routley-Meyer frame,
- v^i: $\mathsf{Prop} \longrightarrow 2^W$ $(i \in \{+, -\})$.

Based on these basics of **FDE**, as well as the above preliminaries, we are now ready to discuss various expansions of **FDE**.

3 Constants

Since **FDE** is a four-valued logic, there are only four constants. Unlike in classical logic, in which two constants (top and bottom) are definable, none of the four constants are definable in **FDE**. Indeed, constants that always take **b** and **n** are not definable since the classical values are closed under the operations of **FDE**. Moreover, constants that always take **t** and **f** are also not definable since **b** and **n** are fixpoints with respect to the operations of **FDE**. Therefore, the addition of any of the constants will strictly strengthen the expressivity of the system.

4 Unary Operations

We now consider unary operations in expansions of **FDE**.

4.1 "Affirmative" Operators

We first turn our attention to operators that either imply the plain sentence, or are implied by the plain sentence. This classification is, needless to say, not meant to be rigorous, but just a rough classification.

Extensional operators In [5], Matthias Baaz added a unary operator, now known as Baaz' delta, to infinitely many-valued Gödel logic. Based on the fact that this operator can be intuitively read as "…is designated", the following unary operator is added to (first-order) **FDE** in [42].

x	$\triangle x$
t	**t**
b	**t**
n	**f**
f	**f**

This expansion, called **BD**$\triangle$ in [42], turns out to have a rather nice property from a proof-theoretic perspective. More specifically, it is proved that given any extensional expansion of **BD**$\triangle$, we obtain a natural deduction system in a mechanical manner.

Another unary operation that is studied in the literature on tetravalent modal algebras and tetravalent modal logics (see [14]) is the following connective:

x	$\Box x$
t	**t**
b	**f**
n	**f**
f	**f**

As one may expect from the notation, the operator is introduced as kind of a necessity operator, see also [8]. Font and Rius [14] suggest to read $\Box A$ as "the available information confirms that A is true". The truth and falsity conditions in Dunn semantics for this operation are as follows:

- $\Box Ar1$ iff ($Ar1$ and not $Ar0$),
- $\Box Ar0$ iff ($Ar0$ or not $Ar1$).

Note that this operator avoids some negative results for Łukasiewicz's truth-functional necessity operator observed by Font and Hájek in [13]. For a recent discussion on the many-valued approach to modality, see [21,22].

Intensional operators There is quite a literature on modal many-valued logics, and various expansions of **FDE** by intensional modalities have been suggested. The smallest normal modal expansion of **FDE** in the language $\mathcal{L}_{\{\Box,\Diamond\}}$ is known as $\mathbf{K_{FDE}}$; see [34, Chapter 11a], where a straightforward expansion of $\mathcal{T}$**FDE** is shown to be sound and complete with respect to Kripke models $\mathcal{M} = \langle W, R, v^+, v^- \rangle$. Intuitively, the interpretation relation r of the Dunn semantics is parametrized by possible worlds to obtain an interpretation relation r_w for all possible worlds $w \in W$. More precisely, the truth/falsity conditions of formulas $\Box A$ and $\Diamond A$ are as follows, as may be expected:

- $\mathcal{M}, w \models^+ \Box A$ iff for all w' such that wRw', $\mathcal{M}, w' \models^+ A$,
- $\mathcal{M}, w \models^- \Box A$ iff for some w' such that wRw', $\mathcal{M}, w' \models^- A$,
- $\mathcal{M}, w \models^+ \Diamond A$ iff for some w' such that wRw', $\mathcal{M}, w' \models^+ A$,
- $\mathcal{M}, w \models^- \Diamond A$ iff for all w' such that wRw', $\mathcal{M}, w' \models^- A$.

> *A note on the papers in the special issue.* The joint work of Sergei Odintsov and Heinrich Wansing offers a systematic investigation into the relationships between the logics **BK** (an expansion of both the smallest normal modal propositional logic **K** and **FDE**), **KN4** (an expansion of **FDE** with Łukasiewicz's implication, see §5.1, and $\Box$), and **MBL**, modal bilattice logic. For that purpose another modal expansion of **FDE** is introduced, viz. the Fischer Servi–style modal logic $\mathbf{BK^{FS}}$, which is defined as the set of all modal formulas valid under a modified standard translation into first-order **FDE**.

In the study of interrelations between modalities and strong negation, $\sim$, it has been emphasized that a distinction may be drawn between semantical and formal duality, where semantical duality means that $\Box$ and $\Diamond$ are interpreted with respect to one and the same accessibility relation R, whereas formal (or syntactical) duality means that the familiar duality axioms are provable in the presence of a suitable conditional. In [25] it is shown that there are expansions of **FDE** by intuitionistic implication that satisfy semantical (formal) duality but fail to satisfy formal (semantical) duality.

4.2 "Negative" Operators

We now turn to more negation-related operators.

