Henryk Woźniakowski and the Complexity of Continuous Problems

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Henryk Woźniakowski was presented with an honorary doctoral degree by the University of Jena. The celebration took place on June 6, 2008, in the Aula of the Friedrich-Schiller-Universität Jena. This paper is an expanded and translated version of my laudatio.

1 Introduction

Henryk Woźniakowski is a fascinating colleague and friend. This short paper can only describe a very small part of what Henryk has done.

Henryk was born on August 31, 1946, in Lublin, Poland. Lublin is about 150 kilometers south east of Warsaw and today has about 350 000 inhabitants. His family moved to Warsaw in 1950. Henryk studied mathematics and computer science at the University of Warsaw and got his diploma in 1969, Ph.D. in 1972 and habilitation in 1976. From 1972 to 1977, he was assistant professor; in 1977 he became an associate professor in Warsaw.

In 1981, Henryk was elected chairman of the Department of Mathematics, Computer Science and Mechanics. He was running as a Solidarity candidate. In the same year, the Senate of the University of Warsaw decided that Henryk should become full professor. However, he had to wait till 1988 that this decision became a reality because of political reasons—Henryk Woźniakowski was for many years one of the leaders of the Solidarność movement at the University of Warsaw. In 1989, after political changes in Poland, Henryk was elected chairman of Solidarity at the University of Warsaw and served two years. Even before, in 1984, Henryk got a position as a full professor at Columbia University in New York. Since then, he has been teaching in both Warsaw and at Columbia.

Henryk received many prizes, such as the Stanisław Mazur prize of the Polish Mathematics Society in 1988. He had long stays in Berkeley (MSRI and ICSI), at the MGU in Moscow, at Carnegie-Mellon University, and at the University of New South Wales in Sydney. In 2005 Henryk was awarded the Humboldt Research Award and visited from November 2006 till July 2007 the University of Jena. Henryk is a member of the Polish Academy of Sciences.

The University of Jena (FSU) is quite picky with respect to honorary doctoral degrees. Although the FSU celebrated its 450th anniversary in 2008, only three colleagues have received an honorary degree because of their work in mathematics:

- Erna Weber, 1897–1988, for her work in statistics,
- Aleksander Pełczyński, born 1932, for his work in functional analysis,
- Boris Trachtenbrot, born 1921, for his work in theoretical computer science.

After Aleksander Pełczyński, Henryk is the second mathematician from Warsaw who is honored by the University of Jena. This is certainly a very good proof of the high quality of Polish mathematics, as well as the good relations between the Universities in Warsaw and Jena. I cite from the diploma:

> In Anerkennung seiner grundlegenden Arbeiten zur Numerischen Mathematik. Besonders hervorgehoben seien die tiefen Einsichten durch die neue Disziplin „Information-Based Complexity“ und die Arbeiten zum Fluch der Dimension, mit deren Hilfe man erstmals versteht, welche hochdimensionalen Probleme lösbar sind.[1]

[1] Translation: In recognition of his fundamental work in numerical mathematics. We emphasize in particular the deep insights by the new discipline “Information-Based Complexity” and the work on the curse of dimensionality. With this work we understand for the first time which high-dimensional problems are tractable.

2 Early Work

Henryk is an excellent mathematician with a great creative urge and power. His first paper [36] appeared in 1969. Henryk was 23 years old and got his diploma in the same year. During the next ten years Henryk published many papers about the numerical solution of linear and nonlinear equations.

Several of these papers, as well as his Ph.D. dissertation, deal with the maximal order of methods for the solution of nonlinear equations. In particular, Woźniakowski proved a conjecture of Traub and Kung concerning the maximal order of multi-point iterations without memory, see Woźniakowski [37, 38, 39], Traub and Woźniakowski [26], and the paper by Joseph Traub in this book.

Several of his papers about the numerical stability for solving equations appeared in Numerische Mathematik and in BIT in the years 1977 and 1978. These papers [7, 40, 41, 42] are still cited quite often.

3 A General Theory of Optimal Algorithms

Henryk was decisively involved with the creation of two big theories—the second one is his own child, other colleagues collaborating only later.

Together with Joseph Traub, Henryk built a complexity theory for continuous problems around 1977. The discrete world of the Turing machine is too narrow for many applications. We want to understand efficient algorithms for numerical integration, for the solution of differential equations, and for many other problems that involve real- or complex-valued functions on intervals or more complicated domains.

Of course there is a long tradition of studying algorithms for continuous problems and some of the algorithms even bear the name of their inventors, such as Newton's method, Gaussian quadrature formula, or Lagrange interpolation. More recent algorithms include the Metropolis algorithm or the Jenkins-Traub algorithm.

Also the complexity, i.e., the cost of optimal algorithms, was studied, sometimes for a restricted class of algorithms and always only for a specific problem.

Hence there existed something what could be called "pieces of the puzzle" and it is fair to mention many fathers of the theory of optimal algorithms for continuous problems, such as Babushka, Bakhvalov, Kolmogorov, Nikolskij, Smolyak and Sobolev in the east and Golomb, Kiefer, Sard and Weinberger in the west. This list is certainly far from complete!

But the flow of information between the west and the east was sometimes slow and, more important, there only existed somewhat isolated results for specific problems. These results became a part of a comprehensive theory only later.

Hence the "first black book" by Traub and Woźniakowski, *A General Theory of Optimal Algorithms*, was a sensation. It was published in 1980 by Academic Press. This book described for the first time a comprehensive theory for continuous problems. The book also contains, as Part C, a brief history of the field and a long annotated bibliography. There the reader may find all the references that are missing here.

