INTRACTABILITY RESULTS FOR INTEGRATION AND DISCREPANCY

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ABSTRACT. We mainly study multivariate (uniform or Gaussian) integration defined for integrand spaces F_d such as weighted Sobolev spaces of functions of d variables with smooth mixed derivatives. The weight γ_j moderates the behavior of functions with respect to the jth variable. For $\gamma_j \equiv 1$, we obtain the classical Sobolev spaces whereas for decreasing γ_j's the weighted Sobolev spaces consist of functions with diminishing dependence on the jth variables. We study the minimal errors of quadratures that use n function values for the unit ball of the space F_d. It is known that if the smoothness parameter of the Sobolev space is one, then the minimal error is the same as the discrepancy. Let $n(\varepsilon, F_d)$ be the smallest n for which we reduce the initial error, i.e., the error with $n = 0$, by a factor ε. The main problem studied in this paper is to determine whether the problem is intractable, i.e., whether $n(\varepsilon, F_d)$ grows faster than polynomially in ε^{-1} or d. In particular, we prove intractability of integration (and discrepancy) if $\limsup_d \sum_{j=1}^d \gamma_j / \ln d = \infty$. Previously, such results were known only for restricted classes of quadratures.

For $\gamma_j \equiv 1$, the following bounds hold for discrepancy

- with boundary conditions

$$1.0628^d(1 + o(1)) \le n(\varepsilon, F_d) \le 1.5^d \, \varepsilon^{-2}, \qquad \text{as } d \to \infty,$$

- without boundary conditions

$$1.0463^d(1 + o(1)) \le n(\varepsilon, F_d) \le 1.125^d \, \varepsilon^{-2}, \qquad \text{as } d \to \infty.$$

These results are obtained by analyzing arbitrary linear tensor product functionals I_d defined over weighted tensor product reproducing kernel Hilbert spaces F_d of functions of d variables. We introduce the notion of a *decomposable* kernel. For reproducing kernels that have a decomposable part we prove intractability of all I_d with non-zero subproblems with respect to the decomposable part of the kernel, as long as the weights satisfy the condition mentioned above.

1. INTRODUCTION

There has been an increasing interest in studying the complexity of linear multivariate problems defined on spaces F_d of functions of d variables with large d. The primary example is integration where d can be hundreds or even thousands as happens in financial mathematics, see e.g., [17, 20] and references listed there. We are especially interested in the tractability of such problems.

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By a linear multivariate problem, we mean the approximation of a linear operator I_d by computing a finite number of function values. The (worst case) complexity $n(\varepsilon, F_d)$ is roughly understood as the minimal number of function values needed to reduce the initial error by a factor ε. Here, the initial error is equal to the norm of I_d and is the error which can be achieved without computing any function values. Tractability means that the complexity $n(\varepsilon, F_d)$ may grow at most polynomially in ε^{-1} and d. Strong tractability means that $n(\varepsilon, F_d)$ has a polynomial bound in ε^{-1} that is independent of d.

Tractability and strong tractability seem very demanding properties. One may therefore suspect that only trivial problems may be tractable or strongly tractable. On the other hand, there is compelling empirical evidence that integration is strongly tractable for many finance problems, see again [17, 20].

It is therefore a challenging problem for theory to understand which multivariate problems are tractable or strongly tractable. This issue is addressed in a number of papers, see [2, 7, 9, 10, 11, 15, 16, 22, 23, 24, 25, 26, 27].

In particular, much is known about tractability and strong tractability for linear tensor product problems whose range is at least two dimensional, see [22, 26]. It is somehow surprising that the remaining case of linear tensor product functionals is not fully understood. The reason is that this case is rich in possibilities. For example, it is known, see [11], that there are Hilbert spaces for which all linear tensor product functionals are tractable and other Hilbert spaces for which all non-trivial linear tensor product functionals are intractable. In both cases, however, the Hilbert spaces are quite esoteric. For typical Hilbert spaces, the trouble is that we do not know whether a particular linear functional, such as integration, is tractable or intractable.

To some extent, this paper provides a solution to this problem. In particular, for many classical Hilbert spaces we are able to prove intractability of weighted integration. By weighted integration, we mean an approximation of the integral of f with the weight ϱ. If ϱ is the indicator function of a compact set, such as $[0,1]^d$, we have *uniform* integration, and if ϱ is the density of Gaussian measure, we have *Gaussian* integration. In particular, uniform and Gaussian integration in the Sobolev spaces of functions with smooth mixed derivatives are intractable. If the smoothness parameter is one, then intractability of integration is equivalent to intractability of the star or centered discrepancy, depending on details how the norm of the Sobolev space is defined. The centered discrepancy was studied by Hickernell [6], see [3, 8] for a detailed discussion of discrepancy. We also prove intractability of centered discrepancy when the norm is understood in the L_p sense for $p \in [1, \infty]$. The star discrepancy is only studied in this paper in the L_2 sense, and we prove intractability of the L_2-star discrepancy. The choice of the norm is essential, since the L_∞-star discrepancy is tractable with $n(\varepsilon, F_d) \le C\, d\, \varepsilon^{-2}$ for some positive C, see [5].

For the L_2-star discrepancy we consider two cases. The first case is for the Sobolev space with a boundary condition ($f(0) = 0$ for $d = 1$, and $f(x) = 0$ if at least one component of x is zero for $d \ge 1$). Then for a fixed $\varepsilon \in (0, 1)$, we have the following bounds on the L_2-star discrepancy

$$1.0628^d(1 + o(1)) \;\le\; n(\varepsilon, F_d) \;\le\; 1.5^d\, \varepsilon^{-2}, \qquad \text{as } d \to \infty. \tag{1}$$

The upper bound is easy to obtain and follows from the well known facts that the average L_2-star discrepancy is at most $2^{-d/2}n^{-1/2}$, whereas the initial L_2-star discrepancy is $3^{-d/2}$.

The second case is for the Sobolev space without a boundary condition. Then for a fixed $\varepsilon \in (0,1)$, we have

$$1.0463^d(1+o(1)) \le n(\varepsilon, F_d) \le 1.125^d\,\varepsilon^{-2}, \qquad \text{as } d \to \infty. \tag{2}$$

Again, the upper bound is easy to obtain since the average L_2-star discrepancy is now at most $1.5^{d/2}n^{-1/2}$, whereas the initial L_2-star discrepancy is now $(4/3)^{d/2}$. Note that the base 1.5 in (1) is now replaced by 1.125. This is due to the fact that, in particular, the initial L_2-star discrepancy with the boundary condition is exponentially *small* in d, whereas the initial L_2-star discrepancy without the boundary condition is exponentially *large* in d.

The lower bounds in (1) and (2) are exponentially large in d. They do not, however, address the dependence in ε^{-1}. It is known that for a fixed d and ε^{-1} tending to infinity, we have $n(\varepsilon, F_d)$ of order $\varepsilon^{-1}\ln^{(d-1)/2}\varepsilon^{-1}$, see [13] for a lower bound, and [14, 4] for an upper bound. We also mention that better lower bounds on $n(\varepsilon, F_d)$ are known if we consider only non-negative coefficients in the L_2-star discrepancy formula. Then 1.0628^d in (1) can be replaced by $1.125^d(1-\varepsilon^2)$, and 1.0463^d in (2) can be replaced by $1.0563^d(1-\varepsilon^2)$, see [27] and [12].

The approach in this paper is rather general. As in [16] and [23], we consider weighted classes of spaces where the weight γ_j moderates the behavior of functions with respect to the jth variable. If all γ_j are 1, then we have the unweighted case with the classical Sobolev spaces as the main example. If γ_j is decreasing with j, then we have spaces with diminishing dependence on the jth variable. We provide sufficient conditions for the lack of strong tractability or tractability of linear functionals in terms of γ_j. The typical result is that $\sum_{j=1}^{\infty}\gamma_j = \infty$ implies the lack of strong tractability, whereas $\limsup_d \sum_{j=1}^{d}\gamma_j/\ln d = \infty$ implies intractability.

For the Sobolev space with the smoothness parameter one these conditions are also necessary, see [16]. Hence, integration is strongly tractable iff $\sum_{j=1}^{\infty}\gamma_j < \infty$, and tractable iff $\limsup_d \sum_{j=1}^{d}\gamma_j/\ln d < \infty$. These results were previously known for classes of quasi-Monte Carlo algorithms (quadratures), see [16], and for classes of algorithms with nonnegative weights, see [9, 27]. Our results hold for the unrestricted class of algorithms.

We now describe the approach of our paper in more detail. Although we study arbitrary linear tensor product functionals, our main emphasis is on multivariate weighted integration, which in the special case reduces to the star or centered discrepancy. This explains why the title of our paper refers only to integration and discrepancy.

We consider tensor product reproducing kernel Hilbert spaces. The kernel is denoted by K_d, and is fully determined by the kernel K_1 for the case of univariate functions. Our point of departure is the notion of a *decomposable* kernel. We say that the kernel K_1 is decomposable if $K_1(x,t) = 0$ for all $x \le a \le t$ for some a. Examples of decomposable kernels are kernels of the Sobolev spaces of smooth functions that vanish at a. For such kernels, the problem of approximating the linear functional I_d naturally decomposes into 2^d subproblems. These subproblems are not trivial iff the two subproblems for $d = 1$ are not trivial. This is usually the case when the point a belongs to the interior of the function domain. If so, we find that $n(\varepsilon, F_d)$ depends exponentially on d, and so the problem is intractable.

We then extend our analysis to certain non-decomposable kernels. We assume first that the kernel $K_1 = R_1 + \gamma\, R_2$ where R_i are kernels of Hilbert spaces $H(R_i)$ such that $H(R_1) \cap H(R_2) = \{0\}$, and R_2 is decomposable. For the d dimensional case, we take the weighted tensor product Hilbert space whose kernel is

$$K_d(x,t) = \prod_{j=1}^{d} \left(R_1(x_j,t_j) + \gamma_j R_2(x_j,t_j)\right),$$

where x_j and t_j denote the successive components of x and t. We stress that we may use different weights γ_j for different components. Assuming, as before, that I_d has non-trivial subproblems with respect to $H(R_2)$, we prove that $\sum_{j=1}^{\infty} \gamma_j = \infty$ implies that I_d is not strongly tractable, and that $\limsup_d \sum_{j=1}^{d} \gamma_j / \ln\, d = \infty$ implies that I_d is intractable.

This form of the reproducing kernel is satisfied for the Sobolev spaces without the condition that functions vanish at a as long as a does not belong to the boundary of the domain. This holds for the centered discrepancy, which is also analyzed in the Banach case where the norm is taken in the L_p sense with $p \in [1, \infty]$.

Finally, we consider the case when $K_1 = R_1 + \gamma R_2$ for which R_2 is not necessarily decomposable but is the sum of two kernels with one decomposable term. The last assumption holds for the Sobolev spaces with or without boundary conditions when the kernel R_2 is modified by an operator of finite rank. In particular, this decomposition holds for the L_2-star discrepancy. In this case we obtain the same sufficient conditions for the lack of strong tractability and intractability of I_d.

We end this introduction by the following remark which leads to an open problem. Our approach requires us to identify a part of the reproducing kernel that is decomposable. As demonstrated by many examples in this paper, this can be done for many classical spaces. Decomposability means, roughly speaking, that F_1 contains two orthogonal subspaces $F_{(0)}$ and $F_{(1)}$ such that

$$f(x) = 0 \quad \text{for} \quad x \notin D_{(0)} \ \forall\, f \in F_{(0)},$$

$$f(x) = 0 \quad \text{for} \quad x \notin D_{(1)} \ \forall\, f \in F_{(1)},$$

where the set $D_{(0)} \cap D_{(1)}$ has at most one element and both the $D_{(i)}$ have positive Lebesgue measure. If the spaces $F_{(i)}$ are non-trivial (i.e., they contain elements different from zero) then our results are non-trivial and lead to intractability.

Therefore our theory cannot be applied to spaces of analytic functions, since these functions cannot be decomposed as above. For example, take

$$K_1(x,t) \;=\; \sum_{j=1}^{\infty} \frac{x^j t^j}{(j!)^2},$$

which corresponds to a Hilbert space of entire functions. Can we say anything about intractability of linear functionals for such spaces? Also some spaces discussed in [9, 10] are spaces of analytic functions, and the results of the present paper do not apply to these spaces.

2. Tractability

In this section, we define the problem of approximating continuous linear functionals over spaces of multivariate functions of d variables. We also recall the concepts of tractability and strong tractability.

Let F_d be a normed space of functions $f : D_d \to \mathbb{R}$, where $D_d \subset \mathbb{R}^d$. The norm of F_d will be denoted by $\|\cdot\|_{F_d}$. Let $I_d : F_d \to \mathbb{R}$ be a continuous linear functional. For all f from the unit ball of F_d, we want to approximate $I_d(f)$ by computing a number of function values $f(z_i)$ for some (deterministic) sample points z_i from D_d and $i = 1, 2, \dots, n$. It is known, see e.g., [19], that it suffices to consider linear algorithms $Q_{n,d}(f) = \sum_{i=1}^n a_i f(z_i)$ for some real coefficients a_i. The worst case error of the algorithm $Q_{n,d}$ is defined by its worst case performance over the unit ball of F_d,

$$e(Q_{n,d}) \;=\; \sup_{f \in F_d,\, \|f\|_{F_d} \le 1} \big| I_d(f) - Q_{n,d}(f) \big| \;=\; \|I_d - Q_{n,d}\|. \tag{3}$$

For a given n, we would like to find coefficients a_i and sample points z_i such that the error of the algorithm $Q_{n,d}$ is minimized. Let

$$e(n, F_d) \;=\; \inf_{a_i,\, z_i, i=1,2,\dots,n} \; \sup_{f \in F_d,\, \|f\|_{F_d} \le 1} \left| I_d(f) - \sum_{i=1}^n a_i f(z_i) \right| \tag{4}$$

be the nth minimal error when we use n function values. For $n = 0$, we formally set $Q_{0,d} = 0$ and

$$e(0, F_d) \;=\; \sup_{f \in F_d,\, \|f\|_{F_d} \le 1} |I_d(f)| \;=\; \|I_d\|$$

is the initial error. This is the a priori error in approximating I_d without sampling the function.

We would like to reduce the initial error by a factor ε, where $\varepsilon \in (0, 1)$. We are looking for the smallest $n = n(\varepsilon, F_d)$ for which there exists an algorithm $Q_{n,d}$ such that $e(Q_{n,d}) \le \varepsilon\, e(0, F_d)$. That is,

$$n(\varepsilon, F_d) \;=\; \min \{\, n \;:\; e(n, F_d) \;\le\; \varepsilon\, e(0, F_d) \,\}. \tag{5}$$

As in [26], we say that the problem of approximating I_d for a sequence F_d of spaces is *tractable*, shortly I_d is tractable in F_d, iff there exist nonnegative C, q and p such that

$$n(\varepsilon, F_d) \;\le\; C\, d^q\, \varepsilon^{-p} \qquad \forall\, d = 1, 2, \dots \;\; \forall\, \varepsilon \in (0, 1). \tag{6}$$

Tractability means that we can reduce the initial error by a factor ε by using a number of function values that is polynomial in d and ε^{-1}.

If $q = 0$ in (6), then we say that the problem of approximating I_d for a sequence $\{F_d\}$ of spaces is *strongly tractable*, shortly I_d is strongly tractable in F_d. In this case, the number of samples is independent of d and depends polynomially on ε^{-1}. The smallest (or the infimum of) p in (6) is called now the *strong exponent* of tractability.

In this paper we mainly study the problem of *multivariate weighted integration*, for which the operator I_d is of the form

$$I_d(f) = \int_{D_d} \varrho_d(t)\, f(t)\, dt \tag{7}$$

for some weight function $\varrho_d : D_d \to \mathbb{R}_+$. For $D_d = [0,1]^d$ and $\varrho_d(t) \equiv 1$, we obtain *uniform integration*, and for $D_d = \mathbb{R}_d$ and $\varrho_d(t) = (2\pi)^{-d/2} \exp(-(t_1^2 + \cdots + t_d^2)/2)$, we obtain *Gaussian integration*. Here and elsewhere, $t = [t_1, t_2, \ldots, t_d]$.

As we shall see later, multivariate weighted integration is *intractable* for many natural spaces F_d that are tensor products of Sobolev spaces of smooth functions of one variable. On the other hand, multivariate weighted integration becomes *strongly* tractable for many spaces F_d which are *weighted* tensor products of Sobolev spaces, see [7, 9, 16, 27].

3. Decomposable Kernels

We present lower bounds on the minimal errors $e(n, F_d)$. From these bounds we will conclude intractability for certain tensor product linear functionals I_d and tensor product spaces F_d.

More precisely, we proceed as follows. For $d = 1$, we assume that F_1 is a class of univariate functions defined over $D_1 \subset \mathbb{R}$, which is a reproducing kernel Hilbert space whose reproducing kernel is $K_1 : D_1 \times D_1 \to \mathbb{R}$, see [1, 21]. The inner product of F_1 is denoted by $\langle \cdot, \cdot \rangle_{F_1}$. The continuous linear functional I_1 takes now the form $I_1(f) = \langle f, h_1 \rangle_{F_1}$ for some function h_1 from F_1.