Extensional operators One of the connectives one might think of adding to **FDE** is a unary operation that behaves like the negation in classical logic. How to characterize this operation, however, is itself an interesting question. From a proof-theoretic perspective, one would expect both the law of excluded middle and the law of explosion. From a semantic perspective, one would expect that the classical negation of a sentence is designated iff the negated sentence is not designated. This, however, will *not* pin down a unique operation. Indeed, from a proof-theoretic perspective there are a few options regarding as to how the classical negation interacts with the negation of **FDE**, and from a semantic perspective, there are a few options as to how we set out the falsity condition. More specifically, there are 16 unary operations that may be referred to as classical negation, according to an observation in [9], but here we will only highlight the following two operations.

x	$\neg_b x$	$\neg_e x$
t	**f**	**f**
b	**n**	**f**
n	**b**	**t**
f	**t**	**t**

Note here that the subscripts b and e stand for *boolean* negation and *exclusion* negation respectively. Note also that **BD**$\triangle$ is expressively equivalent to **FDE** expanded by the exclusion negation.

Intensional operators Once a semantics for **FDE** is generalized to accommodate intensional notions by introducing possible worlds and accessibility relations (or neighborhood functions), **FDE** can be expanded by negative modalities, such as the negative alethic modalities of impossibility and unnecessity. Also, intuitionistic negation may be seen as a negative modality that can be added to **FDE**, but probably one would want to consider it together with intuitionistic implication, cf. §5.1. Whilst there is a literature on negation as a modal operator (see, e.g., the references given in [17]), it seems that so far no compelling motivation for adding negative modalities to **FDE** has given rise to studies of such **FDE**-based systems.

4.3 Others

Here, we discuss some operators that are neither affirmative nor negative.

Extensional operators Paul Ruet in [41] considers a unary operation $\circlearrowright$ with the following truth table:

x	$\circlearrowright x$
t	**b**
b	**f**
n	**t**
f	**n**

It is then claimed that the algebra resulting from adding $\circlearrowright$ to $\mathcal{BD}$ is functionally complete. Here we revisit the notion of functional completeness by following a presentation due to Grzegorz Malinowski in [19] with some slight modifications.

Definition 24. For any natural number $n \geq 2$, we denote by U_n any algebra of the form $U_n = (E_n, \mathcal{F})$ where $E_n = \{1, 2, \ldots, n\}$ and $\mathcal{F}$ is a set of finitary operations on E_n. Then,

- g *defines* the k-place operation f *in* $\mathcal{F}$ if g is a composition of some of the operations from $\mathcal{F}$ such that: $f(\vec{x}) = g(\vec{x})$ for all $\vec{x} \in E_n^k$.
- f is *definable in* $\mathcal{F}$ if g defines f in $\mathcal{F}$ for some g.
- U_n is *functionally complete* if every finitary mapping $f : E_n^k \to E_n$ $(k \in \omega)$ is definable in $\mathcal{F}$.

For stating a result due to Słupecki, the following definition is useful.

Definition 25. Let U_n be an algebra, and f be a binary operation defined in $\mathcal{F}$. Then, f is *unary reducible* iff for some unary operation g definable in $\mathcal{F}$, $f(x, y) = g(x)$ for all $x, y \in U_n$ or $f(x, y) = g(y)$ for all $x, y \in U_n$. And f is *essentially binary* if f is not unary reducible.

Intuitively speaking, an essentially binary operation has a truth table "in which at least one line [= row] and one column do not have all elements identical", and this is exactly the way how Słupecki stated the condition (cf. [44, p.154]).

Theorem 5. (Słupecki [44]). *U_n $(n \geq 3)$ is functionally complete iff in U_n*

(i) *all unary operations on E_n are definable, and*

(ii) *at least one surjective and essentially binary operation on E_n is definable.*

In case of expansions of the algebra related to **FDE**, we can simplify even further.

Theorem 6. (Omori & Sano [30]). *Given any expansion $\mathcal{F}$ of $\mathcal{BD}$, the following claims are equivalent:*

(i) *$\mathcal{F}$ is functionally complete.*

(ii) *All of the δ_as as well as C_as ($a \in \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$) of the following tables are definable.*

x	$\delta_{\mathbf{t}}(x)$	$\delta_{\mathbf{b}}(x)$	$\delta_{\mathbf{n}}(x)$	$\delta_{\mathbf{f}}(x)$	$\mathsf{C}_a(x)$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	a
$\mathbf{b}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	a
$\mathbf{n}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	a
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	a

Based on this theorem, it remains to be proved that eight unary operations are definable, and this is indeed possible only by adding the unary operation of Ruet (details are spelled out in [30]).

Remark 26. We may also think of a generalized notion of functional completeness along the two-valued Dunn semantics. For details, see [30].