Ko-Wei Lih writes in the Mathematical Reviews an excellent report that ends with "... the authors should be congratulated on their magnificent product which elevates the study of the approximate to a higher dimension." Actually, this report is still very informative and this is why it is reprinted here in full length:

> This monograph is a report on work in progress in the theory of analytic computational complexity which is the study of optimal algorithms for problems solved approximately. Such a line of investigation had its inception around 1950 with the work of Kiefer, Sard, and Nikolskij on optimal algorithms for locating the maximum, for integration, for approximation, etc. This stream of research generated mainly results concerning specific problems. In 1961 Traub initiated a second stream of research with the study of solutions to nonlinear equations by iterative methods. The possibility of unification of these two streams into one general and necessarily more abstract framework was first shown by the authors in two long reports ["General theory of optimal error algorithms and analytic complexity", parts A and B; per bibl.]. This monograph includes extended and improved material from these two reports. A central concern of the computer scientist is the selection of the best algorithm for solving a problem. However, selection of the best is subject to multivariate criteria such

as time and space complexity, ease of implementation, robustness, and stability. The authors only deal with time complexity here. Nevertheless, conclusions could be easily adjusted to work for space complexity. Also, the authors study only problems which cannot be solved exactly with finite complexity or problems which one chooses to solve approximately for reasons of efficiency. The final theory includes algebraic complexity as a special case. The generality and simplicity achieved by this theory has its cornerstone on the notion of information operator. Adversary arguments based on the information used by an algorithm lead to lower bound theorems. This has its practical application in the rationalization of the synthesis of algorithms. Traditional ad hoc algorithms are revealed to be paying high penalty without the use of optimal information. The authors propose 20 general questions to be studied. The following is a sample of some of them. 1. What is a lower bound on the error of any algorithm for solving a problem using given information? 2. In general is there an algorithm which gets arbitrarily close to this lower bound? 3. What is the optimal information for solving a problem? 4. Given a specific problem, how do we characterize and construct an optimal algorithm for its solution? 5. Can it be established that one problem is intrinsically harder than another? 6. Compare the power of adaptive and non-adaptive algorithms. 7. Compare the power of linear and nonlinear information operators. 8. What is the class of all problems which can be solved by iteration using linear information?

This monograph is divided into three parts. Part A has ten chapters and deals with a general information model. The basic concepts are first formalized. The notions of optimal error algorithm and optimal complexity algorithms are introduced. Then a large portion is devoted to the study of linear problems using linear information. It is shown that adaptive information is not more powerful than non-adaptive information for a linear problem. A linear problem is also constructed to possess no linear optimal error algorithms. However, natural problems are immune from such pathology. Algorithms optimal in the sense of Sard and Nikolskij are shown to be optimal error algorithms. There exist linear problems with essentially arbitrary complexity. So there are no "gaps" in the complexity function.

In Chapter 6, the theory is applied to the solution of many different linear problems including approximation, interpolation, integration, and the solution of linear partial differential equations. Finally, the theory of nonlinear information is developed and applications given. In the general information setting the class of nonlinear information operators is actually too powerful to be of interest. In the last two chapters, a partial hierarchy of complexity is presented and other models of computation are briefly discussed. Part B consists of one chapter with 11 sections. It deals with the iterative information model and is built on some 20 years of research on iterative complexity initiated by Traub. The deepest question studied is: what problems can be solved by iteration using iterative linear information? For one-point stationary iterations using iterative linear information, it is shown that the class of iterative algorithms is empty for a problem unless the "index" of the problem is finite. A conjecture characterizing problems with finite index is posed to the effect that a positive solution implies that only nonlinear equations can be solved by iteration. Part C provides a brief history of the theory of analytic computational complexity and an annotated bibliography of over 300 papers and books covering both the eastern European and the Western literature. The authors supply numerous conjectures and open problems throughout the book. They also recommend eight tracks for various readers with particular interests such as researchers interested in open problems, researchers interested in the literature on history, theoretical computer scientists, mathematicians, numerical analysts, scientists and engineers. Some of these readers will definitely find that the study of this book is a quite strenuous task. However, the authors should be congratulated on their magnificent product which elevates the study of the approximate to a higher dimension. [MR0584446 (84m:680410)]

4 Information-Based Complexity

There are two more monographs, written jointly with Joseph Traub and Grzegorz Wasilkowski. The third monograph, *Information-Based Complexity*, is certainly a special highlight.

The first monograph did *not* discuss the average case setting and did *not* study randomized algorithms. These are two major new subjects of the "second black book", that appeared again with Academic Press, see [24]. Of course the book contains many more results, for example also a section on linear PDEs written by Arthur Werschulz. Again I cite the complete report from the Mathematical Reviews, written by M. I. Dekhtyar.