For $d > 1$, we take $F_d = F_1 \otimes F_1 \otimes \cdots \otimes F_1$ (d times) as a tensor product of F_1. Then F_d is a class of multivariate functions defined over $D_d = D_1 \times D_1 \times \cdots \times D_1$ (d times). The space F_d is a Hilbert space whose reproducing kernel $K_d : D_d \times D_d \to \mathbb{R}$ is given by

$$K_d(x,t) = \prod_{j=1}^{d} K_1(x_j, t_j).$$

The inner product of F_d is denoted by $\langle \cdot, \cdot \rangle_{F_d}$. Finally, the continuous linear functional $I_d = I_1 \otimes \cdots \otimes I_1$ is a tensor product of I_1. This means that

$$I_d(f) = \langle f, h_d \rangle_{F_d} \qquad \text{with} \qquad h_d(t) = h_1(t_1) h_1(t_2) \cdots h_1(t_d).$$

Clearly, $e(0,d) = \|I_d\| = \|h_d\|_{F_d} = \|h_1\|_{F_1}^d$. Hence, $e(0,d) = 0$ only for trivial problems with $h_1 \equiv 0$.

For $a^* \in \mathbb{R}$, define $D_{(0)} = \{x \in D_1 :\ x \le a^*\}$ and $D_{(1)} = \{x \in D_1 :\ x \ge a^*\}$. Obviously $D_1 = D_{(0)} \cup D_{(1)}$ and $D_{(0)} \cap D_{(1)}$ is either empty or consists of one point a^* if $a^* \in D_1$. We also have

$$D_1^2 = D_{(0)} \times D_{(0)} \cup D_{(0)} \times D_{(1)} \cup D_{(1)} \times D_{(0)} \cup D_{(1)} \times D_{(1)}.$$

We say that the kernel K_1 is *decomposable* iff there exists a number $a^* \in \mathbb{R}$ such that

$$K_1(x,t) \;=\; 0 \quad \text{for } (x,t) \in D_{(0)} \times D_{(1)} \cup D_{(1)} \times D_{(0)}. \tag{8}$$

The essence of this property is that the function K_1 may take nonzero values only if x and t belong to the same quadrant $D_{(0)}^2$ or $D_{(1)}^2$.

Observe that if K_1 is decomposable and $a^* \in D_1$ then $K_1(\cdot, a^*) = 0$. This implies that all functions in F_1 vanish at a^* since $f(a^*) = \langle f, K_1(\cdot, a^*)\rangle_{F_1} = 0$. For arbitrary d, we also have $K_d(\cdot, x) = 0$ if $x_i = a^*$ for some i and all functions from F_d vanish at $x \in D_1^d$ if $x_i = a^*$ for some i.

Furthermore, $K_d(x,t) = 0$ if x and t belong to different quadrants. More precisely, for a Boolean vector $b \in \{0,1\}^d$, define

$$D_{(b)} \;=\; D_{(b_1)} \times D_{(b_2)} \times \cdots \times D_{(b_d)}.$$

Then $D_d = \bigcup_{b\in\{0,1\}^d} D_{(b)}$. If $x \in D_{(b_x)}$ and $t \in D_{(b_t)}$ for different Boolean vectors b_x and b_t then $K_d(x,t) = 0$.

If K_1 is decomposable, then the space F_1 can be decomposed as the direct sum of Hilbert spaces $F_{(0)}$ and $F_{(1)}$ of univariate functions defined by

$$F_{(i)} \;=\; \overline{\operatorname{span}}\{K_1(\cdot,t) \;:\; t \in D_{(i)}\,\}.$$

Indeed, functions of the form $f = \sum_{j=1}^k \beta_j K_1(\cdot, t_j)$ with real β_j and $t_j \in D_1$ are dense in F_1. Then for all $t \in D_1$ we have

$$f(t) = \sum_{j=1}^k \beta_j K_1(t,t_j) = \sum_{(t,t_j)\in D_{(0)}^2} \beta_j K_1(t,t_j) + \sum_{(t,t_j)\in D_{(1)}^2} \beta_j K_1(t,t_j) = f_{(0)}(t) + f_{(1)}(t),$$

where $f_{(0)} \in F_{(0)}$ and $f_{(1)} \in F_{(1)}$. For $f \in F_{(i)}$ we have $f(t) = 0$ for $t \in D_{(1-i)}$ and the subspaces $F_{(0)}$ and $F_{(1)}$ are orthogonal. Hence

$$\|f\|_{F_1}^2 \;=\; \|f_{(0)}\|_{F_1}^2 + \|f_{(1)}\|_{F_1}^2 \qquad \forall\, f \in F_1.$$

Similarly we can decompose the space F_d if K_1 is decomposable. For any Boolean vector b, we define

$$F_{(b)} \;=\; \overline{\operatorname{span}}\{K_d(\cdot,t) \;:\; t \in D_{(b)}\,\}$$

as the Hilbert space of multivariate functions defined over D_d which vanish outside of $D_{(b)}$. Then $F_{(b)}$ is a subset of F_d. Furthermore the spaces $F_{(b)}$ are mutually orthogonal and $F_d = \oplus_{b\in\{0,1\}^d} F_{(b)}$. If $f_{(b)} = f\big|_{D_{(b)}}$ denotes the restriction of the function f from F_d to the domain $D_{(b)}$, and $f_{(b)}(x) = 0$ for $x \notin D_{(b)}$, then $f_{(b)} \in F_{(b)}$. Moreover $f = \sum_{b\in\{0,1\}^d} f_{(b)}$ and

$$\|f\|_{F_d}^2 \;=\; \sum_{b\in\{0,1\}^d} \|f_{(b)}\|_{F_d}^2 \qquad \forall\, f \in F_d. \tag{9}$$

We now apply the last formula to the function h_d that defines the problem I_d. For $d = 1$, we have $h_1 = h_{1,(0)} + h_{1,(1)}$ with $\|h_1\|_{F_1}^2 = \|h_{1,(0)}\|_{F_1}^2 + \|h_{1,(1)}\|_{F_1}^2$. For $d > 1$, we use the fact that the function h_d is the tensor product of the function h_1 and obtain

$$h_{d,(b)}(x) \,=\, \prod_{j\,:\,b_j=0} h_{1,(0)}(x_j) \prod_{j\,:\,b_j=1} h_{1,(1)}(x_j).$$

From this we get

$$\|h_{d,(b)}\|_{F_d} \,=\, \|h_{1,(0)}\|_{F_1}^{0(b)}\, \|h_{1,(1)}\|_{F_1}^{1(b)}, \tag{10}$$

where $0(b) = |\{j\,:\, b_j = 0\}|$ and $1(b) = |\{j\,:\, b_j = 1\}|$ are defined as the number of zeros and ones in the vector b. Obviously, $0(b) + 1(b) = d$ for any $b \in \{0,1\}^d$.

The essence of (9) is that the problem of approximating $I_d(f) = \langle f, h_d\rangle_{F_d}$ can be decomposed into 2^d independent subproblems of approximating $I_{d,(b)}(f) = \langle f, h_{d,(b)}\rangle_{F_d}$ whose initial errors are given by (10). Clearly, if we use n function values with $n < 2^d$ then at least $2^d - n$ of these subproblems can be approximated only by zero. If all $h_{d,(b)}$ are non-zero this yields intractability of I_d in F_d. The formal proof is presented below.

Theorem 1. *If K_1 is decomposable then*

$$e(n, F_d) \,\ge\, \left(1 \,-\, n\,\alpha^d\right)_+^{1/2}\, e(0, F_d),$$

where

$$\alpha \,=\, \frac{\max\left(\|h_{1,(0)}\|_{F_1}^2, \|h_{1,(1)}\|_{F_1}^2\right)}{\|h_{1,(0)}\|_{F_1}^2 + \|h_{1,(1)}\|_{F_1}^2}, \quad (0/0 = 1).$$

Hence, if both $h_{1,(0)}$ and $h_{1,(1)}$ are non-zero, then I_d in F_d is intractable since $\alpha < 1$ and

$$n(\varepsilon, d) \,\ge\, (1-\varepsilon^2)\left(\frac{1}{\alpha}\right)^d$$

is exponential in d. Furthermore,

$$\lim_{d\to\infty} \frac{e(d^q, F_d)}{e(0, F_d)} \,=\, 1 \quad \forall\, q = 1, 2, \dots, .$$

Proof. First of all note that the lower bound on $e(n, F_d)$ is trivial if $n\alpha^d \ge 1$. Hence, it is enough to prove it for n such that $n\alpha^d < 1$. Since $\alpha \ge 1/2$ this means that $n < 2^d$.

Take an arbitrary algorithm $Q_{n,d}(f) = \sum_{j=1}^n a_j f(z_j)$. It is known, and can be easily checked, that the square of the worst case error of $Q_{n,d}$ in the Hilbert space is given by

$$e^2(Q_{n,d}) \,=\, \|h_d\|_{F_d}^2 - 2\sum_{j=1}^n a_j h_d(z_j) + \sum_{i,j=1}^n a_i a_j K_d(z_i, z_j).$$

Note that although the sets $D_{(b)}$ do not have to be disjoint, the only elements that belong to their intersections are vectors with at least one component equal to a^*. For such vectors the value of h_d, as well as of K_d, is zero. Therefore without loss of generality we may assume that no component of any sample point z_i equals a^*. We have

$$\sum_{j=1}^{n} a_j h_d(z_j) \;=\; \sum_{b\in\{0,1\}^d}\ \sum_{j\,:\,z_j\in D_{(b)}} a_j h_d(z_j).$$

This and decomposability of K_1 yield

$$\sum_{i,j} a_i a_j K_d(z_i,z_j) \;=\; \sum_{b\in\{0,1\}^d}\ \sum_{i,j\,:\,z_i,z_j\in D_{(b)}^2} a_i a_j K_d(z_i,z_j).$$

From (9) we can rewrite $e^2(Q_{n,d})$ as

$$e^2(Q_{n,d}) \;=\; \sum_{b\in\{0,1\}^d} e(b),$$

where

$$e(b) \;=\; \|h_{d,(b)}\|_{F_d}^2 - 2\sum_{j\,:\,z_j\in D_{(b)}} a_j h_d(z_j) + \sum_{i,j\,:\,z_i,z_j\in D_{(b)}^2} a_i a_j K_d(z_i,z_j).$$

Since $h_d(z_j)=h_{d,(b)}(z_j)$ for $z_j\in D_{(b)}$, we see that $e(b)$ is just the square of the worst case error of approximating the continuous linear functional $I_{d,(b)}(f)=\left\langle f,h_{d,(b)}\right\rangle_{F_{(b)}}=\left\langle f,h_{d,(b)}\right\rangle_{F_d}$ by using sample points $z_j\in D_{(b)}$. Let $n_{(b)}=|\{j\,:\,z_j\in D_{(b)}\}|$ be the cardinality of the sample points z_j in the set $D_{(b)}$. Clearly, $n=\sum_{b\in\{0,1\}^d} n_{(b)}$. If we define $e(n_{(b)},F_{(b)})$ as the $n_{(b)}$th minimal error of approximating the functional $I_{d,(b)}$ then clearly $e(b)\ge e^2(n_{(b)},F_{(b)})$. Observe also that $e^2(0,F_{(b)})=\|h_{d,(b)}\|_{F_d}^2$ which, due to (10), is equal to $a_0^{0(b)}a_1^{1(b)}$ with $a_0=\|h_{1,(0)}\|_{F_1}^2$ and $a_1=\|h_{1,(1)}\|_{F_1}^2$. Hence,

$$e^2(Q_{n,d}) \;\ge\; \sum_{b\in\{0,1\}^d} e^2(n_{(b)},F_{(b)})$$

The last sum has 2^d terms and $n<2^d$. Therefore at least 2^d-n numbers $n_{(b)}$ must be equal to zero. From this we conclude

$$e^2(Q_{n,d}) \;\ge\; \sum_{b\in\{0,1\}^d} e^2(0,F_{(b)}) \;-\; n\max_{b\in\{0,1\}^d} e^2(0,F_{(b)}).$$

The first sum is just $\|h_d\|_{F_d}^2=e^2(0,F_d)=(a_0+a_1)^d$, whereas

$$\max_{b\in\{0,1\}^d} e^2(0,F_{(b)}) \;=\; \max_{b\in\{0,1\}^d} a_0^{0(b)}a_1^{1(b)} \;=\; \max(a_0,a_1)^d.$$

This proves that

$$e^2(Q_{n,d}) \;\ge\; \left(1-n\left(\frac{\max(a_0,a_1)}{a_0+a_1}\right)^d\right) e^2(0,F_d) \;=\; \left(1-n\,\alpha^d\right)\,e^2(0,F_d).$$

Since this holds for any algorithm that uses n function values we conclude that the same lower bound holds for $e^2(n,F_d)$, as claimed.

If both a_0 and a_1 are positive, then α is strictly less than one; moreover the initial error $e(0,F_d)$ is positive. Therefore if we want to reduce the initial error by a factor

ε, then $n(\varepsilon,d) \geq (1-\varepsilon^2)\alpha^{-d}$, as claimed. This means that I_d is intractable in F_d. For $\alpha < 1$, we have

$$1 \;\geq\; e(d^q, F_d)/e(0,F_d) \;\geq (1-d^q\alpha^d)_+^{1/2} \;\to\; 1 \qquad \text{as } d \to \infty,$$

which completes the proof. □

As already remarked in the proof of Theorem 1, the number α belongs to $[0.5, 1]$. In fact, α can take any value from this interval as we shall see later in examples. For $\alpha = 1$, Theorem 1 is trivial, since the lower bound of $e(n, F_d)$ is zero for $n \geq 1$. This should come as no surprise, since there exist trivial problems with $\alpha = 1$. For instance, take $I_1(f) = f(a)$ for an arbitrary $a \in D_1$. Then $I_d(f) = f(a, a, \dots, a)$ can be solved exactly using one function value. That means that $e(n, F_d) = 0$ for all $n \geq 1$. For this problem we have $h_1 = K_1(\cdot, a)$ and $h_{1,(0)} = 0$ if $a \leq a^*$, and $h_{1,(1)} = 0$ otherwise. In either case $\alpha = 1$.

If α is smaller than one but is close to one then the exponential dependence on d becomes less drastic since d must be sufficiently large to be hurt by α^{-d}. The worst case for α is when $\alpha = 0.5$. We now show that this holds for decomposable K_1 when the set D_1, the kernel K_1 and the function h_1 are *symmetric* with respect to a^*. That is, when

$$\begin{array}{rcl} a^* - x \in D_1 & \text{implies} & a^* + x \in D_1, \\ K_1(a^*-x, a^*-t) & = & K_1(a^*+x, a^*+t), \\ h_1(a^*-x) & = & h_1(a^*+x). \end{array}$$

Indeed, symmetry of K_1 implies that the spaces $F_{(0)}$ and $F_{(1)}$ are essentially the same. More precisely, define the mapping $P(x) = 2a^* - x$ for $x \in D_1$. Then $P^2(x) = x$ and $P(D_{(0)}) = D_{(1)}$, $P(D_{(1)}) = D_{(0)}$. Then $f \in F_{(0)}$ iff $fP \in F_{(1)}$, and $\|f\|_{F_{(0)}} = \|fP\|_{F_{(1)}}$ for $f \in F_{(0)}$. For symmetric functions $f \in F_1$ we have $\|f\big|_{D_{(0)}}\|_{F_{(0)}} = \|f\big|_{D_{(1)}}\|_{F_{(1)}}$. Since h_1 is symmetric we have $\|h_{1,(0)}\|_{F_1} = \|h_{1,(1)}\|_{F_1} = \|h_1\|_{F_1}/\sqrt{2}$ and for non-zero h_1 we have $\alpha = 0.5$, as claimed. We summarize this in a corollary.

Corollary 1. *Let D_1 and h_1 be symmetric, and K_1 be decomposable and symmetric. Then*

$$e(n, F_d) \;\geq\; \left(1 - n\,2^{-d}\right)_+^{1/2}\; e(0,d).$$

If $e(0,d)$ is positive then I_d in F_d is intractable, since

$$n(\varepsilon, F_d) \;\geq\; (1-\varepsilon^2)\,2^d.$$

We now illustrate Theorem 1 and Corollary 1 by several examples. In particular, we show that the estimates of Theorem 1 are the best possible.

Example: Weighted Integration

We take $F_1 = W_0^r(\mathbb{R})$ as the Sobolev space of functions defined over $\mathbb{R}$ whose $(r-1)$st derivatives are absolutely continuous, with the rth derivatives belonging to

$L_2(\mathbb{R})$ and their derivatives up to the $(r-1)$st at zero being zero. That is, $D_1 = \mathbb{R}$ and

$$F_1 \,=\, \{f : \mathbb{R} \to \mathbb{R} \,:\, f^{(i)}(0) = 0,\ i \in [0, r-1],\ f^{(r-1)} \text{ abs. cont. and } f^{(r)} \in L_2(\mathbb{R})\}.$$

The inner product of F_1 is given as

$$\langle f, g \rangle_{F_1} \,=\, \int_{\mathbb{R}} f^{(r)}(t) g^{(r)}(t)\, dt.$$

It is not hard to check that this Hilbert space has the reproducing kernel

$$K_1(x,t) \,=\, 1_M(x,t) \int_{\mathbb{R}_+} \frac{(|t|-u)_+^{r-1}\, (|x|-u)_+^{r-1}}{((r-1)!)^2}\, du,$$

where 1_M is the characteristic (indicator) function of the set $M = \{(x,t) :\ xt \ge 0\}$.

Observe that this kernel is *decomposable* and $a^* = 0$. Indeed, $D_{(0)} = \{x \in \mathbb{R} : x \le 0\}$ and $D_{(1)} = \{x \in \mathbb{R} :\ x \ge 0\}$, and $M = D_{(0)}^2 \cup D_{(1)}^2$. The kernel K_1 is also symmetric since $K_1(x,t) = K_1(-x,-t)$.