Another unary operation that deserves special attention is *conflation* with the following truth table:

x	$-x$
$\mathbf{t}$	$\mathbf{t}$
$\mathbf{b}$	$\mathbf{n}$
$\mathbf{n}$	$\mathbf{b}$
$\mathbf{f}$	$\mathbf{f}$

The truth and falsity conditions in Dunn semantics for conflation are as follows:

- $-Ar1$ iff not $Ar0$,
- $-Ar0$ iff not $Ar1$.

From an intuitive point of view, it is hard to grasp its meaning based on these conditions. This is not a surprise, however, since the importance of this operation becomes much clearer in the context of bilattices which is briefly reviewed in §5.4, and we will reintroduce conflation in Definition 29. Here we only note that $\sim -x$ (or $- \sim x$) defines the boolean negation.

Finally, we discuss the *classicality operator* which has the following truth table:

x	$\circ x$
t	**t**
b	**f**
n	**f**
f	**t**

The truth and falsity conditions in Dunn semantics for the classicality operator are as follows:

- $\circ Ar1$ iff ($Ar1$ and not $Ar0$) or (not $Ar1$ and $Ar0$),
- $\circ Ar0$ iff ($Ar1$ and $Ar0$) or (not $Ar1$ and not $Ar0$).

This operator may be seen as a generalized version of the characteristic connective in the tradition of paraconsistent logics known as Logics of Formal Inconsistency (LFIs), now controlling the behavior of not only gluts but also gaps. An interesting property of this operation is that it does not force the definability of classical negation. For more details on the expansion of **FDE** by the classicality operator, see [29].

Intensional operators We may also consider a constructive version of the classicality operator. More specifically, based on a Nelson model $\mathcal{M}$, the following truth and falsity conditions give rise to a constructive variant of the classicality operator:

- $\mathcal{M}, w \models^+ \circ A$ iff for all $w' \geq w$: ($\mathcal{M}, w' \models^+ A$ and $\mathcal{M}, w' \not\models^- A$) or ($\mathcal{M}, w' \not\models^+ A$ and $\mathcal{M}, w' \models^- A$),
- $\mathcal{M}, w \models^- \circ A$ iff for all $w' \geq w$: $\mathcal{M}, w' \models^+ A$ iff $\mathcal{M}, w' \models^- A$.

Again, for more details on this expansion of **FDE**, see [29].

5 Binary Operations

We now turn to the binary operations.

5.1 Conditional

FDE is known to lack a "decent" conditional. Of course we can define the material conditional in terms of negation and disjunction, but many of the properties that may be expected to hold for a genuine conditional are lost: not only modus ponens and transitivity, but even identity, namely "if A then

A" is not valid. This motivates one to expand the language by various conditionals. We will give an overview of the most important and/or interesting ones known in the literature.

Extensional operators There are a number of conditionals that can be considered in the context of **FDE**. Here, we focus on the following four conditionals:

$x \to_e y$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**b**	**n**	**f**
n	**t**	**t**	**t**	**t**
f	**t**	**t**	**t**	**t**

$x \to_b y$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**t**	**n**	**n**
n	**t**	**b**	**t**	**b**
f	**t**	**t**	**t**	**t**

$x \to_l y$	**t**	**b**	**n**	**f**
t	**t**	**f**	**n**	**f**
b	**t**	**b**	**n**	**f**
n	**t**	**n**	**t**	**n**
f	**t**	**t**	**t**	**t**

$x \to_c y$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**b**	**n**	**f**
n	**b**	**b**	**b**	**b**
f	**b**	**b**	**b**	**b**

The most well-known, as well as well-studied, implication is $\to_e$. This may be also seen as a material implication defined in terms of exclusion negation and disjunction (this justifies our choice of the subscript e). We may also define the material implication in terms of boolean negation and disjunction, and leads us to $\to_b$ (subscript b, of course, for boolean).

> *A note on the papers in the special issue.* Norihiro Kamide's paper explores a variant of the expansion of **FDE** by $\to_e$ (called $\mathbf{B}_4^{\to}$ in [23]), in which double-negation simulates classical negation in a very unusual manner.

One of the features missing in the above conditionals $\to_e$ and $\to_b$ is the contraposition with respect to $\sim$. This is sometimes handled by considering a conjunction of the implication and its contraposed form. If we apply this to $\to_e$, then we obtain $\to_l$ which may be seen as a four-valued generalization of Łukasiewicz's implication (this again justifies our subscript being l). Note also that $\to_e$ can be defined modulo $\mathcal{L}_{\{\to_l\}}$ as follows: $x \to_e y \ := \ (x \to_l (x \to_l y)) \vee y$.

Finally, much less studied, but very interesting, is the connexive implication $\to_c$. See [50] for connexive logics in general, and [50, §2.5] and [28, §3.5] for systems with $\to_c$.

