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Coin Tossing Algorithms for Integral Equations and Tractability

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Abstract

Integral equations with Lipschitz kernels and right-hand sides are intractable for deterministic methods, the complexity increases exponentially in the dimension d. We study coin tossing algorithms or restricted Monte Carlo methods, where only random bits are allowed. We construct a restricted Monte Carlo method with error ε that uses roughly ε^{-2} function values and only $d \log^2 \varepsilon$ random bits. The number of arithmetic operations is of the order $\varepsilon^{-2} + d \log^2 \varepsilon$. Hence, the cost of our algorithm increases only mildly with the dimension d, we obtain the upper bound $C \cdot (\varepsilon^{-2} + d \log^2 \varepsilon)$ for the complexity. In particular, the problem is tractable for coin tossing algorithms.

1 Introduction and Results

We study the "local" solution of Fredholm integral equations of the second type

$$u - \int_{[0,1]^d} k(.,t)\, u(t)\, dt = f,$$

where $k : [0,1]^{2d} \to \mathbb{R}$ and $f : [0,1]^d \to \mathbb{R}$ are given and one function value $u(s)$ of u is to be computed, up to some error ε. We assume that both k and f are Lipschitz continuous, as well as $\|k\|_\infty \le \gamma < 1$ and $\|f\|_\infty \le 1$. Under these conditions, the solution $u(s)$ can be any number from the interval

$$\left[-\frac{1}{1-\gamma}, \frac{1}{1-\gamma} \right], \tag{1}$$

so that the problem is scaled in the same way for any dimension d.

To compute the solution $u(s)$ with an optimal deterministic method, one needs about

$$C_d\, \varepsilon^{-2d}$$

function values (of k and f) and the total number of arithmetic operations is also of this order. This is a result of Emelyanov and Ilin (1967), the authors use two-grid methods to prove the upper bound.

Observe that the complexity of deterministic algorithms increases exponentially with the dimension d, hence the problem is intractable (for deterministic algorithms). In general a problem is called tractable if the complexity is bounded from above like

$$\mathrm{comp}^{\mathrm{det}}(\varepsilon, d) \leq C \cdot d^{\alpha} \cdot \varepsilon^{-\beta} \tag{2}$$

with certain constants C, α, and β that do not depend on d or ε. A necessary but not sufficient condition for tractability is an upper bound of the form $\mathrm{comp}^{\mathrm{det}}(\varepsilon, d) \leq C_d \cdot \varepsilon^{-\beta}$. See [6] for a recent survey on tractability, mainly of integration.

The lack of tractability in the deterministic setting (also called the curse of dimension) is a reason to study Monte Carlo methods or randomized algorithms. Monte Carlo methods with the (dimension independent) rate ε^{-2} are well known, hence we have an upper bound for the complexity in the randomized setting of the form $C_d \cdot \varepsilon^{-2}$. Even the optimal rate of convergence is known, see Heinrich and Mathé (1993), it is of the form

$$\mathrm{comp}^{\mathrm{ran}}(\varepsilon, d) \approx C_d \cdot \varepsilon^{-2d/(d+1)}. \tag{3}$$

In this paper we deviate from Heinrich and Mathé (1993) in two ways:

- We want to allow only a very restricted randomness, namely random bits or random elements from $\{0, 1\}$. This leads to the class of "coin tossing algorithms" or "restricted Monte Carlo methods", see [5]. How many random bits, arithmetic operations, and function values (of k and f) are needed to achieve an error ε?
- As in (2) we want to have explicit bounds, without constants C_d that depend in an uncontrolled way on d.

We construct a restricted Monte Carlo method with error ε that uses roughly ε^{-2} function values and only $d \log^2 \varepsilon$ random bits. The number of arithmetic operations is of the order $\varepsilon^{-2} + d \log^2 \varepsilon$. In these bounds there are no constants that depend on d or ε. Hence, the cost of our algorithm increases only mildly with the dimension d, we obtain the upper bound

$$\mathrm{comp}^{\mathrm{coin}}(\varepsilon, d) \leq C \cdot (\varepsilon^{-2} + d \log^2 \varepsilon) \tag{4}$$

for restricted Monte Carlo methods. In particular, the problem is tractable in this setting. We use the results of [4] on the summation problem.

