A Multiscale Method for the Evaluation of Wiener Integrals

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Abstract. Many applications require approximate values of Wiener integrals. A typical approach is to approximate the path integral by a high dimensional integral and apply a Monte Carlo (randomized) method. Here we develop (deterministic) quadrature formulas for the Wiener measure, the 'knots' are piecewise linear functions. Our construction is based on polynomial interpolation and uses ideas of Smolyak as well as the multiscale decomposition of the Wiener measure, due to Lévy and Ciesielski. Numerical examples indicate that our method seems to work well if the integrand is smooth.

§1. Introduction

The Wiener measure w is a Borel probability measure on $C = C[0,1]$, equipped with the topology of uniform convergence. It is uniquely determined by the following properties, see, e.g., [8]:

1) $\int_C x(t)\, dw(x) = 0$ for every $t \in [0,1]$,
2) $\int_C x(s) \cdot x(t)\, dw(x) = \min(s,t)$ for all $s, t \in [0,1]$,
3) for all $d \in \mathbb{N}$ and $t_1, \ldots, t_d \in [0,1]$ the joint distribution of $x(t_1), \ldots, x(t_d)$ with respect to w is Gaussian.

The Wiener measure is probably the most widely used measure on an infinite-dimensional (function) space. It plays a central role in the theory of stochastic processes and it is used in many applications, see [5]. Integrals

$$I_\infty(f) = \int_C f(x)\, dw(x) \tag{1}$$

of w-integrable functionals f on C occur, for instance, in the theory of partial differential equations, in mathematical finance, and in several branches of physics.

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These integrals usually cannot be evaluated explicitly, and therefore efficient quadrature formulas are needed. As in the finite dimensional case, we use the notion of quadrature formula for finite linear combinations of values of the integrand. Thus a quadrature formula

$$Q_\infty^n(f) = \sum_{i=1}^{n} a_i \cdot f(x_i) \tag{2}$$

for the Wiener measure is defined by 'knots' $x_1, \ldots, x_n \in C$ and coefficients $a_1, \ldots, a_n \in \mathbb{R}$.

Typically, Wiener integrals are first approximated by d-dimensional integrals for large d. These high-dimensional integrals are then evaluated by Monte Carlo methods. Alternatively, quasi Monte Carlo methods are used, see [1]. The worst case complexity of integration with respect to the Wiener measure, and, more generally, of path integration for Gaussian measures, is studied in [15].

Since the Wiener measure is Gaussian, the high-dimensional integrals are Gaussian, too, for all linear discretizations. Interpolatory quadrature formulas for d-dimensional Gaussian integrals are constructed in [6,10]. These formulas work well, say, up to dimension $d = 20$, but much larger dimensions are needed for the approximate calculation of Wiener integrals.

We present a modification of the approach from [10], based on a multiscale decomposition of the Wiener measure, due to Lévy and Ciesielski. One obtains for instance a formula with $n = 79\,537$ 'knots' and approximating dimension $d = 1024$. We demonstrate the efficiency of the new quadrature formulas by means of two examples, one of which is dealing with the valuation of a mortgage backed security.

§2. Approximation by Finite Dimensional Integrals

A natural approach for the approximate calculation of $I_\infty(f)$ is to replace the infinite dimensional integral (1) by a d-dimensional integral and then to apply a d-dimensional quadrature formula. Specifically, we use integrals with respect to joint distributions of $x(1/d), x(2/d), \ldots, x(1)$ for suitable dimensions d. For $Y \in \mathbb{R}^d$ let $\ell_d(Y)$ denote the piecewise linear function with breakpoints i/d and

$$\ell_d(Y)(i/d) = (Y_1 + \cdots + Y_i)/\sqrt{d}$$

for $i = 1, \ldots, d$ as well as $\ell_d(Y)(0) = 0$. Define

$$I_d(g) = \int_{\mathbb{R}^d} g(Y) \cdot \rho(Y_1) \cdots \rho(Y_d)\, dY, \tag{3}$$