Remark 27. Here are precise explications for two modal logics based on **FDE**. The smallest modal expansion **BK** of **FDE** from [26] is formulated in the language $\mathcal{L}_{\{\bot,\rightarrow_e,\Box,\Diamond\}}$, where in this case $\bot$ stands for the constantly false proposition, and $\Box$ and $\Diamond$ are interpreted as in $\mathbf{K}_{\mathbf{FDE}}$. The modal logic **KN4** from [16] is presented in the language $\mathcal{L}_{\{\rightarrow_l,\Box\}}$, where the necessity operator $\Box$ is interpreted as in **BK**.

Intensional operators One of the most popular expansions of **FDE** is obtained by adding the conditional of intuitionistic logic. This results in a system nowadays known as **N4** ([18,24,47]) which has the following truth and falsity conditions within a given Nelson model $\mathcal{M}$:

- $\mathcal{M}, w \models^+ (A \rightarrow B)$ iff for all $w' \geq w$: $\mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^+ B$,
- $\mathcal{M}, w \models^- (A \rightarrow B)$ iff $\mathcal{M}, w \models^+ A$ and $\mathcal{M}, w \models^- B$.

Note also that there is a connexive variant of **N4** obtained by replacing the above falsity condition by the following one:

- $\mathcal{M}, w \models^- (A \rightarrow B)$ iff for all $w' \geq w$: $\mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^- B$.

For details of the resulting system **C**, see [48].

> *A note on the papers in the special issue.* Melvin Fitting's paper devises a justification logic closely related to **N4**. Norihiro Kamide's paper explores a variant of **N4** in which double-negation simulates intuitionistic negation, again in a very unusual manner.

Another famous and important expansion of **FDE** is obtained by adding a conditional interpreted by means of a ternary relation R. More specifically, if we add the following truth condition within the star semantics, then we obtain the relevance logic à la *Australian* plan:

- $I(w, A \rightarrow B) = 1$ iff for all x, y such that $Rwxy$: $(I(x, A) \neq 1$ or $I(y, B) = 1)$.

On the other hand, if we add the following truth and falsity conditions within a given Routley-Meyer model $\mathcal{M}$, then we obtain the relevance logic à la *American* plan:

- $\mathcal{M}, w \models^+ A \rightarrow B$ iff for all x, y such that $Rwxy$: $(\mathcal{M}, x \not\models^+ A$ or $\mathcal{M}, y \models^+ B)$,
- $\mathcal{M}, w \models^- A \rightarrow B$ iff for some x, y such that $Rwxy$: $(\mathcal{M}, x \models^+ A$ and $\mathcal{M}, y \models^- B)$.

Note that there are a number of falsity conditions one may take. See for example [27,36].

Finally, one may also add a strict implication on top of **FDE**. For example, the underlying logic for the logic of intentionality developed by Priest in

[35] deploys the following truth and falsity conditions within a given Kripke model $\mathcal{M}$ such that the binary relation R is the universal relation:

- $\mathcal{M}, w \models^+ A \to B$ iff for all x such that wRx: $(\mathcal{M}, x \not\models^+ A$ or $\mathcal{M}, x \models^+ B)$,
- $\mathcal{M}, w \models^- A \to B$ iff for some x such that wRx: $(\mathcal{M}, x \models^+ A$ and $\mathcal{M}, x \models^- B)$.

Again, one may consider a number of falsity conditions for the conditional.

5.2 Coimplication

Many implications (but not all, such as the relevant implication), residuate the extensional conjunction, namely the following holds:

$$A \wedge B \vdash C \text{ iff } A \vdash B \to C.$$

It is then not unnatural to introduce a connective that residuates disjunction in the following manner:

$$C \vdash A \vee B \text{ iff } C \leftarrow B \vdash A.$$

It is this connective $\leftarrow$ that is called *coimplication*, and sometimes it is also called "pseudo-difference".

Extensional operators In the extensional case one may consider "dual" versions of the truth tables from §5.1:

$x \leftarrow_e y$	**t**	**b**	**n**	**f**
t	**f**	**b**	**n**	**t**
b	**f**	**b**	**n**	**t**
n	**f**	**f**	**f**	**f**
f	**f**	**f**	**f**	**f**

$x \leftarrow_b y$	**t**	**b**	**n**	**f**
t	**f**	**b**	**n**	**t**
b	**f**	**f**	**n**	**n**
n	**f**	**b**	**f**	**b**
f	**f**	**f**	**f**	**f**

$x \leftarrow_l y$	**t**	**b**	**n**	**f**
t	**f**	**t**	**n**	**t**
b	**f**	**b**	**n**	**t**
n	**f**	**n**	**f**	**n**
f	**f**	**f**	**f**	**f**

$x \leftarrow_c y$	**t**	**b**	**n**	**f**
t	**f**	**b**	**n**	**t**
b	**f**	**b**	**n**	**t**
n	**b**	**b**	**b**	**b**
f	**b**	**b**	**b**	**b**

To the best of our knowledge, these connectives have not yet been investigated.