Let us describe the problem more formally. Set $D = [0,1]^d$. As kernels and right-hand sides of the integral equation we consider Lipschitz continuous functions. We use the Lipschitz constant

$$\mathrm{Lip}(g) := \sup\left\{ \frac{|g(x) - g(y)|}{\|x - y\|_\infty} : \ x, y \in D^m, \ x \neq y \right\},$$

where $g : D^m \to \mathbb{R}$. Fix $\gamma \in (0,1)$, and $s \in D$. Let Id denote the identity on the space $C(D)$ of continuous functions on D, endowed with the supremum norm.

Furthermore, let, for $k \in C(D^2)$, T_k denote the Fredholm integral operator on $C(D)$, that is,

$$T_k v = \int_D k(.,t)v(t)\,dt \qquad\qquad (v \in C(D)).$$

Let finally the real mapping S on the set

$$\begin{aligned} F := \big\{(k,f) :\ & k \in C(D^2),\ \|k\|_\infty \le \gamma,\ \mathrm{Lip}(k) \le 1, \\ & f \in C(D),\ \|f\|_\infty \le 1,\ \mathrm{Lip}(f) \le 1\big\} \end{aligned}$$

be given by

$$S(k,f) := \big((\mathrm{Id} - T_k)^{-1} f\big)(s).$$

The mapping S is well defined, since the operator norm of T_k satisfies $\|T_k\| \le \gamma < 1$. Hence, $(\mathrm{Id} - T_k)^{-1}$ exists, with $\big\|(\mathrm{Id} - T_k)^{-1}\big\| \le 1/(1-\gamma)$. Using the constant functions $k = \gamma$ and $f = \alpha$, where $\alpha \in [-1, 1]$, we see that (1) holds.

We study the approximation of the solution operator S on the set F of problem elements by restricted randomized algorithms. For such algorithms we allow in addition to the operations of the real number model of computation with an oracle the instruction "choose a random bit", that is, "choose an element of $\{0,1\}$ according to the equidistribution". See [5] for a formal description of this model of computation.

Let $(\Omega, \mathcal{B}, \mathbf{P})$ be the countable infinite product of the probability space that corresponds to the coin tossing instruction. To a restricted randomized algorithm A we associate for each $\omega = (\omega_1, \omega_2, \omega_3, \ldots) \in \Omega$ a (partial) mapping $A_\omega : F \to \mathbb{R}$ as follows. Given a problem element $(k,f) \in F$, we apply A to (k,f) taking, when necessary, ω_i as the ith random bit. If the algorithm terminates, we set $A_\omega(k,f)$ equal to the output of the algorithm and $e(S,A,(k,f),\omega) := |S(k,f) - A_\omega(k,f)|$, and we let $\mathrm{cost}(A,(k,f),\omega)$ be the sum of the number of function evaluations, of the number of used entries of ω, and of the number of performed arithmetic operations.

In the case that the mappings $e(S,A,(k,f),.)$ and $\mathrm{cost}(A,(k,f),.)$ are defined almost everywhere on Ω and are measurable for each $(k,f) \in F$, we call the quantities

$$e(S,A,F) := \sup_{(k,f)\in F} e(S,A,(k,f)) := \sup_{(k,f)\in F} \Big(\mathbf{E}\big(e(S,A,(k,f),.)^2\big)\Big)^{1/2}$$

and

$$\mathrm{cost}(A,F) := \sup_{(k,f)\in F} \mathrm{cost}(A,(k,f)) := \sup_{(k,f)\in F} \mathbf{E}\big(\mathrm{cost}(A,(k,f),.)\big)$$

the error and cost, respectively, of A on F. Here, $\mathbf{E}$ denotes the expectation with respect to $\mathbf{P}$.

