Some Results on the Complexity of Numerical Integration

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Abstract We present some results on the complexity of numerical integration. We start with the seminal paper of Bakhvalov (1959) and end with new results on the curse of dimensionality and on the complexity of oscillatory integrals.
This survey paper consists of four parts:

1. Classical results till 1971
2. Randomized algorithms
3. Tensor product problems, tractability and weighted norms
4. Some recent results: C^k functions and oscillatory integrals

1 Classical Results till 1971

I start with a warning: We do *not* discuss the complexity of path integration and infinite-dimensional integration on $\mathbb{R}^{\mathbb{N}}$ or other domains although there are exciting new results in that area, see [8, 14, 21, 22, 23, 41, 43, 44, 53, 69, 77, 90, 96, 121, 123]. For parametric integrals see [16, 17], for quantum computers, see [48, 49, 80, 115].

We mainly study the problem of numerical integration, i.e., of approximating the integral

$$S_d(f) = \int_{D_d} f(x)\,\mathrm{d}x \tag{1}$$

over an open subset $D_d \subset \mathbb{R}^d$ of Lebesgue measure $\lambda^d(D_d) = 1$ for integrable functions $f\colon D_d \to \mathbb{R}$. The main interest is on the behavior of the minimal number of function values that are needed in the worst case setting to achieve an error at most $\varepsilon > 0$. Note that classical examples of domains D_d are the unit cube $[0,1]^d$ and the

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normalized Euclidean ball (with volume 1), which are closed. However, we work with their interiors for definiteness of certain derivatives.

We state our problem. Let F_d be a class of integrable functions $f\colon D_d \to \mathbb{R}$. For $f \in F_d$, we approximate the integral $S_d(f)$, see (1), by algorithms of the form

$$A_n(f) = \phi_n(f(x_1), f(x_2), \ldots, f(x_n)),$$

where $x_j \in D_d$ can be chosen adaptively and $\phi_n\colon \mathbb{R}^n \to \mathbb{R}$ is an arbitrary mapping. Adaption means that the selection of x_j may depend on the already computed values $f(x_1), f(x_2), \ldots, f(x_{j-1})$. We define $N\colon F_d \to \mathbb{R}^n$ by $N(f) = (f(x_1), \ldots, f(x_n))$. The (worst case) error of the algorithm A_n is defined by

$$e(A_n) = \sup_{f \in F_d} |S_d(f) - A_n(f)|,$$

the optimal error bounds are given by

$$e(n, F_d) = \inf_{A_n} e(A_n).$$

The information complexity $n(\varepsilon, F_d)$ is the minimal number of function values which is needed to guarantee that the error is at most ε, i.e.,

$$n(\varepsilon, F_d) = \min\{n \mid \exists\, A_n \text{ such that } e(A_n) \le \varepsilon\}.$$

We minimize n over all choices of adaptive sample points x_j and mappings ϕ_n.

In this paper we give an overview on some of the basic results that are known about the numbers $e(n, F_d)$ and $n(\varepsilon, F_d)$. Hence we concentrate on complexity issues and leave aside other important questions such as implementation issues.

It was proved by Smolyak and Bakhvalov that as long as the class F_d is convex and balanced we may restrict the minimization of $e(A_n)$ by considering only non-adaptive choices of x_j and linear mappings ϕ_n, i.e., it is enough to consider A_n of the form

$$A_n(f) = \sum_{i=1}^{n} a_i f(x_i). \tag{2}$$

Theorem 0 (Bakhvalov [6]). *Assume that the class F_d is convex and balanced. Then*

$$e(n, F_d) = \inf_{x_1, \ldots, x_n} \sup_{\substack{f \in F_d \\ N(f)=0}} S_d(f) \tag{3}$$

and for the infimum in the definition of $e(n, F_d)$ it is enough to consider linear and nonadaptive algorithms A_n of the form (2)*.*

In this paper we only consider convex and balanced F_d and then we can use the last formula for $e(n, F_d)$.