Intensional operators Coimplication appears in the language of dual-intuitionistic logic (and in that of bi-intuitionistic logic, also called Heyting–Brouwer logic). As an intensional operator, coimplication is interpreted on intuitionistic Kripke models $\mathcal{M}$. The support of truth of a formula A at

$w \in W$ ($\mathcal{M}, w \models A$) is defined recursively; for implication and coimplication the defining clauses are:

- $\mathcal{M}, w \models (A \rightarrow B)$ iff for all $w' \geq w$: $\mathcal{M}, w' \not\models A$ or $\mathcal{M}, w' \models B$,
- $\mathcal{M}, w \models (A \leftarrow B)$ iff for some $w' \leq w$: $\mathcal{M}, w' \models A$ and $\mathcal{M}, w' \not\models B$.

If intuitionistic implication and dual-intuitionistic co-implication are added to $\mathcal{L}$, then we make use of Nelson models $\mathcal{M}$. The positive clauses are as expected:

- $\mathcal{M}, w \models^+ (A \rightarrow B)$ iff for all $w' \geq w$: $\mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^+ B$,
- $\mathcal{M}, w \models^+ (A \leftarrow B)$ iff for some $w' \leq w$: $\mathcal{M}, w' \models^+ A$ and $\mathcal{M}, w' \not\models B$.

Although there is a standard, "classical" understanding of the falsity conditions of a conditional, there are indeed several options for defining such conditions, including the so-called *connexive* understanding of negated conditionals. In [49] sixteen different expansions of **FDE** are considered and proof-theoretically characterized by display sequent calculi. These logics are obtained as combinations of the following support of falsity conditions of implications:

- $\mathcal{M}, w \models^- (A \rightarrow B)$ iff $\mathcal{M}, w \models^+ A$ and $\mathcal{M}, w \models^- B$,
- $\mathcal{M}, w \models^- (A \rightarrow B)$ iff for all $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^- B$,
- $\mathcal{M}, w \models^- (A \rightarrow B)$ iff for some $w' \leq w : \mathcal{M}, w' \models^+ A$ and $\mathcal{M}, w' \not\models^+ B$,
- $\mathcal{M}, w \models^- (A \rightarrow B)$ iff for some $w' \leq w : \mathcal{M}, w' \not\models^- A$ and $\mathcal{M}, w' \models^- B$,

and co-implications:

- $\mathcal{M}, w \models^- (A \leftarrow B)$ iff $\mathcal{M}, w \models^- A$ or $\mathcal{M}, w \models^+ B$,
- $\mathcal{M}, w \models^- (A \leftarrow B)$ iff for some $w' \leq w : \mathcal{M}, w' \models^- A$ and $\mathcal{M}, w' \not\models^+ B$,
- $\mathcal{M}, w \models^- (A \leftarrow B)$ iff for all $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^+ B$,
- $\mathcal{M}, w \models^- (A \leftarrow B)$ iff for all $w' \geq w : \mathcal{M}, w' \models^- A$ or $\mathcal{M}, w' \not\models^- B$.

5.3 Conjunction and Disjunction

Since we already have conjunction and disjunction in the basic language of **FDE**, the demand for additional conjunctions and disjunctions is not very high. Still there are some important connectives considered in the literature.

Extensional operators First, we should mention the "informational" conjunction and disjunction which have the following truth tables.

$x \otimes y$	**t**	**b**	**n**	**f**
t	**t**	**t**	**n**	**n**
b	**t**	**b**	**n**	**f**
n	**n**	**n**	**n**	**n**
f	**n**	**f**	**n**	**f**

$x \oplus y$	**t**	**b**	**n**	**f**
t	**t**	**b**	**t**	**b**
b	**b**	**b**	**b**	**b**
n	**t**	**b**	**n**	**f**
f	**b**	**b**	**f**	**f**

In terms of truth and falsity conditions in Dunn semantics, we obtain the following conditions:

- $A \otimes Br1$ iff $Ar1$ and $Br1$,
- $A \otimes Br0$ iff $Ar0$ and $Br0$,
- $A \oplus Br1$ iff $Ar1$ or $Br1$,
- $A \oplus Br0$ iff $Ar0$ or $Br0$.

These connectives play a very important role in the context of bilattices, which will be briefly overviewed in the next subsection.

> *A note on the papers in the special issue.* The contribution by Ofer Arieli and Arnon Avron offers a detailed examination of the expressivity of languages with $\oplus$ and $\otimes$, as well as a systematic study of proof systems in the style of Hilbert and Gentzen.