> There are two main branches of computational complexity theory. The first is combinatorial complexity, which considers problems for which the information is complete, exact, and free. The second, which deals with problems for which the information is partial, noisy, and priced and for which solutions are not exact, is called information-based complexity and is the subject of the book under review. The authors summarize and present a number of results that are concerned with various definitions of the cost and the error of algorithms. The book may be viewed as a continuation and extension of two previous books [Traub and Woźniakowski, A general theory of optimal algorithms, Academic Press, New York, 1980; the authors, Information, uncertainty, complexity, Addison-Wesley, Reading, MA, 1983].
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> The book consists of twelve chapters and two appendices. Chapter 1 is an introduction. In Chapter 2, the basic concepts of information-based complexity are illustrated by the example of continuous binary search.
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> In Chapter 3 an abstract formulation of an information-based theory is presented. A problem is defined as a solution operator $S : F \to G$, where F is a set and G is a normed linear space over the scalar field of real or complex numbers. Elements $f \in F$ are called problem elements, and the $S(f)$ are called solution elements. Computation of an approximation $U(f)$ of $S(f)$ consists

of two steps. The first is to obtain information about f:

$$N(f) = [L_1(f), L_2(f; y_1), \cdots, L_{n(f)}(f; y_1, \cdots, y_{n(f)-1})],$$

where $y_i = L_i(f; y_1, \cdots, y_{i-1})$, and L_i is a permissible information operation. Information N is called non-adaptive if $y_i = L_i(f)$. The second step is to evaluate the approximation by $N(f) \mapsto U(f) = \varphi(N(f))$, where φ is a mapping (algorithm): $N(F) \to G$. Then the cost of computing $U(f)$ is given by $\mathrm{cost}(U, f) = \mathrm{cost}(N, f) + \mathrm{cost}(\varphi, N(f))$. The main results presented in the book deal with the first item of this sum. Three definitions of the error $e(U)$ are considered: (i) the worst case setting: $e(U) = \{\sup \|S(f) - U(f)\|\ f \in F\}$; (ii) the average case setting: $e(U) = (\int_F \|S(f) - U(f)\|(dF))^{1/2}$; (iii) the probabilistic setting: let $\delta \in [0, 1]$; then $e(U) = \inf\{\{\sup \|S(f) - U(f)\|\ f \in F - A\}\ (A) \leq \delta\}$. In Chapter 4 theoretical results for the worst case setting are presented. The radius of information is introduced; it is a sharp lower bound on the error of any algorithm using this information. The minimal cardinality of information with radius at most ε is denoted by $m(\varepsilon)$. If c is the cost of one information operation then $c\,m(\varepsilon)$ is a lower bound on the ε-complexity. Conditions under which this bound is almost sharp are investigated. Special attention is paid to the class of linear problems. It is shown that the use of adaptive information does not decrease ε-complexity for this class. Chapter 5 contains examples of approximation problems to which the results of Chapter 4 are applied to obtain complexity bounds and optimal algorithms. They include integration, function approximation, optimization, etc. For most of them only short sketches of the results are presented and the authors direct the reader to the references cited for detailed analysis.

Chapters 6 and 7 deal with the average case setting. A Gaussian measure is proposed as the probability measure on the set F. The average radius of information and the average minimal cardinality of error $m^{\mathrm{avg}}(\varepsilon)$ are introduced. As in Chapter 4, $c\,m^{\mathrm{avg}}(\varepsilon)$ is a lower bound on the average ε-complexity. It is shown that this bound is tight for linear problems. In Chapter 7 three linear problems are considered and bounds on their average ε-complexity are established.

The probabilistic setting is considered in Chapter 8. The probabilistic radius of information and the probabilistic (ε, δ)-cardinality number $m^{\text{prob}}(\varepsilon, \delta)$ are defined in such a way that $c\, m^{\text{prob}}(\varepsilon, \delta)$ is a lower bound on probabilistic complexity. Complexity of linear problems is analyzed. The probabilistic complexity is compared with the average complexity and some relations are developed.

Chapter 9 contains a comparison between different settings for four problems: the integration of smooth functions, the integration of smooth periodic functions, the approximation of smooth periodic functions, and the approximation of smooth non-periodic functions. For each problem a 5×3 table is presented with formulas for the complexity under five error criteria (absolute error, normalized error, and three kinds of relative errors) in three settings.

In Chapter 10 the asymptotic setting is studied. Two approaches to optimal asymptotic algorithms are considered. Under one of them the best speed of convergence is achieved by algorithms which are optimal in the worst case setting. The other approach leads to a close relation between the asymptotic and average case settings.

The main question investigated in Chapter 11 is the extent to which randomization can lower the worst and the average case complexities. The results presented here show that randomization does not help significantly for linear problems.[2] Some results concerning noisy information are given in Chapter 12. Noisy information about $f \in F$ has the form

$$N(f, \overline{x}) = [L_1(f) + x_1, \cdots, L_n(f; z_1, \cdots, z_{n-1}) + x_n],$$

[2]Remark of the author: It is true that randomization does not help for *some* linear problems. There are other problems, however, where randomization helps a lot; the most popular is the problem of numerical integration.

where $\overline{x} = [x_1, \cdots, x_n]$ is noise, $n = n(f, \overline{x})$, and $z_i = L_i(f; z_1, \cdots, z_{i-1}) + x_i$ is the ith observed piece of information. The relationship between adaptive information and non-adaptive information is discussed. These areas are open for further investigation.

Two appendices contain the main definitions and facts concerning functional analysis and measure theory. The extensive bibliography includes more than 450 items. Almost all chapters and sections are followed by notes and remarks that contain some additional results, comments and references. The book is clearly written and may be used as a handbook by specialists in information-based complexity; it may also be recommended as a textbook for those who want to study this area of computer science. [MR0958691 (90f:68085)]

5 Tractability of Multivariate Problems

Many results in numerical analysis and approximation theory concern the *optimal order of convergence* for a problem and a class of functions. Also many results in the two black books GTOA and IBC deal with this subject. There is a widespread belief that we understand the complexity of a problem if we know the optimal order. This belief is wrong; it was Henryk who first studied the following problem seriously since about 1992:

> Which multivariate problems can be efficiently solved in high dimensions?

