Optimal Approximation of Elliptic Problems by Linear and Nonlinear Mappings IV: Errors in L_2 and Other Norms

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June 2, 2009

Abstract

We study the optimal approximation of the solution of an operator equation $\mathcal{A}(u) = f$ by linear and different types of nonlinear mappings. In our earlier papers we only considered the error with respect to a certain H^s-norm where s was given by the operator since we assumed that $\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega)$ is an isomorphism. The most typical case here is $s = 1$. It is well known that for certain regular problems the order of convergence is improved if one takes the L_2-norm. In this paper we study error bounds with respect to such a weaker norm, i.e., we assume that $H_0^s(\Omega)$ is continuously embedded into a space X and we measure the error in the norm of X. A major example is $X = L_2(\Omega)$ or $X = H^r(\Omega)$ with $r < s$. We prove this better rate of convergence also for non-regular problems.

AMS subject classification: 41A25, 41A46, 41A65, 42C40, 46E35, 65C99

Key Words: Elliptic operator equations, worst case error, Riesz basis, wavelets, linear and nonlinear approximation methods, approximation numbers, best n-term approximation, nonlinear widths, manifold widths, L_2-errors.

*Supported by Deutsche Forschungsgemeinschaft (DFG), Grant DA 360/12–1.
†Supported by Deutsche Forschungsgemeinschaft (DFG), Grant NO 175/17–1.
‡Supported by Deutsche Forschungsgemeinschaft (DFG), Grant SI 487/14–1.

1 Introduction

We continue our work from [9, 10, 11][1]. There we studied the optimal approximation of the solution of an operator equation

$$\mathcal{A}(u) = f, \tag{1}$$

where $\mathcal{A}$ is a linear operator

$$\mathcal{A} : H \to G \tag{2}$$

from a Hilbert space H to another Hilbert space G. We always assume that $\mathcal{A}$ is boundedly invertible, and so (1) has a unique solution for any $f \in G$. We have in mind the more specific situation of an elliptic operator equation which is given as follows. Assume that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and assume that

$$\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega) \tag{3}$$

is an isomorphism, where $s > 0$. A standard case (for second order elliptic boundary value problems for PDEs) is $s = 1$, but also other values of s are of interest. Now we put $H = H_0^s(\Omega)$ and $G = H^{-s}(\Omega)$. Since $\mathcal{A}$ is boundedly invertible, the inverse mapping $S : G \to H$ is well defined. We call S the *solution operator.*

We use linear and (different kinds of) nonlinear mappings S_n for the approximation of the solution $u = S(f)$ for f contained in $F \subset G$. We consider the worst case error

$$e(S_n, F, X) = \sup_{\|f\|_F \le 1} \|S(f) - S_n(f)\|_X, \tag{4}$$

where F is a normed (or quasi-normed) subspace of G and H is continuously embedded into the Banach space X. Here $S_n : F \to X$ denotes an approximation of S and n denotes the degrees of freedom. In our main results, F and X are Sobolev or Besov spaces.[2] Hence we use the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{S_F} & X \\ {\scriptstyle I_1}\big\downarrow & & \big\uparrow{\scriptstyle I_2} \\ G & \xrightarrow{S} & H\,. \end{array}$$

[1]The present paper is complete in the sense that we repeat certain definitions and results. Still we recommend our earlier papers for a more detailed discussion. For the definition of the function spaces, see, e.g., [10]

[2]Formally we deal with Besov spaces. Because of the embeddings $B_1^{-s+t}(L_p(\Omega)) \subset W_p^{-s+t}(\Omega) \subset B_\infty^{-s+t}(L_p(\Omega))$, which hold for $1 \le p \le \infty$, $t \ge s$, see [39], our results are valid also for Sobolev spaces.

Here $I: F \to G$ denotes the identity and S_F the restriction of S to F. In the specific case (3) this diagram is given by

$$\begin{tikzcd} B_q^{-s+t}(L_p(\Omega)) \arrow[r, "S_F"] \arrow[d, "I_1"'] & H^r(\Omega) \\ H^{-s}(\Omega) \arrow[r, "S"] & H_0^s(\Omega) \arrow[u, "I_2"'] \end{tikzcd},$$

where $F := B_q^{-s+t}(L_p(\Omega))$ denotes a Besov space compactly embedded into $H^{-s}(\Omega)$ and S_F the restriction of S to $B_q^{-s+t}(L_p(\Omega))$. We are interested in approximations that have the optimal order of convergence depending on n, where n denotes the *degree of freedom*. In general our results are *constructive in a mathematical sense*, because we can describe optimal approximations S_n in mathematical terms. This does not mean, however, that these descriptions are constructive in a practical sense, since it might be difficult to convert those descriptions into a practical algorithm. As a consequence, most of our results give optimal benchmarks and can serve for the evaluation of old and new algorithms.

We consider the worst case setting for deterministic algorithms. Randomized algorithms and algorithms for the quantum computer where recently studied by Heinrich [21, 22, 23].

This paper is organized as follows. In Section 2, we recall some general results concerning all the different widths that are used in this paper. In Section 3, we study quite general elliptic boundary value problems in Lipschitz domains. It turns out that for regular elliptic problems in Hilbert spaces the linear widths, the nonlinear widths and the manifold widths show the same asymptotic behavior. In contrary to this, nonregular problems behave quite different in the sense that in general no lower bounds can be derived. In Section 4, the results are generalized to the case where the right-hand sides belong to quasi-Banach spaces. Then, in Section 5, the abstract machinery derived so far is applied to a concrete problem, i.e., to the Poisson equation in Lipschitz domains. It turns out that in contrary to the general situation in this case also lower bounds can be derived. Moreover, special emphasis is layed on best n-term wavelet approximation. By employing regularity results for the solution in Besov spaces, we determine the approximation order of best n-term wavelet approximation in various different norms. It turns out, that for a large range of parameters, and in particular for weak norms, best n-term wavelet approximation indeed realizes the optimal order of convergence. This is one of the main results of this paper.

2 General Inequalities

For the definition of the linear widths e_n^{lin}, the manifold widths e_n^{cont} and the the nonlinear widths $e_{n,C}^{\text{non}}$ we refer to the Appendix at the end of this paper.

We start with a result that is well known, see [29].

Proposition 1. *Assume that F and X are Hilbert spaces and $S : F \to X$ is linear and continuous. Then*

$$e_n^{\text{lin}}(S, F, X) = e_n^{\text{cont}}(S, F, X). \tag{5}$$

The following result is an improvement of a similar result (Theorem 1) of [10].

Theorem 1. *Assume that F is quasi-normed and X is a Hilbert space and $S : F \to X$ is linear and compact. Then for all $C \geq 1$ and all $n \in \mathbb{N}$, we have*

$$e_{4n+1}^{\text{cont}}(S, F, X) \leq 2\, C\, e_{n,C}^{\text{non}}(S, F, X)\,. \tag{6}$$

Proof. This is a worst case result for the unit ball of F and we have to prove the following. Assume that there is a Riesz basis $\mathcal{B} \in \mathcal{B}_C$ (for definitions see the appendix) of X such that

$$\sup_{\|f\|_F \leq 1} \|S(f) - \sigma_n(S(f))\| = \alpha,$$

where $\sigma_n(S(f))$ is the best n-term approximation of $S(f)$ by elements from $\mathcal{B}$ in the norm of X. Then we have to prove that there are continuous mappings $N : F \to \mathbb{R}^{4n+1}$ and $\varphi : \mathbb{R}^{4n+1} \to X$ such that

$$\sup_{\|f\|_F \leq 1} \|\varphi(N(f)) - S(f)\| \leq 2C\alpha.$$

For the proof we use Proposition 1 from [10] and apply it to the set $S(F_1) \subset X$, where F_1 is the unit ball of F. We obtain continuous mappings $\widetilde{N} : X \to \mathbb{R}^{4n+1}$ and $\varphi : \mathbb{R}^{4n+1} \to X$ such that

$$\sup_{\|f\|_F \leq 1} \|\varphi(\widetilde{N}(S(f))) - S(f)\| \leq 2C\alpha.$$

Hence we obtain the claim with $N = \widetilde{N} \circ S$. $\square$

In many applications one studies problems with "finite smoothness" and then, as a rule, one has the estimate

$$e_{2n}^{\text{lin}}(S, F, X) \asymp e_n^{\text{lin}}(S, F, X). \tag{7}$$

Formula (7) especially holds for the operator equations that we study in Section 3. Then we conclude that approximation by optimal linear mappings yields the same order of convergence as the best n-term approximation.

Corollary 1. *Assume that $S : F \to X$ with Hilbert spaces F and X, with* (7) *holding. Then, for any $C \geq 1$, we have*

$$e_n^{\mathrm{lin}}(S, F, X) = e_n^{\mathrm{cont}}(S, F, X) \asymp e_{n,C}^{\mathrm{non}}(S, F, X). \tag{8}$$

Remark 1. Just by definition one has

$$\max \left(e_n^{\mathrm{cont}}(S, F, X), e_{n,C}^{\mathrm{non}}(S, F, X) \right) \leq e_n^{\mathrm{lin}}(S, F, X)\,.$$

Now (8) reads as: in the context of Hilbert spaces *optimal* linear methods are as good as the *optimal* nonlinear methods. However, the optimal linear methods are not always of practical relevance, which means, they can not be translated into a good algorithm, for instance because of too much precalculations. For a more detailed discussion we refer to [10].

Finally we recall the multiplicativity of certain s-numbers.

Lemma 1. *Let $m, n \in \mathbb{N}$, and let $S_2 : F \to Y$ and $S_1 : Y \to X$ with quasi-normed linear spaces F, Y, and X. Then*

$$e_{m+n}^{\mathrm{cont}}(S_1 \circ S_2, F, X) \leq e_m^{\mathrm{cont}}(S_2, F, Y)\, e_n^{\mathrm{cont}}(S_1, Y, X) \tag{9}$$

holds. The same inequality holds for the linear widths (approximation numbers) e_n^{lin}.

Remark 2. For the proof of (9) we refer to [14]. There the proof is given in a more specific context, however, the method carries over to the present situation. In this generality the lemma is formulated in [10]. For the linear widths we refer to [30].

3 Elliptic Problems I

In this section, we study the more special case where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and $\mathcal{A} = S^{-1} : H_0^s(\Omega) \to H^{-s}(\Omega)$ is an isomorphism, where $s > 0$. Furthermore, we restrict ourselves to the case where the right-hand side belongs to a space $H^{-s+t}(\Omega)$ $(t > 0)$ and the error will be measured with respect to a suitable $H^r(\Omega)$-norm. For the definition of $H^s(\Omega)$ and $H_0^s(\Omega)$ we refer to [40] and to the appendix in [10]. Obviously, we are working in a Hilbert space context.