For $d > 1$, we obtain $F_d = W_0^{r,r,\dots,r}(\mathbb{R}^d) = W_0^r(\mathbb{R}) \otimes \cdots \otimes W_0^r(\mathbb{R})$ (d times) as the Sobolev space of smooth functions defined over $D_d = \mathbb{R}^d$ such that $D^\alpha f(x) = 0$ if at least one component of x is zero for any multi-index $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d]$ with integers $\alpha_i \in [0, r-1]$. Here D^α is the partial differential operator, $D^\alpha f = \partial^{|\alpha|} f / \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d$. The inner product of F_d is given by

$$\langle f, g \rangle_{F_d} \,=\, \int_{\mathbb{R}^d} D^{[r,r,\dots,r]} f(x)\, D^{[r,r,\dots,r]} g(x)\, dx.$$

Consider the *weighted integration* problem. That is,

$$I_1(f) \,=\, \int_{\mathbb{R}} \varrho(t)\, f(t)\, dt$$

for some measurable non-zero weight function $\varrho : \mathbb{R} \to \mathbb{R}_+$. It is easy to check that I_1 is a continuous linear functional iff the function

$$h_1(x) \,=\, \int_{\mathbb{R}} \varrho(t) K_1(x,t)\, dt$$

belongs to F_1, which holds iff

$$\int_{\mathbb{R}^2} \varrho(t)\varrho(x) K_1(x,t)\, dt\, dx \,<\, \infty. \tag{11}$$

It is easy to check that $K_1(x,t) = O(|t\,x|^{r-1/2})$, and (11) holds if

$$\int_{\mathbb{R}} \varrho(t)\, |t|^{r-1/2}\, dt \,<\, \infty.$$

The last condition imposes a restriction on the behavior of the weight ϱ at infinity. If (11) holds, then

$$I_1(f) \,=\, \langle f, h_1 \rangle_{F_1} \ \text{ and } \|I_1\| \,=\, \|h_1\|_{F_1} \,=\, \left(\int_{\mathbb{R}^2} \varrho(t)\varrho(x) K_1(x,t)\, dt\, dx \right)^{1/2} \,<\, \infty.$$

We also have

$$h_{1,(0)}(x) \;=\; \int_{-\infty}^{0} \varrho(t) K_1(x,t)\,dt \quad \text{and} \quad h_{1,(1)}(x) \;=\; \int_{0}^{\infty} \varrho(t) K_1(x,t)\,dt.$$

For $d > 1$, we have

$$I_d(f) \;=\; \int_{\mathbb{R}^d} \varrho_d(t)\, f(t)\,dt \quad \text{with} \quad \varrho_d(t) \;=\; \varrho(t_1)\varrho(t_2)\cdots\varrho(t_d).$$

We are ready to apply Theorem 1 for weighted integration. Observe that $D_{(0)} = \mathbb{R}_- \cup \{0\}$ and $D_{(1)} = \mathbb{R}_+ \cup \{0\}$. If the weight ϱ does not vanish (in the L_2 sense) over $\mathbb{R}_-$ and $\mathbb{R}_+$ then the norms of $h_{1,(0)}$ and $h_{1,(1)}$ are positive, and we have intractability of weighted integration in F_d. In particular, if we take a nonzero symmetric ϱ, i.e., $\varrho(t) = \varrho(-t)$, then Corollary 1 holds and $\alpha = 0.5$. This is the case for Gaussian integration, since $\varrho(t) = (2\pi)^{-1/2} \exp(-t^2/2)$ is symmetric. Hence, for Gaussian integration, we have

$$n(\varepsilon, F_d) \;\ge\; (1-\varepsilon^2)\, 2^d.$$

We stress that this intractability result holds independently of the assumed smoothness of functions, i.e., this holds for arbitrary large r.

We now take the constant weight ϱ over the interval $[a,b]$, where $a \le 0 < b$. More specifically, we assume that $\varrho(t) = 1$ for $t \in [a,b]$ and $\varrho(t) = 0$ otherwise. For simplicity, we only consider $r = 1$. Then $K_1(x,t) = 1_M(x,t) \min(|x|,|t|)$ and

$$\begin{aligned} h_{1,(0)}(x) &= \int_a^0 K_1(x,t)\,dt \;=\; \big(a \,\max(x,a) - 0.5\, \max(x,a)^2\big)_+, \\ h_{1,(1)}(x) &= \int_0^b K_1(x,t)\,dt \;=\; \big(b \,\min(x,b) - 0.5\, \min(x,b)^2\big)_+. \end{aligned}$$

From this we conclude that

$$\|h_{1,(0)}\|_{F_1}^2 \;=\; |a|^3/3 \quad \text{and} \quad \|h_{1,(1)}\|_{F_1}^2 \;=\; b^3/3.$$

Hence,

$$\alpha \;=\; \frac{\max(|a|^3, b^3)}{|a|^3 + b^3}.$$

Taking $b = 1$ and $|a| = t \in [0,1]$ we obtain $\alpha = \alpha(t) = 1/(1+t^3)$, which varies continuously with t and takes all values from $[0.5, 1]$. This shows that α in Theorem 1 can be an arbitrary number from $[0.5, 1]$. Obviously, for $a = 0$ Theorem 1 is trivial. This means that we do not yet know whether the problem

$$I_d(f) \;=\; \int_{[0,1]^d} f(t)\,dt$$

is intractable in F_d. □

Example: Uniform Integration

We now define $D_1 = [0,1]$ and take $F_1 = W_a^1([0,1])$ as the Sobolev space of absolutely continuous functions whose first derivatives are in $L_2([0,1])$ and whose function values are zero at the point a of the interval $[0,1]$. That is,

$$F_1 = \{f : [0,1] \to \mathbb{R} \;: f(a) = 0,\; f \text{ abs. cont. and } f' \in L_2([0,1])\}$$

with the inner product $\langle f, g\rangle_{F_1} = \int_0^1 f'(t)g'(t)\,dt$. It can be checked that this Hilbert space has the reproducing kernel

$$K_1(x,t) = 1_M(x,t)\cdot\min(|x-a|,|t-a|), \tag{12}$$

where $M = [0,a]\times[0,a]\cup[a,1]\times[a,1]$. We may also write

$$K_1(x,t) = \tfrac{1}{2}(|x-a| + |t-a| - |x-t|). \tag{13}$$

The kernel K_1 is *decomposable* with $a^* = a$. It is symmetric only if $a = \frac{1}{2}$. We have $D_{(0)} = [0,a]$ and $D_{(1)} = [a,1]$.

For $d > 1$, we obtain $F_d = W_a^{1,1,\dots,1}([0,1]^d) = W_a^1([0,1]) \otimes \cdots \otimes W_a^1([0,1])$ (d times) as the Sobolev space of smooth functions f defined over $D_d = [0,1]^d$ such that $f(x) = 0$ if at least one component of x is a. The inner product of F_d is given by

$$\langle f, g\rangle_{F_d} = \int_{[0,1]^d} \frac{\partial^d}{\partial x_1\cdots\partial x_d} f(x)\, \frac{\partial^d}{\partial x_1\cdots\partial x_d} g(x)\,dx.$$

Consider the *uniform integration* problem. That is,

$$I_1(f) = \int_0^1 f(t)\,dt.$$

It is easy to compute

$$\begin{aligned} h_{1,(0)}(x) &= \int_0^a \min(a-x, a-t)\,dt &=& \tfrac{1}{2}\,(a-x)(a+x) \qquad \forall\, x \in [0,a],\\ h_{1,(1)}(x) &= \int_a^1 \min(x-a, t-a)\,dt &=& \tfrac{1}{2}\,(x-a)(2-a-x) \quad \forall\, x \in [a,1]. \end{aligned}$$

Furthermore,

$$\|h_{1,(0)}\|_{F_1} = \frac{1}{3}\,a^3 \quad \|h_{1,(1)}\|_{F_1} = \frac{1}{3}\,(1-a)^3.$$

Hence, we have

$$\alpha = \frac{\max(a^3,(1-a)^3)}{a^3+(1-a)^3}.$$

From Theorem 1 we conclude that the problem I_d in F_d is intractable if $a \in (0,1)$. Observe also that if a is close to zero, say, then $\alpha^{-1} = 1 + a^3 + O(a^4)$ is barely larger than 1. This means that although α^{-d} goes exponentially fast to infinity with d we must take really large d to get large α^{-d}. □

Example: Centered Discrepancy

We now specialize the previous example by taking $a = \frac{1}{2}$. Then the kernel K_1 and the function h_1 are symmetric. Corollary 1 now applies with $\alpha = \frac{1}{2}$ and $\|h\|_{F_1} =$

$12^{-1/2}$. In this special case, we can say more about errors of algorithms, as well as we generalize the error estimates to the L_q norms with $q \in [1, \infty]$.

It can be checked that the error of the algorithm $Q_{n,d}(f) = \sum_{i=1}^n a_i f(z_i)$ is now given by

$$e^2(Q_{n,d}) \;=\; \int_{[0,1]^d} \left| \prod_{j=1}^d \min(x_j, 1-x_j) - \sum_{i=1}^n a_i \cdot 1_{J(b(x),x)}(z_i) \right|^2 dx, \tag{14}$$

where $J(b,x)$ is the cube generated by x and the vertex b of $[0,1]^d$. By $b(x)$ we mean the vertex of $[0,1]^d$ which is closest, in the sup-norm, to x. That is, $x \in D_{(b)}$ iff $b(x) = b$ for almost all $x \in [0,1]^d$.

Essentially the same formulas are presented by Hickernell [6] who considers spaces similar to F_d without assuming the condition $f(\frac{1}{2}) = 0$ and considers algorithms with $a_i = 1/n$. The error of the algorithm $Q_{n,d}$ with $a_i = 1/n$ is called by Hickernell the *centered discrepancy* and denoted by $d_2^c(Q_{n,d})$. We will be using the same name for arbitrary coefficients a_i. For the space F_d with the condition $f(\frac{1}{2}) = 0$ we denote the centered discrepancy by $\tilde{d}_2^c(n,d)$. In the next section we remove this condition and consider the centered discrepancy $d_2^c(Q_{n,d})$ (without the tilde), as originally studied by Hickernell [6].

The minimal centered discrepancy is defined as

$$\tilde{d}_2^c(n,d)^2 \;=\; \inf_{a_i, z_i, i=1,2,\dots,n} \int_{[0,1]^d} \left| \prod_{j=1}^d \min(x_j, 1-x_j) - \sum_{i=1}^n a_i \cdot 1_{J(b(x),x)}(z_i) \right|^2 dx. \tag{15}$$

Corollary 1 states that

$$\tilde{d}_2^c(n,d) \;\ge\; \left(1 - n\, 2^{-d}\right)_+^{1/2} \, \tilde{d}_2^c(0,d) \quad \text{with} \quad \tilde{d}_2^c(0,d) \;=\; 12^{-d/2}. \tag{16}$$

The centered discrepancy (and uniform integration) may be also considered for the space $L_q([0,1]^d)$ with $q \in [1,\infty]$. More precisely, for $d = 1$ we take the space $F_{1,q}$ as the Sobolev space of absolutely continuous functions whose first derivatives are in $L_q([0,1])$ and that vanish at $\frac{1}{2}$. The norm in $F_{1,q}$ is given by $\|f\|_{F_{1,q}} = \left(\int_0^1 |f'(t)|^q \, dt\right)^{1/q}$ for $q < \infty$, and $\|f\|_{F_{1,\infty}} = \operatorname{ess\,sup}_{t\in[0,1]} |f'(t)|$ for $q = \infty$. For $d > 1$, the space $F_{d,q}$ is taken as a tensor product of factors $F_{1,q}$. Then functions from $F_{d,q}$ vanish at x whenever at least one component of x is $\frac{1}{2}$ and their norm is

$$\|f\|_{F_{d,q}} = \|D^{\vec{1}} f\|_{L_q([0,1]^d)},$$

where $\vec{1} = [1,1,\dots,1]$. From

$$I_d(f) - Q_{n,d}(f) \;=\; \int_{[0,1]^d} D^{\vec{1}} f(t)\, D^{\vec{1}} \left(h_d - \sum_{i=1}^n a_i K_d(\cdot, z_i) \right)(t)\, dt$$

we conclude that

$$e(Q_{n,d}) \;:=\; \sup_{f\in F_{d,q}:\, \|f\|_{F_{d,q}}\le 1} \left| I_d(f) - Q_{n,d}(f) \right| \;=\; \tilde{d}_p^c(Q_{n,d}),$$

where $1/p + 1/q = 1$ and $\tilde{d}^c_p(Q_{n,d})$ is the centered p-discrepancy given by

$$\tilde{d}^c_p(Q_{n,d}) = \left(\int_{[0,1]^d} \left| \prod_{j=1}^d \min(x_j, 1-x_j) - \sum_{i=1}^n a_i \cdot 1_{J(b(x),x)}(z_i) \right|^p dx \right)^{1/p}.$$

If $q = 1$ then $p = \infty$ and, as usual, the integral is replaced by the ess sup in the formula above.

Let $e(n, F_{d,q}) = \tilde{d}_p(n, d)$ denote the minimal error, or equivalently the minimal centered p-discrepancy, that can be achieved by using n function values. The initial error, or the initial centered discrepancy, is now given by

$$e(0, F_{d,q}) \;=\; \tilde{d}^c_p(0, d) \;=\; \frac{2^{-d}}{(p+1)^{d/p}}$$

for $q > 1$, and $e(0, F_{d,1}) = 2^{-d}$.

Hence, for all values of q, the initial centered discrepancy is at most 2^{-d}. We are ready to prove the following corollary, which easily follows from the proof of Theorem 1.

Corollary 2. *For $n < 2^d$ and $p < \infty$, we have*

$$\tilde{d}^c_p(n, d) \;\ge\; \left(1 - n\, 2^{-d}\right)^{1/p} \tilde{d}^c_p(0, d).$$

Hence, uniform integration is intractable in $F_{d,q}$ since

$$n(\varepsilon, F_{d,q}) \;\ge\; (1 - \varepsilon^p) 2^d \quad \textit{and} \quad \lim_{d\to\infty} \frac{\tilde{d}^c_p(d^k, d)}{\tilde{d}^c_p(0, d)} \;=\; 1 \quad \forall\, k = 1, 2, \dots, .$$

Proof. As in the proof of Theorem 1, take a Boolean vector $b = [b_1, \dots, b_d] \in \{0,1\}^d$ and an algorithm $Q_{n,d}(f) = \sum_{i=1}^n a_i f(z_i)$. The worst case error of $Q_{n,d}$ is the centered discrepancy $\tilde{d}^c_p(Q_{n,d})$. The formula for $\tilde{d}^c_p(Q_{n,d})$ can be written as

$$\tilde{d}^c_p(Q_{n,d})^p = \sum_{b\in\{0,1\}^d} \int_{D(b)} \left| \prod_{j=1}^d \min(x_j, 1-x_j) - \sum_{i=1}^n a_i \cdot 1_{J(b,x)}(z_i)) \right|^p dx.$$

Observe that for each sample point z_i and for almost all x, there is at most one b such that $z_i \in J(b, x)$. Hence, at least $2^d - n$ terms in the last sum with respect to b have zero contribution from the algorithm $Q_{n,d}$. Since

$$\int_{D(b)} \prod_{j=1}^d \min(x_j, 1-x_j)^p dx \;=\; \left(\frac{2^{-p-1}}{p+1} \right)^d$$

does not depend on b, we obtain

$$\tilde{d}^c_p(Q_{n,d})^p \ge (2^d - n)\, 2^{-d} \left(\frac{2^{-p}}{p+1} \right)^d \;=\; \left(1 - n\, 2^{-d}\right) \tilde{d}^c_p(0, d)^p,$$

as claimed. The formula for $n(\varepsilon, d)$ and the value for the limit easily follow from the lower bound on $\tilde{d}^c_p(n, d)$. This completes the proof. □

Observe that for $q = 1$ we have $p = \infty$, and since $\tilde{d}^c_p(n,d)$ is a nondecreasing function of p we have

$$\tilde{d}^c_\infty(n,d) = \tilde{d}^c_\infty(0,d) = 2^{-d} \quad \forall\, n < 2^d.$$

It is known that $\tilde{d}^c_\infty(n,d)$ goes to zero for fixed d at least like $n^{-1}(\log n)^{d-1}$ as n tends to infinity. In view of the previous property we must wait, however, exponentially long in d to see this rate of convergence. □

Example: Two Function Values

We now show that the estimate of Theorem 1 is sharp. This will be done for a seemingly simple problem I_1 defined by only two function values. More precisely, consider the space F_1 with symmetric D_1 around $a^* = 0$ such that $\{-1,1\} \subset D_1$. Let the reproducing kernel K_1 of F_1 be symmetric and decomposable for $a^* = 0$. We assume that $K_1(1,1) > 0$. Define

$$I_1(f) = f(-1) + f(1).$$

Then $h_1(x) = K_1(x,-1) + K_1(x,1)$, and

$$\begin{aligned} h_{1,(0)}(x) &= K_1(x,-1) \quad \text{with} \qquad \|h_{1,(0)}\|^2_{F_1} = K_1(-1,-1) = K_1(1,1),\\ h_{1,(1)}(x) &= K_1(x,1) \quad \text{with} \qquad \|h_{1,(1)}\|^2_{F_1} = K_1(1,1). \end{aligned}$$

We show that

$$e(n,d) = \left(1 - n\,2^{-d}\right)_+^{1/2} e(0,d).$$

Due to Theorem 1, it is enough to find a matching upper bound, that is, an algorithm whose error $e(Q_{n,d}) \le \left(1 - n\,2^{-d}\right)_+^{1/2} e(0,d)$. Observe that

$$I_d(f) = \sum_{b\in\{-1,1\}^d} f(b)$$

and $h_d(x) = \sum_{b\in\{-1,1\}^d} K_d(x,b)$. From this, we have

$$\|h_d\|^2_{F_d} = \sum_{b\in\{-1,1\}^d} K(b,b) = 2^d K_d(\vec{1},\vec{1}) = \left(2K_1(1,1)\right)^d.$$

Define the algorithm $Q_{n,d}(f) = \sum_{b\in A_n} f(b)$ where A_n is a subset of $\{-1,1\}^d$ and has $\min(n,2^d)$ elements. Then

$$I_d(f) - Q_{n,d}(f) = \sum_{b\in\{-1,1\}^d - A_n} f(b)$$

and

$$e^2(Q_{n,d}) = \sum_{b\in\{-1,1\}^d - A_n} K_d(b,b) = (2^d - \min(n,2^d))K_1^d(1,1) = \left(1 - n\,2^{-d}\right)_+ e^2(0,F_d),$$

as claimed. □

4. Non-decomposable Kernels

In the previous section, we showed intractability for decomposable kernels. In this section, we show similar results for certain non-decomposable kernels.