First we formalize this question according to [45, 46]. Assume that we want to solve a problem S_d for functions $f \in F_d$, where F_d is a class of d-variate functions. An example would be the computation of

$$S_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x, \tag{1}$$

for $f \in F_d$, up to an error ε. Assume that the cost of an optimal algorithm for this problem is $n(\varepsilon, d)$. If

$$n(\varepsilon, d) \le C \cdot \varepsilon^{-\alpha} d^{\beta} \tag{2}$$

for certain $C, \alpha, \beta > 0$ then the problem is called *(polynomially) tractable*. Observe that C is independent of d in the definition (2) of tractability. Therefore the optimal order of convergence does not say much about tractability. We give three examples. Only the first example uses known results about the order of convergence to decide the tractability problem.

5.1 Integration of C^k-Functions

There is a basic result of Bakhvalov from the year 1959 that the optimal order of convergence for the computation of the integral (1) for C^k-functions is $n^{-k/d}$. We conclude that the problem is *not* (polynomially) tractable. Roughly speaking, the cost is exponential in d, this is called the *curse of dimensionality*, after Bellman.

5.2 Integration of smooth periodic functions

We now consider the Korobov space $F_{d,\alpha}$ of complex functions from $L_1([0,1]^d)$, where $\alpha > 1$. This class is defined by controlling the sizes of Fourier coefficients of functions. More precisely, for $h = [h_1, h_2, \dots, h_d]$ with integers h_j, consider the Fourier coefficients

$$\hat{f}(h) = \int_{[0,1]^d} f(x) e^{-2\pi \mathrm{i} h \cdot x} \, \mathrm{d}x,$$

where $\mathrm{i} = \sqrt{-1}$ and $h \cdot x = \sum_{j=1}^d h_j x_j$. Denote $\bar{h}_j = \max(1, |h_j|)$. Then

$$F_{d,\alpha} = \left\{ f \in L_1([0,1]^d) \mid |\hat{f}(h)| \le \left(\bar{h}_1 \bar{h}_2 \cdots \bar{h}_d\right)^{-\alpha} \text{ for all } h \in \mathbb{Z}^d \right\}.$$

Again we consider the integration problem

$$S_d(f) = \int_{[0,1]^d} f(x) \, \mathrm{d}x \qquad \text{for } \; f \in F_{d,\alpha}.$$

We consider algorithms A_n that use n function values, the worst case error $e^{\mathrm{wor}}(A_n)$, the nth minimal worst case error $e(n,d)$, and the minimal number

$$n^{\mathrm{wor}}(\varepsilon, S_d, F_{d,\alpha})$$

of function values needed to approximate the integrals to within ε.

The integration problem for the Korobov class $F_{d,\alpha}$ has been studied in a number of papers and books. It is known that

$$e(n,d) = \mathcal{O}\left(n^{-p}\right) \quad \text{as} \quad n \to \infty, \qquad \text{for all} \quad p < \alpha.$$

For $p = \alpha$ we have

$$e(n,d) = \mathcal{O}\left(n^{-\alpha}\left(\ln n\right)^{\beta(d,\alpha)}\right)$$

where $\beta(d,\alpha)$ is of order d. Such errors can be obtained by lattice rules of rank 1, i.e., by algorithms of the form

$$A_n(f) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(\left\{j\frac{z}{n}\right\}\right),$$

where n is prime and $z \in \{1, 2, \ldots, n-1\}^d$ is a well-chosen integer vector. Here, $\{x\}$ denotes the vector whose jth component is the fractional part of x_j.

Hence, for large α, the optimal order of convergence is also large and roughly equal to α independently of d. This is encouraging, but what can we say about tractability?

The tractability of this integration problem was studied by Sloan and Woźniakowski in [20], where it was proved that

$$e(n,d) = 1 \qquad \text{for} \quad n = 0, 1, \ldots, 2^d - 1, \tag{3}$$

which implies that

$$n^{\mathrm{wor}}(\varepsilon, S_d, F_{d,\alpha}) \geq 2^d \quad \text{for all} \quad \varepsilon \in (0,1).$$

That is, even for arbitrarily large α, despite an excellent order of convergence, this integration problem is *not* tractable. More about this problem can be found in Henryk's paper in this booklet.

5.3 The star-discrepancy

After two negative examples the reader may have the impression that all "interesting" problems are intractable and the curse of dimension is "always" present. This would be a wrong impression, since there are many tractable problems; actually there are many multivariate problems that can be solved in very high dimension.

Discrepancy is a measure of the deviation from uniformity of a set of points. It is desirable that a set of n points be chosen so that the discrepancy is as small as possible. The notion of discrepancy appears in many fields of mathematics.

We begin with the definition of the star discrepancy. Let $x = [x_1, x_2 \ldots, x_d]$ be from $[0,1]^d$. By the box $[0, x)$ we mean the set $[0, x_1) \times [0, x_2) \times \cdots \times [0, x_d)$, whose (Lebesgue) measure is clearly $x_1 x_2 \cdots x_d$. For given points $t_1, t_2, \ldots, t_n \in [0,1]^d$, we approximate the volume of $[0, x)$ by the fraction of the points t_i that are in the box $[0, x)$. The error of such an approximation is called the *discrepancy function*, and is given by

$$\operatorname{disc}(x) = x_1 x_2 \cdots x_d - \frac{1}{n} \sum_{i=1}^{n} 1_{[0,x)}(t_i),$$

where $1_{[0,x)}$ is the indicator (characteristic) function, so that $1_{[0,x)}(t_i) = 1$ if $t_i \in [0, x)$ and $1_{[0,x)}(t_i) = 0$ otherwise.