3.1 Regular Problems

The notion of regularity is important for the theory and the numerical treatment of operator equations, see [20]. For us it will be convenient to use the following definition.

Definition 1. *Let $s, t > 0$. An isomorphism $\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega)$ is H^{s+t}-*regular *if also*

$$\mathcal{A} : H_0^s(\Omega) \cap H^{s+t}(\Omega) \to H^{-s+t}(\Omega) \tag{10}$$

is an isomorphism.

A classical example is the Poisson equation in a C^∞-domain with $s = 1$.

Lemma 2. *Let Ω be a bounded C^∞-domain. Then the associated Poisson problem is given by*

$$\begin{aligned} -\triangle u &= f \quad in \quad \Omega \\ u &= 0 \quad on \quad \partial\Omega\,. \end{aligned}$$

Let $\mathcal{A}$ denote the mapping which sends u to f. Then $\mathcal{A}$ is H^{1+t}-regular for every $t > 0$. Furthermore, with $S := \mathcal{A}^{-1}$ and $C \geq 1$, we have

$$\begin{aligned} e_n^{\mathrm{lin}}(S, H^{-1+t}(\Omega), L_2(\Omega)) &\asymp e_n^{\mathrm{cont}}(S, H^{-1+t}(\Omega), L_2(\Omega)) \qquad (11) \\ &\asymp e_{n,C}^{\mathrm{non}}(S, H^{-1+t}(\Omega), L_2(\Omega)) \asymp n^{(-1-t)/d}\,. \qquad (12) \end{aligned}$$

Remark 3. This result is known. We refer to [2] and [20, Chapter 9] for the regularity part and to [42] for the approximation part. We also refer to the next Theorem 2 and Subsection 5.1, where we prove a more general result.

We shall prove that (11), (12) extend to all H^r-norms with $r < 1 + t$. Moreover, the optimal rate can be obtained by using Galerkin spaces that do not depend on the particular operator $\mathcal{A}$. With nonlinear approximations we cannot obtain a better rate of convergence.

Theorem 2. *Let $\mathcal{A}$ be H^{s+t}-regular and $-\infty < r < s + t$ with $t > 0$. Then, for all $C \geq 1$, we have*

$$\begin{aligned} e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H^r(\Omega)) &\asymp e_n^{\mathrm{cont}}(S, H^{-s+t}(\Omega), H^r(\Omega)) \qquad (13) \\ &\asymp e_{n,C}^{\mathrm{non}}(S, H^{-s+t}(\Omega), H^r(\Omega)) \asymp n^{(r-s-t)/d}\,, \end{aligned}$$

and the optimal order can be obtained by subspaces of $H^r(\Omega)$ that do not depend on the operator $S = \mathcal{A}^{-1}$.

Proof. Consider first the identity (embedding) $I : H^{s+t}(\Omega) \to H^r(\Omega)$. Under the restriction $r < s + t$ it is known that

$$e_n^{\mathrm{lin}}(I, H^{s+t}(\Omega), H^r(\Omega)) \asymp n^{(r-s-t)/d}\,.$$

This is a classical result (going back to Kolmogorov (1936), see [26]) for $s, t \in \mathbb{N}$, see also [31]. For the general case ($s, t > 0$ and arbitrary bounded Lipschitz domains) see [17] and [39]. We obtain the same order for $I : H^{s+t}(\Omega) \cap H_0^s(\Omega) \to H^r(\Omega)$. Here only the estimate from below needs a further comment. Let $B \subset \Omega$ be a ball such that $\operatorname{dist}(B, \partial\Omega) > 0$. The restriction to B of any distribution belonging to $H^s(\Omega)$ belongs to $H^s(B)$ and

$$\| u \,|H^s(B)\| \le \| u \,|H^s(\Omega)\| \,. \tag{14}$$

This implies

$$e_n^{\text{lin}}(I, H^{s+t}(B), H^r(B)) \le e_n^{\text{lin}}(I, H^{s+t}(\Omega) \cap H_0^s(\Omega), H^r(\Omega)) \,.$$

Since

$$e_n^{\text{lin}}(I, H^{s+t}(B), H^r(B)) \asymp n^{(r-s-t)/d}$$

the claimed assertion follows.

We assume (10), and hence $S : H^{-s+t}(\Omega) \to H^{s+t}(\Omega) \cap H_0^s(\Omega)$ is an isomorphism. It is elementary to prove that linear isomorphisms do not change the asymptotic behaviour of linear widths. Consequently we obtain the same order of the e_n^{lin} for I and for $I \circ S_{|H^{-s+t}(\Omega)}$. Together with Corollary 1 this proves (13). Assume that the linear mapping

$$S_n(f) := \sum_{i=1}^n g_i \, L_i(f)$$

is good for the mapping $I : H^{s+t}(\Omega) \cap H_0^s(\Omega) \to H^r(\Omega)$, i.e., we consider a sequence of such approximations with the optimal rate. Here we assume $g_i \in H^r(\Omega)$, $i = 1, \ldots, n$. Then the linear mappings

$$S_n(Sf) = \sum_{i=1}^n g_i \, L_i(Sf)$$

achieve the optimal rate for the mapping $S : H^{-s+t}(\Omega) \to H^{s+t}(\Omega) \hookrightarrow H^r(\Omega)$ since

$$\begin{aligned} \| S(f) - S_n(Sf) \,|H^r(\Omega)\| &\le c\, n^{(r-s-t)/d} \, \| S(f) \,|H^{s+t}(\Omega)\| \\ &\le c\, n^{(r-s-t)/d} \, \| S \| \; \| f \,|H^{-s+t}(\Omega)\| \,, \end{aligned}$$

where c is independent of n. □

Remark 4. The same g_i are good for all $H^{s+t}(\Omega)$-regular problems on $H^{-s+t}(\Omega)$; only the linear functionals, given by $L_i \circ S_{|H^{-s+t}}$, depend on the operator $\mathcal{A}$. For the numerical realization we can use the Galerkin method with the space V_n generated by $g_1, \ldots, g_n$.

3.2 Nonregular Problems

The next result shows that linear approximations also give at least (!) the rate $n^{(r-s-t)/d}$ in the nonregular case. An important difference, however, is the fact that now the Galerkin space may depend on the operator $\mathcal{A}$. Again we allow arbitrary s and $t > 0$ and arbitrary bounded Lipschitz domains. We also prove that nonlinear approximation methods do not yield a better rate of convergence.

Theorem 3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Assume that $r < s$ and $t, s > 0$. Let $S : H^{-s}(\Omega) \to H_0^s(\Omega)$ be an isomorphism, with no further assumptions. Then*

$$e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H^r(\Omega)) \prec n^{(r-s-t)/d}. \tag{15}$$

Proof. Consider first the identity (or embedding) $I : H^{-s+t}(\Omega) \to H^{-s}(\Omega)$. It is known that

$$e_n^{\mathrm{lin}}(I, H^{-s+t}(\Omega), H^{-s}(\Omega)) \asymp n^{-t/d}.$$

Again this is a classical result, for the general case (with $s, t > 0$ and Ω an arbitrary bounded Lipschitz domain), see [39].

By assumption we have that $S : H^{-s}(\Omega) \to H_0^s(\Omega)$ is an isomorphism, so that e_n^{lin} have the same order for I and for $S \circ I$, more exactly

$$e_n^{\mathrm{lin}}(I, H^{-s+t}(\Omega), H^{-s}(\Omega)) \asymp e_n^{\mathrm{lin}}(S \circ I, H^{-s+t}(\Omega), H_0^s(\Omega)) = e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H_0^s(\Omega))$$

$n \in \mathbb{N}$. Next we apply Lemma 1 and obtain

$$\begin{aligned} e_{2n}^{\mathrm{lin}}(I \circ S, H^{-s+t}(\Omega), H^r(\Omega)) &\le e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H_0^s(\Omega))\, e_n^{\mathrm{lin}}(I, H_0^s(\Omega), H^r(\Omega)) \\ &\le c\, n^{-t/d}\, n^{-(s-r)/d}, \qquad n \in \mathbb{N}, \end{aligned}$$

where c does not depend on n. □

It seems to be natural that nonregular problems should be at least as difficult as regular ones and hence we should always have

$$e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H^r(\Omega)) \asymp n^{(r-s-t)/d},$$

as in the regular case. However, this is in general *not* the case and one can construct (artificial) examples where, under the assumptions of Theorem 3, the sequence e_n^{lin} converges to zero arbitrarily fast.

Lemma 3. *Let $s, t > 0$ and assume $r < s$. Let $(\delta_n)_n$ be a sequence of positive numbers tending monotonically to zero. Then there exists a linear isomorphism $S : H^{-s}(\Omega) \to H_0^s(\Omega)$ such that*

$$e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H^r(\Omega)) \le \delta_n, \qquad n \in \mathbb{N}.$$

Proof. Let $F := H^{-s+t}(\Omega)$. As above we can write $S_F : H^{-s+t}(\Omega) \to H^r(\Omega)$ as $S_F = I_2 \circ S \circ I_1$ where $I_1 : H^{-s+t}(\Omega) \to H^{-s}(\Omega)$ and $I_2 : H^s(\Omega) \to H^r(\Omega)$, while S is the isomorphism from $H^{-s}(\Omega)$ to $H_0^s(\Omega)$. It follows that I_1 is of the form

$$I_1 : e_n \mapsto \sigma_n \tilde{e}_n, \quad \sigma_n \asymp n^{-t/d}$$

and

$$I_2 : e_n^* \mapsto \tilde{\sigma}_n \bar{e}_n, \quad \tilde{\sigma}_n \asymp n^{(r-s)/d}\,,$$

where the families $(e_n)_n$, $(\tilde{e}_n)_n$, $(e_n^*)_n$ and $(\bar{e}_n)_n$ are suitable complete orthonormal systems of the spaces H^{-s+t}, H^{-s}, H_0^s and H^r, respectively.

Now it is enough to consider those $S : H^{-s}(\Omega) \to H_0^s(\Omega)$ that are of the form

$$S : \tilde{e}_n \mapsto e^*_{\pi(n)},$$

where $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation. Then the singular values of $S_F = I_2 \circ S \circ I_1$ are given by $\sigma_n \cdot \tilde{\sigma}_{\pi(n)}$ and we can define π in such a way that the (ordered) values are smaller than the given sequence $(\delta_n)_n$ of positive numbers. $\square$

Remark 5. Corollary 1 is applicable in our situation, i.e., if we assume (3) and (7). Then we have $F = H^{-s+t}(\Omega)$ and $X = H^r(\Omega)$ and obtain

$$e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H^r(\Omega)) = e_n^{\mathrm{cont}}(S, H^{-s+t}(\Omega), H^r(\Omega)) \asymp e_{n,C}^{\mathrm{non}}(S, H^{-s+t}(\Omega), H^r(\Omega))\,, \tag{16}$$

i.e., linear approximation is as good as nonlinear approximation.