where

$$\rho(Y_i) = 1/\sqrt{2\pi} \cdot \exp(-Y_i^2/2).$$

Instead of $I_\infty(f)$ we consider the integral $I_d(f \circ \ell_d)$. The use of piecewise linear interpolation is motivated by the fact that $\ell_d(Y)(t)$ is the conditional mean of $x(t)$ with respect to w given $Y = (x(1/d), \dots, x(1))$. Note that every quadrature formula Q_d^n for I_d yields a quadrature formula Q_∞^n for I_∞ by $Q_\infty^n(f) = Q_d^n(f \circ \ell_d)$. The latter formula is based on piecewise linear 'knots' x_i.

Let F denote the class of Lipschitz continuous functionals on C with Lipschitz constant bounded by one. Then

$$\limsup_{d\to\infty} \sup_{f\in F} |I_\infty(f) - I_d(f \circ \ell_d)| \cdot \sqrt{d/\ln d} \le 1/\sqrt{2},$$

which follows from [12]. For $f(x) = \max(x) = \max_{t\in[0,1]} x(t)$ we have

$$\lim_{d\to\infty} |I_\infty(f) - I_d(f \circ \ell_d)| \cdot \sqrt{d} = c$$

with an explicitly known constant $c > 0$, see [2]. Therefore even $d = 100$ might only yield a modest accuracy.

Integrals (3) also arise from the Karhunen-Loève expansion. In this case piecewise linear functions are replaced by Fourier partial sums. See [15] for the analysis of a corresponding algorithm that is based on high-order derivatives of the functionals f.

The construction of approximating finite-dimensional integrals for functional integrals is studied in the monograph [5], see also [3,9]. In [4,7] the Wiener measure and functionals of a specific form are investigated.

§3. Quadrature Formulas for d-Dimensional Gaussian Integrals

For the approximate calculation of (3) we use quadrature formulas for d-dimensional Gaussian integrals, see [6] and [10]. First we select the Gauss-Hermite formulas U^i with $2^{i-1}+1$ knots for $i > 1$ and a single knot for $i = 1$ to approximate $\int_{\mathbf{R}} g(y)\rho(y)\,dy$. Then we use Smolyak's construction to obtain formulas for the d-dimensional integral (3). The Smolyak formulas $A(q,d)$ are linear combinations of product formulas with the following key properties. Only products with a relatively small number of knots are used and the linear combination is chosen in such a way that an interpolation property for $d = 1$ is preserved for $d > 1$. The formula $A(q,d)$ is defined by

$$A(q,d) = \sum_{q-d+1\le|\boldsymbol{i}|\le q} (-1)^{q-|\boldsymbol{i}|} \cdot \binom{d-1}{q-|\boldsymbol{i}|} \cdot (U^{i_1} \otimes \cdots \otimes U^{i_d}), \qquad (4)$$

where $q \ge d$, $\boldsymbol{i} \in \mathbb{N}^d$, and $|\boldsymbol{i}| = i_1 + \cdots + i_d$.

It is proved in [10] that $A(d+k,d)$ has polynomial exactness $2k+1$ and uses, for $d \to \infty$, about $2^k/k! \cdot d^k$ knots. This dependence is almost optimal and also numerical tests show that the formuals $A(q,d)$ are very competitive if the integrand is smooth and if d is of moderate size, say $d = 10$ or $d = 20$.

Assume, however, that $d = 100$ which is not unrealistic, see Section 2. Then we can use only a few of the formulas $A(q,d)$, at most $A(104, 100)$.