In what follows we consider two reproducing kernels R_1 and R_2 each defined over D_1^2. Let $H(R_i)$ denote the Hilbert space with the reproducing kernel R_i. We assume that $H(R_1) \cap H(R_2) = \{0\}$.

Define the reproducing kernel K_1 as the sum of the two kernels,

$$K_1 \;=\; R_1 + R_2. \tag{17}$$

The Hilbert space F_1 with the kernel K_1 is the space of functions of the form $f_1 + f_2$ for $f_i \in H(R_i)$ with the inner product

$$\langle f, g\rangle_{F_1} \;=\; \langle f_1, g_1\rangle_{H(R_1)} + \langle f_2, g_2\rangle_{H(R_2)}\,,$$

see [1] p. 353.

As in the previous section, we define $I_1(f) = \langle f, h_1\rangle_{F_1}$ for some function h_1 from F_1. The function h_1 has the unique decomposition

$$h_1 \;=\; h_{1,1} + h_{1,2}, \quad \text{with } h_i \in H(R_i).$$

We will later assume that the kernel R_2 is decomposable. Hence, the kernel K_1 has one term R_1, which may be non-decomposable, and one decomposable term R_2. As in the previous section, by $h_{1,2,(0)}$ and $h_{1,2,(1)}$ we denote the restriction of the function $h_{1,2}$ to the sets $D_{(0)}$ and $D_{(1)}$, respectively.

We now generalize the problem by considering *weighted* tensor product problems. Define $F_{1,\gamma}$ as the Hilbert space with the weighted reproducing kernel $K_{1,\gamma}$,

$$K_{1,\gamma} \;=\; R_1 \;+\; \gamma\, R_2, \tag{18}$$

where γ is a non-negative parameter. Although, its role in the one dimensional case is not very important, as we shall see, the parameter γ will be crucial in the multivariate case. For $\gamma = 1$, we obtain $K_{1,1} = K_1$ and $F_{1,1} = F_1$. Without loss of generality we assume below that γ is positive. The case $\gamma = 0$ can be recovered by passing in the limit with positive γ to zero.

Observe that for any reproducing kernel R we have $H(\gamma R) = H(R)$, and their inner products are related by

$$\langle f, g\rangle_{H(\gamma R)} \;=\; \gamma^{-1}\,\langle f, g\rangle_{H(R)}\,.$$

We now consider the functional I_1 over the space $F_{1,\gamma}$. This functional is still a continuous linear functional since $F_{1,\gamma} = F_1$ and their norms are equivalent. Hence, there exists $h_{1,\gamma} \in F_{1,\gamma}$ such that

$$I_1(f) \;=\; \langle f, h_{1,\gamma}\rangle_{F_{1,\gamma}} \quad \forall\, f \in F_{1,\gamma}.$$

It is easy to show that

$$h_{1,\gamma} \;=\; h_{1,1} + \gamma h_{1,2}.$$

Observe that

$$\|h_{1,\gamma}\|^2_{F_{1,\gamma}} \;=\; \|h_{1,1}\|^2_{H(R_1)} + \gamma^2\|h_{1,2}\|^2_{H(\gamma R_2)} \;=\; \|h_{1,1}\|^2_{H(R_1)} + \gamma\|h_{1,2}\|^2_{H(R_2)}. \tag{19}$$

We are ready to discuss the multivariate case. For $d > 1$, we take

$$F_{d,\gamma} \;=\; F_{1,\gamma_1} \otimes F_{1,\gamma_2} \otimes \cdots \otimes F_{1,\gamma_d}$$

as the tensor product of F_{1,γ_i} for some sequence of positive weights γ_i. Without loss of generality, we assume that the weights γ_i are ordered, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_d$. The reproducing kernel of $F_{d,\gamma}$ is

$$K_{d,\gamma}(x,t) \;=\; \prod_{j=1}^d \left(R_1(x_j,t_j) \;+\; \gamma_j\, R_2(x_j,t_j)\right).$$

Observe that if $\gamma_i = 1$ for all i, we obtain $F_{d,\gamma} = F_d$ and $K_{d,\gamma} = K_d$ as in the previous section. The continuous linear functional I_d is defined without any change as the d folded tensor products of I_1. That is,

$$I_d(f) \;=\; \langle f, h_d\rangle_{F_d} \;=\; \langle f, h_{d,\gamma}\rangle_{F_{d,\gamma}},$$

where

$$\begin{aligned} h_d(x) \;&=\; \prod_{j=1}^d h_1(x_j) \;=\; \prod_{j=1}^d \left(h_{1,1}(x_j) + h_{1,2}(x_j)\right), \\ h_{d,\gamma}(x) \;&=\; \prod_{j=1}^d h_{1,\gamma_j}(x_j) \;=\; \prod_{j=1}^d \left(h_{1,1}(x_j) + \gamma_j h_{1,2}(x_j)\right). \end{aligned}$$

From (19) we have

$$e^2(0,F_{d,\gamma}) \;=\; \|h_{d,\gamma}\|^2_{F_{d,\gamma}} \;=\; \prod_{j=1}^d \left(\|h_{1,1}\|^2_{H(R_1)} + \gamma_j \|h_{1,2}\|^2_{H(R_2)}\right).$$

We are ready to present a lower bound on the nth minimal error $e(n,F_{d,\gamma})$ of approximating I_d over the unit ball of the space $F_{d,\gamma}$.

Theorem 2. *Assume that $H(R_1) \cap H(R_2) = \{0\}$ and R_2 is decomposable. Then*

$$e^2(n,F_{d,\gamma}) \geq \sum_{k=0}^d C_{d,k} \left(1 \;-\; n\,\alpha^k\right)_+ \alpha_1^{d-k}\alpha_2^k, \quad (0^0 = 1),$$

where

$$C_{d,k} \;=\; \sum_{u \subset \{1,2,\dots,d\}\,:\,|u|=k} \;\prod_{j\in u} \gamma_j, \quad (C_{d,0} = 1),$$

$$\alpha_1 \;=\; \|h_{1,1}\|^2_{H(R_1)},\;\; \alpha_2 \;=\; \|h_{1,2}\|^2_{H(R_2)},\;\; \alpha \;=\; \frac{\max(\|h_{1,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,(1)}\|^2_{H(R_2)})}{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}}.$$

Proof. For $x \in D_1^d$ and $u \subset \{1,2,\dots,d\}$, by x_u we mean the vector from $D_1^{|u|}$ whose components corresponding to indices in u are the same as the components of the vector x, i.e., $x_{u,j} = x_j$ for all $j \in u$. By $x_{\overline{u}}$ we mean $x_{\{1,2,\dots,d\}-u}$.

Define the functions $h_{u,1} : D_1^{d-|u|} \to \mathbb{R}$ and $h_{u,2} : D_1^{|u|} \to \mathbb{R}$ by

$$h_{u,1}(x_{\overline{u}}) \;=\; \prod_{j\notin u} h_{1,1}(x_j), \quad \text{and} \quad h_{u,2}(x_u) \;=\; \prod_{j\in u} \gamma_j h_{1,2}(x_j).$$

For $u = \emptyset$ or $u = \{1, 2, \ldots, d\}$, we take $h_{\emptyset,2} = 1$ and $h_{\{1,2,\ldots,d\},1} = 1$. Let

$$R_{u,1}(x_{\overline{u}}, t_{\overline{u}}) \;=\; \prod_{j\notin u} R_1(x_j, t_j) \quad \text{and} \quad R_{u,2}(x_u, t_u) = \prod_{j\in u} \gamma_j R_2(x_j, t_j)$$

be the reproducing kernels of the Hilbert spaces $H(R_{u,1})$ and $H(R_{u,2})$. We take $R_{\emptyset,2} = 1$ and $R_{\{1,\ldots,d\},1} = 1$. Then $h_{u,i} \in H(R_{u,i})$, and

$$\|h_{u,1}\|_{H(R_{u,1})} \;=\; \|h_{1,1}\|_{H(R_1)}^{d-|u|} \quad \text{and} \quad \|h_{u,2}\|_{H(R_{u,2})} \;=\; \|h_{1,2}\|_{H(R_2)}^{|u|} \prod_{j\in u} \gamma_j^{1/2}.$$

For the function $h_{d,\gamma}$ we have

$$h_{d,\gamma}(x) \;=\; \prod_{j=1}^{d} (h_{1,1}(x_j) + \gamma_j h_{1,2}(x_j)) \;=\; \sum_{u\subset\{1,2,\ldots,d\}} h_{u,1}(x_{\overline{u}})\, h_{u,2}(x_u). \tag{20}$$

Furthermore,

$$\|h_{d,\gamma}\|_{F_{d,\gamma}}^2 \;=\; \prod_{j=1}^{d} \left(\|h_{1,1}\|_{H(R_1)}^2 + \gamma_j \|h_{1,2}\|_{H(R_2)}^2\right) \;=\; \sum_{u\subset\{1,2,\ldots,d\}} \|h_{u,1}\|_{H(R_{u,1})}^2 \|h_{u,2}\|_{H(R_{u,2})}^2. \tag{21}$$

Consider now the reproducing kernel $K_{d,\gamma}$ of the space $F_{d,\gamma}$. We have

$$K_{d,\gamma}(x, t) \;=\; \prod_{j=1}^{d} (R_1(x_j, t_j) + \gamma_j R_2(x_j, t_j)) \;=\; \sum_{u\subset\{1,2,\ldots,d\}} R_{u,1}(x_{\overline{u}}, t_{\overline{u}})\, R_{u,2}(x_u, t_u). \tag{22}$$

We are ready to estimate from below the worst case error of an algorithm $Q_{n,d}(f) = \sum_{i=1}^{n} a_i f(z_i)$. As always we have

$$e^2(Q_{n,d}) \;=\; \|h_{d,\gamma}\|_{F_{d,\gamma}}^2 - 2\sum_{i=1}^{n} a_i h_{d,\gamma}(z_i) + \sum_{i,j=1}^{n} a_i a_j K_{d,\gamma}(z_i, z_j).$$

Using (20), (21) and (22) we get $e^2(Q_{n,d}) = \sum_{u\subset\{1,2,\ldots,d\}} e_u$ where

$$\begin{aligned} e_u \;&=\; \|h_{u,1}\|_{H(R_{u,1})}^2 \|h_{u,2}\|_{H(R_{u,2})}^2 - 2\sum_{i=1}^{n} a_i h_{u,1}((z_i)_{\overline{u}}) h_{u,2}((z_i)_u) \\ &+\; \sum_{i,j=1}^{n} a_i a_j R_{u,1}((z_i)_{\overline{u}}, (z_j)_{\overline{u}}) R_{u,2}((z_i)_u, (z_j)_u). \end{aligned}$$

There exists an orthonormal basis $r_k = r_{k,u}$ of the space $H(R_{u,1})$, where $k \in \mathcal{I}$ and $\mathcal{I}$ is an index set that is at most countable if the space is separable. It is known that the reproducing kernel $R_{u,1}$ may be written as

$$R_{u,1}(x_{\overline{u}},t_{\overline{u}}) \;=\; \sum_{k\in\mathcal{I}} r_k(x_{\overline{u}})\, r_k(t_{\overline{u}}).$$

Even for an uncountable set $\mathcal{I}$ the last series has always at most a countable number of non-zero terms. Furthermore,

$$h_{u,1} \;=\; \sum_{k\in\mathcal{I}} \langle h_{u,1},r_k\rangle_{H(R_{u,1})}\, r_k \quad\text{and}\quad \|h_{u,1}\|^2_{H(R_{u,1})} \;=\; \sum_{k\in\mathcal{I}} \langle h_{u,1},r_k\rangle^2_{H(R_{u,1})}\,.$$

We can thus rewrite e_u as $e_u=\sum_{k\in\mathcal{I}} e_{u,k}$ with

$$\begin{aligned} e_{u,k} \;&=\; \langle h_{u,1},r_k\rangle^2_{H(R_{u,1})}\,\|h_{u,2}\|^2_{H(R_{u,2})} - 2\sum_{i=1}^n a_i\,\langle h_{u,1},r_k\rangle_{H(R_{u,1})}\, r_k((z_i)_{\overline{u}})h_{u,2}((z_i)_u) \\ &+\; \sum_{i,j=1}^n a_ia_jr_k((z_i)_{\overline{u}})r_k((z_j)_{\overline{u}})R_{u,2}((z_i)_u,(z_j)_u). \end{aligned}$$

Assume for a moment that $\langle h_{u,1},r_k\rangle_{H(R_{u,1})}\neq 0$, and define

$$a_i' \;=\; \frac{a_ir_k((z_i)_{\overline{u}})}{\langle h_{u,1},r_k\rangle_{H(R_{u,1})}}.$$

Then $e_{u,k}=\langle h_{u,1},r_k\rangle^2_{H(R_{u,1})}\,e'_{u,k}$ with

$$e'_{u,k} \;=\; \|h_{u,2}\|^2_{H(R_{u,2})} - 2\sum_{i=1}^n a_i'h_{u,2}((z_i)_u) + \sum_{i,j=1}^n a_i'a_j'R_{u,2}((z_i)_u,(z_j)_u).$$

Observe that $e'_{u,k}$ is the square of the worst case error of approximating $I_u(f)=\langle f,h_{u,2}\rangle_{H(R_{u,2})}$ in the unit ball of the space $H(R_{u,2})$. The reproducing kernel $R_{u,2}$ is a tensor product of R_2 which is decomposable. Therefore we may apply Theorem 1 to obtain

$$e_{u,k} \;\ge\; \langle h_{u,1},r_k\rangle^2_{H(R_{u,1})}\,\left(1\,-\,n\,\alpha^{|u|}\right)_+\,\|h_{u,2}\|^2_{H(R_{u,2})}.$$

Observe that the last inequality is also true if $\langle h_{u,1},r_k\rangle_{H(R_{u,1})}=0$.

Summing up with respect to k we obtain

$$e_u \;\ge\; \|h_{u,1}\|^2_{H(R_{u,1})}\,\left(1\,-\,n\,\alpha^{|u|}\right)_+\,\|h_{u,2}\|^2_{H(R_{u,2})} \;=\; \alpha_1^{d-|u|}\,\left(1\,-\,n\,\alpha^{|u|}\right)_+\,\alpha_2^{|u|}\prod_{j\in u}\gamma_j.$$

Summing up with respect to u we obtain

$$e^2(Q_{n,d}) \;\ge\; \sum_{k=0}^d\left(\sum_{u:\,|u|=k}\;\prod_{j\in u}\gamma_j\right)\,\left(1-n\,\alpha^k\right)_+\alpha_1^{d-k}\alpha_2^k.$$

Since this holds for any algorithm $Q_{n,d}$, the same lower bound holds for the nth minimal error $e^2(n,F_{d,\gamma})$. This completes the proof. □

Observe that Theorem 2 generalizes Theorem 1. Indeed, it is enough to take $h_{1,1} = 0$ (which always holds for $R_1 = 0$). Then $\alpha_1 = 0$ and the sum in Theorem 2 consists of only one term for $k = d$. We also have $C_{d,d} = \prod_{j=1}^{d} \gamma_j$ and $C_{d,d}\alpha_2^d = e^2(0, F_{d,\gamma})$. Thus

$$e(n, F_{d,\gamma}) \;\ge\; \left(1 - n\,\alpha^d\right)_+^{1/2}\, e(0, F_{d,\gamma}), \tag{23}$$

as in Theorem 1. If any γ_j is zero then the problem is trivial since $e(0, F_{d,\gamma}) = 0$; otherwise the problem is intractable for $\alpha < 1$.

For $h_{1,1} = 0$, the role of the weights γ_j is insignificant since the minimal errors are proportional to $\prod_{j=1}^{d} \gamma_j^{1/2}$, and the ratio $e(n, F_{d,\gamma})/e(0, F_{d,\gamma})$ does not depend on γ_j. As we shall see, the role of γ_j is very important if $h_{1,1} \neq 0$.

Theorem 2 presents a lower bound on the nth minimal error. It is easy to relate this lower bound to the initial error. We have

$$e^2(0, F_{d,\gamma}) \;=\; \prod_{j=1}^{d} (\alpha_1 + \gamma_j \alpha_2) \;=\; \sum_{u \subset \{1,2,\dots,d\}} \alpha_1^{d-|u|} \alpha_2^{|u|} \prod_{j \in u} \gamma_j \;=\; \sum_{k=0}^{d} C_{d,k} \alpha_1^{d-k} \alpha_2^k.$$

If we assume that $n \le a\alpha^{-m}$ for some $a \in (0,1)$ with $m < d$ we can estimate $(1 - n\alpha^k)_+$ by zero for $k < m$ and by $1 - a$ for $k \ge m$. This yields the following corollary.