Theorem 3 yields also upper bounds for e_n^{cont} and $e_{n,C}^{\mathrm{non}}$, respectively. Lemma 3 makes clear that without further assumptions concerning S there is no hope for lower bounds, hence the bounds of Theorem 3 are not always optimal but we still have (16).

4 Elliptic Problems II

Now we are leaving the Hilbert space context. In contrast to Section 3 we allow now that our right-hand side f belongs to a Besov space $B_q^{-s+t}(L_p(\Omega))$ under certain restrictions on t. We consider the commutative diagram

$$\begin{array}{ccc} B_q^{-s+t}(L_p(\Omega)) & \xrightarrow{S_F} & H^r(\Omega) \\ {\scriptstyle I_1}\big\downarrow & & \big\uparrow{\scriptstyle I_2} \\ H^{-s}(\Omega) & \xrightarrow{S} & H_0^s(\Omega)\,, \end{array}$$

where $F := B_q^{-s+t}(L_p(\Omega))$ and I_1 and I_2 are identity operators. To make this diagram meaningful we need to have the continuity of the embeddings

$$B_q^{-s+t}(L_p(\Omega)) \hookrightarrow H^{-s}(\Omega) \qquad \text{and} \qquad H_0^s(\Omega) \hookrightarrow H^r(\Omega)\,,$$

respectively. This is guaranteed by

$$t > d \max\Big(0, \frac{1}{p} - \frac{1}{2}\Big) \qquad \text{and} \qquad s \geq r\,. \tag{17}$$

Theorem 4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathcal{A}$ as in* (3)*, $S = \mathcal{A}^{-1}$. Let $0 < p, q \leq \infty$, $s > 0$, and let* (17) *be satisfied. Then there exists a constant c such that*

$$e_n^{\text{lin}}(S, B_q^{-s+t}(L_p(\Omega)), H^r(\Omega)) \leq c \begin{cases} n^{-\frac{t+s-r}{d}} & \text{if} \quad 2 \leq p \leq \infty\,, \\ n^{-\frac{t+s-r}{d}+1/p-1/2} & \text{if} \quad 0 < p < 2\,, \end{cases}$$

as well as

$$e_n^{\text{cont}}(S, B_q^{-s+t}(L_p(\Omega)), H^r(\Omega)) \leq c\, n^{-\frac{t+s-r}{d}}$$

holds for all $n \in \mathbb{N}$.

Proof. Obviously, $S_F = I_2 \circ S \circ I_1$. The linear widths as well as the manifold widths of $S \circ I_1$ have been estimated in [10]. The multiplicativity of these numbers, see Lemma 1, and

$$e_n^{\text{lin}}(I_1, H_0^s(\Omega), H^r(\Omega)) \asymp n^{(r-s)/d}$$

as well as

$$e_n^{\text{cont}}(I_1, H_0^s(\Omega), H^r(\Omega)) \asymp n^{(r-s)/d}\,,$$

see Lemma 1, yield the claim. $\square$

Remark 6. Since $r < s$ and $B_q^{-s+t}(L_p(\Omega)) \hookrightarrow H^{-s}(\Omega)$ is compact Theorem 1 yields

$$e_{4n+1}^{\text{cont}}(S, B_q^{-s+t}(L_p(\Omega)), H^r(\Omega)) \leq c\, e_{n,C}^{\text{non}}(S, B_q^{-s+t}(L_p(\Omega)), H^r(\Omega))\,,$$

$n \in \mathbb{N}$. This can be complemented by the obvious inequality $e_{n,C}^{\text{non}} \leq e_n^{\text{lin}}$.

5 The Poisson Equation

In this section we discuss our results for the specific case of the Poisson equation

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{18}$$

on a bounded Lipschitz domain Ω contained in $\mathbb{R}^d$, $d \geq 2$. Here, as always in this paper, we understand Lipschitz domain in the sense of Stein's notion of domains with minimal smooth boundary, cf. [34, VI.3].
In the particular situation of the Poisson problem the scale $H^s_0(\Omega)$ (defined to be the closure of the test functions in Ω with respect to the norm $\|\cdot\|_{H^s(\Omega)}$), is not the correct one, at least in general. Here we are forced to work with the scale

$$H^s_{\partial\Omega}(\Omega) := \Big\{ u \in H^s(\Omega) : \quad \mathrm{tr}\, u = 0 \Big\}\,, \tag{19}$$

where tr means the trace with respect to $\partial\Omega$. In such a generality the definition of the trace needs some care. Here we follow [25] and [38], see also [24]. First we associate to $u \in H^s(\Omega)$ a function $\mathcal{E}u \in H^s(\mathbb{R}^d)$, an extension of u, and afterwards we take the restriction of $\mathcal{E}u$ to the boundary $\partial\Omega$. The technical details of this procedure, even in a more general context, are explained, e.g., in [25, pp. 205-209], [38, 9.1] or [41, 5.1.1]. For Lipschitz domains Ω the boundary $\partial\Omega$ is a so-called d-set with $d = n-1$. It turns out that this procedure is reasonable if $s > 1/2$. Similarly, if $s > 1/p$, one defines the more general scales $B^s_{q,\partial\Omega}(L_p(\Omega))$ and $H^s_{p,\partial\Omega}(\Omega)$, respectively. Here, by $H^s_p(\Omega)$ we denote the classical Bessel potential spaces, for their definition and basic properties we refer to [37] and [24].
Concerning the relations between the two scales $H^s_0(\Omega)$ and $H^s_{\partial\Omega}(\Omega)$ we remark the following. Obviously, we always have

$$H^s_0(\Omega) \hookrightarrow H^s_{\partial\Omega}(\Omega)\,.$$

Under more restrictive conditions we even have equality. Let $\mathring{H}^s_p(\Omega)$ and $\mathring{B}^{s+1/p}_q(L_p(\Omega))$ denote the closure of the test functions in Ω with respect to the corresponding norms.

Proposition 2. *Let Ω be a bounded Lipschitz domain. Let $1 < p < \infty$, $1 \leq q < \infty$ and $0 < s < 1$. Then*

$$\mathring{H}^{s+1/p}_p(\Omega) = H^{s+1/p}_{p,\partial\Omega}(\Omega) \qquad \text{and} \qquad \mathring{B}^{s+1/p}_q(L_p(\Omega)) = B^{s+1/p}_{q,\partial\Omega}(L_p(\Omega))\,. \tag{20}$$

Remark 7. (i) As a consequence of Proposition 2 we obtain in case $s \geq 1$

$$H^s_{p,\partial\Omega}(\Omega) = H^s_p(\Omega) \cap \mathring{H}^1_p(\Omega)\,,$$

which means for $p = 2$

$$H^s_{\partial\Omega}(\Omega) = H^s_{2,\partial\Omega}(\Omega) = H^s(\Omega) \cap H^1_0(\Omega)\,,$$

(all to be understood in the sense of equivalent norms).
(ii) A proof of the above proposition may be found in [38, Prop. 19.5]. However, it is based on some results of Netrusov, see [1, Sect. 10] and the references given there. For smooth domains we refer to [16] and [41, Thm. 5.21].

5.1 The Poisson Equation in Smooth Domains

For a better understanding and for later use we first consider the Poisson problem in C^∞ domains. The following is an extension of Lemma 2.

Theorem 5. *Let Ω be a bounded C^∞-domain.*
(i) *Let $t > -1/2$ and $r < t+1$. Then for the associated Poisson problem* (18) *it holds: the mapping $\mathcal{A}: \ u \to f$ is a linear isomorphism of $H^{t+1}_{\partial\Omega}(\Omega)$ onto $H^{t-1}(\Omega)$. Furthermore, with $S = \mathcal{A}^{-1}$ and $C \geq 1$ arbitrary we have*

$$\begin{aligned} e_n^{\mathrm{lin}}(S, H^{t-1}(\Omega), H^r(\Omega)) &= e_n^{\mathrm{cont}}(S, H^{t-1}(\Omega), H^r(\Omega)) \\ &\asymp e_{n,C}^{\mathrm{non}}(S, H^{t-1}(\Omega), H^r(\Omega)) \asymp n^{(r-t-1)/d} \,. \end{aligned}$$

(ii) *Let $0 < p, q \leq \infty$,*

$$t > \frac{1}{p} - 1 + (d-1)\, \max\Big(0, \frac{1}{p} - 1\Big) \tag{21}$$

and

$$r < t + 1 - d\, \max\Big(0, \frac{1}{p} - \frac{1}{2}\Big)\,. \tag{22}$$

Then for the solution operator S of the Poisson problem (18) *it holds: the mapping $\mathcal{A}: \ u \to f$ is a linear isomorphism of $B^{t+1}_{q,\partial\Omega}(L_p(\Omega))$ onto $B^{t-1}_q(L_p(\Omega))$. Furthermore,*

$$e_n^{\mathrm{lin}}(S, B^{t-1}_q(L_p(\Omega)), H^r(\Omega)) \asymp \begin{cases} n^{(r-t-1)/d+1/p-1/2} & \textit{if} \quad 0 < p \leq 2\,, \\ n^{(r-t-1)/d} & \textit{if} \quad 2 \leq p \leq \infty\,, \end{cases}$$

as well as

$$e_n^{\mathrm{cont}}(S, B^{t-1}_q(L_p(\Omega)), H^r(\Omega)) \asymp n^{(r-t-1)/d}\,.$$

Proof. The fact that the mapping $\mathcal{A}: \ u \to f$ is a linear isomorphism under the given restrictions is well-known in the literature. We refer to [32, Thm. 3.5.3], but see also [18], [37, 4.3.3] and the references given there.
The rest of the proof is oriented on that one given for Theorem 2. Under the restriction (22)

$$e_n^{\mathrm{lin}}(I, B^{t+1}_q(L_p(\Omega)), H^r(\Omega)) \asymp \begin{cases} n^{(r-t-1)/d+1/p-1/2} & \text{if} \quad 0 < p \leq 2\,, \\ n^{(r-t-1)/d} & \text{if} \quad 2 \leq p \leq \infty\,, \end{cases}$$

see e.g. [39]. We obtain the same order for $I : \mathring{B}^{t+1}_q(L_p(\Omega)) \to H^r(\Omega)$ by reasonings as used in proof of Theorem 2. Because of

$$\mathring{B}^{t+1}_q(L_p(\Omega)) \hookrightarrow B^{t+1}_{q,\partial\Omega}(L_p(\Omega)) \hookrightarrow B^{t+1}_q(L_p(\Omega))$$

we also obtain the same order with respect to $I : B^{t+1}_{q,\partial\Omega}(L_p(\Omega)) \to H^r(\Omega)$. Since $S : B^{t-1}_q(L_p(\Omega)) \to B^{t+1}_{q,\partial\Omega}(L_p(\Omega))$ is an isomorphic map we conclude that the order of $e^{\rm lin}_n$ for I and for $I \circ S_F$ coincide. The arguments with respect to $e^{\rm cont}_n$ are the same starting with

$$e^{\rm cont}_n(I, B^{t+1}_q(L_p(\Omega)), H^r(\Omega)) \asymp n^{(r-t-1)/d}\,,$$

see [10]. It follows that this remains true if $B^{t+1}_q(L_p(\Omega))$ is replaced by $B^{t+1}_{q,\partial\Omega}(L_p(\Omega))$. This proves (ii). Part (i) can be proved in the same way. In addition one has to use Corollary 1 to derive the behaviour of $e^{\rm non}_{n,C}$. □

We also have a supplement to part (ii) of Theorem 5. For this we need an additional restriction concerning the admissible domains Ω.