We give a rough idea of our algorithm. Let, for $(k,f) \in F$ and $m \in \mathbb{N}$, the function $g_m^{(k,f)} : D^m \to \mathbb{R}$ be defined by

$$\begin{aligned} g_m^{(k,f)}(x) &= g_m^{(k,f)}(x_1,\ldots,x_m) \\ &:= k(s,x_1) \cdot \prod_{i=1}^{m-1} k(x_i,x_{i+1}) \cdot f(x_m) \qquad (x_1,\ldots,x_m \in D). \end{aligned}$$

Since $(\mathrm{Id} - T_k)^{-1} = \mathrm{Id} + \sum_{m=1}^{\infty} T_k^m$ by Neumann's series, we obtain

$$S(k,f) = f(s) + \sum_{m=1}^{\infty} \int_{D^m} g_m^{(k,f)}(x)\, dx. \tag{5}$$

We replace the infinite series of integrals with a finite series of sums and approximate each of these sums using an algorithm of [4]. This approach results in the restricted randomized algorithm A_n whose precise definition we give in Section 2, as well as the proof of the following assertions about its error and cost.

Theorem 1.1 *There are constants $c_1, c_2, c_3 > 0$ depending only on γ such that*

$$e(S, A_n, F) \le c_1\, n^{-1/2} \qquad \text{and} \qquad \mathrm{cost}(A_n, F) \le c_2\, n + c_3\, d\, \log^2 n$$

for $n \ge 2$.

Hence, the cost of A_n increases only mildly with the dimension d. In particular, the approximation of S on F in the restricted randomized setting is tractable. It is clear that (4) follows from Theorem 1.1. We add in passing that the "global" solution of Fredholm equations (compute u, not only $u(s)$) is studied in [1] and [2].

2 Proofs

First, we precisely define the restricted randomized algorithm A_n. We make use of tensor products Q_l^d of the midpoint rule Q_l^1, given by

$$Q_l^1(g) := \frac{1}{l} \sum_{i=1}^{l} g\Big(\frac{2i-1}{2l}\Big)$$

for $g : [0,1] \to \mathbb{R}$. It is well known that Q_l^d is an optimal deterministic method for the approximation of the integration functional I^d on the class $\{g \in C(D) : \mathrm{Lip}(g) \le 1\}$; it holds

$$\sup_{\substack{g \in C(D) \\ \mathrm{Lip}(g) \le 1}} \big| I^d(g) - Q_l^d(g) \big| = \frac{d}{2d+2}\, l^{-1}. \tag{6}$$

Here, optimality refers to the relation between the number of function evaluations and the worst case error, see Sukharev (1979).

Furthermore, we make use of the restricted randomized algorithm $\widetilde{A}_n^4$ from [4]. It provides an approximation of the mean of a finite sequence of real numbers and does this for the components of a vector $h = \big(h(0), \ldots, h(N-1)\big) \in \mathbb{R}^N$ in the following way.

Let P be a prime with $N \le P < 2N$. (According to Bertrand's Postulate, such a prime exists.) Associate to h the vector $\widetilde{h} = \big(h(0), \ldots, h(N-1), 0, \ldots, 0\big) \in \mathbb{R}^P$. Then $\widetilde{A}_n^4$ has the form

$$\widetilde{A}_{n,\omega}^4(h) = \frac{P}{N\big(n + \sqrt{n(P-n)/(P-1)}\big)} \sum_{k=1}^{n} \widetilde{h}(i_k^{\omega})\,,$$

where the i_k^ω are defined as follows. Choose $x^\omega \in \{0, 1, \ldots, P-1\}$ and $y^\omega \in \{1, 2, \ldots, P-1\}$ independently according to the respective uniform distributions and put
$$i_k^\omega := x^\omega + (k-1) \cdot y^\omega \mod P.$$

On the average $c \log N$ random bits and the same number of arithmetic operations suffice to realize x^ω and y^ω. Here, c is a positive constant not depending on N or P. In addition, the computation of the i_k^ω and of the corresponding sum requires $4n$ arithmetic operations[1] and the evaluation of n components of $\widetilde{h}$. As to the error of $\widetilde{A}_n^4$, it holds

$$\mathbf{E}\biggl(\frac{1}{N}\sum_{i=0}^{N-1} h(i) - \widetilde{A}_n^4(h)\biggr)^2 \leq \sqrt{2}\, n^{-1/2} \frac{1}{N}\sum_{i=0}^{N-1} h(i)^2. \tag{7}$$

See [4] for more details.