Corollary 3. *Let $n \le a\alpha^{-m}$ for some $a \in (0,1)$ with $m < d$. Under the assumptions of Theorem 2 we have*

$$e(n, F_{d,\gamma}) \;\ge\; (1 - a)^{1/2} \left(1 - \frac{\sum_{k=0}^{m-1} C_{d,k} \alpha_1^{d-k} \alpha_2^k}{\sum_{k=0}^{d} C_{d,k} \alpha_1^{d-k} \alpha_2^k} \right)^{1/2} e(0, F_{d,\gamma}).$$

We now discuss tractability of I_d in $F_{d,\gamma}$. It is evident from Corollary 3 that tractability depends on the weights γ_j. In particular, for the extreme case $\gamma_j = 0$ for all j, the only non-zero $C_{d,k}$ is for $k = 0$ and the lower bounds in Theorem 2 and Corollary 3 trivialize to zero for $n \ge 1$. On the other hand, if we take $\gamma_j = 1$ for all j then $C_{d,k} = \binom{d}{k}$ and it is easy to conclude intractability from Corollary 3 for positive α_1 and α_2. The next theorem addresses how the behavior of the weights γ_j implies a lack of either strong tractability or tractability.

Theorem 3. *Assume that $H(R_1) \cap H(R_2) = \{0\}$ and R_2 is decomposable. Assume also that both $h_{1,2,(0)}$ and $h_{1,2,(1)}$ are non-zero.*

(1) *If $h_{1,1} = 0$ then I_d in $F_{d,\gamma}$ is intractable for positive γ_j.*
(2) *If $h_{1,1}$ is non-zero and $\sum_{j=1}^{\infty} \gamma_j = \infty$ then*

$$\lim_{d \to \infty} \frac{e(n, F_{d,\gamma})}{e(0, F_{d,\gamma})} = 1 \quad \forall\, n,$$

and I_d in $F_{d,\gamma}$ is not strongly tractable.

(3) *If* $h_{1,1}$ *is non-zero and* $\lim^*_{d\to\infty} \frac{\sum_{j=1}^d \gamma_j}{\ln\, d} = \infty$ *then*

$$\lim^*_{d\to\infty} \frac{e(d^q, F_{d,\gamma})}{e(0, F_{d,\gamma})} = 1 \quad \forall\, q = 1, 2, \dots ,$$

and I_d *in* $F_{d,\gamma}$ *is intractable. Here,* $\lim^* \in \{\lim, \limsup\}$.

(4) *If* $h_{1,1}$ *is non-zero and* $\gamma_j \ge \gamma_1^* > 0$ *then*

$$\lim_{d\to\infty} \frac{e(\lfloor b^d \rfloor, F_{d,\gamma})}{e(0, F_{d,\gamma})} = 1 \quad \forall\, b \in (1, \alpha^{-c}),$$

where

$$\alpha = \frac{\max(\|h_{1,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,(1)}\|^2_{H(R_2)})}{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}},$$

and c, $c \in (0,1)$, *satisfies the two inequalities*

$$c \le \alpha_3\gamma_1^* \ \text{and}\ (1 + \ln\, \alpha_3\gamma_1^* - \ln\, c)c < \ln(1 + \alpha_3\gamma_1^*),$$

with

$$\alpha_3 = \|h_{1,2}\|^2_{H(R_2)}\|h_{1,1}\|^{-2}_{H(R_1)}.$$

Proof. The case (1), $h_{1,1} = 0$, follows from (23). For $h_{1,1} \ne 0$, Theorem 2 and the form of the initial error yield

$$1 \ge \frac{e^2(n, F_{d,\gamma})}{e^2(0, F_{d,\gamma})} \ge \frac{\sum_{k=0}^d C_{d,k}\alpha_3^k(1 - n\alpha^k)_+}{\sum_{k=0}^d C_{d,k}\alpha_3^k}.$$

Define $\gamma_j' = \alpha_3\gamma_j$ and

$$C'_{d,k} = \sum_{u\subset\{1,2,\dots,d\},\ |u|=k} \prod_{j\in u} \gamma_j' = \alpha_3^k C_{d,k}.$$

Then we have

$$1 \ge \frac{e^2(n, F_{d,\gamma})}{e^2(0, F_{d,\gamma})} \ge \frac{\sum_{k=0}^d C'_{d,k}(1 - n\alpha^k)_+}{\sum_{k=0}^d C'_{d,k}}. \tag{24}$$

To prove (2) and (3), we take n fixed or $n = d^q$ for an arbitrary integer q. Then for any $a \in (0,1)$ there exist non-negative β_1 and β_2 such that for $k(d,\beta) = \lceil \beta_2 + \beta_1\, \ln\, d \rceil$ we have

$$n\alpha^k \le a \quad \text{for } k \in [k(d,\beta), d].$$

Here, we take $\beta_1 = 0$ for a fixed n and d so large that $d > k(d,\beta)$. From (24) we conclude

$$\frac{e^2(n, F_{d,\gamma})}{e^2(0, F_{d,\gamma})} \ge (1-a)\, \frac{\sum_{k=k(d,\beta)+1}^d C'_{d,k}}{\sum_{k=0}^d C'_{d,k}} = (1-a)\,(1 - \alpha_{d,\beta})\,,$$

where

$$\alpha_{d,\beta} \;=\; \frac{\sum_{k=0}^{k(d,\beta)} C'_{d,k}}{\sum_{k=0}^{d} C'_{d,k}}.$$

To prove (2) and (3) it is enough to show that $\alpha_{d,\beta}$ goes to zero. Indeed, this implies that for large d the ratio $e(n, F_{d,\gamma})/e(0, F_{d,\gamma})$ is greater than roughly $1-a$. Hence, the limit must be at least $1-a$ and since a can be arbitrary small, the limit must be 1. This also means that $n = \lfloor C d^r \varepsilon^{-p} \rfloor$ function values are not enough to solve the problem to within ε, which contradicts strong tractability (if $r=0$) or tractability (with arbitrary r).

To prove that $\alpha_{d,\beta}$ goes to zero, observe that $C'_{d,0} = 1$ and $C'_{d,1} = s_d := \sum_{j=1}^d \gamma'_j$. We now show that

$$C'_{d,k} \;\le\; \frac{s_d^k}{k!}.$$

Indeed, we have

$$s_d^k \;=\; \sum_{i_1,i_2,\dots,i_k \in \{1,2,\dots,d\}} \prod_{j=1}^k \gamma'_{i_j}.$$

Note that each term in $C'_{d,k}$ is indexed by $u = \{u_1, u_2, \dots, u_k\}$ for distinct u_i's from $\{1,2,\dots,d\}$. Such terms appear in s_d^k. In fact, each term appears $k!$ times for all permutations of u_i's. Therefore $k! C'_{d,k} \le s_d^k$, as claimed. Furthermore, the last inequality is not sharp for $k \ge 2$ since there are terms in s_d^k with no distinct i_j's that are not present in $C'_{d,k}$. Hence, we have

$$\sum_{k=0}^{k(d,\beta)} C'_{d,k} \;\le\; \sum_{k=0}^{k(d,\beta)} \frac{s_d^k}{k!} \;\le\; \sum_{k=0}^{k(d,\beta)} s_d^k.$$

Since s_d goes to infinity, $s_d \ge 2$ for large d and we can estimate the last sum by $s_d^{k(d,\beta)+1}$. Therefore

$$\sum_{k=0}^{k(d,\beta)} C'_{d,k} \;\le\; s_d^{k(d,\beta)+1} \;\le\; (d\gamma_1\alpha_3)^{\beta_2+2+\beta_1 \ln d}.$$

Observe also that

$$\sum_{k=0}^{d} C'_{d,k} \;=\; \prod_{j=1}^{d}(1+\gamma'_j) \;=\; \exp\left(\sum_{j=1}^{d} \ln\left(1+\gamma'_j\right)\right).$$

Assume first that $\lim_{j\to\infty} \gamma_j \ge \gamma > 0$. Then

$$\sum_{k=0}^{d} C'_{d,k} \;\ge \exp\left(d \,\ln(1+\gamma\alpha_3)\right).$$

Therefore

$$\alpha_{d,\beta} \;\le\; \frac{(d\gamma_1\alpha_3)^{\beta_2+2+\beta_1 \ln d}}{\exp\left(d\,\ln(1+\gamma\alpha_3)\right)} \;\to\; 0 \quad \text{as } d \to \infty.$$

Assume next that $\lim_{j\to\infty}\gamma_j = 0$. We now have

$$\exp\left(\sum_{j=1}^d \ln\left(1+\gamma_j'\right)\right) \;=\; \Theta\left(\exp(s_d)\right),$$

and

$$\frac{s_d^{\beta_2+2+\beta_1\,\ln\,d}}{\exp(s_d)} \;=\; \exp\left(-s_d + (\beta_2+2+\beta_1\,\ln\,d)\,\ln\,s_d\right).$$

To prove the lack of strong tractability take $\beta_1 = 0$. Then

$$\alpha_{d,\beta} \;=\; O\left(\exp\left(-s_d\left(1-(\beta_2+2)(\ln\,s_d)/s_d\right)\right)\right)$$

goes to zero as d goes to infinity.

Assume finally that $\lim_d^* s_d/\ln\,d = \infty$. We now use

$$\sum_{k=0}^{k(d,\beta)} C'_{d,k} \;\le\; \sum_{k=0}^{k(d,\beta)} \frac{s_d^k}{k!}.$$

Observe that $s_d^k/k!$ is an increasing function of k over the interval $[0,k^*]$ as long as $s_d \ge k^*$. Since $k(d,\beta) = O(\ln\,d)$, we have $s_d \ge k(d,\beta)$ for large d. Hence

$$\sum_{k=0}^{k(d,\beta)} \frac{s_d^k}{k!} \;\le\; k(d,\beta)\,\frac{s_d^{k(d,\beta)}}{k(d,\beta)!} = \exp\left(k(d,\beta)\,\ln\,s_d \;-\; \ln\left(k(d,\beta)-1\right)!\right).$$

Using Stirling's formula for factorials, we get

$$\begin{aligned}\sum_{k=0}^{k(d,\beta)} C'_{d,k} \;&\le\; \exp\left(k(d,\beta)\,\ln\,s_d \;-\; (k(d,\beta)-1)\,\ln\left(k(d,\beta)-1\right) \;+\; o(1)\right)\\ &\le\; \exp\left(k(d,\beta)\,\ln\,\frac{s_d}{k(d,\beta)-1} \;+\; O(\ln\,d)\right)\\ &\le\; \exp\left(O\left(\ln\,d\left(\ln\,\frac{s_d}{\ln\,d} \;+\; O(1)\right)\right)\right).\end{aligned}$$

Therefore

$$\begin{aligned}\alpha_{d,\beta} \;&=\; O\left(\exp\left(-s_d + O\left(\ln\,d\left(\ln\,\frac{s_d}{\ln\,d} + O(1)\right)\right)\right)\right)\\ &=\; O\left(\exp\left(-\ln\,d\left(\frac{s_d}{\ln\,d} + O\left(\ln\,\frac{s_d}{\ln\,d} + O(1)\right)\right)\right)\right)\end{aligned}$$

goes to zero for some subsequence d_i as i goes to infinity in the case $\lim^* = \limsup$, and simply goes to zero if $\lim^* = \lim$. This completes the proof of (2) and (3).

To prove (4), we take $a \in (0,1)$, $n = \lfloor b^d \rfloor$ and $k(d) = \lfloor cd \rfloor$. Then

$$n\alpha^k \;\le\; b^d\alpha^{cd-1} \;=\; \alpha^{-1}\,(b\alpha^c)^d \;\le\; a \quad \forall\, k \in [k(d), d],$$

for sufficiently large d. From (24) we have

$$\frac{e^2(n,F_{d,\gamma})}{e^2(0,F_{d,\gamma})} \ge (1-a)\,\frac{\sum_{k=k(d)+1}^d C'_{d,k}}{\sum_{k=0}^d C'_{d,k}} \;=\; (1-a)\,(1-\alpha_d),$$

where

$$\alpha(d) \;=\; \frac{\sum_{k=0}^{k(d)} C'_{d,k}}{\sum_{k=0}^{d} C'_{d,k}}.$$

For $k=0$ we have $C'_{d,0}=1$, and for $k=1,2,\dots,d$, we have

$$\begin{aligned} C'_{d,k} \;&=\; \gamma'_d \sum_{u\subset\{1,\dots,d-1\},\,|u|=k-1}\ \prod_{j\in u}\gamma'_j \;+\; \sum_{u\subset\{1,\dots,d-1\},\,|u|=k}\ \prod_{j\in u}\gamma'_j \\ &=\; \gamma'_d C'_{d-1,k-1} \;+\; C'_{d-1,k} \quad \text{with } C'_{d-1,d}=0. \end{aligned}$$

Therefore

$$\begin{aligned} \alpha(d) \;&=\; \frac{1+\gamma'_d\sum_{k=1}^{k(d)} C'_{d-1,k-1} + \sum_{k=1}^{k(d)} C'_{d-1,k}}{1+\gamma'_d\sum_{k=1}^{d} C'_{d-1,k-1} + \sum_{k=1}^{d-1} C'_{d-1,k}} \\ &=\; \frac{\gamma'_d\sum_{k=0}^{k(d)-1} C'_{d-1,k} + \sum_{k=0}^{k(d)} C'_{d-1,k}}{\gamma'_d\sum_{k=0}^{d-1} C'_{d-1,k} + \sum_{k=0}^{d-1} C'_{d-1,k}}. \end{aligned}$$

From this formula it easily follows that $\alpha(d)$ is a non-increasing function of γ'_d. Since all γ'_j play the same role in $\alpha(d)$, we conclude that $\alpha(d)$ is maximized for the smallest values of γ'_j. Since $\gamma'_j=\alpha_3\gamma_j\ge\alpha_3\gamma_1^*$ we obtain

$$\alpha(d) \;\le\; \frac{\sum_{k=0}^{k(d)} C^*_{d,k}}{\sum_{k=0}^{d} C^*_{d,k}},$$

where

$$C^*_{d,k} \;=\; \sum_{u\subset\{1,2,\dots,d\},\,|u|=k\}}\ \prod_{j\in u}\alpha_3\gamma_1^* \;=\; (\alpha_3\gamma_1^*)^k\binom{d}{k}.$$

Therefore

$$\alpha(d) \;\le\; \frac{\sum_{k=0}^{k(d)}(\alpha_3\gamma_1^*)^k\binom{d}{k}}{(1+\alpha_3\gamma_1^*)^d}.$$

Observe that $(\alpha_3\gamma_1^*)^k\binom{d}{k}\le(\alpha_3\gamma_1^* d)^k/k!$. The last function of k is increasing for $k\le\alpha_3\gamma_1^* d$. Since $k\;\le\;\lfloor cd\rfloor\;<\alpha_3\gamma_1^* d$, we have

$$\alpha(d) \;\le\; \frac{(\lfloor cd\rfloor+1)(\alpha_3\gamma_1^* d)^{\lfloor cd\rfloor}}{\lfloor cd\rfloor!(1+\alpha_3\gamma_1^*)^d}.$$

By Stirling's formula, we obtain for large d,

$$\alpha(d) \;\le\; \frac{(\lfloor cd\rfloor+1)}{\sqrt{2\pi}\,\lfloor cd\rfloor^{1/2}}\left(\frac{\alpha_3\gamma_1^* d}{\lfloor cd\rfloor}\right)^{\lfloor cd\rfloor}\frac{\exp(\lfloor cd\rfloor)}{(1+\alpha_3\gamma_1^*)^d}\,(1+o(1))\,.$$

This goes to zero since $(1+\ln\,\alpha_3\gamma_1^*-\ln\,c)c<\ln(1+\alpha_3\gamma_1^*)$. Hence, for any δ there exists $d(\delta)$ such that for all $d\ge d(\delta)$ we have

$$1 \;\ge\; \frac{e^2(n,F_{d,\gamma})}{e^2(0,F_{d,\gamma})} \;\ge\; 1-a-\delta.$$

Since a and δ can be arbitrarily small, the proof of (4) is complete. $\square$

We now illustrate Theorems 2 and 3 by continuing the examples of the previous section.

Example: Weighted Integration (continued)

We now remove the conditions $f^{(j)}(0)$ from the Sobolev space $W_0^r(\mathbb{R})$ and consider $F_{1,\gamma} = W^r(\mathbb{R})$ as the Sobolev space equipped with the inner product

$$\langle f, g \rangle_{F_{1,\gamma}} \;=\; \sum_{j=0}^{r-1} f^{(j)}(0) g^{(j)}(0) \;+\; \gamma^{-1} \int_{\mathbb{R}} f^{(r)}(t) g^{(r)}(t)\, dt.$$

The reproducing kernel $K_{1,\gamma}$ takes now the form

$$K_{1,\gamma}(x,t) \;=\; R_1(x,t) \;+\; \gamma\, 1_M(x,t)\, R_2(x,t),$$

where 1_M is the characteristic (indicator) function of the set $M = \{(x,t) :\; xt \geq 0\}$, and

$$\begin{aligned} R_1(x,t) &\;=\; \sum_{j=0}^{r-1} \frac{x^j}{j!} \frac{t^j}{j!}, \\ R_2(x,t) &\;=\; 1_M(x,t) \int_{\mathbb{R}_+} \frac{(|t|-u)_+^{r-1} \, (|x|-u)_+^{r-1}}{((r-1)!)^2}\, du. \end{aligned}$$

The space $H(R_1)$ is the space of polynomials of degree at most $r-1$ and has dimension r. The space $H(R_2) = W_0^r(\mathbb{R})$ is the Sobolev space with the conditions $f^{(j)}(0) = 0$ for $j = 0, 1, \dots, r-1$. Clearly $H(R_1) \cap H(R_2) = \{0\}$. The kernel R_2 is decomposable with $a^* = 0$ (and also symmetric), so all the assumptions of Theorem 2 hold.