The *star discrepancy* of the points $t_1, \ldots, t_n \in [0,1]^d$ is defined by the L_∞-norm of the discrepancy function disc

$$\operatorname{disc}(t_1, t_2, \ldots, t_n) = \sup_{x \in [0,1]^d} \left| x_1 x_2 \cdots x_d - \frac{1}{n} \sum_{i=1}^{n} 1_{[0,x)}(t_i) \right|. \tag{4}$$

The main problem associated with star discrepancy is that of finding points $t_1, t_2, \ldots, t_n$ that minimize disc, and to study how this minimum depends on d and n. We now show that the star discrepancy is intimately related to multivariate integration. Let $W_1^{\mathbf{1}} := W_1^{(1,1,\ldots,1)}([0,1]^d)$ be the Sobolev space of functions defined on $[0,1]^d$ that are once differentiable in each variable and whose derivatives have finite L_1-norm. We consider first the subspace of functions that satisfy the boundary conditions $f(x) = 0$ if at least one component of x is 1, and define the norm

$$\|f\|_{d,1}^* = \int_{[0,1]^d} \left| \frac{\partial^d}{\partial x} f(x) \right| \mathrm{d}x.$$

Here, $\partial x = \partial x_1 \partial x_2 \cdots \partial x_d$.

That is, we consider the class

$$F_d^* = \left\{ f \in W_1^{\mathbf{1}} \;\middle|\; f(x) = 0 \text{ if } x_j = 1 \text{ for some } j \in [1,d], \text{ and } \|f\|_{d,1}^* \le 1 \right\}.$$

Consider the multivariate integration problem

$$S_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x \qquad \text{for } f \in F_d^*.$$

We approximate $S_d(f)$ by *quasi-Monte Carlo* algorithms, which are of the form

$$Q_{d,n}(f) = \frac{1}{n} \sum_{j=1}^{n} f(t_j)$$

for some points $t_j \in [0,1]^d$. We stress that the points t_j are chosen non-adaptively and deterministically. The name "quasi-Monte Carlo" is widely used, since these algorithms are similar to the Monte Carlo algorithm which takes the same form but for which the points t_j are randomly chosen, usually as independent uniformly distributed points over $[0,1]^d$.

We also stress that we use especially simple coefficients n^{-1}. This means that if $f(t_1), f(t_2), \ldots, f(t_n)$ are already computed then the computation of $Q_{d,n}(f)$ requires just $n-1$ additions and one division. Since the points $t_1, t_2, \ldots, t_n$ are non-adaptive, $Q_{d,n}f$ can be evaluated very efficiently in parallel since each $f(t_j)$ can be computed on a different processor. Obviously, $Q_{d,n}$ integrates constant functions exactly, even though $1 \notin F_d^*$.

The quality of the algorithm $Q_{n,d}$ depends on the points t_j. There is a deep and beautiful theory about how the points t_j should be chosen. We add that quasi-Monte Carlo algorithms have been used very successfully for many applications, including mathematical finance applications, for d equal 360 or even larger.

We now recall Hlawka and Zaremba's identity, which states that for $f \in W_1^{\mathbf{1}}$, we have

$$S_d(f) - Q_{d,n}(f) = \sum_{\emptyset \ne \mathfrak{u} \subseteq \{1,2,\ldots,d\}} (-1)^{|\mathfrak{u}|} \int_{[0,1]^{|\mathfrak{u}|}} \mathrm{disc}(x_{\mathfrak{u}}, 1) \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, 1) \mathrm{d}x_{\mathfrak{u}}.$$

Here, we use the following standard notation. For any subset $\mathfrak{u}$ of $\{1,2,\ldots,d\}$ and for any vector $x \in [0,1]^d$, we let $x_{\mathfrak{u}}$ denote the vector from $[0,1]^{|\mathfrak{u}|}$, where

$|\mathfrak{u}|$ is the cardinality of $\mathfrak{u}$, whose components are those components of x whose indices are in $\mathfrak{u}$. For example, for $d = 5$ and $\mathfrak{u} = \{2, 4, 5\}$ we have $x_{\mathfrak{u}} = [x_2, x_4, x_5]$. Then $\partial x_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \partial x_j$ and $\mathrm{d}x_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \mathrm{d}x_j$. By $(x_{\mathfrak{u}}, 1)$ we mean the vector from $[0, 1]^d$ with the same components as x for indices in $\mathfrak{u}$ and with the rest of components being replaced by 1. For our example, we have $(x_{\mathfrak{u}}, 1) = [1, x_2, 1, x_4, x_5]$. Note that

$$\mathrm{disc}(x_{\mathfrak{u}}, 1) = \prod_{k\in\mathfrak{u}} x_k - \frac{1}{n}\sum_{j=1}^{n} 1_{[0,x_{\mathfrak{u}})}\left((t_j)_{\mathfrak{u}}\right).$$

For $f \in F_d^*$, the boundary conditions imply that all terms in Hlawka and Zaremba identity vanish except the term for $\mathfrak{u} = \{1, 2, \ldots, d\}$. Hence, for $f \in F_d^*$ we have

$$S_d(f) - Q_{d,n}(f) = (-1)^d \int_{[0,1]^d} \mathrm{disc}(x)\frac{\partial^d}{\partial x} f(x)\mathrm{d}x.$$

Applying the Hölder inequality, we obtain that the worst case error of $Q_{n,d}$ is

$$e^{\mathrm{wor}}(Q_{d,n}) = \sup_{f\in F_d^*} |S_d(f) - Q_{d,n}(f)| = \mathrm{disc}(t_1, t_2, \ldots, t_n),$$

which is the star discrepancy for the points $t_1, t_2, \ldots, t_d$ that are used by the quasi-Monte Carlo algorithm $Q_{d,n}$.