We now can define the restricted randomized algorithm A_n. Set, for $n \geq 2$, $l := \lceil n^{1/2} \rceil$, $M := \lceil (\log \gamma^{-1})^{-1} \log l \rceil$, and $n_m := \lceil \gamma^m n \rceil$ for $m = 1, \ldots, M$. Let $x_{l,m}^1, \ldots, x_{l,m}^{l^{md}}$ denote the nodes of Q_l^{md}, and let the mapping $\Gamma_{l,m} : F \to \mathbb{R}^{Md}$ be given by

$$\Gamma_{l,m}(k,f) = \Bigl(\underbrace{g_m^{(k,f)}(x_{l,m}^1), \ldots, g_m^{(k,f)}(x_{l,m}^{l^{md}}), \ldots, g_m^{(k,f)}(x_{l,m}^1), \ldots, g_m^{(k,f)}(x_{l,m}^{l^{md}})}_{l^{(M-m)d}\ \text{times}} \Bigr).$$

Hence, the mean of the components of $\Gamma_{l,m}(k,f)$ is equal to $Q_l^{md}(g_m^{(k,f)})$. We finally set

$$A_{n,\omega}(k,f) = f(s) + \sum_{m=1}^{M} \widetilde{A}_{n_m,\omega}^4\bigl(\Gamma_{l,m}(k,f)\bigr).$$

Let us turn to the proof of Theorem 1.1. We use positive constants c_i' that depend on γ, but not on the dimension d or the point s. We first show the assertion about the error of A_n. To this end, we introduce the mapping $\widetilde{S}_l : F \to \mathbb{R}$,

$$\widetilde{S}_l(k,f) = f(s) + \sum_{m=1}^{M} Q_l^{md}(g_m^{(k,f)}).$$

Then the triangle inequality yields

$$e\bigl(S, A_n, (k,f)\bigr) \leq \Bigl| S(k,f) - \widetilde{S}_l(k,f) \Bigr| + \sum_{m=1}^{M} \Bigl(\mathbf{E}\bigl(Q_l^{md}(g_m^{(k,f)}) - \widetilde{A}_{n_m}^4(\Gamma_{l,m}(k,f))\bigr)^2 \Bigr)^{1/2} \tag{8}$$

for each $(k,f) \in F$. The first summand on the right-hand side corresponds to the error made by replacing the infinite series $\sum_{m=1}^{\infty} I^{md}(g_m^{(k,f)})$ with the finite series $\sum_{m=1}^{M} Q_l^{md}(g_m^{(k,f)})$, the other summands correspond to the respective error made by approximating the mean $Q_l^{md}(g_m^{(k,f)})$ by $\widetilde{A}_{n_m}^4$.

We estimate these errors using relations (6) and (7). This requires knowledge of the Lipschitz constants and of the norms of the $g_m^{(k,f)}$.

[1] We assume here that the modulo operation is allowed and has unit cost.

Lemma 2.1 *Let $(k,f) \in F$ and $m \in \mathbb{N}$. Then*

$$\operatorname{Lip}\big(g_m^{(k,f)}\big) \le (m+1)\gamma^{m-1} \qquad \text{and} \qquad \Big\| g_m^{(k,f)} \Big\|_\infty \le \gamma^m.$$

Proof. The second inequality is an immediate consequence of the definition of $g_m^{(k,f)}$ and of the assumptions on the norms of k and f. To prove the first one, we define, for $(k,f) \in F$ and $m \in \{2,3,4,\dots\}$, the function

$$h_m^{(k,f)} : D^m \to \mathbb{R}, \quad x = (x_1,\dots,x_m) \mapsto \prod_{i=1}^{m-1} k(x_i,x_{i+1}) \cdot f(x_m).$$

For fixed $(k,f) \in F$, we show by induction that

$$\operatorname{Lip}\big(h_m^{(k,f)}\big) \le m\gamma^{m-2}, \qquad m \in \{2,3,4,\dots\}. \tag{9}$$

Since $g_{m-1}^{(k,f)} = h_m^{(k,f)}(s,.)$, the first inequality follows.