For $d > 1$, $F_{d,\gamma}$ is the Sobolev space $W^r(\mathbb{R}) \otimes \cdots \otimes W^r(\mathbb{R})$ (d times) whose inner product $\langle f, g \rangle_{F_{d,\gamma}}$ is a sum of $(r+1)^d$ terms. In the case $r = 1$ and $d = 2$ we obtain

$$\begin{aligned} \langle f, g \rangle_{F_{2,\gamma}} \;=\; & f(0)\, g(0) \;+\; \gamma_1^{-1} \int_{\mathbb{R}} \frac{\partial f}{\partial x_1}(x_1, 0)\, \frac{\partial g}{\partial x_1}(x_1, 0)\, dx_1 \;+ \\ & \gamma_2^{-1} \int_{\mathbb{R}} \frac{\partial f}{\partial x_2}(0, x_2)\, \frac{\partial g}{\partial x_2}(0, x_2)\, dx_2 \;+ \\ & \gamma_1^{-1} \gamma_2^{-1} \int_{\mathbb{R}^2} \frac{\partial^2 f}{\partial x_1\, \partial x_2}(x_1, x_2)\, \frac{\partial^2 g}{\partial x_1\, \partial x_2}(x_1, x_2)\, dx_1\, dx_2. \end{aligned}$$

Consider the *weighted integration* problem,

$$I_1(f) \;=\; \int_{\mathbb{R}} \varrho(t)\, f(t)\, dt \quad \text{with} \quad \int_{\mathbb{R}} \varrho(t)\, |t|^r\, dt \;<\; \infty.$$

We now have

$$h_{1,1}(x) \;=\; \sum_{j=0}^{r-1} \frac{x^j}{j!} \int_{\mathbb{R}} \frac{t^j}{j!}\, \varrho(t)\, dt,$$

and $h_{1,2} = h_{1,2,(0)} + h_{1,2,(1)}$ with

$$h_{1,2,(0)}(x) \;=\; \int_{-\infty}^{0} \varrho(t) R_2(x,t)\, dt \quad \text{and} \quad h_{1,2,(1)}(x) \;=\; \int_{0}^{\infty} \varrho(t) R_2(x,t)\, dt.$$

Hence, $h_{1,1}$ is a polynomial of degree at most $r-1$. Note that $h_{1,1}$ is not zero since the coefficient of $h_{1,1}$ for $j=0$ is $\int_{\mathbb{R}} \varrho(t)\, dt$, which is positive. We assume that ϱ is non-zero over $\mathbb{R}_-$ and $\mathbb{R}_+$, which yields that both of the $h_{1,2,(i)}$ are also non-zero and $\alpha < 1$. For symmetric ϱ we have $\alpha = \frac{1}{2}$.

Theorem 3 states that weighted integration in $F_{d,\gamma}$ is intractable as long as the sum of the weights $s_d = \sum_{j=1}^{d} \gamma_j$ is such that $\ln s_d / \ln d$ goes to infinity. This holds, in particular, for equal weights, as well as for $\gamma_j = j^{-a}$ with $a < 1$.

For $r = 1$, Theorem 3 and the results of [16] yield necessary and sufficient conditions on γ_j for the lack of strong tractability or tractability of weighted integration with non-zero $h_{1,2,(i)}$. For $r > 1$, these conditions are only sufficient; we do not know if they are also necessary. □

Example: Uniform Integration (continued)

We now remove the condition $f(a) = 0$ from the example of uniform integration studied in the previous section. We now have $D_1 = [0,1]$ and $F_{1,\gamma}$ is the Sobolev space $W^1([0,1])$ with the inner product

$$\langle f, g \rangle_{F_{1,\gamma}} \;=\; f(a)g(a) + \gamma^{-1} \int_0^1 f'(x) g'(x)\, dx,$$

where $\gamma > 0$. The reproducing kernel is

$$K_{1,\gamma}(x,t) \;=\; 1 + \gamma\, 1_M(x,t) \min(|x-a|, |t-a|)$$

with $M = [0,a] \times [0,a] \cup [a,1] \times [a,1]$.

For $d > 1$, we take $F_{d,\gamma} = W^{(1,1,\dots,1)}([0,1]^d)$ as the tensor product of $W^1([0,1])$. The inner product of $F_{d,\gamma}$ is now given by

$$\langle f, g \rangle_{F_{d,\gamma}} \;=\; \sum_{u \subset \{1,2,\dots,d\}} \gamma_u^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{|u|}}{\partial x_u} f(x_u, a)\, \frac{\partial^{|u|}}{\partial x_u} g(x_u, a)\, dx_u.$$

Here $\gamma_\emptyset = 1$ and $\gamma_u = \prod_{j \in u} \gamma_j$ for nonempty u, and (x_u, a) is the vector $x \in [0,1]^d$ with all components whose indices not in u are replaced by a.

We consider uniform integration $I_d(f) = \int_{[0,1]^d} f(t)\, dt$. We now have $h_{1,1} = 1$. The functions $h_{1,2,(0)} = h_{1,(0)}$ and $h_{1,2,(1)} = h_{1,(1)}$ are as in the previous sections. Therefore all the assumptions of Theorem 3 are satisfied as long as $a \in (0,1)$. Hence, uniform integration in $W^{1,1,\dots,1}([0,1]^d)$ is not strongly tractable as long as $\sum_{j=1}^{\infty} \gamma_j = \infty$. In fact, this condition on γ_j is also necessary, as shown in [16]. Furthermore, if $\limsup_d \ln s_d / \ln d = \infty$ then uniform integration in $W^{1,1,\dots,1}([0,1]^d)$ is intractable. This condition is also necessary, since $\limsup_d \ln s_d / \ln d < \infty$ implies tractability, see [16]. □

Example: Centered Discrepancy (continued)

We now remove the condition $f(\frac{1}{2})=0$ from the example of centered discrepancy studied in the previous section. As before, for this special case we study the more general case of the L_q norm. That is, $D_1=[0,1]$ and $F_{1,q,\gamma}$ is the Sobolev space $W_q^1([0,1])$ with the norm

$$\|f\|_{F_{1,q,\gamma}} \;=\; \left(\left|f(\tfrac{1}{2})\right|^q+\gamma^{-q/2}\int_0^1 |f'(x)|^q\,dx\right)^{1/q},$$

where $\gamma>0$. Observe that for $q=\infty$ we have

$$\|f\|_{F_{1,\infty,\gamma}} \;=\; \max\left(|f(\tfrac{1}{2})|,\gamma^{1/2}\operatorname{ess\,sup}_{t\in[0,1]}|f'(t)|\right).$$

For $q=2$, we have the Hilbert space with the kernel

$$K_{1,\gamma}=1+\gamma\,1_M(x,t)\min(|x-\tfrac{1}{2}|,|t-\tfrac{1}{2}|).$$

For $d>1$, we take $F_{d,q,\gamma}=W_q^{(1,1,\dots,1)}([0,1]^d)$ as the tensor product of $W_q^1([0,1])$. The norm in $F_{d,q,\gamma}$ is given by

$$\|f\|_{F_{d,q,\gamma}} \;=\; \left(\sum_{u\subset\{1,2,\dots,d\}}\gamma_u^{-q/2}\int_{[0,1]^{|u|}}\left|\frac{\partial^{|u|}}{\partial x_u}f(x_u,1/2)\right|^q dx_u\right)^{1/q}. \tag{25}$$

We consider uniform integration $I_d(f)=\int_{[0,1]^d}f(t)\,dt$. The formula for the error of the algorithm $Q_{n,d}(f)=\sum_{i=1}^n a_i f(z_i)$ takes the form, see [6],

$$I_d(f)-Q_{n,d}(f)=\sum_{u\subset\{1,2,\dots,d\}}\int_{[0,1]^{|u|}}\frac{\partial^{|u|}}{\partial x_u}f(x_u,\tfrac{1}{2})\frac{\partial^{|u|}}{\partial x_u}\left(h_d-\sum_{i=1}^n a_iK_d(\cdot,z_i)\right)(x_u,\tfrac{1}{2})\,dx_u,$$

where h_d and the kernel K_d are given as before by

$$\begin{aligned} h_d(x) &= 2^{-d}\prod_{j=1}^d\left(|x_j-\tfrac{1}{2}|-|x_j-\tfrac{1}{2}|^2\right),\\ K_d(x,t) &= 2^{-d}\prod_{j=1}^d\left(|x_j-\tfrac{1}{2}|+|t_j-\tfrac{1}{2}|-|x_j-t_j|\right). \end{aligned}$$

Applying the Hölder inequality for integrals and sums to $I_d(f)-Q_{n,d}(f)$ we conclude (see again [6] for the details) that

$$e(Q_{n,d}) := \sup_{f\in F_{d,q,\gamma}:\,\|f\|_{F_{d,q,\gamma}}\le 1}\left|I_d(f)-Q_{n,d}(f)\right| \;=\; d_{p,\gamma}^c(Q_{n,d}),$$

where $1/p+1/q=1$ and the weighted centered p-discrepancy $d_{p,\gamma}^c(Q_{n,d})$ is given by

$$d_{p,\gamma}^c(Q_{n,d})=\left(\sum_{u\subset\{1,2,\dots,d\}}\gamma_u^{p/2}\int_{[0,1]^{|u|}}|\operatorname{disc}^c(x_u,1/2)|^p\,dx_u\right)^{1/p}, \tag{26}$$

with

$$\operatorname{disc}^c(x_u, 1/2) = \prod_{\ell \in u} \min(x_\ell, 1 - x_\ell) - \sum_{i=1}^{n} a_i \cdot 1_{J(a(x_u), x_u)}(t_i)_u.$$

Let $e(n, F_{d,q,\gamma}) = d_{p,\gamma}(n, d)$ denote the minimal error, or equivalently the minimal weighted centered p-discrepancy, that can be achieved by using n function values. The initial error, or the initial centered weighted p-discrepancy, is now given by

$$e(0, F_{d,q}) \;=\; d^c_{p,\gamma}(0, d) \;=\; \left(\sum_{u \subset \{1,2,\ldots,d\}} \gamma_u^{p/2} \left(\frac{2^{-p}}{p+1} \right)^{|u|} \right)^{1/p}.$$

For $q = 1$, we have $p = \infty$ and

$$e(0, F_{d,1}) \;=\; d^c_{\infty,\gamma}(0, d) \;=\; \max_{k=0,1,\ldots,d} \left(2^{-k} (\gamma_1 \gamma_2 \cdots \gamma_k)^{1/2} \right).$$

From Corollary 2 we conclude that

$$d^c_{p,\gamma}(n, d) \geq \left(\sum_{k=0}^{d} C_{d,p,k} \left(\frac{2^{-p}}{p+1} \right)^k \cdot (1 - n\, 2^{-k})_+ \right)^{1/p},$$

where

$$C_{d,p,k} \;=\; \sum_{u \subset \{1,2,\ldots,d\} \,:\, |u| = k} \gamma_u^{p/2}.$$

For $n \leq 2^m < 2^d$, this can be rewritten for $p < \infty$ as

$$d^c_{p,\gamma}(n, d) \geq 2^{-1/p} \left(1 - \frac{\sum_{k=0}^{m} C_{d,p,k} (\frac{2^{-p}}{p+1})^k}{\sum_{k=0}^{d} C_{d,p,k} (\frac{2^{-p}}{p+1})^k} \right)^{1/p} d^c_{p,\gamma}(0, d),$$

and for $p = \infty$, as

$$d^c_{\infty,\gamma} \geq 2^{-(m+1)} \left(\gamma_1 \gamma_2 \cdots \gamma_{m+1} \right)^{1/2}.$$

For $q > 1$, i.e., $p < \infty$, we can check the lack of strong tractability and tractability of I_d in $F_{d,q,\gamma}$ by using the proof of Theorem 3.

Corollary 4. *Let $q > 1$. If $\sum_{j=1}^{\infty} \gamma_j^{p/2} = \infty$ then*

$$\lim_{d \to \infty} \frac{d^c_{p,\gamma}(n, d)}{d^c_{p,\gamma}(0, d)} \;=\; 1 \qquad \forall\, n,$$

and uniform integration in $F_{d,q,\gamma}$ is not strongly tractable. If

$$\lim{}^*_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_j^{p/2}}{\ln\, d} \;=\; \infty$$

then

$$\lim{}^*_{d \to \infty} \frac{d^c_{p,\gamma}(d^k, d)}{d^c_{p,\gamma}(0, d)} \;=\; 1 \quad \forall\, k = 1, 2, \ldots,$$

and uniform integration in $F_{d,q,\gamma}$ is intractable, where $\lim^* \in \{\lim, \limsup\}$.

As we shall see later, for some applications the kernel $K_{1,\gamma}$ is of the form (18) but the kernel R_2 is not decomposable. However, R_2 may be represented as the sum of two kernels, one of which is decomposable. Hence, we need to generalize (18) and consider kernels $K_{1,\gamma}$ of the form

$$K_{1,\gamma} \;=\; R_1 + \gamma R_3 + \gamma R_2, \tag{27}$$

where R_i is a reproducing kernel of the Hilbert space $H(R_i)$ and we assume that $(H(R_i) \oplus H(R_j)) \cap H(R_k) \;=\; \{0\}$ for pairwise different i, j, and k. Later we will assume that, as before, R_2 is decomposable.

The Hilbert space $F_{1,\gamma} = H(K_{1,\gamma})$ consists of functions $f \;=\; f_1 + \gamma f_3 + \gamma f_2$ with $f_i \in H(R_i)$ and this representation of f is unique. The inner product of $F_{1,\gamma}$ is given by

$$\langle f, g\rangle_{F_{1,\gamma}} \;=\; \langle f_1, g_1\rangle_{H(R_1)} + \gamma\, \langle f_3, g_3\rangle_{H(R_3)} + \gamma\, \langle f_2, g_2\rangle_{H(R_2)}\,.$$

As before, for $d > 1$ we take

$$F_{d,\gamma} \;=\; F_{1,\gamma_1} \otimes F_{1,\gamma_2} \otimes \cdots \otimes F_{1,\gamma_d}$$

as the tensor product of F_{1,γ_i} for some sequence of weights γ_i with $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_d$. The reproducing kernel of $F_{d,\gamma}$ is

$$K_{d,\gamma}(x,t) \;=\; \prod_{j=1}^{d} \left(R_1(x_j,t_j) \;+\; \gamma_j\, R_3(x_j,t_j) \;+\; \gamma_j\, R_2(x_j,t_j)\right).$$

Observe that for $\gamma_i \equiv$ const, the kernels (27) are not more general than the kernels (18) since we can just rename $R_1 + \gamma R_3$ as R_1. The essence of (27) is when we use varying γ_i. Then the decomposable part $\gamma_i R_2$, as well as the non-decomposable part $R_1 + \gamma_i R_3$, of the kernel K_{1,γ_i} varies.

The continuous linear functional $I_d(f) = \langle f, h_{d,\gamma}\rangle_{F_{1,\gamma}}$ is defined as before as tensor products of I_1's. That is,

$$h_{d,\gamma}(x) \;=\; \prod_{j=1}^{d} h_{1,\gamma_j}(x_j) \;=\; \prod_{j=1}^{d} \left(h_{1,1}(x_j) + \gamma_j h_{1,3}(x_j) + \gamma_j h_{1,2}(x_j)\right).$$

We now have

$$e^2(0, F_{d,\gamma}) \;=\; \|h_{d,\gamma}\|^2_{F_{d,\gamma}} \;=\; \prod_{j=1}^{d} \left(\, \alpha_1 + \gamma_j \alpha_3 + \gamma_j \alpha_2 \,\right),$$

where

$$\alpha_i \;=\; \|h_{1,i}\|^2_{H(R_i)}, \quad i = 1,2,3.$$

Note that our notation is consistent with the notation of Theorem 2. Also as before we denote

$$\alpha \;=\; \frac{\max(\|h_{1,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,(1)}\|^2_{H(R_2)})}{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}}.$$

We are ready to prove the following theorem by a slight modification of the proofs of Theorems 2 and 3.

Theorem 4. *Assume that* $(H(R_i) \oplus H(R_j)) \cap H(R_k) = \{0\}$ *for pairwise different* i, j, *and* k *with a decomposable* R_2. *Then*

$$e^2(n, F_{d,\gamma}) \geq \sum_{u \subset \{1,2,\ldots,d\}} \left(1 - n\alpha^{|u|}\right)_+ \alpha_2^{|u|} \prod_{j \in u} \gamma_j \prod_{j \notin u} (\alpha_1 + \gamma_j \alpha_3), \quad (0^0 = 1).$$

Assume also that both $h_{1,2,(0)}$ *and* $h_{1,2,(1)}$ *are non-zero.*

(1) *If* $h_{1,1} = 0$ *then* I_d *in* $F_{d,\gamma}$ *is intractable for positive* γ_j.

(2) *If* $h_{1,1}$ *is non-zero and* $\sum_{j=1}^{\infty} \gamma_j = \infty$ *then*

$$\lim_{d \to \infty} \frac{e(n, F_{d,\gamma})}{e(0, F_{d,\gamma})} = 1, \quad \forall\, n,$$

and I_d *in* $F_{d,\gamma}$ *is not strongly tractable.*

(3) *If* $h_{1,1}$ *is non-zero and* $\lim^*_{d \to \infty} \frac{\sum_{j=1}^d \gamma_j}{\ln d} = \infty$ *then*

$$\lim{}^*_{d \to \infty} \frac{e(d^q, F_{d,\gamma})}{e(0, F_{d,\gamma})} = 1 \quad \forall\, q = 1, 2, \ldots\,,$$

and I_d *in* $F_{d,\gamma}$ *is intractable, with* $\lim^* \in \{\lim, \limsup\}$.