For such small values of q we obtain completely wrong results for many functionals f because all 'knots' x_i in the respective quadrature formula $Q^n_\infty(f) = A(q,d)(f \circ l_d)$ have a too small L_∞-norm. For instance, if $d = 100$ and $q = 104$ then $\|x_i\|_\infty \le 0.49$ for all i, while $I_\infty(\max) = \sqrt{2/\pi}$. Therefore we cannot use this method though it works well for $d \le 20$. In the next section we apply a modified ('weighted') version of (4) to a transformed version of $I_d(f \circ l_d)$ to obtain much better formulas.

§4. Using the Lévy-Ciesielski Representation

For $\mu \in \mathbb{N}$ put $m(\mu) = 2^{\mu-2}$ if $\mu \ge 2$ and $m(1) = 1$. We define the Schauder functions $S_{\mu,\nu}$ for $\nu = 1, \dots, m(\mu)$ in the usual way. Put $S_{1,1}(t) = t$. For $\mu \ge 2$ let $S_{\mu,\nu}$ denote the piecewise linear function with breakpoints $\xi/(2m(\mu))$ such that

$$S_{\mu,\nu}(\xi/(2m(\mu))) = \delta_{2\nu-1,\xi}$$

for $\xi = 0, \dots, 2m(\mu)$. Finally, define $\sigma(1) = 1$ and

$$\sigma(\mu) = 2^{-\mu/2}$$

for $\mu \ge 2$. Henceforth let $d = 2^k$ for $k \in \mathbb{N}_0$ and put

$$Y = (Y_{1,1}, \dots, Y_{\mu,1}, \dots, Y_{\mu,m(\mu)}, \dots, Y_{k+1,m(k+1)}) \in \mathbb{R}^d.$$

We define

$$h_d(Y) = \sum_{\mu=1}^{k+1} \sigma(\mu) \sum_{\nu=1}^{m(\mu)} Y_{\mu,\nu} \cdot S_{\mu,\nu}.$$

In the limit, with a sequence of independent standard normal random variables $Y_{\mu,\nu}$, this so-called Lévy-Ciesielski construction yields a multiscale decomposition of the Wiener measure, see [8, Section 2.3.i]. For finite d we have

$$I_d(f \circ \ell_d) = I_d(f \circ h_d).$$

This representation was used in [1] to compute integrals by quasi-Monte Carlo methods, see also Section 6.

Observe that different variables $Y_{\mu,\nu}$ influence the integrand $f \circ h_d$ in a different way, according to the standard deviations $\sigma(\mu)$. Hence one could study weighted norms, see [13]. In the following we suggest a modification of (4). For simplicity we use the same Gauss-Hermite formulas U^i as in Section 3. However, a multiindex $\boldsymbol{i} \in \mathbb{N}^d$ now corresponds to a product formula $U^{i_1-\beta_1} \otimes \cdots \otimes U^{i_d-\beta_d}$. The sequence $\beta_1, \beta_2, \dots$ is essentially defined by the following conditions. For the second variable $Y_{2,1}$ we take $\beta_2 = 0$. Suppose that $j = 2^{\mu-2} + \nu$ with $1 \le \nu \le m(\mu)$ or that $j = \mu = 1$. Then $U^{i_j-\beta_j}$ should use about $\sigma(\mu)/\sigma(2)$ times as many knots as U^{i_j}. After suitable rounding we end up with the definition

$$\beta_1 = -1, \qquad \beta_2 = \beta_3 = \beta_4 = 0, \qquad \beta_{2^{\mu-2}+\nu} = \lfloor \mu/2 \rfloor - 1.$$

Hence we obtain, for $d = 2^k$,

$$A(q,d) = \sum_{q-d+1 \leq |\boldsymbol{i}| \leq q} (-1)^{q-|\boldsymbol{i}|} \cdot \binom{d-1}{q-|\boldsymbol{i}|} \cdot (U^{i_1-\beta_1} \otimes \cdots \otimes U^{i_d-\beta_d}),$$

where $\boldsymbol{i} \in \mathbb{N}^d$ and $U^j = U^1$ for $j < 1$.