Consider $m = 2$. The assumptions on k and f imply

$$\begin{aligned}
\Big| h_2^{(k,f)}(x_1,x_2) - h_2^{(k,f)}(x_1',x_2') \Big| &= |k(x_1,x_2)f(x_2) - k(x_1',x_2')f(x_2')| \\
&\le |k(x_1,x_2)| \cdot |f(x_2) - f(x_2')| \\
&\quad + |k(x_1,x_2) - k(x_1',x_2')| \cdot |f(x_2')| \\
&\le \gamma \cdot \|x_2 - x_2'\|_\infty + \|(x_1,x_2) - (x_1',x_2')\|_\infty \cdot 1 \\
&\le 2\gamma^{2-2} \cdot \|(x_1,x_2) - (x_1',x_2')\|_\infty .
\end{aligned}$$

Hence, relation (9) is valid for $m = 2$.

Suppose now that (9) holds for an $m \ge 2$. Using

$$\begin{aligned}
&h_{m+1}^{(k,f)}(x_1,\dots,x_{m+1}) - h_{m+1}^{(k,f)}(x_1',\dots,x_{m+1}') \\
&\qquad = k(x_1,x_2) \cdot \big(h_m^{(k,f)}(x_2,\dots,x_{m+1}) - h_m^{(k,f)}(x_2',\dots,x_{m+1}') \big) \\
&\qquad\qquad + \big(k(x_1,x_2) - k(x_1',x_2') \big) \cdot h_m^{(k,f)}(x_2',\dots,x_{m+1}')
\end{aligned}$$

we conclude that

$$\begin{aligned}
&\Big| h_{m+1}^{(k,f)}(x_1,\dots,x_{m+1}) - h_{m+1}^{(k,f)}(x_1',\dots,x_{m+1}') \Big| \\
&\qquad \le \gamma \cdot m\gamma^{m-2} \big\|(x_2,\dots,x_{m+1}) - (x_2',\dots,x_{m+1}')\big\|_\infty \\
&\qquad\qquad + \|(x_1,x_2) - (x_1',x_2')\|_\infty \cdot \gamma^{m-1} \\
&\qquad \le (m+1)\gamma^{(m+1)-2} \big\|(x_1,\dots,x_{m+1}) - (x_1',\dots,x_{m+1}')\big\|_\infty .
\end{aligned}$$

Consequently, (9) holds also for $m+1$. The lemma is proved. ∎

Now we can estimate the summands on the right-hand side of (8).

Lemma 2.2 *(i) There is a positive constant $c_1' = c_1'(\gamma)$ such that*

$$\Big| S(k,f) - \widetilde{S}_l(k,f) \Big| \le c_1' l^{-1}, \qquad (k,f) \in F. \tag{10}$$

(ii) For $(k,f) \in F$ and $m = 1, \ldots, M$ holds

$$\Big(\mathbf{E}\big(Q_l^{md}(g_m^{(k,f)}) - \widetilde{A}^4_{n_m}(\Gamma_{l,m}(k,f))\big)^2\Big)^{1/2} \le \sqrt{2}\, n_m^{-1/2}\, \gamma^m.$$

Proof. Let $(k,f) \in F$. Using, for $m \in \mathbb{N}$, the bound for the Lipschitz constant of $g_m^{(k,f)}$ in Lemma 2.1 and the linearity of the functionals I^{md} and Q_l^{md} we conclude from (6) that

$$\Big|I^{md}(g_m^{(k,f)}) - Q_l^{md}(g_m^{(k,f)})\Big| \le (m+1)\gamma^{m-1} \cdot \frac{md}{2md+2}\, l^{-1} \le (m+1)\gamma^{m-1} l^{-1}.$$

Furthermore, we have $\Big|I^{md}(g_m^{(k,f)})\Big| \le \gamma^m$ by the second inequality of Lemma 2.1. Consequently, we obtain

$$\begin{aligned}
\Big|S(k,f) - \widetilde{S}_l(k,f)\Big| &\le \sum_{m=1}^{M} \Big|I^{md}(g_m^{(k,f)}) - Q_l^{md}(g_m^{(k,f)})\Big| + \sum_{m=M+1}^{\infty} \Big|I^{md}(g_m^{(k,f)})\Big| \\
&\le \sum_{m=1}^{\infty} (m+1)\gamma^{m-1} l^{-1} + \sum_{m=M+1}^{\infty} \gamma^m \\
&= \gamma^{-1}\Big(\frac{1}{(1-\gamma)^2} - 1\Big) l^{-1} + \frac{\gamma}{1-\gamma}\gamma^M.
\end{aligned}$$

Since $\gamma^M \le l^{-1}$ by the definition of M, the assertion (i) follows.