Remark 0. a) For a proof of Theorem 0 see, for example, [89, Theorem 4.7]. This result is not really about complexity (hence it got its number), but it helps to prove complexity results.

b) A linear algorithm A_n is called a quasi Monte Carlo (QMC) algorithm if $a_i = 1/n$ for all i and is called a positive quadrature formula if $a_i > 0$ for all i. In general it may happen that optimal quadrature formulas have some negative weights and, in addition, we cannot say much about the position of good points x_i.

c) More on the optimality of linear algorithms and on the power of adaption can be found in [15, 79, 89, 113, 114]. There are important classes of functions that are *not* balanced and convex, and where Theorem 0 can not be applied, see also [13, 95]. □

The *optimal order of convergence* plays an important role in numerical analysis. We start with a classical result of Bakhvalov (1959) for the class

$$F_d^k = \{f\colon [0,1]^d \to \mathbb{R} \mid \|D^\alpha f\|_\infty \le 1,\ |\alpha| \le k\},$$

where $k \in \mathbb{N}$ and $|\alpha| = \sum_{i=1}^d \alpha_i$ for $\alpha \in \mathbb{N}_0^d$ and $D^\alpha f$ denotes the respective partial derivative. For two sequences a_n and b_n of positive numbers we write $a_n \asymp b_n$ if there are positive numbers c and C such that $c < a_n/b_n < C$ for all $n \in \mathbb{N}$.

Theorem 1 (Bakhvalov [5]).

$$e(n, F_d^k) \asymp n^{-k/d}. \tag{4}$$

Remark 1. a) For such a complexity result one needs to prove an upper bound (for a particular algorithm) and a lower bound (for all algorithms). For the upper bound one can use tensor product methods based on a regular grid, i.e., one can use the $n = m^d$ points x_i with coordinates from the set $\{1/(2m), 3/(2m), \dots, (2m-1)/(2m)\}$.

The lower bound can be proved with the technique of "bump functions": One can construct $2n$ functions $f_1, \dots, f_{2n}$ with disjoint supports such that all 2^{2n} functions of the form $\sum_{i=1}^{2n} \delta_i f_i$ are contained in F_d^k, where $\delta_i = \pm 1$ and $S_d(f_i) \ge c_{d,k}\, n^{-k/d-1}$. Since an algorithm A_n can only compute n function values, there are two functions $f^+ = \sum_{i=1}^{2n} f_i$ and $f^- = f^+ - 2\sum_{k=1}^{n} f_{i_k}$ such that $f^+, f^- \in F_d^k$ and $A_n(f^+) = A_n(f^-)$ but $|S_d(f^+) - S_d(f^-)| \ge 2nc_{d,k}n^{-k/d-1}$. Hence the error of A_n must be at least $c_{d,k}n^{-k/d}$. For the details see, for example, [78].

b) Observe that we can not conclude much on $n(\varepsilon, F_d^k)$ if ε is fixed and d is large, since Theorem 1 contains *hidden factors* that depend on k and d. Actually the lower bound is of the form

$$e(n, F_d^k) \ge c_{d,k} n^{-k/d},$$

where the $c_{d,k}$ decrease with $d \to \infty$ and tend to zero.

c) The proof of the upper bound (using tensor product algorithms) is easy since we assumed that the domain is $D_d = [0,1]^d$. The optimal order of convergence is known for much more general spaces (such as Besov and Triebel-Lizorkin spaces) and arbitrary bounded Lipschitz domains, see [86, 116, 119]. Then the proof of the upper bounds is more difficult, however.

d) Integration on fractals was recently studied by Dereich and Müller-Gronbach [18]. These authors also obtain an optimal order of convergence $n^{-k/\alpha}$. The definition of S_d must be modified and α coincides, under suitable conditions, with the Hausdorff dimension of the fractal. □

By the *curse of dimensionality* we mean that $n(\varepsilon, F_d)$ is exponentially large in d. That is, there are positive numbers c, ε_0 and γ such that

$$n(\varepsilon, F_d) \geq c\,(1+\gamma)^d \quad \text{for all} \quad \varepsilon \leq \varepsilon_0 \quad \text{and infinitely many} \quad d \in \mathbb{N}. \tag{5}$$

If, on the other hand, $n(\varepsilon, F_d)$ is bounded by a polynomial in d and ε^{-1} then we say that the problem is *polynomially tractable*. If $n(\varepsilon, F_d)$ is bounded by a polynomial in ε^{-1} alone, i.e., $n(\varepsilon, F_d) \leq C\varepsilon^{-\alpha}$ for $\varepsilon < 1$, then we say that the problem is *strongly polynomially tractable*.