Theorem 6. *Let $0 < p, q \le \infty$, and t, r as in* (21), (22). *Let Ω be a bounded C^∞-domain such that $B^{t+1}_q(L_p(\Omega))$ as well as $H^r(\Omega)$ can be discretized by one common wavelet system $\mathcal{B}^*$ belonging to $\mathcal{B}_{C^*}$ for some $1 \le C^* < \infty$. Then, if $C \ge C^*$,*

$$e^{\rm non}_{n,C}(S, B^{t-1}_q(L_p(\Omega)), H^r(\Omega)) \asymp n^{(r-t-1)/d}\,.$$

Proof. Theorem 1 and Theorem 5 yield the lower bound. The upper bound follows by applying the same arguments as in proof of Theorem 5 together with [10, Lem. 3, Thm. 10]. □

Remark 8. (i) In all the assertions in this subsection where Besov spaces are involved the microscopic parameter q does not play an important role. Only the constants behind $\asymp$ depend on it. As a consequence of simple continuous embeddings we obtain

$$e^{\rm lin}_n(S, F^{t-1}_q(L_p(\Omega)), H^r(\Omega)) \asymp \begin{cases} n^{(r-t-1)/d+1/p-1/2} & \text{if} \quad 0 < p \le 2\,, \\ n^{(r-t-1)/d} & \text{if} \quad 2 \le p \le \infty\,, \end{cases}$$

as well as

$$e^{\rm non}_{n,C}(S, F^{t-1}_q(L_p(\Omega)), H^r(\Omega)) \asymp e^{\rm cont}_n(S, F^{t-1}_q(L_p(\Omega)), H^r(\Omega)) \asymp n^{(r-t-1)/d}\,,$$

where $F^t_q(L_p(\Omega))$ denotes the Triebel-Lizorkin space. Specialization to $q = 2$ and restriction to $1 < p < \infty$ leads to corresponding results for the scale of Bessel potential spaces $H^t_p(\Omega)$ since $H^t_p(\Omega) = F^t_2(L_p(\Omega))$ in the sense of equivalent norms.
(ii) Based on the regularity theory for the Poisson equation in the framework of Besov-Lizorkin-Triebel spaces in [18] and [37, Chapt. 4], see also [32, Chapt. 3], one can extend the results from Theorem 5 also to the situation where $H^r(\Omega)$ is replaced by a general Besov space. But this would be restricted to the quantities $e^{\rm lin}_n$ and $e^{\rm cont}_n$.

5.2 The Poisson Equation in Lipschitz Domains

It is a classical assertion that (18) also in the context of Lipschitz domains fits into our setting with $s = 1$. Indeed, if we consider the weak formulation of this problem, it can be checked that (18) induces a boundedly invertible operator $\mathcal{A} = \triangle : \ H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$, see [20, Chapter 7.2] for details. However, in this specific situation much more can be said.

5.2.1 Estimates from Below

To obtain estimates from below we make a comparison with the Poisson problem for C^∞ domains.

Lemma 4. *Let Ω be a bounded Lipschitz domain and let $C \geq 1$. Then, under the same conditions as in Theorem 5*

$$e_n^{\mathrm{lin}}(S, B_q^{t-1}(L_p(\Omega)), H^r(\Omega)) \geq c \begin{cases} n^{(r-t-1)/d+1/p-1/2} & \text{if} \quad 0 < p \leq 2\,, \\ n^{(r-t-1)/d} & \text{if} \quad 2 \leq p \leq \infty\,, \end{cases} \tag{23}$$

as well as

$$e_n^{\mathrm{cont}}(S, B_q^{t-1}(L_p(\Omega)), H^r(\Omega)) \geq c\, n^{(r-t-1)/d} \tag{24}$$

and

$$e_{n,C}^{\mathrm{non}}(S, B_q^{t-1}(L_p(\Omega)), H^r(\Omega)) \geq c\, n^{(r-t-1)/d} \tag{25}$$

for some positive c independent of $n \in \mathbb{N}$.

Proof. Let $B \subset \Omega$ be a ball such that $\operatorname{dist}(B, \partial\Omega) = a > 0$. Furthermore, let $\mathcal{E} : \ B_q^{t-1}(L_p(B)) \to B_q^{t-1}(L_p(\mathbb{R}^d))$ be a linear and continuous extension operator, see e.g. [33]. Let $\widetilde{B} := \{x \in \mathbb{R}^d : \ \operatorname{dist}(x, B) < a/2\}$ and let $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi(x) = 1$ if $x \in B$ and $\psi(x) = 0$ if $x \notin \widetilde{B}$. If u is the solution of (18) on B with right-hand side $\widetilde{f} \in B_q^{t-1}(L_p(B))$ then the function $h := \psi \cdot \mathcal{E}u$ solves the Poisson problem

$$-\Delta h = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad h = 0 \quad \text{on} \quad \partial\Omega$$

with some f. Of course, since $h = \psi \cdot \mathcal{E}u = u$ on B we obtain by using the Poisson equation

$$\widetilde{f} = f_{|B}\,.$$

To derive $f \in B_q^{t-1}(L_p(\Omega))$ we apply the elliptic regularity theory with respect to B, see the previous subsection. This implies $u \in B_q^{t+1}(L_p(B))$, hence $\psi\, \mathcal{E}u \in B_q^{t+1}(L_p(\mathbb{R}^d))$ since ψ is a pointwise multiplier $for B_q^{t+1}(L_p(\mathbb{R}^d))$, see e.g. [32, 4.7.1] or [37, 2.8.2]. Consequently f belongs to $B_q^{t-1}(L_p(\Omega))$. Let S_Ω denote the solution

operator with respect to our Poisson problem on Ω and similarly S_B with respect to B. Now we turn to the approximation of these operators.
Step 1. Let $S_n : B^{t-1}_q(L_p(\Omega)) \to H^r(\Omega)$ be an element of $\mathcal{L}_n$, see the Appendix for a definition of the set $\mathcal{L}_n$. Hence

$$S_n f = \sum_{i=1}^n L_i(f)\, h_i\,,$$

where $h_i \in H^r(\Omega)$ and the L_i are linear functionals defined on $B^{t-1}_q(L_p(\Omega))$. Then we define its restriction $\widetilde{S_n}$ as

$$\widetilde{S}_n \widetilde{f} := \sum_{i=1}^n L_i\Big(-\Delta(\psi\, \mathcal{E}(S_B \widetilde{f}))\Big)\, h_{i_{|B}}\,.$$

By construction $(S_n f)_{|B} = \widetilde{S}_n \widetilde{f}$. Furthermore, $\widetilde{S}_n$ belongs to the same class as S_n itself (of course, with respect to the new pair $(B^{t-1}_q(L_p(B)), H^r(B))$. Because of $S_\Omega f_{|B} = u = S_B \widetilde{f}$ we find

$$(S_\Omega f - S_n f)_{|B} = S_B \widetilde{f} - \widetilde{S}_n \widetilde{f}\,.$$

From this we conclude

$$\begin{aligned}
\|(S_\Omega f - S_n f)_{|B}\, |H^r(B)\| &= \inf\Big\{ \|g|H^r(\mathbb{R}^d)\| :\ g \in H^r(\mathbb{R}^d)\,,\ g_{|B} = (S_\Omega f - S_n f)_{|B} \Big\} \\
&\le \inf\Big\{ \|g|H^r(\mathbb{R}^d)\| :\ g \in H^r(\mathbb{R}^d)\,,\ g_{|\Omega} = S_\Omega f - S_n f \Big\} \\
&= \|S_\Omega f - S_n\, |H^r(\Omega)\|\,.
\end{aligned}$$

But this implies

$$e_n^{\mathrm{lin}}(S, B^{t-1}_q(L_p(B)), H^r(B)) \le e_n^{\mathrm{lin}}(S, B^{t-1}_q(L_p(\Omega)), H^r(\Omega))\,.$$

Step 2. Let $S_n : B^{t-1}_q(L_p(\Omega)) \to H^r(\Omega)$ be an element of $\mathcal{C}_n$, see the Appendix for a definition of the set $\mathcal{C}_n$. Then $S_n f = \varphi(N_n(f))$, where $N_n :\ B^{t-1}_q(L_p(\Omega)) \to \mathbb{R}^n$ and $\varphi :\ \mathbb{R}^n \to H^r(\Omega)$ are continuous mappings. We define

$$\widetilde{S}_n \widetilde{f} := \varphi\Big(N_n\Big(-\Delta(\psi\, \mathcal{E}(S_B \widetilde{f}))\Big)\Big)_{|B}\,.$$

With

$$\widetilde{\varphi} := \varphi_{|B} \qquad \text{and} \qquad \widetilde{N}_n := N_n\Big(-\Delta(\psi\, \mathcal{E}(S_B\, \cdot\,))$$

we immediately see that $\widetilde{S}_n = \widetilde{\varphi} \circ \widetilde{N}_n \in \mathcal{C}_n$. As above it follows $(S_n f)_{|B} = \widetilde{S}_n \widetilde{f}$. Now we may argue as in Step 1.
Next we apply Theorem 5. This proves (23) as well as (24). Finally, we employ Theorem 1 and (24) to conclude (25). $\square$

5.2.2 Estimates from Above

It is well-known that the solution operator S is an isomorphism of $H^{t-1}(\Omega)$ onto $H_0^{t+1}(\Omega)$ as long as $-1/2 < t < 1/2$, see [24, Thm. 0.5] and Proposition 2. We consider the commutative diagram

$$\begin{array}{ccc} B_q^{t-1}(L_p(\Omega)) & \xrightarrow{S_F} & H^r(\Omega) \\ {\scriptstyle I_1}\big\downarrow & & \big\uparrow{\scriptstyle I_2} \\ H^{t-u-1}(\Omega) & \xrightarrow{S} & H_0^{t+1-u}(\Omega)\,, \end{array}$$

where $F := B_q^{t-1}(L_p(\Omega))$ and I_1 and I_2 are identity operators. This diagram becomes meaningful if

$$u > d\,\max\Big(0, \frac{1}{p} - \frac{1}{2}\Big) \qquad \text{and} \qquad r < t + 1 - u\,, \tag{26}$$

since these two inequalities are guaranteeing the embeddings $B_q^{u+t-1}(L_p(\Omega)) \hookrightarrow H^{t-1-u}(\Omega)$ and $H_0^{t+1-u}(\Omega) \hookrightarrow H^r(\Omega)$.