The second inequality of Lemma 2.1 and relation (7) yield assertion (ii). ∎

Substituting the results of Lemma 2.2 into (8) we obtain

$$e\big(S, A_n, (k,f)\big) \le c_1' l^{-1} + \sqrt{2} \sum_{m=1}^{M} n_m^{-1/2} \cdot \gamma^m.$$

The definitions of the n_m and of l imply

$$\begin{aligned}
e\big(S, A_n, (k,f)\big) &\le c_1' l^{-1} + \sqrt{2} \sum_{m=1}^{M} \gamma^{-m/2} n^{-1/2} \cdot \gamma^m \\
&\le c_1' l^{-1} + \sqrt{2}\, n^{-1/2} \sum_{m=0}^{\infty} \gamma^{m/2} \\
&\le c_2' n^{-1/2}.
\end{aligned}$$

This establishes the assertion about the error of A_n.

We now prove the claim about the cost of A_n, which splits into the information cost, the randomness cost, and the arithmetic cost. We first consider the information cost. Since a function value of $g_m^{(k,f)}$ is made up of m function values of k and of one of f, we have

$$\mathrm{cost}_{\mathrm{info}}(A_n, (k,f)) \le 1 + \sum_{m=1}^{M} (m+1) n_m.$$

Taking the definition of the n_m into account, we obtain

$$\begin{aligned}
\mathrm{cost}_{\mathrm{info}}(A_n,(k,f)) &\le 1 + \sum_{m=1}^{M}(m+1)(\gamma^m \cdot n + 1) \\
&\le 1 + n\sum_{m=1}^{\infty}(m+1)\gamma^m + \sum_{m=1}^{M}(m+1) \\
&= 1 + n\cdot\Big(\frac{1}{(1-\gamma)^2} - 1\Big) + \frac{M(M+1)}{2} + M.
\end{aligned}$$

Since $M^2 \le c_3' \log^2 n \le c_3' n$, it follows that

$$\mathrm{cost}_{\mathrm{info}}(A_n,(k,f)) \le c_4' n.$$

Let us turn to the randomness cost. According to the construction of A_n, one computes from a realization of x^ω and y^ω, for $m = 1,\dots,M$, the respective components of $\Gamma_{l,m}(g_m^{(k,f)})$ that have to be sampled. As mentioned above, on the average $c\log N$ random bits suffice to realize x^ω and y^ω. Since $N = l^{Md}$ we get

$$\mathrm{cost}_{\mathrm{coin}}(A_n,(k,f)) \le cMd\log l.$$

Because of $M \le c_5' \log n$ and $\log l \le 2\log n$ we obtain

$$\mathrm{cost}_{\mathrm{coin}}(A_n,(k,f)) \le c_6' d\, \log^2 n.$$

We finally consider the arithmetic cost. As also mentioned above, on the average $c\log N$ arithmetic operations suffice for the realization of x^ω and y^ω. For $m = 1,\dots,M$, the computation of the i_k^ω, that of the corresponding components of $\Gamma_{l,m}(g_m^{(k,f)})$, and that of their sum can be accomplished in $(3+m+1)n_m$ arithmetic operations. At last, we have to compute the sum of the $\widetilde{A}^4_{n_m,\omega}(\Gamma_{l,m}(g_m^{(k,f)}))$ and of $f(s)$. Hence, we obtain

$$\mathrm{cost}_{\mathrm{ari}}(A_n,(k,f)) \le cMd\log l + \sum_{m=1}^{M}(3+m+1)n_m + M.$$

The right hand side can be bounded similarly as above and we obtain

$$\mathrm{cost}_{\mathrm{ari}}(A_n,(k,f)) \le c_7' n + c_6' d\log^2 n.$$

Summing up these three cost estimates, we get

$$\mathrm{cost}(A_n,(k,f)) \le c_8' n + c_9' d\log^2 n.$$

This completes the proof of Theorem 1.1.

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