The quadrature formulas $Q^n_\infty(f) = A(d+k,d)(f \circ h_d)$ do not depend on d if $d \geq 4^k$. This follows from the fact that

$$(U^{i_1} \otimes \cdots \otimes U^{i_d} \otimes U^1 \otimes \cdots \otimes U^1)(f \circ h_{2d}) = (U^{i_1} \otimes \cdots \otimes U^{i_d})(f \circ h_d).$$

Hence we define

$$A(k)(f) = A(4^k + k, 4^k)(f \circ h_d)$$

for $k \in \mathbb{N}_0$ and obtain a sequence of quadrature formulas for the Wiener measure with $n(k)$ 'knots'. One gets $n(0) = 3$, $n(1) = 25$, $n(2) = 201$, $n(3) = 1561$, $n(4) = 11\,393$, and $n(5) = 79\,537$, and so on. In Figure 1 we show the 25 'knots' of $A(1)$.

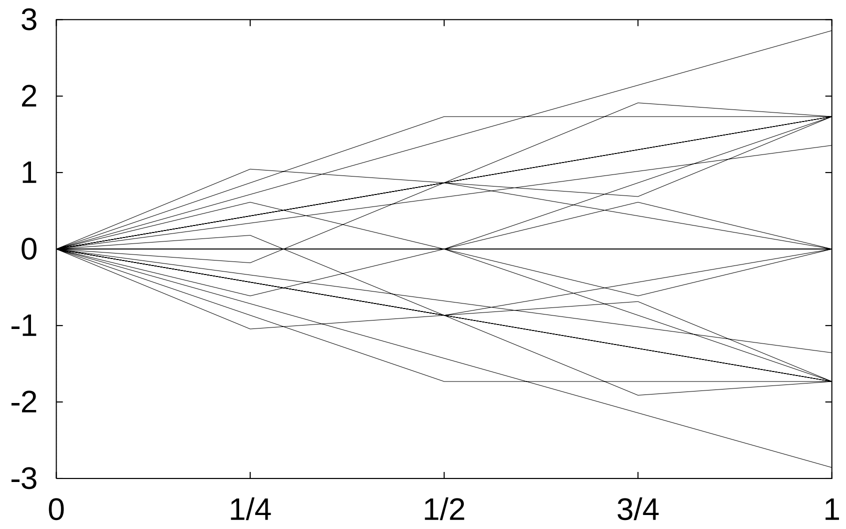

Fig. 1. The 25 'knots' of $A(1)$.

§5. A Numerical Example

As an example, we study

$$f(x) = \exp(2 \int_0^1 x(t)\, dt). \tag{5}$$

The random variable $\int_0^1 x(t)\, dt$ is normally distributed with mean zero and variance 1/3. Hence $I_\infty(f) = \exp(2/3)$ and

$$\sigma^2(f) = \exp(8/3) - \exp(4/3) \approx 10.6$$

for the variance $\sigma^2(f)$ of f. An idealized form of the classical Monte Carlo method is

$$M_n(f) = \frac{1}{n}\sum_{i=1}^{n} f(x_i),$$

where the 'knots' x_i are chosen independently according to the Wiener measure. This method has a standard deviation ('error') $\sigma(f)/\sqrt{n}$.

In Figure 2 we show the accuracy of $A(k)$ and M_n. The accuracy ('number of correct digits') is defined by the negative logarithm (to the basis 10) of the relative error. This accuracy is plotted against the logarithm of the number n of 'knots'. We also show the accuracy of antithetic sampling, which is only slightly better than the 'crude' Monte Carlo method.