We now remove the boundary conditions and consider the class

$$F_d = \left\{ f \in W_1^{\mathbf{1}} \;\; | \;\|f\|_{d,1} \le 1 \right\},$$

where the norm is given by

$$\|f\|_{d,1} = \sum_{\mathfrak{u}\subseteq\{1,2,\ldots,d\}} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, 1)\right| \mathrm{d}x_{\mathfrak{u}}.$$

The term for $\mathfrak{u} = \emptyset$ corresponds to $|f(1)|$.

We return to the Hlawka and Zaremba identity and again apply the Hölder inequality, this time for integrals and sums, and conclude that the worst case error again is

$$e^{\mathrm{wor}}(Q_{d,n}) = \sup_{f\in F_d} |S_d(f) - Q_{d,n}(f)| = \mathrm{disc}(t_1, t_2, \ldots, t_n).$$

The multivariate problem is properly scaled for both classes F_d^* and F_d since the initial error is 1. Then

$$n(\varepsilon,d) = \min\{n \mid \exists\, t_1,t_2,\ldots,t_n \in [0,1]^d \ \text{ such that } \ \mathrm{disc}(t_1,t_2,\ldots,t_n) \le \varepsilon\}$$

is the same for both classes; this is just the inverse of the star-discrepancy.

Hence, tractability of multivariate problems depends on how the inverse of the star discrepancy behaves as a function of ε and d. Based on many negative results for classical spaces and on the fact that all variables play the same role for the star discrepancy, it would be natural to expect an exponential dependence on d, i.e., intractability. Therefore it was quite a surprise when a positive result was proved in [5]. More precisely, let

$$\mathrm{disc}(n,d) = \inf_{t_1,t_2,\ldots,t_n \in [0,1]^d} \mathrm{disc}(t_1,t_2,\ldots,t_n)$$

denote the minimal star discrepancy that can be achieved with n points in the d-dimensional case. Then there exists a positive number C such that

$$\mathrm{disc}(n,d) \le C d^{1/2} n^{-1/2} \quad \text{for all } \ n,d \in \mathbb{N}.$$

The proof of this bound follows directly from deep results of the theory of empirical processes. The proof is unfortunately non-constructive, and we do not know points for which this bound holds. The slightly worse upper bound

$$\mathrm{disc}(n,d) \le 2\sqrt{2}\, n^{-1/2} \left(d \ln \left(\left\lceil \frac{d n^{1/2}}{2(\ln 2)^{1/2}} \right\rceil + 1 \right) + \ln 2 \right)^{1/2}$$

follows from Hoeffding's inequality and is quite elementary. Also this proof is non-constructive. However, using a probabilistic argument, it is easy to show that many points $t_1,t_2,\ldots,t_n$ satisfy both bounds modulo a multiplicative factor greater than one, see [5] for details.

The upper bounds on $\mathrm{disc}(n,d)$ can be easily translated into upper bounds on $n(\varepsilon,d)$. In particular, we have

$$n(\varepsilon,d) \le \left\lceil C^2 d \left(\frac{1}{\varepsilon} \right)^2 \right\rceil \quad \text{for all } \ \varepsilon \in (0,1) \text{ and } \ d \in \mathbb{N}. \tag{5}$$

This means that we have *polynomial tractability*. Furthermore it was also shown in [5] that there exists a positive number c such that

$$n(\varepsilon,d) \ge c\, d \ln \varepsilon^{-1} \quad \text{for all } \ \varepsilon \in (0,1/64] \ \text{ and } \ d \in \mathbb{N}.$$

In fact, this lower bound holds not only for quasi-Monte Carlo algorithms, but also in full generality for all algorithms.

The theory of tractability of multivariate problems was initiated by Henryk and also mainly developed by Henryk—sometimes together with colleagues and friends. The recent paper [47] of Henryk is an excellent survey and also describes the history of this young field.

We now understand why certain multivariate problems are tractable or not. We also know how intractable problems can be modified to obtain tractable problems. But, again, with a new theory, there are also new problems. Actually, the recent book [15] contains 30 Open Problems that, hopefully, are a challenge for many mathematicians.[3]

6 Why is the work of Henryk so fascinating?

There are two reasons. First, the persistence of Henryk who attacks, from quite different angles, similar questions again and again: How can we describe and find good or even optimal algorithms for different continuous problems of mathematics? What properties do those algorithms have? Henryk wants to understand this by a comprehensive theory. This is visible already in Henryk's early work and gets even more prominent later when Henryk's interests cover the whole range of numerical mathematics, as well as other subjects, such as computational physics and quantum computers.

Secondly, the strength and patience that are needed to study problems in their detail: For this it was necessary to work in many different areas of mathematics.

It is not enough to develop a general theory. Also in mathematics a theory gets thought-provoking only through laborious investigations of many single problems that need many different skills and lots of energy. Henryk got a lot of deep results by studying such problems and examples; he had to study many parts of mathematics to obtain these results. In this way he influenced many fields, as can be seen by studying the Mathematical Reviews.