In the next theorem we need an u that satisfies (26) as well as $-1/2 < t-u < 1/2$. Such an u clearly exists if

$$\max\Big(t - \frac{1}{2}, d\,\max\Big(0, \frac{1}{p} - \frac{1}{2}\Big)\Big) < \min\Big(t + \frac{1}{2}, t + 1 - r\Big). \tag{27}$$

Theorem 7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $0 < p, q \le \infty$, and let (27) be satisfied.*
(i) *Then*

$$e_n^{\mathrm{lin}}(S, B_q^{t-1}(L_p(\Omega)), H^r(\Omega)) \asymp \begin{cases} n^{-(t+1-r)/d} & \text{if} \quad 2 \le p \le \infty\,, \\ n^{-(t+1-r)/d+1/p-1/2} & \text{if} \quad 0 < p < 2\,, \end{cases} \tag{28}$$

as well as

$$e_n^{\mathrm{cont}}(S, B_q^{t-1}(L_p(\Omega)), H^r(\Omega)) \asymp n^{-(t+1-r)/d}\,. \tag{29}$$

(ii) *If in addition $2 \le p \le \infty$, then we also have*

$$e_{n,C}^{\mathrm{non}}(S, B_q^{t-1}(L_p(\Omega)), H^r(\Omega)) \asymp n^{-(t+1-r)/d}\,, \tag{30}$$

for all $C \ge 1$.

Proof. In part (i) the estimates from above are consequences of the multiplicativity of e_n^{lin} and e_n^{cont}, see Lemma 1. The estimates from below in part (i) are consequences of Lemma 4. Finally, part (ii) follows from Theorem 1 and the obvious inequality $e_{n,C}^{\mathrm{non}} \le e_n^{\mathrm{lin}}$. □

Here is another variant of Theorem 7 but restricted to Bessel potential spaces. Our point of departure is the following fundamental result of Jerison and Kenig, see [24, Thm. 1.1]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$. There exists μ, $0 < \mu \leq 1$, depending only on the Lipschitz character of Ω such that for every $f \in H_p^{t-1}(\Omega)$ there is a unique solution $u \in \mathring{H}_p^{t+1}(\Omega)$ to the Poisson problem (18) provided the pair $(t+1, 1/p)$ belongs to the open hexagon H_μ. Here $(t+1, 1/p) \in H_\mu$ if one of the following holds:

(a) $\quad p_0 < p < p_0'$ and $\frac{1}{p} < t+1 < 1 + 1/p$;

(b) $\quad 1 < p \leq p_0$ and $\frac{3}{p} - 1 - \mu < t+1 < 1 + \frac{1}{p}$;

(c) $\quad p_0' \leq p < \infty$ and $\frac{1}{p} < t+1 < \frac{3}{p} + \mu$,

see also Figure 1 below. The value of p_0 is fixed by

$$\frac{1}{p_0} := \frac{1}{2} + \frac{\mu}{2}\,.$$

Moreover, the estimate

$$\| \, u \, | H_p^{t+1}(\Omega) \| \leq c \, \| \, f \, | H_p^{t-1}(\Omega) \|$$

holds with c independent of f. A similar result holds true with $d = 2$, see [24, Thm. 1.3], but with a different definition of H_μ. For $d = 2$ and $0 < \mu \leq 1/2$ the set H_μ is defined to be the collection of all pairs $(t+1, 1/p)$ such that one of the following holds:

(a) $\quad p_0 < p < p_0'$ and $\frac{1}{p} < t+1 < 1 + 1/p$;

(b) $\quad 1 < p \leq p_0$ and $\frac{2}{p} - \frac{1}{2} - \mu < t+1 < 1 + \frac{1}{p}$;

(c) $\quad p_0' \leq p < \infty$ and $\frac{1}{p} < t+1 < \frac{2}{p} + \frac{1}{2} + \mu$

and the value of p_0 is fixed by $\frac{1}{p_0} := \frac{1}{2} + \mu$.

We wish to add two comments. The first one concerns μ in case of C^1 domains. In both cases ($d \geq 3$ as well as $d = 2$) p_0 may be chosen to be 1, see [24, Thm. 1.1, Thm. 1.3]. The second one concerns the spaces $\mathring{H}_p^{t+1}(\Omega)$. It is easily seen that the hexagon H_μ is a subset of the strip

$$\{(s, 1/p) : \quad 1 < p < \infty, \; 1/p < s < 1 + 1/p\}\,.$$

Hence $\mathring{H}_p^{t+1}(\Omega) = H_{p,\partial\Omega}^{t+1}(\Omega)$, see Proposition 2.

Since our mapping $\mathcal{A}$ is an isomorphism under the given restriction we can apply the same type of arguments as in proof of Theorem 5.

Theorem 8. *Let Ω be a bounded Lipschitz domain.*
(i) *Suppose $(t+1, 1/2) \in H_\mu$. Then, with $S := \mathcal{A}^{-1}$, and $C \geq 1$ arbitrary we have*

$$\begin{aligned} e_n^{\mathrm{lin}}(S, H^{t-1}(\Omega), H^r(\Omega)) &= e_n^{\mathrm{cont}}(S, H^{t-1}(\Omega), H^r(\Omega)) \\ &\asymp e_n^{\mathrm{non}}(S, H^{t-1}(\Omega), H^r(\Omega)) \asymp n^{(r-t-1)/d} . \end{aligned}$$

(ii) *Let $0 < p, q \leq \infty$. Suppose $(t+1, 1/p) \in H_\mu$ and*

$$r < t + 1 - d \, \max(0, \frac{1}{p} - \frac{1}{2}) . \tag{31}$$

Then for the solution operator S of the Poisson problem (18) *it holds*

$$e_n^{\mathrm{lin}}(S, H_p^{t-1}(\Omega)), H^r(\Omega)) \asymp \begin{cases} n^{(r-t-1)/d+1/p-1/2} & \textit{if} \quad 0 < p \leq 2 , \\ n^{(r-t-1)/d} & \textit{if} \quad 2 \leq p \leq \infty , \end{cases}$$

as well as

$$e_n^{\mathrm{cont}}(S, H_p^{t-1}(\Omega), H^r(\Omega)) \asymp n^{(r-t-1)/d} .$$

Proof. Again we use the factorization of $S : \ H_p^{t-1}(\Omega) \to H_0^r(\Omega)$ into $S : \ H_p^{t-1}(\Omega) \to H_{p,\partial\Omega}^{t+1}(\Omega) = \mathring{H}_p^{t+1}(\Omega)$ and $I : \ \mathring{H}_p^{t+1}(\Omega) \to H_0^r(\Omega)$. The result follows from

$$e_n^{\mathrm{lin}}(I, \mathring{H}_p^{t+1}(\Omega), H^r(\Omega)) \asymp \begin{cases} n^{(r-t-1)/d+1/p-1/2} & \text{if} \quad 1 < p \leq 2 , \\ n^{(r-t-1)/d} & \text{if} \quad 2 \leq p < \infty , \end{cases}$$

and

$$e_n^{\mathrm{cont}}(I, \mathring{H}_p^{t+1}(\Omega), H^r(\Omega)) \asymp n^{(r-t-1)/d} .$$

For spaces without the $\circ$ on the top these estimates can be found e.g. in [39] and [10]. The result with $\circ$ can be proved by considering the spaces defined on a ball contained in Ω (estimate from below). The estimate from above is obvious. $\square$

Again we also have a supplement to part (ii) of Theorem 5. As above we need an additional restriction concerning the admissible domains Ω.

Theorem 9. *Let Ω be a bounded Lipschitz domain. Suppose $(t+1, 1/p) \in H_\mu$ and r as in* (31). *Furthermore, we assume that $B_q^{t+1}(L_p(\Omega))$ as well as $H^r(\Omega)$ can be discretized by one common wavelet system $\mathcal{B}^*$ belonging to $\mathcal{B}_{C^*}$ for some $1 \leq C^* < \infty$. Then, if $C \geq C^*$,*

$$e_{n,C}^{\mathrm{non}}(S, H_p^{t-1}(\Omega), H^r(\Omega)) \asymp n^{(r-t-1)/d} .$$

Proof. Theorem 1 and Theorem 8 yield the lower bound. The upper bound follows by applying the same arguments as in proof of Theorem 5 together with [10, Lem. 3, Thm. 10] and

$$B_1^{t+1}(L_p(\Omega)) \hookrightarrow H_p^{t-1}(\Omega) \hookrightarrow B_\infty^{t+1}(L_p(\Omega)),$$

see [37, 2.3.2, 2.5.6]. □

Remark 9. It seems to be easier to characterize Besov spaces by wavelets instead of Bessel potential spaces with $p \neq 2$. For that reason we formulated Theorem 9 by using Besov spaces as well.

5.3 Best n-term Wavelet Approximation of the Solution of the Poisson Equation

For non-smooth domains optimal Galerkin spaces may depend on the operator $\mathcal{A}$. This is inconvenient. In this subsection we will investigate the approximation power of best n-term approximation with respect to one fixed wavelet system. This does not mean that we have an algorithm realizing this order of approximation. A few further remarks will be given at the end of this subsection.

5.3.1 Besov Regularity of the Solution of the Poisson Equation

First we investigate additional regularity properties of the solution of the Poisson equation with respect to Besov spaces with small p, sometimes called Besov regularity of the solution.

It makes sense to decompose the $(\alpha, 1/q)$-plane in dependence of the regularity of the right-hand side f. We concentrate on the case $d \geq 3$. For given Ω the associated hexagon H_μ is given by the following collection of points $ABCDEF$:

$$\begin{aligned} A &:= (0,0), \quad B := (1/p_0, 1/p_0), \quad C := (1, 2-\mu), \\ D &:= (1,1), \quad E := (1/p_0', 1 + 1/p_0'), \quad F := (0, \mu), \end{aligned}$$

see Subsection 5.2.2.