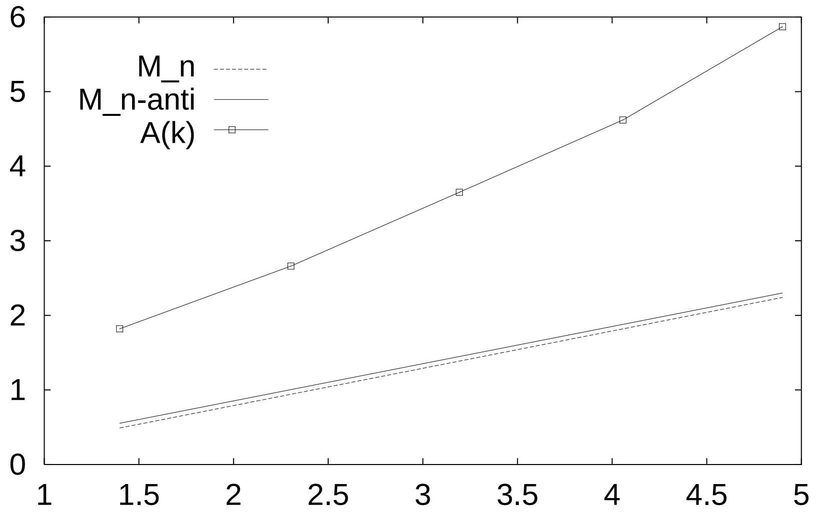

Fig. 2. Test results for the functional (5).

§6. An Example from Finance

High dimensional integration is needed to compute the value of financial derivatives, such as mortgage backed securities, see [1,11]. The geometric Brownian motion is the canonical model for the interest rate, and the value of the security is given as an expectation. For mortgage backed securities the integrand is rather complicated, due to a prepayment option. Hence the integral cannot be computed analytically. A discretization is used, and a typical security of length 360 months yields a 360-dimensional integral.

Here we consider the example (4.5) from [1]. We use the method from Section 4, with minor modifications due to the fact that $d = 360$ is not a power of four. In Figure 3 we also present (as in Figure 2) results from [1] for four different methods. Namely, a crude Monte Carlo method (MC) and a Monte Carlo method with antithetic sampling (MC-anti), both being based on pseudo-random numbers. Furthermore, two quasi-Monte Carlo methods, (QR,LC) and (QR-anti,LC), that are based on the Sobol sequence and the

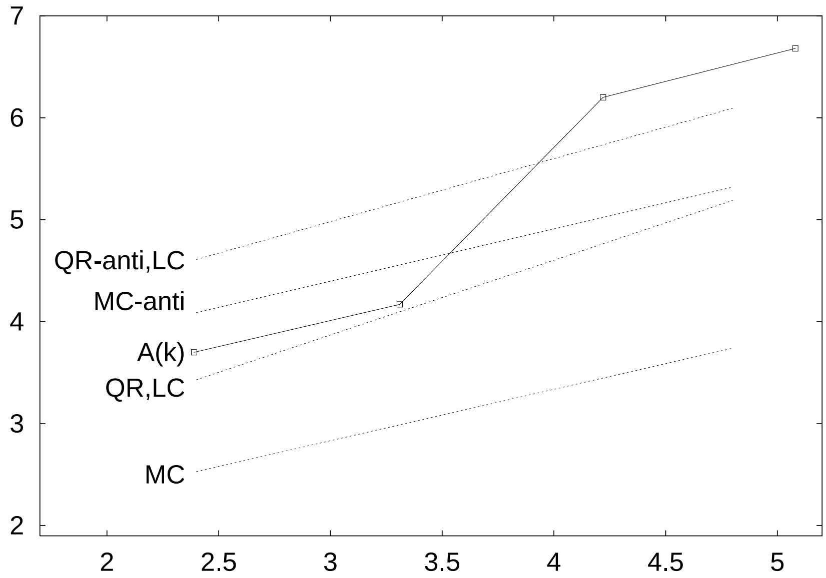

Fig. 3. An Example from Finance.

Lévy-Ciesielski representation. The method QR-anti,LC uses antithetic sampling, in addition.

We stress that even better results are obtained by taking into account the special structure of the integrand or by using second-order derivatives of the integrand. See [1,14] for details. Here, however, we have only considered general purpose methods.

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