Most of us publish papers in a relative small field, and probably are happy to work a small amount in a second field. Henryk published results

[3]Open Problems 18 and 25 of [15] have been already solved by Stefan Heinrich.

in numerical analysis and in many other fields, such as computability, number theory, linear algebra, measure theory, interpolation and approximation, Fourier analysis, functional analysis, operator theory, probability theory and stochastic processes, statistics and quantum theory.

There do not exist many other mathematicians with a similar versatility. All these excursions into different parts of mathematics are still strongly related to the main basic question that was always studied by Henryk: How can we construct and understand optimal algorithms for numerical problems?

Today, Henryk is a main leader all over the world and is a great communicator who works together with excellent colleagues in many countries.

7 Four of Henryk's Papers

It is difficult to select only four papers of Henryk. Probably it would be an interesting game to select the four "most influential" or the four "best" papers. To avoid this intractable problem I mention four papers that are cited most often—as can be checked with the Mathematical Reviews.[4]

Average case complexity of multivariate integration [43], 1991

Henryk studies the average case complexity of multivariate integration for the class $C([0,1]^d)$ equipped with the classical Wiener sheet measure. To derive the average case complexity one needs to obtain optimal sample points. This design problem was open for a long time. In this paper Henryk proves that the optimal design is closely related to discrepancy theory. The respective L_2-discrepancy problem was solved by K. F. Roth (the lower and the upper bounds being published in 1954 and 1980, respectively) and by K. K. Frolov (who also proved the upper bound), who showed that optimal sample points are given by shifted Hammersley points $z_1^*, z_2^*, \dots, z_n^*$. Henryk showed that $1 - z_1^*, 1 - z_2^*, \dots, 1 - z_n^*$ are the optimal sample points for the quadrature problem, and that the ε-complexity of the problem is of the order

[4]This kind of selection discriminates against older papers as well as very young papers. This is obvious for very fresh papers. But also older papers have a disadvantage since most colleagues do not bother to cite a paper from the early eighties if they can also cite the IBC book. Therefore all four papers were published between 1991 and 2000.

$\Theta(\varepsilon^{-1}(\ln \varepsilon^{-1})^{(d-1)/2})$.[5]

Hence this paper is in the intersection of complexity theory, stochastic processes, discrepancy theory, number theory and numerical analysis. In particular, the paper proves that this intersection is non-empty.

Explicit cost bounds of algorithms for multivariate tensor product problems [29], with Grzegorz Wasilkowski, 1995

The authors study explicit error bounds for the Smolyak algorithm in the worst case setting and in the average case setting for multivariate tensor product problems.

In 1963, Smolyak introduced an algorithm for tensor product problems and proved bounds of the form

$$n(\varepsilon, d) \le C_d\, \varepsilon^{-\beta_1} (\ln \varepsilon^{-1})^{\beta_2(d-1)}. \tag{6}$$

The interesting thing is that β_1, the order of convergence, does not depend on d. The constant C_d is, however, not known and therefore this is a typical classical result about the order of convergence. The authors prove bounds of the form

$$n(\varepsilon, d) \le C \left(\beta_1 + \beta_2 \frac{\ln \varepsilon^{-1}}{d-1}\right)^{\beta_3(d-1)} \varepsilon^{-\beta_4}, \tag{7}$$

where all the constants $C, \beta_1, \beta_2, \beta_3$ and β_4 are known and can be computed from error bounds for $d = 1$.

When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? [21], with Ian Sloan, 1998[6]

This paper is of fundamental importance for the understanding of high dimensional problems. It gives a partial answer to why quasi-Monte Carlo algorithms are successful, even in huge dimension. The authors define *weighted spaces* of functions using the idea that, for many applications, the number d

[5] Observe that this paper studies the *optimal order of convergence*, not tractability. The tractability of the L_2-discrepancy problem was studied later.

[6] This is the paper of Henryk that recently has been cited most often, according to Mathematical Reviews. By the way, it is is also the paper of Ian Sloan that is most often cited.

of variables is huge, however, not all variables play the same role and some variables are "less important" than others. The idea of *weighted spaces* is central for the recent theory of tractability of multivariate problems.

Integration and approximation in arbitrary dimensions [6], with Fred Hickernell, 2000

The authors study several multivariate integration and approximation problems. They consider algorithms for classes $f \in F_d$ using function values. Let $n(\varepsilon, d)$ be the minimal number of function values needed for a worst case error ε in the dimension d for the class F_d. The authors are mainly interested in spaces and problems with the property

$$n(\varepsilon, d) \leq C\, \varepsilon^{-p}, \tag{8}$$

where C and p do not depend on d. Problems with this property are called *strongly (polynomially) tractable.* The authors prove that integration and approximation are strongly tractable for certain weighted Korobov and Sobolev spaces.

For the approximation problem the authors also consider algorithms that use arbitrary linear functionals instead of function values. The main result is that (under some assumptions) this much more general information is "not much" better, i.e., the ε-exponents stay the same.

8 Other Directions

Here I just mention very few other of Henryk's research directions. Again my choice is very selective.

- Linear Optimization. Traub and Woźniakowski study in [27] the ellipsoid algorithm and observe that, even if this algorithm is a polynomial-time algorithm within the bit number model, it has unbounded cost in the real number model. The authors conjecture that there does not exist a polynomial-time algorithm for the linear inequalities problem. This problem is still open today, in 2008.

- Computation of Fixed Points. It is known from work of Nemirovskii that it is impossible to improve the efficiency of the simple iteration whenever the dimension of the domain of contractive functions is large. However for a modest dimension, Sikorski, Tsay and Woźniakowski [18] exhibit a fixed point ellipsoid algorithm that is much more efficient than the simple iteration for mildly contractive functions. This algorithm is based on Khachiyan's construction of minimal volume ellipsoids used for solving linear programming.