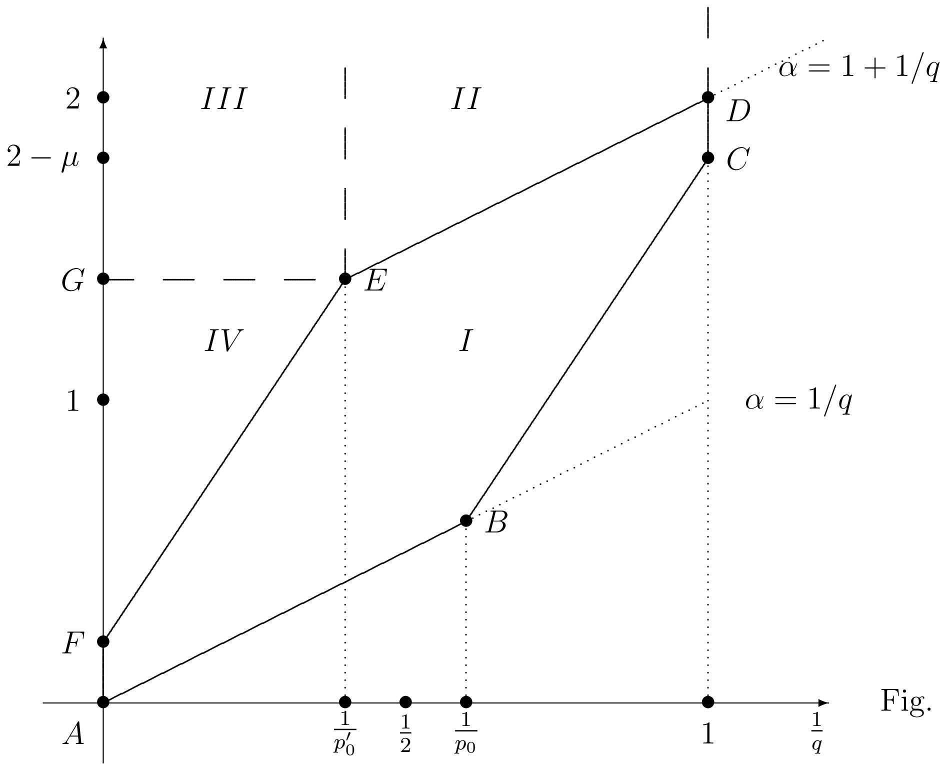

Fig. 1

We shall decompose our considerations into four cases indicated by the regions $I - IV$ in Figure 1. The starting point will be always the regularity of the right-hand side $f \in H_p^{t-1}(\Omega)$. However, the hexagon reflects the regularity of the solution. This means we consider

- Case I: the pair $(t+1, 1/p) \in H_\mu = I$;
- Case II: the pair $(t+1, 1/p) \in II$;
- Case III: the pair $(t+1, 1/p) \in III$;
- Case IV: the pair $(t+1, 1/p) \in IV$.

The simplest case is Case I. Then Theorem 1.1 in [24] and the chain of continuous embeddings

$$u \in H_p^{t+1}(\Omega) \hookrightarrow B_\infty^{t+1}(L_\tau(\Omega)) \hookrightarrow B_\tau^{t+1-\varepsilon}(L_\tau(\Omega)), \qquad 0 < \tau \leq p, \quad \varepsilon > 0,$$

yield the following.

Lemma 5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and H_μ the associated hexagon. Let $\varepsilon > 0$. Then the solution u of the Poisson problem* (18) *with right-hand side $f \in H_p^{t-1}(\Omega)$, $(t+1, 1/p) \in H_\mu$, belongs to all spaces $B_\tau^{\alpha-\varepsilon}(L_\tau(\Omega))$, where $\alpha \leq t+1$ and $0 < \tau \leq p$.*

More interesting and more complicated is the situation with respect to the other regions. The most interesting Case II has been investigated in [7, Thm. 4.1]. Cases III and IV can be reduced to Case II and Case I, respectively, by using obvious embeddings (monotonicity of $H^{t-1}_p(\Omega)$ with respect to p).

Lemma 6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let H_μ denote the associated hexagon and p_0 the specific number occurring in the definition of H_μ. Let $\varepsilon > 0$.*
(i) *Case II. Let $1 < p < p_0'$ and let $t \geq 1/p$. Then the solution u of the Poisson problem* (18) *with right-hand side $f \in H^{t-1}_p(\Omega)$ belongs to all spaces $B^{\alpha-\varepsilon}_\tau(L_\tau(\Omega))$, where*

$$\begin{aligned}(\alpha, 1/\tau) \;\; \in \;\; & \Big(\Big\{(\beta, 1/q) : \; \beta \leq \min(t+1, 1+1/q)\,, \quad \frac{d-1}{d+1} < q \leq p\Big\} \\ & \cup \;\; \Big\{(\beta, 1/q) : \; \beta \leq \min\Big(t+1, \frac{2d}{d-1}\Big)\,, \quad 0 < q \leq \frac{d-1}{d+1}\Big\}\Big)\,.\end{aligned}$$

(ii) *Case III. Let $d \geq 3$. Let $p_0' \leq p < \infty$ and suppose*

$$t+1 \geq 1 + \frac{1}{p_0'}\,.$$

Then the solution u of the Poisson problem (18) *with right-hand side $f \in H^{t-1}_p(\Omega)$ belongs to all spaces $B^{\alpha-\varepsilon}_\tau(L_\tau(\Omega))$, where*

$$\begin{aligned}(\alpha, 1/\tau) \;\; \in \;\; & \Big(\Big\{(\beta, 1/q) : \; \beta \leq \min(t+1, 1+1/q)\,, \quad \frac{d-1}{d+1} < q \leq p_0'\Big\} \\ & \cup \;\; \Big\{(\beta, 1/q) : \; \beta \leq \max\Big(t+1, \frac{2d}{d-1}\Big)\,, \quad 0 < q \leq \frac{d-1}{d+1}\Big\}\Big)\,.\end{aligned}$$

(iii) *Case IV. Let $d \geq 3$. Let $p_0' \leq p < \infty$ and suppose*

$$1 + \frac{1}{p_0'} > t+1 \geq \frac{3-\mu}{1+\mu}\frac{1}{p} + \mu\,.$$

Then the solution u of the Poisson problem (18) *with right-hand side $f \in H^{t-1}_p(\Omega)$ belongs to all spaces $B^{\alpha-\varepsilon}_\tau(L_\tau(\Omega))$, where*

$$(\alpha, 1/\tau) \in \Big(\Big\{(\beta, 1/q) : \; \beta \leq t+1, \quad 0 < q \leq q^*\Big\}\,,$$

where

$$\frac{1}{q^*} := (t+1-\mu)\,\frac{1+\mu}{3-\mu}\,.$$

Remark 10. The upper boundary is always given by a polygon in the $(\alpha, 1/q)$-plane. We shall call this boundary *polygon of maximal regularity* of u.

5.3.2 Best n-term Approximation of the Solution of the Poisson Equation

The regularity information about u will be combined with the following result on best n-term wavelet approximation. Here we always consider a pair of spaces. The function u belongs to some Besov space $B^\beta_q(L_p(\Omega))$ and we want to approximate it with respect to the norm of the space $H^r(\Omega)$. As a general assumption we use that both spaces can be characterized by one common wavelet system $\mathcal{B}^*$, see [10, Section 5.10]. By assumption such a wavelet system belongs to $\mathcal{B}_{C^*}$ for some $1 \le C^* < \infty$. Sufficient conditions for certain special domains are known, we refer to [3, 41].

Proposition 3. *Let Ω and $\mathcal{B}^*$ be as above. Let $0 < \tau \le \infty$, $r \in \mathbb{R}$ and*

$$u > d \max\left(0, \frac{1}{\tau} - \frac{1}{2}\right).$$

Then we have

$$\sup\left\{\sigma_n(u, \mathcal{B}^*)_{H^r(\Omega)} : \quad \| u|B^{r+u}_\tau(L_\tau(\Omega))\| \le 1\right\} \asymp n^{-u/d}\,.$$

Remark 11. In this form the proposition is proved in [10] (recall $H^r(\Omega) = B^r_2(L_2(\Omega))$ in the sense of equivalent norms). Rather extended surveys on nonlinear approximation are [12] and [35, 36], but see also [28].

Our strategy consists in using Proposition 3 with

$$u = \varepsilon + d\left(\frac{1}{\tau} - \frac{1}{2}\right), \qquad \tau < 2\,,$$

and $\varepsilon > 0$ small. We consider the half-line

$$\left\{(\beta, 1/q) : \quad \beta = r + d\left(\frac{1}{q} - \frac{1}{2}\right), \quad 0 < q < 2\right\}.$$

This time we have to study where this half-line and the polygon of maximal regularity of u meet in a $(\beta, 1/q)$-plane, see Lemma 5, 6. There will be several different cases. To make the situation more transparent we only consider the case $p = 2$. However, all other cases can be treated in the same way. First we combine Proposition 3 and Lemma 5.

Case 1. Let $-1/2 < t < 1/2$, i.e. $(t+1, 1/2) \in H_\mu$. Let $r < t + 1$. Then the line $\beta = r + d(1/q - 1/2)$ meets the line $\beta = t + 1$ for some $q < 2$. This implies

$$\sup\left\{\sigma_n(u, \mathcal{B}^*)_{H^r(\Omega)} : \quad \| f|H^{t-1}(\Omega)\| \le 1\right\} \le c\, n^{-\frac{t+1-r}{d}+\varepsilon}\,,$$

for some c independent of n.

Case 2. We continue with $t \ge 1/2$. Lemma 6 (i) tells us that in case $p = q = 2$ the regularity of u in limited by $3/2$ in the H^u-scale. However, in [24] is proved, that for any $u > 3/2$ there exists a Lipschitz domain Ω such that $f \in C^\infty(\overline{\Omega})$ but $u \notin H^u(\Omega)$. So we restrict us to $r < 3/2$. We need some further decompositions. First we treat t large.