From the proof of Theorem 1 we can not conclude whether the curse of dimensionality holds for the classes F_d^k or not; see Theorem 11. Possibly Maung Zho Newn and Sharygin [74] were the first who published (in 1971) a complexity result for arbitrary d with explicit constants and so proved the curse of dimensionality for Lipschitz functions.

Theorem 2 (Maung Zho Newn and Sharygin [74]). *Consider the class*

$$F_d = \{f\colon [0,1]^d \to \mathbb{R} \mid |f(x) - f(y)| \leq \max_i |x_i - y_i|\}.$$

Then

$$e(n, F_d) = \frac{d}{2d+2} \cdot n^{-1/d}$$

for $n = m^d$ with $m \in \mathbb{N}$.

Remark 2. One can show that for $n = m^d$ the regular grid (points x_i with coordinates from the set $\{1/(2m), 3/(2m), \ldots, (2m-1)/(2m)\}$) and the midpoint rule $A_n(f) = n^{-1} \sum_{i=1}^n f(x_i)$ are optimal. See also [3, 4, 12, 108] for this result and for generalizations to similar function spaces. □

2 Randomized Algorithms

The integration problem is difficult for all deterministic algorithms if the classes F_d of inputs are too large, see Theorem 2. One may hope that randomized algorithms make this problem much easier.

Randomized algorithms can be formalized in various ways leading to slightly different models. We do not explain the technical details and only give a reason why it makes sense to study different models for upper and lower bounds, respectively; see [89] for more details.

- Assume that we want to construct and to analyze concrete algorithms that yield upper bounds for the (total) complexity of given problems including the arithmetic cost and the cost of generating random numbers. Then it is reasonable to consider a rather restrictive model of computation where, for example, only the standard arithmetic operations are allowed. One may also restrict the use of random numbers and study so-called *restricted Monte Carlo methods*, where only random bits are allowed; see [51].
- For the proof of lower bounds we take the opposite view and allow general randomized mappings and a very general kind of randomness. This makes the lower bounds stronger.

It turns out that the results are often very robust with respect to changes of the computational model. For the purpose of this paper, it might be enough that a randomized algorithm A is a random variable $(A^\omega)_{\omega\in\Omega}$ with a random element ω where, for each fixed ω, the algorithm A^ω is a (deterministic) algorithm as before. We denote by μ the distribution of the ω. In addition one needs rather weak measurability assumptions, see also the textbook [73]. Let $\bar{n}(f,\omega)$ be the number of function values used for fixed ω and f.

The number

$$\tilde{n}(A) = \sup_{f\in F} \int_\Omega \bar{n}(f,\omega)\,\mathrm{d}\mu(\omega)$$

is called the *cardinality* of the randomized algorithm A and

$$e^{\mathrm{ran}}(A) = \sup_{f\in F} \left(\int_\Omega^* \|S(f) - \phi_\omega(N_\omega(f))\|^2\,\mathrm{d}\mu(\omega) \right)^{1/2}$$

is the *error* of A. By $\int^*$ we denote the upper integral. For $n\in\mathbb{N}$, define

$$e^{\mathrm{ran}}(n,F_d) = \inf\{e^{\mathrm{ran}}(A) : \tilde{n}(A)\le n\}.$$

If $A\colon F\to G$ is a (measurable) deterministic algorithm then A can also be treated as a randomized algorithm with respect to a Dirac (atomic) measure μ. In this sense we can say that deterministic algorithms are special randomized algorithms. Hence the inequality

$$e^{\mathrm{ran}}(n,F_d) \le e(n,F_d) \tag{6}$$

is trivial.

The number $e^{\mathrm{ran}}(0,F_d)$ is called the *initial error in the randomized setting*. For $n=0$, we do not sample f, and $A^\omega(f)$ is independent of f, but may depend on ω. It is easy to check that for a linear S and a balanced and convex set F, the best we can do is to take $A^\omega = 0$ and then

$$e^{\mathrm{ran}}(0,F_d) = e(0,F_d).$$

This means that for linear problems the initial errors are the same in the worst case and randomized setting.