Intensional operators Intensional variants of conjunction and disjunction are the fusion and fission operators from relevance logic, referred to in linear logic as multiplicative conjunction and disjunction, respectively, which we here also write as $\otimes$ and $\oplus$. In the star semantics with a ternary relation R, one obtains the following semantic clauses:

- $I(w, A{\otimes}B){=}1$ iff for some y, z: $Ryzw, I(y, A){=}1$ and $I(z, B){=}1$,
- $I(w, A{\oplus}B){=}1$ iff for all y, z: if $Rwyz$ and $I(y^*, A){=}0$ then $I(z, B){=}1$.

The intensional conjunction also appears in substructural subsystems of the **FDE**-based logic **N4** (cf. [18,46]).

5.4 Interlude: Bilattice Semantics

Given that we have mentioned the informational connectives, it is now a good moment to introduce some basic notions related to bilattices (see [4, 11, 15]).

DEFINITION 28. A *bilattice* is a structure $\mathcal{B} = \langle B, \leq_t, \leq_k, \sim \rangle$ where

- B is a non-empty set that contains at least two elements,

- $\langle B, \leq_t \rangle$, $\langle B, \leq_k \rangle$ are complete lattices,
- $\sim$ is a unary operation on B that has the following properties:
  - if $a \leq_t b$ then $\sim b \leq_t \sim a$,
  - if $a \leq_k b$ then $\sim a \leq_k \sim b$,
  - $\sim\sim a = a$.

Definition 29. A *conflation*, $-$, is a unary operation on a bilattice $\mathcal{B}$ that has the following properties:

- if $a \leq_t b$ then $-a \leq_t -b$,
- if $a \leq_k b$ then $-b \leq_k -a$,
- $--a = a$,
- $\sim -a = - \sim a$.

Definition 30. A *bifilter* of a bilattice $\mathcal{B} = \langle B, \leq_t, \leq_k, \sim \rangle$ is a non-empty subset $\mathcal{F} \subsetneq B$ such that:

- $a \wedge b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$,
- $a \otimes b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$.

Moreover, a bifilter is *prime* iff it also satisfies:

- $a \vee b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$,
- $a \oplus b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

Definition 31. A *logical bilattice* is a pair $(\mathcal{B}, \mathcal{F})$, in which $\mathcal{B}$ is a bilattice, and $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

Definition 32. A formula A is a *bilattice semantic consequence* of Γ ($\Gamma \models_{bl} A$) iff for all logical bilattices $(\mathcal{B}, \mathcal{F})$, if $v(C) \in \mathcal{F}$ for all $C \in \Gamma$ then $v(A) \in \mathcal{F}$.

Remark 33. Note that the definition of the notion of a bilattice itself does not assume additional connectives beside $\sim$, but for logical bilattices, the informational conjunction and disjunction are required, which implies that strict expansions of **FDE** are concerned in the discussion. Note, moreover, that the implications $\rightarrow_e$ and $\rightarrow_l$ coincide with the weak and strong implications considered in the logic of bilattices, see [4]. The theory of bilattices has been generalized to a study of tri- and multilattices, cf. [43].

A note on the papers in the special issue. The contribution by Thomas Ferguson focuses on trilattices, and reports some new results on cut-down connectives, originally introduced by Fitting in the context of bilattices. In particular, Ferguson establishes some interesting connections to containment logics of Harry Deutsch and Richard Angell. The logics of multilattices expanded by bi-intuitionistic and connexive vocabulary are introduced and some basic results are established in the paper by Norihiro Kamide, Yaroslav Shramko and Heinrich Wansing.

Remark 34. The language of the modal bilattice logic **MBL** from [40] extends the language of the logic of logical bilattices from [4] with the four constants from **FDE**, $\Box$, and $\Diamond$ (as a defined operator). However, the semantics of the modal operators is significantly modified insofar as the binary relation R is required to be four-valued, with the effect that **MBL** is not a normal modal logic.

6 Quantifiers

It is desirable to extend the language of propositional **FDE** by first- and higher-order universal and particular quantifiers. In the first-order case the language comprises a denumerable supply of individual variables, x, y, z, x_1, x_2, etc., individual constants, a, b, c, a_1, a_2, etc., n-place predicate symbols P, Q, P_1, P_2, etc. (for all natural numbers $n > 0$), with the tacit assumption of using them with appropriate arity, and quantifier prefixes of the form $\forall x$ and $\exists x$. The set $\mathsf{Q_1FORM}$ of all formulas of first-order quantified **FDE** is defined as usual (and the focus on studying the addition of quantifiers may justify neglecting function symbols and a distinguished primitive identity predicate).

6.1 Semantics

We first spell out several semantics for first-order **FDE**.

Dunn semantics The extension of the relational semantics of **FDE** from [34, chapter 22] can be presented as follows. A model for quantified **FDE** is a pair $\langle D, v\rangle = \mathcal{M}$, where D is a non-empty set of individuals and v is a function such that for all individual constants a, $v(a) \in D$ and for all n-ary predicate symbol P, $v(P)$ is a pair $\langle v^+(P), v^-(P)\rangle$, where $v^+(P)$ (the extension of P) and $v^-(P)$ (the anti-extension of P) are subsets of D^n. In order to avoid variable assignments and to define, for a given model $\mathcal{M}$, an interpretation relation between *closed* formulas and the values 1 and 0, the language is extended by constants $\mathbf{a}$, for all $a \in D$.