- Testing Operators. Together with David Lee, Henryk wrote several papers about testing and verification of linear and nonlinear operators, see, e.g., [12]. For the testing problem, A is an implementation of a specification S, both are mappings from a compact metric space F into a metric space G. Given $\varepsilon > 0$, one is allowed to compute Af and Sf for a finite number of f and has to decide whether $d(Af, Sf) \leq \varepsilon$ for all $f \in F$. It is shown that asymptotically correct sequences of guesses can be arranged. Sharp upper and lower bounds on the number of tests are given in terms of the Kolmogorov entropy of F. Probabilistic testing methods are developed and analyzed.

- Tractability of Path Integration. In [30], Wasilkowski and Woźniakowski analyze the complexity of computing integrals $\int_X f(x)\, \mu(\mathrm{d}x)$, where μ is a Gaussian measure on a Banach space X. For r times differentiable functions on X, the integration problem is tractable for deterministic algorithms iff the covariance operator is of finite rank. Hence for measures μ with infinite-dimensional covariance operator, Monte Carlo integration is superior to deterministic algorithms. For certain classes of entire functions on X, it is shown that the problem becomes tractable in the deterministic setting.

  Plaskota, Wasilkowski and Woźniakowski [17] suggest a new algorithm for the computation of Feynman-Kac path integrals. This algorithm has a very small cost, which gives a dramatic improvement of earlier results. However, there is also a problem since the new algorithm needs a lot of precomputation and therefore can (so far) only be used if the error requirements are moderate.

- Quantum Computers. Henryk studied quantum computation and wrote several papers about optimal numerical algorithms in the quantum setting. Kwas and Woźniakowski [11] prove sharp error bounds for the Boolean summation problem. Traub and Woźniakowski [28] study path integration on a quantum computer.

- Weighted Problems and Finite-Order Weights. As already mentioned, the work [21] by Sloan and Woźniakowski has been continued by many colleagues. Dick, Sloan, Wang and Woźniakowski [1] discuss general weights in order to give recommendations for choosing the weights in practice. They defined in [2] finite-order weights. Such weights seem to be appropriate for many applications and they model functions of d variables that can be expressed as a sum of functions of k variables with k independent of d. The authors also prove new lower and upper bounds for the tractability of quasi-Monte Carlo algorithms for the computation of integrals for functions from weighted Sobolev spaces. For finite-order weights, we usually have tractability bounds depending polynomially on $d^{\beta(k)}$ with $\beta(k)$ linearly dependent on k.

- Lower bounds. To determine the complexity of a problem, we also need good lower bounds. This is straightforward for linear operators $S : F \to G$ between Hilbert spaces, if we consider algorithms that use arbitrary linear functionals in the worst case setting. Then we have to study the singular values of S. The proof of lower bounds is much more difficult in the randomized setting and/or if we consider algorithms that use function values. Many papers of Henryk deal with lower bounds, we mention [14, 20, 30].

- Good Lattice Rules. By lattice rule algorithms we mean algorithms that are based on function values at (sometimes shifted) lattice sample points. Such algorithms can be used for integration and approximation and in different settings. Some tractability results were first proved in a non-constructive way and today can be proved (by the work of Kuo, Joe, Sloan and others) in a constructive way, see also work of Cools and Nuyens. We mention [2, 8, 9, 22], where the reader can find more references.

- Smolyak Algorithm and Generalizations. The algorithm of Smolyak has been generalized to the concept of weighted tensor product algorithms to prove, in a constructive way, the tractability of many tensor product problems, see [31].

- The Power of Standard Information. Often we know upper bounds for algorithms based on arbitrary linear information, based on estimates of singular values. It is then a challenge to prove similar upper bounds for algorithms based on function values—or to prove that such algorithms do not exist. Here we mention the paper [34], which deals with the randomized setting, and the paper [10], which deals with the worst case setting.

- Generalized Tractability. In this short survey we discussed polynomial tractability that is defined by the requirement

$$n(\varepsilon, d) \leq C \cdot \varepsilon^{-\alpha} d^{\beta} \tag{9}$$

  for certain $C, \alpha, \beta > 0$ and all $\varepsilon > 0$ and $d \in \mathbb{N}$. There are different notions of tractability, since one might be interested in different tractability domains (for example, only d is large while ε^{-1} is modest) and different tractability functions, instead of polynomials. Then (9) is replaced by

$$n(\varepsilon, d) \leq C \cdot T(\varepsilon^{-1}, d)^t, \tag{10}$$

  for all $(\varepsilon^{-1}, d) \in \Omega$. This was studied by Henryk together with Gnewuch, see [3, 15].

- Quasilinear Problems. Many IBC results have been proved for linear problems, for example for linear operator equations, where the solution u depends linearly on the right hand side f. Together with Arthur Werschulz, Henryk studied the tractability of quasilinear problems. Many problems of mathematical physics belong to this class of problems. The paper [34] is the first paper in a series of papers.

9 Concluding Remarks

I want to conclude with a few personal remarks. I have known Henryk since 1985, when Joseph Traub and Henryk invited me to a conference to New York. We immediately started discussions—which have not ended so far. Our first joint paper appeared in 1992. Henryk was always a good friend—actually, he is the nicest guy you can imagine. I thank him for 1000 suggestions and also for his sympathetic warmth.

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