The main advantage of randomized algorithms is that the curse of dimensionality is not present even for certain large classes of functions. With the standard Monte Carlo method we obtain

$$e^{\rm ran}(n,F_d) \le \frac{1}{\sqrt{n}},$$

when F_d is the unit ball of $L_p([0,1]^d)$ and $2 \le p \le \infty$. Mathé [72] proved that this is almost optimal and the optimal algorithm is

$$A_n^\omega(f) = \frac{1}{n+\sqrt{n}} \sum_{i=1}^n f(X_i)$$

with i.i.d. random variables X_i that are uniformly distributed on $[0,1]^d$. It also follows that

$$e^{\rm ran}(n,F_d) = \frac{1}{1+\sqrt{n}},$$

when F_d is the unit ball of $L_p([0,1]^d)$ and $2 \le p \le \infty$. In the case $1 \le p < 2$ one can only achieve the rate $n^{-1+1/p}$, for a discussion see [50].

Bakhvalov [5] found the optimal order of convergence already in 1959 for the class

$$F_d^k = \{f\colon [0,1]^d \to \mathbb{R} \mid \|D^\alpha f\|_\infty \le 1,\ |\alpha| \le k\},$$

where $k \in \mathbb{N}$ and $|\alpha| = \sum_{i=1}^d \alpha_i$ for $\alpha \in \mathbb{N}_0^d$.

Theorem 3 (Bakhvalov [5]).

$$e^{\rm ran}(n,F_d^k) \asymp n^{-k/d-1/2}. \tag{7}$$

Remark 3. A proof of the *upper bound* can be given with a technique that is often called *separation of the main part* or also *control variates*. For $n = 2m$ use m function values to construct a good L_2 approximation f_m of $f \in F_d^k$ by a deterministic algorithm. The optimal order of convergence is

$$\|f - f_m\|_2 \asymp m^{-k/d}.$$

Then use the unbiased estimator

$$A_n^\omega(f) = S_d(f_m) + \frac{1}{m} \sum_{i=1}^m (f - f_m)(X_i)$$

with i.i.d. random variables X_i that are uniformly distributed on $[0,1]^d$. See, for example, [73, 78] for more details. We add in passing that the optimal order of convergence can be obtained for many function spaces (Besov spaces, Triebel-Lizorkin spaces) and for arbitrary bounded Lipschitz domains $D_d \subset \mathbb{R}^d$; see [86], where the approximation problem is studied. To obtain an explicit randomized algorithm with the optimal rate of convergence one needs a random number generator for the set D_d. If it is not possible to obtain efficiently random samples from the uniform distri-

bution on D_d one can work with Markov chain Monte Carlo (MCMC) methods, see Theorem 5.

All known proofs of *lower bounds* use the idea of Bakhvalov (also called Yao's Minimax Principle): study the average case setting with respect to a probability measure on F and use the theorem of Fubini. For details see [45, 46, 47, 73, 78, 90]. □

We describe a problem that was studied by several colleagues and solved by Hinrichs [58] using deep results from functional analysis. Let $H(K_d)$ be a reproducing kernel Hilbert space of real functions defined on a Borel measurable set $D_d \subseteq \mathbb{R}^d$. Its reproducing kernel $K_d : D_d \times D_d \to \mathbb{R}$ is assumed to be integrable,

$$C_d^{\text{init}} := \left(\int_{D_d} \int_{D_d} K_d(x,y)\, \rho_d(x)\, \rho_d(y)\, \mathrm{d}x\, \mathrm{d}y \right)^{1/2} < \infty.$$

Here, ρ_d is a probability density function on D_d. Without loss of generality we assume that D_d and ρ_d are chosen such that there is no subset of D_d with positive measure such that all functions from $H(K_d)$ vanish on it.

The inner product and the norm of $H(K_d)$ are denoted by $\langle \cdot, \cdot \rangle_{H(K_d)}$ and $\| \cdot \|_{H(K_d)}$. Consider multivariate integration

$$S_d(f) = \int_{D_d} f(x)\, \rho_d(x)\, \mathrm{d}x \quad \text{for all} \quad f \in H(K_d),$$

where it is assumed that $S_d : H(K_d) \to \mathbb{R}$ is continuous.