DEFINITION 35. Let $\mathcal{M} = \langle D, v\rangle$ be a model for quantified **FDE**. For closed atomic formulas, the interpretation relation r for $\mathcal{M}$ is defined as follows:

- $P(a_1, \ldots, a_n)r1$ iff $\langle v(a_1), \ldots, v(a_n)\rangle \in v^+(P)$,
- $P(a_1, \ldots, a_n)r0$ iff $\langle v(a_1), \ldots, v(a_n)\rangle \in v^-(P)$.

The relation r is extended to a relation between all closed formulas and the values 1 and 0 by the clauses from Definition 3 and the following stipulations, where $A(\mathbf{a}/x)$ is the result of replacing all free occurrences of x in A by $\mathbf{a}$:

- $\forall xAr1$ iff for all $a \in D$, $A(\mathbf{a}/x)r1$,
- $\forall xAr0$ iff for some $a \in D$, $A(\mathbf{a}/x)r0$,
- $\exists xAr1$ iff for some $a \in D$, $A(\mathbf{a}/x)r1$,
- $\exists xAr0$ iff for all $a \in D$, $A(\mathbf{a}/x)r0$.

DEFINITION 36. If $\Gamma \cup \{A\} \subseteq \mathsf{Q_1FORM}$ is a set of closed formulas, then A is a two-valued semantic consequence of Γ ($\Gamma \models_2 A$) iff for all models $\mathcal{M}$ for quantified **FDE**, if $Br1$ for all $B \in \Gamma$ then $Ar1$.

Algebraic semantics There seems to be no worked out algebraic semantics for first-order **FDE**, at least to the best of our knowledge.

Bilattice semantics An extension of the bilattice semantics to the first-order case is briefly mentioned in [4, Sect 3.5]. The semantics of the universal and of the particular quantifier are defined in terms of the infimum and supremum, respectively, of the truth-order. Note also that one can also introduce quantifiers for the information order.

6.2 Proof Systems

We now turn to proof systems.

Hilbert style At the time being we are not aware of a sound and complete Hilbert-style proof system for first-order **FDE**.

Gentzen style The following Gentzen-style sequent calculus for first-order **FDE** is taken from [2][5] (notation adjusted). In the axiom and rules of the calculus, J, K, L, and M range over finite, possibly empty sequences of formulas.

[5] Anderson and Belnap treat $\wedge$ and $\forall$ as defined in the standard way: $A \wedge B =_{\mathrm{def}} {\sim}({\sim} A \vee {\sim} B)$; $\forall xA =_{\mathrm{def}} {\sim}\exists x{\sim} A$.

Definition 37. Let $\mathcal{G}$**QFDE** be the Gentzen-style system consisting of the axiom $J, A, K \to L, A, M$, in which A is an atomic formula or a negated atomic formula, and the following sequent rules:

$$\frac{K, A, L \to J}{K, \sim\sim A, L \to J}\ (R1) \qquad \frac{J \to K, A, L}{J \to K, \sim\sim A, L}\ (R2)$$

$$\frac{J, A, K \to L \quad J, B, K \to L}{J, A \vee B, K \to L}\ (R3) \qquad \frac{J \to K, A, B, L}{J \to K, A \vee B, L}\ (R4)$$

$$\frac{J, \sim A, \sim B, K \to L}{J, \sim(A \vee B), K \to L}\ (R5) \qquad \frac{J \to K, \sim A, L \quad J \to K, \sim B, L}{J \to K, \sim(A \vee B), L}\ (R6)$$

$$\frac{J, Ay, K \to L}{J, \exists x Ax, K \to L}\ (R7) \qquad \frac{J \to K, Ay, L, \exists x Ax}{J \to K, \exists x Ax, L}\ (R8)$$

$$\frac{J, \sim Ay, K, \sim\exists x Ax \to L}{J, \sim\exists x Ax, K \to L}\ (R9) \qquad \frac{J \to K, \sim Ay, L}{J \to K, \sim\exists x Ax, L}\ (R10)$$

with the restriction in (R7) and (R10) that y does not occur free in the conclusion sequent. Moreover, a rule for alphabetic change of bound variables is assumed.

We write $\Gamma \vdash_{\mathbf{g}} A$ iff there is a sequence of closed formulas $B_1, \ldots, B_n \in \Gamma$ $(n \geq 0)$ such that the sequent $B_1, \ldots, B_n \to A$ is derivable in $\mathcal{G}$**QFDE**. In order to prove soundness of $\mathcal{G}$**QFDE** with respect to the Dunn semantics, the definition of semantical consequence is extended (cf. [45, Definition 2.8.1]).