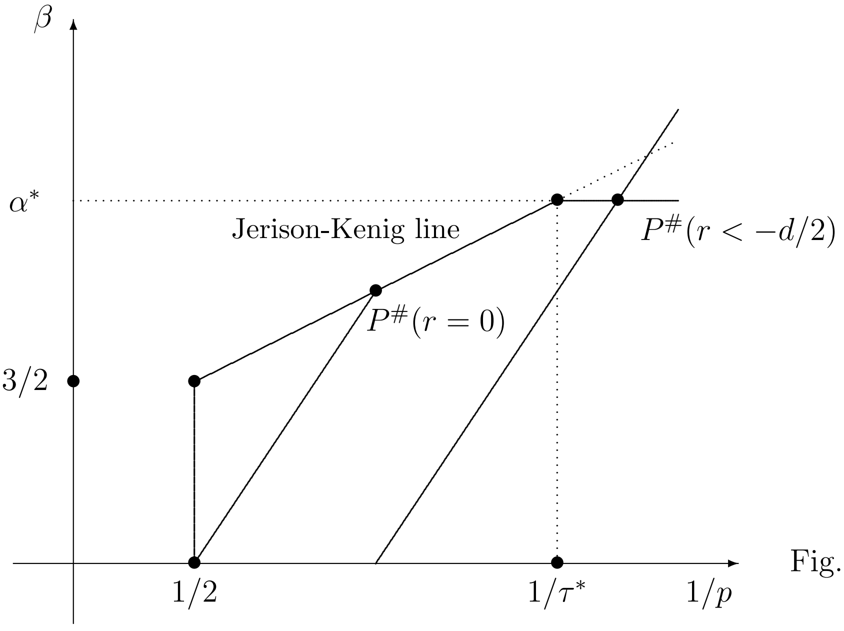

Fig. 2

Case 2.1. Let $t > 1 + 2/(d-1)$, i.e., $t + 1 > 2d/(d-1)$. This means the polygon of maximal regularity is given by

$$\beta = \begin{cases} 1 + 1/q & \text{if } \frac{d-1}{d+1} \le q \le 2\,, \\ \frac{2d}{d-1} & \text{if } 0 < q \le \frac{d-1}{d+1}, \end{cases}$$

see Lemma 6(ii). Let

$$P^* := (\alpha^*, 1/\tau^*)\,, \qquad \alpha^* := \frac{2d}{d-1}\,, \quad \frac{1}{\tau^*} := \frac{d+1}{d-1}\,.$$

At this point the Jerison-Kenig line $\beta = 1 + 1/q$, $0 < q < \infty$, and the line $\beta = d(\frac{1}{q} - 1)$, $0 < q < \infty$, intersect. Next we study the intersection of the half-line $\beta = r + d(1/q - 1/2)$, $0 < q < 2$, with the Jerison-Kenig line $\beta = 1 + 1/q$. The cross $P^\# = (\alpha, 1/\tau)$ has the coordinates

$$\alpha := \frac{\frac{3}{2}d - r}{d-1} \qquad \text{and} \qquad \frac{1}{\tau} := \frac{1 - r + d/2}{d-1}\,.$$

Case 2.1.1. If $-d/2 \le r < 3/2$ the point $P^\#$ is located to the left of P^*. By means of Proposition 3 it follows

$$\sup\Big\{\sigma_n(u, \mathcal{B}^*)_{H^r(\Omega)} \;:\quad \| f | H^{t-1}(\Omega)\| \le 1\Big\} \le c\, n^{-\frac{\frac{3}{2}-r}{d-1}+\varepsilon}\,.$$

In Figure 2 above we have plotted the case $r = 0$ (Case 2.1.1) and $r < -d/2$ (Case 2.1.2).
Case 2.1.2. Next we consider the case $-\infty < r < -d/2$. Now $P^\#$ is located to the right of P^*. In this situation Proposition 3 yields

$$\sup\Big\{\sigma_n(u,\mathcal{B}^*)_{H^r(\Omega)}\ :\quad \| f|H^{t-1}(\Omega)\| \le 1\Big\} \le c\, n^{-\frac{2}{d-1}+\frac{r}{d}+\varepsilon}\,.$$

Case 3. Let $1/2 \le t \le 1 + 2/(d-1)$. This time the polygon of maximal regularity is given by

$$\beta = \begin{cases} 1 + 1/q & \text{if } \ \frac{1}{t} \le q \le 2\,, \\ t+1 & \text{if } \ 0 < q \le \frac{1}{t}, \end{cases}$$

see Lemma 6 (ii). The line $\beta = t + 1$ meets the Jerison-Kenig line at the point $(t + 1, t)$. This implies the splitting into the following two cases, see Figure 3 below.
Case 3.1. Let $t + 1 + d(1/2 - t) \le r < 3/2$. The line $\beta = r + d(1/q - 1/2)$ meets the Jerison-Kenig line in the point $(\beta, 1/q)$ where

$$\frac{1}{q} := \frac{1 - r + d/2}{d-1} \qquad \text{and} \qquad \beta := 1 + \frac{1-r+d/2}{d-1}\,.$$

This implies (by calculating $(\beta - r)/d$)

$$\sup\Big\{\sigma_n(u,\mathcal{B}^*)_{H^r(\Omega)}\ :\quad \| f|H^{t-1}(\Omega)\| \le 1\Big\} \le c\, n^{-\frac{\frac{3}{2}-r}{d-1}+\varepsilon}\,.$$

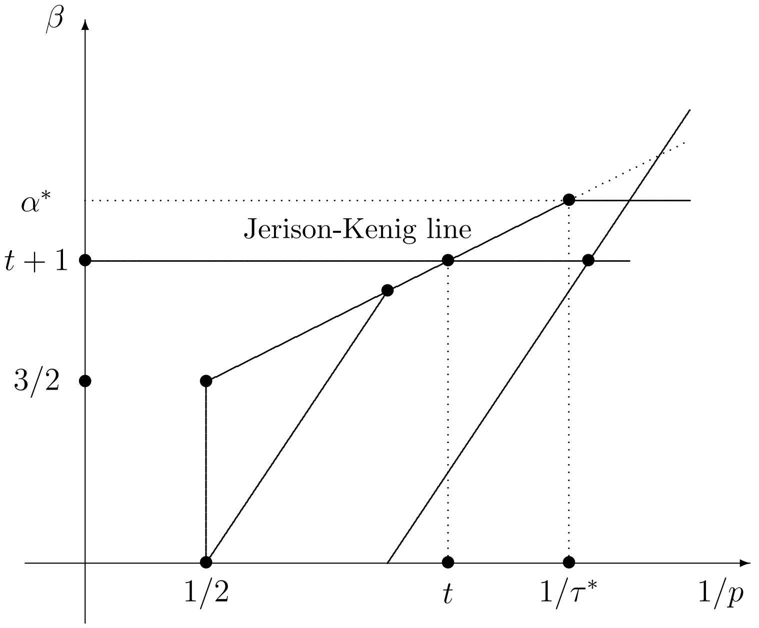

Fig. 3

Case 3.2. Let $-\infty < r < t+1+d(1/2-t)$. Then the line $\beta = r + d(1/q - 1/2)$ meets the line $\beta = t+1$ before it crosses the Jerison-Kenig line. This implies

$$\sup\Big\{\sigma_n(u,\mathcal{B}^*)_{H^r(\Omega)} : \quad \| f|H^{t-1}(\Omega)\| \le 1\Big\} \le c\, n^{-\frac{t+1-r}{d}+\varepsilon}\,.$$

A part of these observations is collected in the following theorem.

Theorem 10. *Let S denote the solution operator for the problem* (18). *Let $\mathcal{B}^*$ be a wavelet system satisfying the conditions mentioned at the beginning of this subsection. Let either*

$$1/2 \le t \le 1 + 2/(d-1) \qquad \textit{and} \qquad -\infty < r < t+1+d(1/2-t)$$

or

$$-1/2 < t < 1/2 \qquad \textit{and} \qquad -\infty < r < t+1\,.$$

Then, for any $\varepsilon > 0$ and sufficiently large C best n-term wavelet approximation with respect to the $H^r(\Omega)$ yields

$$\begin{aligned} e^{\rm non}_{n,C}(S, H^{t-1}(\Omega), H^r(\Omega)) &\le \sup\Big\{\sigma_n(u,\mathcal{B}^*)_{H^r(\Omega)} : \quad \| f|H^{t-1}(\Omega)\| \le 1\Big\} \\ &\le c\, n^{-\frac{t+1-r}{d}+\varepsilon} \end{aligned}$$

where c does not depend on $n \in \mathbb{N}$.

Most interesting are the special cases $r = 1$ (approximation in the energy norm) and $r = 0$ (approximation in the L_2-norm).

Corollary 2. *Let S denote the solution operator for the problem* (18). *Let $\mathcal{B}^*$ be a wavelet system satisfying the conditions mentioned at the beginning of this subsection. Then, for any $\varepsilon > 0$ and sufficiently large C best n-term wavelet approximation with respect to the $H^1(\Omega)$ yields*

$$\begin{aligned} e^{\rm non}_{n,C}(S, H^{t-1}(\Omega), H^1(\Omega)) &\le \sup\Big\{\sigma_n(u,\mathcal{B}^*)_{H^1(\Omega)} : \quad \| f|H^{t-1}(\Omega)\| \le 1\Big\} \\ &\le c \begin{cases} n^{-\frac{1}{2(d-1)}+\varepsilon} & \textit{if} \quad t \ge \frac{1}{2}\,\frac{d}{d-1}\,, \\ n^{-\frac{t}{d}+\varepsilon} & \textit{if} \quad 0 < t \le \frac{1}{2}\,\frac{d}{d-1}\,, \end{cases} \end{aligned}$$

where c does not depend on $n \in \mathbb{N}$.

Corollary 3. *Let S denote the solution operator for the problem* (18). *Let $\mathcal{B}^*$ be a wavelet system satisfying the conditions mentioned at the beginning of this subsection. Then, for any $\varepsilon > 0$ and sufficiently large C best n-term wavelet approximation* *with respect to the $L_2(\Omega)$ yields*

$$\begin{aligned} e_{n,C}^{\text{non}}(S, H^{t-1}(\Omega), L_2(\Omega)) &\le \sup\Big\{\sigma_n(u,\mathcal{B}^*)_{L_2(\Omega)} \ : \quad \| f|H^{t-1}(\Omega)\| \le 1\Big\} \\ &\le c \begin{cases} n^{-\frac{3}{2(d-1)}+\varepsilon} & \text{if} \quad t \ge \frac{2+d}{2d-2}\,, \\ & \\ n^{-\frac{t+1}{d}+\varepsilon} & \text{if} \quad -\frac{1}{2} < t < \frac{2+d}{2d-2}\,, \end{cases} \end{aligned}$$

where c does not depend on $n \in \mathbb{N}$.

However it seems to be also interesting that we have some convergence results in norms which are stronger than the energy norm.