We approximate $S_d(f)$ in the randomized setting using *importance sampling*. That is, for a positive probability density function τ_d on D_d we choose n random sample points $x_1, x_2, \ldots, x_n$ which are independent and distributed according to τ_d and take the algorithm

$$A_{n,d,\tau_d}(f) = \frac{1}{n} \sum_{j=1}^{n} \frac{f(x_j)\, \rho_d(x_j)}{\tau_d(x_j)}.$$

The error of A_{n,d,τ_d} is then

$$e^{\text{ran}}(A_{n,d,\tau_d}) = \sup_{\|f\|_{H(K_d)} \le 1} \left(\mathbb{E}_{\tau_d} \left(S_d(f) - A_{n,d,\tau_d}(f) \right)^2 \right)^{1/2},$$

where the expectation is with respect to the random choice of the sample points x_j.

For $n = 0$ we formally take $A_{0,d,\tau_d} = 0$ and then

$$e^{\text{ran}}(0, H(K_d)) = C_d^{\text{init}}.$$

Theorem 4 (Hinrichs [58]). *Assume additionally that $K_d(x,y) \ge 0$ for all $x, y \in D_d$. Then there exists a positive density function τ_d such that*

$$e^{\mathrm{ran}}(A_{n,d,\tau_d}) \le \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{\sqrt{n}}\, e^{\mathrm{ran}}(0, H(K_d)).$$

Hence, if we want to achieve $e^{\mathrm{ran}}(A_{n,d,\tau_d}) \le \varepsilon\, e^{\mathrm{ran}}(0,H(K_d))$ *then it is enough to take*

$$n = \left\lceil \frac{\pi}{2} \left(\frac{1}{\varepsilon}\right)^2 \right\rceil.$$

Remark 4. In particular, such problems are strongly polynomially tractable (for the normalized error) if the reproducing kernels are pointwise nonnegative and integrable. In [91] we prove that the exponent 2 of ε^{-1} is sharp for tensor product Hilbert spaces whose univariate reproducing kernel is *decomposable* and univariate integration is not trivial for the two parts of the decomposition. More specifically we have

$$n^{\mathrm{ran}}(\varepsilon, H(K_d)) \ge \left\lceil \frac{1}{8} \left(\frac{1}{\varepsilon}\right)^2 \right\rceil \quad \text{for all} \quad \varepsilon \in (0,1) \ \text{ and } \ d \ge \frac{2\ln \varepsilon^{-1} - \ln 2}{\ln \alpha^{-1}},$$

where $\alpha \in [1/2, 1)$ depends on the particular space.

We stress that these estimates hold independently of the smoothness of functions in a Hilbert space. Hence, even for spaces of very smooth functions the exponent of strong polynomial tractability is 2. □

Sometimes one cannot sample easily from the "target distribution" π if one wants to compute an integral

$$S(f) = \int_D f(x)\, \pi(\mathrm{d}x).$$

Then Markov chain Monte Carlo (MCMC) methods are a very versatile and widely used tool.

We use an average of a finite Markov chain sample as approximation of the mean, i.e., we approximate $S(f)$ by

$$S_{n,n_0}(f) = \frac{1}{n} \sum_{j=1}^{n} f(X_{j+n_0}),$$

where $(X_i)_{n\in\mathbb{N}_0}$ is a Markov chain with stationary distribution π. The number n determines the number of function evaluations of f. The number n_0 is the *burn-in* or *warm up* time. Intuitively, it is the number of steps of the Markov chain to get close to the stationary distribution π.

We study the mean square error of S_{n,n_0}, given by

$$e_\nu(S_{n,n_0}, f) = \left(\mathbb{E}_{\nu,K} |S_{n,n_0}(f) - S(f)|\right)^{1/2},$$

where ν and K indicate the initial distribution and the transition kernel of the chain; we work with the spaces $L_p = L_p(\pi)$. For the proof of the following error bound we refer to [99, Theorem 3.34 and Theorem 3.41].