Definition 38. Let $\Gamma \cup \{A\} \subseteq \mathsf{Q_1FORM}$, let $\{x_{i_1}, \ldots, x_{i_2}, \ldots\}$ be the set of free variables occurring in formulas from $\Gamma \cup \{A\}$, and let $\mathcal{M} = \langle D, v\rangle$ be a model for quantified **FDE** with interpretation relation r. If $\overline{a}$ is a sequence $a_1, a_2, \ldots$ of elements from D, then $\Gamma(\overline{a})$, respectively $A(\overline{a})$, is the result of simultaneously replacing the x_{i_j} by $\mathbf{a_j}$ (for $j \geq 1$) in all formulas from Γ, respectively in A. Then $\Gamma(\overline{a})r1$ iff $Br1$ for all $B \in \Gamma(\overline{a})$. Moreover, $\Gamma \models_2 A$ iff for all models $\mathcal{M}$ and sequences $\overline{a}$, if $\Gamma(\overline{a})r1$ then $A(\overline{a})r1$.

Theorem 7. *For any set of closed formulas* $\Gamma \cup \{A\} \subseteq \mathsf{Q_1FORM}$*,* $\Gamma \vdash_{\mathbf{g}} A$ *iff* $\Gamma \models_2 A$.[6]

[6]It suffices to show (i) that if a sequent $K \to L$ is provable, then $\bigwedge K \models_2 \bigvee L$ and (ii) in view of Theorem 8, that any natural deduction derivation of A from finite Δ can be transformed into a proof of $J \to A$ in $\mathcal{G}$**QFDE**, where J is a sequence consisting of all members of Δ.

Natural deduction Providing a natural deduction system is straightforward.

Definition 39. The natural deduction proof system $\mathcal{N}$**QFDE** for **QFDE** is obtained by augmenting $\mathcal{N}$**FDE** with the following rules [33, p. 331 ff.]:

$$\frac{B \vee A(c/x)}{B \vee \forall x A}\ \forall I \qquad \frac{\forall x A}{A(c/x)}\ \forall E \qquad \frac{A(c/x)}{\exists x A}\ \exists I \qquad \frac{\exists x A \quad \begin{array}{c}\overline{A(c/x)}\\ \vdots \\ B\end{array}}{B}\ \exists E$$

$$\frac{\forall x \sim A}{\overline{\sim \exists x A}} \qquad \frac{\exists x \sim A}{\overline{\sim \forall x A}}$$

Note that rules $\forall I$ and $\exists E$ are applicable provided that c does not occur in B, or in any undischarged assumption on which the premise depends.

Syntactic consequence for $\mathcal{N}$**QFDE**, $\vdash_{\mathbf{n}}$, is defined as in Definition 14.

Theorem 8. (Priest). *For any set of closed formulas* $\Gamma \cup \{A\} \subseteq \mathsf{Q_1FORM}$, $\Gamma \vdash_{\mathbf{n}} A$ *iff* $\Gamma \models_2 A$.

Tableaux Providing a tableau calculus is also quite straightforward.

Definition 40. The tableau calculus $\mathcal{T}$**QFDE** for **QFDE** is the result of extending $\mathcal{T}$**FDE** by the following rules ([34, Chap. 22]):

$$\begin{array}{cccccc} \forall x A, + & \forall x A, - & \sim\forall x A, \pm & \exists x A, + & \exists x A, - & \sim\exists x A, \pm \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ A(a/x), + & A(c/x), - & \exists x \sim A, \pm & A(c/x), + & A(a/x), - & \forall x \sim A, \pm \end{array}$$

where $\pm$ stands uniformly for either $+$ or $-$, a is any constant on the branch, or a new constant if there is none, and c is a constant new to the branch. Tableau consequence, $\vdash_{\mathbf{t}}$, for $\mathsf{Q_1FORM}$ is defined as in Definition 15.

Theorem 9. (Priest). *For any finite set of closed formulas* $\Gamma \cup \{A\} \subset \mathsf{Q_1FORM}$, $\Gamma \vdash_{\mathbf{t}} A$ *iff* $\Gamma \models_2 A$.

7 Open Problems

There are, of course, topics for future research; they include, for example:

- investigation of an intensional disjunction that residuates relevant co-implication,

- the study of quantifiers defined in bilattice semantics with respect to the information order,
- reassurance for 2nd-order minimal **FDE**, cf. [33],
- axiomatizing the relevance logics à la American plan, cf. [27,36].

8 Conclusion

40 years after the birth of **FDE**, it seems that much progress has been made in the field. **FDE** is a well-understood core system of many-valued, relevance, and paraconsistent logic. It is naturally arrived at from different proof-theoretic and semantical perspectives and its expansions by various additional operators and quantifiers invite further investigation. The papers from this special issue contribute to that enterprise.

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