(4) *If* $h_{1,1}$ *is non-zero and* $\gamma_j \geq \gamma_1^* > 0$ *then*

$$\lim_{d \to \infty} \frac{e(\lfloor b^d \rfloor, F_{d,\gamma})}{e(0, F_{d,\gamma})} = 1 \quad \forall\, b \in (1, \alpha^{-c}),$$

where c, $c \in (0,1)$, *satisfies the two inequalities*

$$c \leq \alpha_3^* \gamma_1^* \text{ and } (1 + \ln \alpha_3^* \gamma_1^* - \ln c)c < \ln(1 + \alpha_3^* \gamma_1^*),$$

with

$$\alpha_3^* = \frac{\alpha_2}{\alpha_1 + \alpha_3 \gamma_1^*}.$$

Proof. The lower bound on $e^2(n, F_{d,\gamma})$ is obtained by repeating the proof of Theorem 2 with

$$h_{u,1}(x_{\overline{u}}) = \prod_{j \notin u} \left(h_{1,1}(x_j) + \gamma_j h_{1,3}(x_j)\right).$$

If $h_{1,1} = 0$ then $\alpha_1 = 0$ and

$$\begin{aligned} e^2(n, F_{d,\gamma}) &\geq \prod_{j=1}^d \gamma_j \sum_{u \subset \{1,2,\ldots,d\}} \alpha_3^{d-|u|} \alpha_2^{|u|} \left(1 - n\alpha^{|u|}\right)_+ \\ &= \prod_{j=1}^d \gamma_j \sum_{k=0}^d \binom{d}{k} \alpha_3^{d-k} \alpha_2^k \left(1 - n\alpha^k\right)_+. \end{aligned}$$

Since $e^2(0, F_{d,\gamma}) = \prod_{j=1}^d \gamma_j \,(\alpha_2 + \alpha_3)^d$ we see that the ratio $e(n, F_{d,\gamma})/e(0, F_{d,\gamma})$ does not depend on γ_j, and intractability follows from Theorem 3 with the change of $h_{1,1}$ to $h_{1,3}$ and with $\gamma_j = 1$.

Assume now that $h_{1,1} \neq 0$, i.e., $\alpha_1 > 0$. Then, as in the proof of Theorem 3, we take n fixed or $n = d^q$ and note that $n\alpha^k \le a$ for all $k \in [k(d,\beta), d]$. We have

$$\frac{e^2(n, F_{d,\gamma})}{e(0, F_{d,\gamma})} \;\ge\; (1-a)\;(1-\alpha_{d,\beta})\,,$$

with

$$\alpha_{\beta,d} \;=\; \frac{\sum_{u:\,|u|\le k(d,\beta)} \alpha_1^{d-|u|}\alpha_2^{|u|} \prod_{j\notin u}(1+\gamma_j\alpha_4)\;\prod_{j\in u}\gamma_j}{\sum_u \alpha_1^{d-|u|}\alpha_2^{|u|} \prod_{j\notin u}(1+\gamma_j\alpha_4)\;\prod_{j\in u}\gamma_j},$$

and $\alpha_4 = \alpha_3/\alpha_1$. Setting

$$\gamma_j' \;=\; \frac{\alpha_2\gamma_j}{\alpha_1(1+\gamma_j\alpha_4)}$$

we first observe that the sequence of γ_j' satisfies the same conditions as the sequence of γ_j, since

$$\frac{\alpha_2}{\alpha_1(1+\gamma_1\alpha_4)}\,\gamma_j \;\le\; \gamma_j' \;\le\; \frac{\alpha_2}{\alpha_1}\,\gamma_j.$$

Furthermore,

$$\alpha_{\beta,d} \;=\; \frac{\sum_{k=0}^{k(d,\beta)} C_{d,k}'}{\sum_{k=0}^{d} C_{d,k}'}$$

with $C_{d,k}' = \sum_{u\,:\,|u|=k}\prod_{j\in u}\gamma_j'$. From the proof of Theorem 3 it follows that $\alpha_{\beta,d}$ goes to zero and this completes the proof. □

In the next two sections we present examples where Theorem 4 is used.

5. L_2-Star Discrepancy

It is well-known that the L_2-star discrepancy is related to the worst case error of algorithms $Q_{n,d}$ for uniform integration defined on the Sobolev spaces of functions that are once differentiable with respect to all variables. This holds regardless of whether boundary conditions are imposed. We now show how to apply the results of the previous section to conclude intractability of uniform integration and obtain interesting bounds on the L_2-star discrepancy.

We begin with the boundary case. That is, we take the space $F_d = W_0^{1,1,\dots,1}([0,1]^d)$ as in the uniform integration example with the parameter $a = 0$. For an algorithm $Q_{n,d}(f) = \sum_{i=1}^n a_i f(\vec{1}-t_i)$ with $\vec{1} = [1,1,\dots,1] \in [0,1]^d$ we have $e(Q_{n,d}) = \tilde{d}_2^*(Q_{n,d})$, where the ($L_2$ boundary) star discrepancy is

$$\tilde{d}_2^*(Q_{n,d}) = \left(\int_{[0,1]^d} \operatorname{disc}^2(x)\,dx\right)^{1/2} \tag{28}$$

with

$$\operatorname{disc}(x) \;=\; x_1\cdots x_d - \sum_{i=1}^n a_i\cdot 1_{[0,x)}(t_i).$$

Let $\tilde{d}_2^*(n,d)$ denote the minimal L_2-star discrepancy. For $n=0$, we have $\tilde{d}_2(0,d) = 3^{-d/2}$.

The reproducing kernel of the space F_1 is

$$K_1(x,t) \;=\; \min(x,t).$$

This kernel is formally decomposable with $a^*=0$. Then $D_{1,(0)}=\{0\}$ and $h_{1,(0)}=0$ since all functions in F_1 vanish at zero. Therefore Theorem 1 is trivial in this case.

However, Theorem 2 offers an alternative approach. If we can present the kernel K_1 as the sum of the reproducing kernels

$$K_1 \;=\; R_{1,a^*} + R_{2,a^*}$$

with a decomposable R_{2,a^*} for $a^* \in (0,1)$ we may have a chance to conclude intractability of uniform integration for the Sobolev space $W_0^{1,1,\dots,1}([0,1]^d)$ and obtain interesting bounds for the L_2-star discrepancy.

We now show that this approach works. For an arbitrary $a^* \in (0,1)$, consider the subspace F_{a^*} of F_1 by taking functions for which $f(a^*)=0$. That is,

$$F_{a^*} \;=\; \{\, f \in W_0^1([0,1]) \;:\; f(a^*)=0\,\}.$$

Note that the projection $f - f(a^*)K_1(\cdot,a^*)/K_1(a^*,a^*)$ belongs to F_{a^*} for any $f \in F_1$.

This implies that $F_{a^*}=H(R_{2,a^*})$ is the Hilbert space with the reproducing kernel

$$R_{2,a^*}(x,t) \;=\; K_1(x,t) \;-\; \frac{K_1(x,a^*)K_1(t,a^*)}{K_1(a^*,a^*)}.$$

What is important for us is that

$$R_{2,a^*}(x,t) \;=\; 0 \quad \forall\,(x,t) \in [0,a^*]\times[a^*,1] \cup [a^*,1]\times[0,a^*].$$

Hence, R_{2,a^*} is *decomposable* with a^*. We also have

$$R_{1,a^*}(x,t) \;=\; K_1(t,a^*)K_1(x,a^*)/K(a^*,a^*)$$

and the one-dimensional Hilbert space $H(R_{1,a^*})$ consists of functions of the form

$$a\min(\cdot,a^*) \qquad \text{with } a \in \mathbb{R}.$$

Obviously $H(R_{1,a^*}) \cap H(R_{2,a^*}) = \{0\}$ since all functions in $H(R_{2,a^*})$ vanish at a^* and the only function from $H(R_{1,a^*})$ that vanishes at a^* is the zero function. Clearly, $F_1 = H(R_{1,a^*}) \oplus H(R_{2,a^*})$.

For uniform integration the function h_1 is given by $h_1(x)=x-x^2/2$. We thus have

$$\begin{aligned} h_{1,1}(x) &= h_1(a^*)\frac{K_1(x,a^*)}{K_1(a^*,a^*)} = \left(1-\tfrac{1}{2}a^*\right)\min(x,a^*),\\ h_{1,2,(0)}(x) &= 1_{[0,a^*]}(x)\,\left(\tfrac{1}{2}a^*x-\tfrac{1}{2}x^2\right),\\ h_{1,2,(1)}(x) &= 1_{[a^*,1]}(x)\,\left(x-\tfrac{1}{2}x^2-a^*\left(1-\tfrac{1}{2}a^*\right)\right). \end{aligned}$$

From this we compute

$$\begin{aligned} \|h_{1,1}\|^2_{H(R_{1,a^*})} &= \left(1-\tfrac{1}{2}a^*\right)^2 a^*,\\ \|h_{1,2,(0)}\|^2_{H(R_{2,a^*})} &= \frac{(a^*)^3}{12},\\ \|h_{1,2,(1)}\|^2_{H(R_{2,a^*})} &= \frac{(1-a^*)^3}{3}. \end{aligned}$$

For $\gamma_j = 1$, all the assumptions of Theorems 2 and 3 hold and uniform integration is intractable. In particular, we may choose a^* such that $(a^*)^3 = 4(1-a^*)^3$, which corresponds to $a^* = 4^{1/3}/(1+4^{1/3}) = 0.613511790\dots$. Then the parameters of Theorems 2 and 3 are $\alpha = \frac{1}{2}$ and $\alpha_1 = (1-a^*/2)^2\, a^*$ and $\alpha_2 = (a^*)^3/6$ and $\alpha_3 = (a^*)^2/6(1-a^*/2)^2$ and we obtain

$$\alpha_1 = 0.294846029\dots, \quad \alpha_2 = 0.0384873039\dots, \quad \alpha_3 = 0.130533567\dots.$$

We now have $\gamma_1^* = 1$ and point 4 of Theorem 3 holds for $c \in (0,1)$ with $c \le \alpha_3$ and

$$(1 + \ln \alpha_3 - \ln c)\, c < \ln(1 + \alpha_3)$$

This implies that point 4 of Theorem 3 holds for all $c < 0.087978$ and $b < 1.06288$.

We now turn to the case without boundary conditions. That is, we have now the space $F_{d,\gamma} = W^{1,1,\dots,1}([0,1]^d)$ as in the uniform integration example of Section 4 with $a = 0$. The L_2-star discrepancy still has the property that $e(Q_{n,d}) = d^*_{2,\gamma}(Q_{n,d})$, with

$$d^*_{2,\gamma}(Q_{n,d}) = \left(\sum_{u \subset D} \gamma_u \int_{[0,1]^{|u|}} \operatorname{disc}^2(x_u, 0)\, dx_u \right)^{1/2}, \tag{29}$$

see [16]. Let $d^*_{2,\gamma}(n,d)$ denote the minimal L_2-star discrepancy. For $n = 0$, we have

$$d^*_{2,\gamma}(0,d) \;=\; \prod_{j=1}^{d} \left(1 + \frac{1}{3}\gamma_j\right).$$

The reproducing kernel of the space F_1 is now

$$K_{1,\gamma}(x,t) \;=\; 1 + \gamma \min(x,t).$$

We can modify the representation of $K_{1,\gamma}$ as

$$K_{1,\gamma} \;=\; (1 + \gamma\, R_{1,a^*}) \;+\; \gamma\, R_{2,a^*}.$$

From this form we conclude that $K_{1,\gamma}$ satisfies the assumptions of Theorem 4 with $R_1 = 1$, $R_3 = R_{1,a^*}$ and $R_2 = R_{2,a^*}$.

For uniform integration we now have $h_1(x) = 1 + x - x^2/2$ and $h_{1,1}(x) = 1$, $h_{1,3}(x) = (1 - \frac{1}{2}a^*) \min(x, a^*)$, and $\|h_{1,3}\|^2_{H(R_3)} = (1 - a^*/2)^2 a^*$. The functions $h_{1,2}$, $h_{1,2,(0)}$ and $h_{1,2(1)}$ are unchanged. The norm of $h_{1,1}$ is now 1.

For $a^* = 4^{1/3}/(1 + 4^{1/3})$ and $\gamma_1^* = 1$, Theorem 4 holds with

$$\alpha = \tfrac{1}{2}, \quad \alpha_1 = 1, \quad \alpha_2 = 0.0384873039\dots, \quad \alpha_3 = 0.294846029\dots,$$

and

$$\alpha_3^* \;=\; \frac{\alpha_2}{\alpha_1 + \alpha_3} \;=\; 0.0297234598\dots.$$

Point 4 of Theorem 4 holds for $c \le \alpha^*$ if

$$(1 + \ln \alpha_3^* - \ln c)\, c < \ln(1 + \alpha_3^*)$$

This implies that point 4 of Theorem 4 holds for all $c < 0.065363$ and $b < 1.04635$.

We summarize the discussion of this section in the following theorem.

Theorem 5. *Uniform integration in the Sobolev space* $W_0^{1,1,\dots,1}([0,1]^d)$ *is intractable. In particular,*

$$\lim_{d\to\infty} \frac{\tilde{d}_2^*(\lfloor 1.0628^d \rfloor, d)}{\tilde{d}_2^*(0,d)} = 1.$$

If $\sum_{j=1}^\infty \gamma_j = \infty$ *then uniform integration in the Sobolev space* $W^{1,1,\dots,1}([0,1]^d)$ *is not strongly tractable and*

$$\lim_{d\to\infty} \frac{d_{2,\gamma}^*(n,d)}{d_2^*(0,d)} = 1 \quad \forall\, n.$$

If

$$\lim\nolimits^*_{d\to\infty} \sum_{j=1}^d \gamma_j / \ln\, d = \infty$$

then uniform integration in the Sobolev space $W^{1,1,\dots,1}([0,1]^d)$ *is intractable and*

$$\lim\nolimits^*_{d\to\infty} \frac{d_{2,\gamma}^*(d^q,d)}{d_2^*(0,d)} = 1 \quad \forall\, q = 1,2,\dots\,,$$

where $\lim^* \in \{\lim, \limsup\}$.

If $\gamma_j \geq 1$ *then*

$$\lim_{d\to\infty} \frac{d_{2,\gamma}^*(\lfloor 1.0463^d \rfloor, d)}{d_2^*(0,d)} = 1.$$

Remark 1. It is interesting to compare the estimates for the exponent of intractability obtained by different techniques. We begin with the space $W_0^{1,1,\dots,1}([0,1]^d)$. Using a technique for quasi Monte Carlo algorithms, $a_i = n^{-1}$, or more general for quadrature formulas with positive weights, the lower bound $(9/8)^d = 1.125^d$ is proved in [27]. Our weaker bound 1.0628^d holds for *arbitrary* quadrature formulas. The best upper bound is

$$n(\varepsilon, W_0^{1,1,\dots,1}([0,1]^d)) \leq 1.5^d\, \varepsilon^{-2}.$$

This upper bound is non-constructive and follows from a probabilistic argument, see for example [12].

The technique of [27] can also be applied to the space $W^{1,1,\dots,1}([0,1]^d)$. In the case $\gamma_j = 1$ one obtains the lower bound 1.0563^d (for positive quadrature formulas) instead of the new lower bound 1.0463^d which is valid for *arbitrary* quadrature formulas. Here the upper bound is of the form, see [12],

$$n(\varepsilon, W^{1,1,\dots,1}([0,1]^d)) \leq 1.125^d\, \varepsilon^{-2}.$$

Using Theorem 5, the upper bounds presented in [16] for quasi-Monte Carlo algorithms, and the bounds presented in [7], we get the following corollary for the Sobolev space $W^{1,1,\dots,1}([0,1]^d)$.

Corollary 5. *Uniform integration in the Sobolev space* $W^{1,1,\dots,1}([0,1]^d)$ *is strongly tractable iff* $\sum_{j=1}^\infty \gamma_j < \infty$. *If so the strong exponent of tractability belongs to* $[1,2]$. *If* $\sum_{j=1}^\infty \gamma_j^{1/2} < \infty$ *then the strong exponent of tractability is* 1, *see* [7].

Uniform integration in the Sobolev space $W^{1,1,\dots,1}([0,1]^d)$ is tractable iff

$$\limsup_d \sum_{j=1}^d \gamma_j / \ln\, d < \infty.$$

6. Further examples

In this section we provide a few further examples of weighted and uniform integration for which the kernel K_1 of the space F_1 is not decomposable but can be rewritten as $R_1 + R_2$ or as $R_1 + \gamma R_3 + \gamma R_2$ with a decomposable R_2. Then we may apply the results of Section 4 to conclude intractability of integration.

Example: Rank One Modification

The main idea of the L_2-star discrepancy section was to express the original kernel K_1 by a rank one modification,

$$\begin{aligned} R_1(x,t) &= g_1(x)g_1(t), \\ R_2(x,t) &= K_1(x,t) - g_1(x)g_1(t), \end{aligned}$$

where $g_1(x) = K_1(x,a^*)/\sqrt{K_1(a^*,a^*)}$ with $K_1(a^*,a^*) > 0$. Then

$$g_1(a^*) \;=\; \sqrt{K_1(a^*,a^*)} \;=\; \|K_1(\cdot,a^*)\|_{F_1} \quad \text{and} \quad \|g_1\|_{F_1} = 1.$$

The space $H(R_1) = \operatorname{span}(g_1)$ is one dimensional and $H(R_1) \cap H(R_2) = \{0\}$.