Corollary 4. *Let S denote the solution operator for the problem* (18)*. Let $\mathcal{B}^*$ be a wavelet system satisfying the conditions mentioned at the beginning of this subsection. Let $1 < r < 3/2$. Then, for any $\varepsilon > 0$ and sufficiently large C best n-term wavelet approximation with respect to the $H^r(\Omega)$ yields*

$$\begin{aligned} e_{n,C}^{\text{non}}(S, H^{t-1}(\Omega), H^r(\Omega)) &\le \sup\Big\{\sigma_n(u,\mathcal{B}^*)_{H^r(\Omega)} \ : \quad \| f|H^{t-1}(\Omega)\| \le 1\Big\} \\ &\le c \begin{cases} n^{-\frac{\frac{3}{2}-r}{d-1}+\varepsilon} & \text{if} \quad 1+\frac{d}{2}-(d-1)t \le r\,, \\ & \\ n^{-\frac{t+1-r}{d}+\varepsilon} & \text{if} \quad r \le 1+\frac{d}{2}-(d-1)t\,, \end{cases} \end{aligned}$$

where c does not depend on $n \in \mathbb{N}$.

Remark 12. In all three special cases we have a similar behaviour. As long as the smoothness t is below of some barrier depending on r and d, the rate of approximation is the expected one ($= (t+1-r)/d + \varepsilon$). For large values of t we have only suboptimal rates of convergence.

Remark 13. Theorem 10 implies that best n-term wavelet approximation for the solution of the Poisson equation is optimal for a huge scale of (weak) norms. Moreover, we gain approximation order as the norms get weaker. This can be interpreted as a nonlinear analogue to the classical Aubin-Nitsche trick for uniform approximation schemes, see. e.g., [20] for details. However, the reader should observe that this is still a quite theoretical result since the concrete design of an optimal numerical approximation scheme is an open question. It is well-known that adaptive wavelet schemes indeed realize the convergence order of best n-term wavelet approximation, but this is only true for approximations with respect to the energy norm, see, e.g. [4, 5] for details. Convergence with respect to stronger norms can only be established in specific settings, i.e., for adaptive schemes based on Gabor frames [8]. To our knowledge, no convergence results for weaker norms exist so far.

Let us finally remark that for specific domains, i.e., for polygonal domains contained in $\mathbb{R}^2$, much more far-reaching results can be shown. It turns out that in this case the restrictions caused by the Jerison-Kenig line simply disappear. This means that for polygonal domains best n-term wavelet approximation is optimal for *all* values of t.

Theorem 11. *Let S denote the solution operator for the problem* (18) *in a bounded polygonal domain Ω contained in $\mathbb{R}^2$ Let $\omega_l, l = 1, \ldots, N$ denote the measures of the interior angles of Ω and suppose that $t \neq m\pi/\omega_l$ for all $l = 1, \ldots, N, m \geq 1$. Let $\mathcal{B}^*$ be a wavelet system satisfying the conditions mentioned at the beginning of this subsection. Then, for any $\varepsilon > 0$, $r < 3/2$ and sufficiently large C best n-term wavelet approximation with respect to the $H^r(\Omega)$ yields*

$$\begin{aligned} e^{\mathrm{non}}_{n,C}(S, H^{t-1}(\Omega), H^r(\Omega)) &\leq \sup\left\{\sigma_n(u, \mathcal{B}^*)_{H^r(\Omega)} : \quad \| f | H^{t-1}(\Omega)\| \leq 1\right\} \\ &\leq c\, n^{-\frac{t+1-r}{d}+\varepsilon} \end{aligned}$$

where c does not depend on $n \in \mathbb{N}$.

Proof. The proof is based on the fact that in polygonal domains the solution u to (18) can be decomposed into a regular part u_R and a singular part $u_S, u = u_R + u_S$, where $u_R \in H^{t+1}(\Omega)$ and u_S depends only on the shape of the domain and can be computed explicitely, see [19] for details. It has been shown in [6] that the singular part u_S is contained in all the spaces $B^\alpha_\tau(L_\tau(\Omega))$ where

$$(\alpha, 1/\tau) \in \left\{(\beta, 1/q) : \beta < \frac{2}{q} + \frac{1}{2}, \quad 0 < q \leq 2\right\}.$$

Consequently, u is contained in the Besov spaces corresponding to the set

$$(\alpha, 1/\tau) \in \left\{(\beta, 1/q) : \beta < \min\left(t+1, \frac{2}{q} + \frac{1}{2}\right), \quad 0 < q \leq 2\right\}.$$

Now another application of Proposition 3 yields the result. □

Now we turn to the case that we assume that $f \in H^{t-1}_p(\Omega)$, $1 < p < \infty$, $p \neq 2$. It is not our aim to treat the most general case. We concentrate on two situations where best n-term approximation is optimal.

Theorem 12. *Let S denote the solution operator for the problem* (18). *Let $\mathcal{B}^*$ be a wavelet system satisfying the conditions mentioned at the beginning of this subsection.*

(i) *Let $(t+1, 1/p) \in H_\mu$. Further we assume*

$$r < t + 1 - d \max\left(0, \frac{1}{p} - \frac{1}{2}\right). \tag{32}$$

Then, for any $\varepsilon > 0$ and sufficiently large C best n-term wavelet approximation with respect to the $H^r(\Omega)$ yields

$$\begin{aligned} e_{n,C}^{\mathrm{non}}(S, H_p^{t-1}(\Omega), H^r(\Omega)) &\le \sup\Big\{\sigma_n(u,\mathcal{B}^*)_{H^r(\Omega)} : \quad \| f|H^{t-1}(\Omega)\| \le 1\Big\} \\ &\le c\, n^{-\frac{t+1-r}{d}+\varepsilon} \qquad (33) \end{aligned}$$

where c does not depend on $n \in \mathbb{N}$.
(ii) *Let $(t+1, 1/p) \in II$ (see Subsection 5.3.1). Further we assume*

$$\max\Big(\frac{1}{2},\frac{1}{p}\Big) \le t \le 1 + 2/(d-1) \qquad \textit{and} \qquad -\infty < r < t+1-d\Big(t-\frac{1}{2}\Big)\,. \quad (34)$$

Then, for any $\varepsilon > 0$ and sufficiently large C best n-term wavelet approximation with respect to the $H^r(\Omega)$ yields the same estimate as in (33).

Proof. Step 1. Proof of (i). Observe that (32) is guaranteeing that the lines $\beta = t+1$ and $\beta = r + d(\frac{1}{q} - \frac{1}{2})$ intersect at a point $(t+1, 1/q*)$ such that $q^* \le p$. Now it is enough to combine Lemma 5 with Proposition 3.
Step 2. Proof of (ii). The point of intersection of the lines $\beta = r + d(\frac{1}{q} - \frac{1}{2})$ and $\beta = t+1$ has the coordinates

$$\alpha^* := t+1 \qquad \text{and} \qquad \frac{1}{q^*} := \frac{t+1-r}{d} + \frac{1}{2}\,.$$

By assumption (34) we obtain $t \ge 1/2$ and hence $r < t+1$. This shows that $0 < q^* < 2$. Furthermore, since $t - 1/2 \ge 1/p - 1/2$ we conclude

$$r < t+1-d\Big(t-\frac{1}{2}\Big) \le t+1-d\Big(\frac{1}{p}-\frac{1}{2}\Big)\,.$$

But this implies $q^* < p$. Now we can argue as in proof of Theorem 10. □

Remark 14. (i) Also in the general context we have the phenomenon that if r and t are not too large then the approximation order of best n-term wavelet approximation is the optimal one.
(ii) The part (ii) in Theorem 12 represents an improvement of Theorem 10 if $p > 2$. Under weaker assumptions as in Theorem 10 we end up with the same order of best n-term approximation.

6 Appendix

For convenience of the reader we collect the definitions of the various different widths that are used in this paper.

6.1 Linear Widths

We consider the class $\mathcal{L}_n$ of all continuous linear mappings $S_n : F \to X$,

$$S_n(f) = \sum_{i=1}^{n} L_i(f)\, h_i$$

with arbitrary $h_i \in X$ and the L_i are linear functionals defined on F. The worst case error of optimal linear mappings is given by the *approximation numbers* or *linear widths*

$$e_n^{\mathrm{lin}}(S, F, X) = \inf_{S_n \in \mathcal{L}_n} e(S_n, F, X).$$

6.2 Nonlinear Widths

For a given basis $\mathcal{B}$ of X we consider the class $\mathcal{N}_n(\mathcal{B})$ of all (linear or nonlinear) mappings of the form

$$S_n(f) = \sum_{k=1}^{n} c_k\, h_{i_k},$$

where the c_k and the i_k depend in an arbitrary way on f. We also allow that the basis $\mathcal{B}$ to be chosen in a nearly arbitrary way. Then the *nonlinear widths* $e_{n,C}^{\mathrm{non}}(S, F, X)$ are given by

$$e_{n,C}^{\mathrm{non}}(S, F, X) = \inf_{\mathcal{B} \in \mathcal{B}_C} \inf_{S_n \in \mathcal{N}_n(\mathcal{B})} e(S_n, F, X).$$

Here $\mathcal{B}_C$ denotes a set of Riesz bases for X where C indicates the stability of the basis. Hence we assume here that X is a Hilbert space. Then a sequence $h_1, h_2, \ldots$ of elements of X is called a Riesz basis for X if there exist positive constants A and B such that, for every sequence of scalars $\alpha_1, \alpha_2, \ldots$ with $\alpha_i \neq 0$ for only finitely many i, we have

$$A \Big(\sum_k |\alpha_k|^2 \Big)^{1/2} \le \Big\| \sum_k \alpha_k\, h_k \Big\|_X \le B \Big(\sum_k |\alpha_k|^2 \Big)^{1/2} \tag{35}$$

and the vector space of finite sums $\sum \alpha_k\, h_k$ is dense in X. In what follows

$$\mathcal{B} = \{ h_i \mid i \in \mathbb{N} \} \tag{36}$$

will always denote a Riesz basis of X and A and B will be the corresponding optimal constants in (35). For a real number $C \ge 1$ we define

$$\mathcal{B}_C := \Big\{ \mathcal{B} :\ B/A \le C \Big\}. \tag{37}$$

6.3 Manifold Widths

Let $\mathcal{C}_n$ be the class of continuous mappings, given by arbitrary continuous mappings $N_n : F \to \mathbb{R}^n$ and $\varphi_n : \mathbb{R}^n \to X$. Again we define the worst case error of optimal continuous mappings by

$$e_n^{\mathrm{cont}}(S, F, X) = \inf_{S_n \in \mathcal{C}_n} e(S_n, F, X),$$

where $S_n = \varphi_n \circ N_n$. These numbers are called *manifold widths* of S. We refer to [13, 14, 15, 27] and [9, 10] for further information.

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