We will use the same approach for three further examples. The first one is for the Sobolev space $F_1 = W^1(\mathbb{R})$ with the inner product

$$\langle f, g\rangle_{F_1} \;=\; \int_{\mathbb{R}} f(t)g(t)\,dt \;+\; \int_{\mathbb{R}} f'(t)g'(t)\,dt.$$

It is shown in [18] that the reproducing kernel of F_1 is

$$K_1(x,t) \;=\; \frac{1}{2}\exp\left(-|x-t|\,\right).$$

This kernel is not decomposable since it always takes positive values. Take $a^* = 0$. Then

$$R_2(x,t) \;=\; \frac{1}{2}\left[\exp\left(-|x-t|\right) \;-\; \exp\left(-|x| - |t|\right)\right]$$

is decomposable since for $x \le 0 \le t$ we have $R_2(x,t) = 0$.

For $d > 1$, we obtain $F_d = W^{1,1,\dots,1}(\mathbb{R}^d)$ with the inner product

$$\langle f, g\rangle_{F_d} \;=\; \sum_{u \subset \{1,2,\dots,d\}} \int_{\mathbb{R}^d} \frac{\partial^{|u|}}{\partial\, x_u} f(x) \frac{\partial^{|u|}}{\partial\, x_u} g(x)\,dx. \tag{30}$$

The kernel of F_d is

$$K_d(x,t) \;=\; \prod_{j=1}^d \left(\frac{1}{2}\,\exp\left(-|x_j - t_j|\right)\right).$$

Consider weighted integration $I_1(f) = \int_{\mathbb{R}} \varrho(t) f(t)\,dt$ with a weight ϱ such that

$$\int_{\mathbb{R}^2} \varrho(t)\varrho(x)\exp(-|x-t|)\,dt \,<\, \infty.$$

We have

$$h_1(x) \,=\, \frac{1}{2}\int_{\mathbb{R}} \varrho(t)\exp(-|x-t|)\,dt$$

and

$$h_{1,2}(x) \,=\, h_1(x) - \frac{h_1(0)}{\sqrt{K_1(0,0)}}\, g_1(x) \,=\, h_1(x) - h_1(0)\exp(-|x|).$$

Hence, if $h_{1,2}$ is not identically zero for negative x and also is not identically zero for positive x, then $h_{1,2,(0)}$ and $h_{1,2,(1)}$ are non-zero and we may apply Theorem 3 (with $\gamma_j \equiv 1$) to conclude intractability of weighted integration in F_d.

Observe that for Gaussian integration we have $\varrho(t) = (2\pi)^{-1/2}\exp(-t^2/2)$ and

$$h_1(x) = c\left(\exp(-x)\int_{-\infty}^{x} \exp\left(-\frac{(t-1)^2}{2}\right)\,dt + \exp(x)\int_{x}^{\infty} \exp\left(-\frac{(t+1)^2}{2}\right)\,dt\right),$$

with $c = \exp(\frac{1}{2})/(2\sqrt{2\pi})$, and obviously both $h_{1,2,(0)}$ and $h_{1,2,(1)}$ are non-zero. This proves the following corollary.

Corollary 6. *Gaussian integration in the Sobolev space $W^{1,1,\dots,1}(\mathbb{R}^d)$ with the inner product (30) is intractable.*

We now consider two examples of Sobolev spaces of periodic functions. Both of them are for once-differentiable functions, with and without boundary conditions. The first example is for $F_1 = \widetilde{W}_0^1([0,1])$, i.e.,

$$F_1 \,=\, \{f \in W^1([0,1]) \,:\, f(0) = f(1) = 0\,\}$$

with the inner product

$$\langle f, g\rangle_{F_{1,\gamma}} \,=\, \int_0^1 f'(t)g'(t)\,dt.$$

The reproducing kernel is now

$$K_1(x,t) \,=\, \min(x,t) - xt.$$

Take $a^* \in (0,1)$. Then

$$\begin{aligned} R_{2,a^*}(x,t) &= \min(x,t) - xt - R_{1,a^*}(x,t), \\ R_{1,a^*}(x,t) &= \frac{(\min(x,a^*) - a^*x)(\min(t,a^*) - a^*t)}{a^*(1-a^*)}. \end{aligned}$$

The kernel R_{2,a^*} is decomposable since for $x \le a^* \le t$ we have $R_{2,a^*}(x,t) = 0$.

For $d > 1$, we have the Sobolev space $F_d = \widetilde{W}^{1,1,\dots,1}([0,1]^d)$ of periodic functions f such that $f(x) = 0$ if at least one component of x is zero or one. The inner product of F_d is

$$\langle f, g\rangle_{F_d} \,=\, \int_{[0,1]^d} \frac{\partial^d}{\partial x_1 \cdots \partial x_d}\, f(x)\, \frac{\partial^d}{\partial x_1 \cdots \partial x_d}\, g(x)\,dx$$

and the reproducing kernel is

$$K_d(x,t) \;=\; \prod_{j=1}^{d} \left(\min(x_j,t_j) - x_j t_j\right).$$

Consider uniform integration, $I_1(f) = \int_0^1 f(t)\,dt$. We have

$$h_1(x) \;=\; x(1-x)/2 \quad \text{and} \quad h_{1,2}(x) \;=\; \left(x(1-x) - \min(x,a^*) + a^*x\right)/2.$$

From this we obtain

$$h_{1,2,(0)}(x) \;=\; x(a^*-x)_+/2 \quad \text{and} \quad h_{1,2,(1)}(x) \;=\; (x-a^*)_+(1-x)/2.$$

Clearly, both $h_{1,2,(i)}$ are non-zero and therefore uniform integration is intractable in F_d.

We now remove the boundary conditions and consider the Sobolev class of periodic functions

$$F_{1,\gamma} \;=\; \widetilde{W}^1([0,1]) \;=\; \{f \in W^1([0,1]) \;:\; f(0) = f(1)\,\}$$

with the inner product

$$\langle f,g\rangle_{F_{1,\gamma}} \;=\; f(0)g(0) + \gamma^{-1}\int_0^1 f'(t)g'(t)\,dt.$$

For $d>1$, we have the Sobolev space $F_d = \widetilde{W}^{1,1,\dots,1}([0,1]^d)$ with the inner product

$$\langle f,g\rangle_{F_{d,\gamma}} \;=\; \sum_{u\subset\{1,2,\dots,d\}} \prod_{j\in u} \gamma_j^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{|u|}}{\partial x_u} f(x_u,0)\,\frac{\partial^{|u|}}{\partial x_u} g(x_u,0)\,dx_u. \tag{31}$$

The reproducing kernel is now

$$K_{1,\gamma}(x,t) \;=\; 1 + \gamma^{-1}\left(\min(x,t) - xt\right).$$

We can also express $K_{1,\gamma}$ as

$$K_{1,\gamma} \;=\; 1 + \gamma R_{1,a^*} + \gamma R_{2,a^*}.$$

Theorem 4 now applies with $R_1 = 1$, $R_3 = R_{1,a^*}$ and $R_2 = R_{2,a^*}$. For uniform integration we now have $h_1(x) = 1 + x(1-x)/2$ and $h_{1,1} = 1$, $h_{1,3}(x) = (\min(a^*,x) - a^*x)/2$ with the unchanged $h_{1,2}$ and non-zero $h_{1,2,(i)}$.

We summarize the discussion of this section in the following corollary.

Corollary 7. *Uniform integration in the Sobolev space $F_d = \widetilde{W}_0^{1,1,\dots,1}([0,1]^d)$ of periodic functions with boundary conditions is intractable. In particular,*

$$\lim_{d\to\infty} \frac{e(d^q,F_d)}{e(0,F_d)} \;=\; 1 \quad \forall\, q = 1,2,\dots\,.$$

If $\sum_{j=1}^{\infty} \gamma_j = \infty$ then uniform integration in the Sobolev space $F_{d,\gamma} = \widetilde{W}^{1,1,\dots,1}([0,1]^d)$ of periodic function without boundary conditions and with the inner product (31) is not strongly tractable and

$$\lim_{d\to\infty} \frac{e(n,F_{d,\gamma})}{e(0,F_{d,\gamma})} \;=\; 1 \quad \forall\, n.$$

If $\lim^*_{d\to\infty}\sum_{j=1}^d \gamma_j/\ln d = \infty$ *then uniform integration in the Sobolev space* $F_{d,\gamma} = \widetilde{W}^{1,1,\dots,1}([0,1]^d)$ *is intractable and*

$$\lim^*_{d\to\infty}\frac{e(d^q,F_{d,\gamma})}{e(0,F_{d,\gamma})} = 1 \quad \forall\, q=1,2,\dots\,,$$

where $\lim^* \in \{\lim, \limsup\}$.

The next corollary follows from Corollary 7, along with the results from [7, 16].

Corollary 8. *Uniform integration in the Sobolev space* $\widetilde{W}^{1,1,\dots,1}([0,1]^d)$ *is strongly tractable iff* $\sum_{j=1}^\infty \gamma_j < \infty$. *If this holds, then the strong exponent of tractability belongs to* $[1,2]$. *If* $\sum_{j=1}^\infty \gamma_j^{1/2} < \infty$ *then the strong exponent of tractability is* 1, *see* [7].

Uniform integration in the Sobolev space $\widetilde{W}^{1,1,\dots,1}([0,1]^d)$ *is tractable iff*

$$\limsup_d \sum_{j=1}^d \gamma_j/\ln d < \infty.$$

Example: Higher Rank Modifications

We now show that for some spaces we need to represent the original kernel by higher-rank modifications to obtain a decomposable R_2. We will do this for the Sobolev space $F_1 = W_0^r([0,1])$ with boundary conditions at zero and for the Sobolev space $F_1 = W^r([0,1])$ without boundary conditions. We start with the boundary case, for which $f(0)=\cdots=f^{(r-1)}(0)=0$ and

$$\langle f,g\rangle_{F_1} = \int_0^1 f^{(r)}(t)g^{(r)}(t)\,dt.$$

The reproducing kernel is

$$K_1(x,t) = \int_0^1 \frac{(x-u)_+^{r-1}}{(r-1)!}\,\frac{(t-u)_+^{r-1}}{(r-1)!}\,du.$$

Once more, this kernel is not decomposable for $a^*>0$.

We show an explicit form of K_1. It is enough to consider $x\le t$. We use $t-u = t-x+x-u$ and $(t-u)^{r-1} = \sum_{j=0}^{r-1}\binom{r-1}{j}(t-x)^{r-1-j}(x-u)^j$ to conclude by simple integration that

$$K_1(x,t) = \frac{x^r}{r!}\sum_{j=0}^{r-1}\frac{r}{r+j}\,\frac{x^j}{j!}\,\frac{(t-x)^{r-1-j}}{(r-1-j)!}.$$

This means that for a fixed x, the function $K_1(x,\cdot)$ is a polynomial in $t\in[x,1]$ of degree at most $r-1$.

For an arbitrary $a^*\in(0,1)$, consider the subspace F_{a^*} of F_1 consisting of functions f for which $f^{(i)}(a^*)=0$ for $i=0,1,\dots,r-1$. Observe that $f^{(i)}(a^*) =$

$\left\langle f, K_1^{(i,0)}(a^*,\cdot)\right\rangle_{F_1}$, where $K_1^{(i,0)}$ denotes differentiation i times with respect to the first variable. Hence, $f \in F_{a^*}$ iff f is orthogonal to

$$A_{r-1} = \mathrm{span}(K^{(0,0)}(a^*,\cdot), K^{(1,0)}(a^*,\cdot), \dots, K^{(r-1,0)}(a^*,\cdot)).$$

Let $g_i \in A_{r-1}$ be an orthonormal basis of A_{r-1} for $i = 0, 1, \dots, r-1$. Since each $K_1^{(i,0)}(a^*,\cdot)$ is a polynomial of degree at most $r-1$, the same holds for g_i in the interval $[a^*, 1]$.

Let K_{a^*} be the reproducing kernel of F_{a^*}. It is known that

$$K_{a^*}(x,t) \;=\; K_1(x,t) - \sum_{j=0}^{r-1} g_j(x) g_j(t).$$

We claim that K_{a^*} is decomposable with a^*. That is, for $x \le a^* \le t$ we need to show that $K_{a^*}(x,t) = 0$. Indeed, by Taylor's theorem we have

$$K_{a^*}(x,t) \;=\; \sum_{j=0}^{r-1} \frac{1}{j!} K_{a^*}^{(0,j)}(x,a^*)\,(t-a^*)^j \;+\; \int_{a^*}^{t} K_{a^*}^{(0,r)}(x,u) \frac{(t-u)^{r-1}}{(r-1)!}\,du.$$

Observe that the terms of the sum vanish since $K_{a^*}^{(0,j)}(\cdot,a^*) \equiv 0$, due to the fact that

$$0 \;=\; f^{(j)}(a^*) \;=\; \left\langle f, K_{a^*}^{(0,j)}(\cdot,a^*)\right\rangle_{F_{a^*}} \qquad \forall\, f \in F_{a^*}.$$

To show that the last term is also zero, it is enough to prove that $K_{a^*}^{(0,r)}(x,u) = 0$ for all $u \ge a^* \ge x$. This follows from the fact that all g_j are polynomials of degree at most $r-1$ in the interval $[a^*, 1]$, and therefore

$$K_{a^*}^{(0,r)}(x,u) = K_1^{(0,r)}(x,u) = \frac{\partial^r}{\partial u^r} \int_0^x \frac{(x-v)^{r-1}}{(r-1)!} \frac{(u-v)^{r-1}}{(r-1)!}\,dv \;=\; 0,$$

as claimed. Hence, we have $K_1 = R_{1,a^*} + R_{2,a^*}$ with $R_{1,a^*}(x,t) = \sum_{j=1}^{r-1} g_j(x) g_j(t)$ and $R_{2,a^*} = K_{a^*}$ is decomposable.

Consider now uniform integration, $I_1(f) = \int_0^1 f(t)\,dt$. Then

$$h_1(x) \;=\; \int_0^x \frac{(1-u)^r}{r!}\,\frac{(x-u)^{r-1}}{(r-1)!}\,du$$

is a polynomial of degree $2r$. Since $h_{1,2}$ differs from h_1 by a polynomial of degree $r-1$, the function $h_{1,2}$ is a polynomial of degree $2r$. Its restrictions $h_{1,2,(0)}$ and $h_{1,2,(1)}$ are also polynomials of degree $2r$ over $[0,a^*]$ and $[a^*,1]$ and therefore their norms must be non-zero.

Therefore all the assumptions of Theorems 2 and 3 hold with $\gamma_j = 1$. This proves that uniform integration is intractable.

We now remove the boundary conditions and consider the Sobolev space $F_{1,\gamma} = W^r([0,1]^d)$ with the inner product

$$\langle f, g\rangle_{F_{1,\gamma}} \;=\; \sum_{j=0}^{r-1} f^{(j)}(0) g^{(j)}(0) \;+\; \gamma^{-1} \int_0^1 f^{(r)}(t) g^{(r)}(t)\,dt.$$

The kernel is now

$$K_{1,\gamma}(x,t) \;=\; R_1(x,t) + \gamma \int_0^1 \frac{(x-u)_+^{r-1}}{(r-1)!}\,\frac{(t-u)_+^{r-1}}{(r-1)!}\,du$$

with

$$R_1(x,t) \;=\; \sum_{j=0}^{r-1} \frac{x^j}{j!}\,\frac{t^j}{j!}\,.$$

The kernel $K_{1,\gamma}$ can be written as

$$K_{1,\gamma}(x,t) \;=\; R_1(x,t) \;+\gamma \sum_{j=1}^{r-1} g_j(x)g_j(t) \;+\; \gamma\, K_{a^*}(x,t).$$

For uniform integration we have

$$h_1(x) \;=\; \sum_{j=0}^{r-1} \frac{x^j}{j!(j+1)!} \;+\; \gamma \int_0^x \frac{(1-u)^r}{r!}\,\frac{(x-u)^{r-1}}{(r-1)!}\,du.$$

So h_1 is still a polynomial of degree $2r$, and therefore all the assumptions of Theorem 4 hold. We summarize the result of this section in the following corollary.

Corollary 9. *Uniform integration for the Sobolev space $F_d = W_0^{r,r,\dots,r}([0,1]^d)$ is intractable for any $r \ge 1$. In particular,*

$$\lim_{d\to\infty} \frac{e(d^q, F_d)}{e(0, F_d)} \;=\; 1 \quad \forall\, q = 1,2,\dots,\,.$$

If $\sum_{j=1}^{\infty} \gamma_j = \infty$ then uniform integration in the Sobolev space $F_{d,\gamma} = W^{r,r,\dots,r}([0,1]^d)$ is not strongly tractable and

$$\lim_{d\to\infty} \frac{e(n, F_{d,\gamma})}{e(0, F_{d,\gamma})} \;=\; 1 \quad \forall\, n.$$

If $\lim^*_{d\to\infty} \sum_{j=1}^{d} \gamma_j / \ln\, d \,=\, \infty$ then uniform integration in the Sobolev space $F_{d,\gamma} = W^{r,r,\dots,r}([0,1]^d)$ is intractable and

$$\lim\nolimits^*_{d\to\infty} \frac{e(d^q, F_{d,\gamma})}{e(0, F_{d,\gamma})} \;=\; 1 \quad \forall\, q = 1,2,\dots\,,$$

where $\lim^* \in \{\lim, \limsup\}$.

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