Theorem 5 (Rudolf [99]). *Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with reversible transition kernel K, initial distribution ν, and transition operator P. Further, let*

$$\Lambda = \sup\{\alpha\colon \alpha \in \operatorname{spec}(P-S)\},$$

where $\operatorname{spec}(P-S)$ *denotes the spectrum of the operator* $(P-S)\colon L_2 \to L_2$*, and assume that* $\Lambda < 1$*. Then*

$$\sup_{\|f\|_p\le 1} e_\nu(S_{n,n_0},f)^2 \le \frac{2}{n(1-\Lambda)} + \frac{2C_\nu\gamma^{n_0}}{n^2(1-\gamma)^2} \tag{8}$$

holds for $p=2$ and for $p=4$ under the following conditions:

- *for $p=2$, $\frac{d\nu}{d\pi}\in L_\infty$ and a transition kernel K which is L_1-exponentially convergent with (γ,M) where $\gamma<1$, i.e.,*

$$\|P^n - S\|_{L_1\to L_1} \le M\gamma^n$$

  for all $n\in\mathbb{N}$ and $C_\nu = M\left\|\frac{d\nu}{d\pi}-1\right\|_\infty$;
- *for $p=4$, $\frac{d\nu}{d\pi}\in L_2$ and $\gamma = \|P-S\|_{L_2\to L_2} < 1$ where $C_\nu = 64\left\|\frac{d\nu}{d\pi}-1\right\|_2$.*

Remark 5. Let us discuss the results. First observe that we assume that the so called spectral gap $1-\Lambda$ is positive; in general we only know that $|\Lambda|\le 1$. If the transition kernel is L_1-exponentially convergent, then we have an explicit error bound for integrands $f\in L_2$ whenever the initial distribution has a density $\frac{d\nu}{d\pi}\in L_\infty$. However, in general it is difficult to provide explicit values γ and M such that the transition kernel is L_1-exponentially convergent with (γ,M). This motivates to consider transition kernels which satisfy a weaker convergence property, such as the existence of an L_2-spectral gap, i.e., $\|P-S\|_{L_2\to L_2}<1$. In this case we have an explicit error bound for integrands $f\in L_4$ whenever the initial distribution has a density $\frac{d\nu}{d\pi}\in L_2$. Thus, by assuming a weaker convergence property of the transition kernel we obtain a weaker result in the sense that f must be in L_4 rather than L_2.

If we want to have an error of $\varepsilon\in(0,1)$ it is still not clear how to choose n and n_0 to minimize the total amount of steps $n+n_0$. How should we choose the burn-in n_0? One can prove in this setting, see [99], that the choice $n^* = \lceil \frac{\log C_\nu}{1-\gamma}\rceil$ is a reasonable and almost optimal choice for the burn-in.

More details can be found in [84]. For a full discussion with all the proofs see [99]. □

3 Tensor Product Problems and Weights

We know from the work of Bakhvalov already done in 1959 that the optimal order of convergence is $n^{-k/d}$ for functions from the class $C^k([0,1]^d)$. To obtain an order of convergence of roughly n^{-k} for every dimension d, one needs stronger smoothness

conditions. This is a major reason for the study of functions with *bounded mixed derivatives*, or *dominating mixed smoothness*, such as the classes

$$W_p^{k,\mathrm{mix}}([0,1]^d) = \{f : [0,1]^d \to \mathbb{R} \mid \|D^\alpha f\|_p \le 1 \text{ for } \|\alpha\|_\infty \le k\}.$$

Observe that functions from this class have, in particular, the high order derivative $D^{(k,k,\dots,k)} f \in L_p$ and one may hope that the curse of dimensionality can be avoided or at least moderated by this assumption. For $k = 1$ these spaces are closely related to various notions of *discrepancy*, see, for example, [23, 25, 71, 90, 112].

The optimal order of convergence is known for all $k \in \mathbb{N}$ and $1 < p < \infty$ due to the work of Roth [97, 98], Frolov [39, 40], Bykovskii [10], Temlyakov [110] and Skriganov [102], see the survey Temlyakov [112]. The cases $p \in \{1, \infty\}$ are still unsolved. The case $p = 1$ is strongly related to the star discrepancy, see also Theorem 10.

Theorem 6. *Assume that $k \in \mathbb{N}$ and $1 < p < \infty$. Then*

$$e(n, W_p^{k,\mathrm{mix}}([0,1]^d)) \asymp n^{-k} (\log n)^{(d-1)/2}.$$

Remark 6. The upper bound was proved by Frolov [39] for $p = 2$ and by Skriganov [102] for all $p > 1$. The lower bound was proved by Roth [97] and Bykovskii [10] for $p = 2$ and by Temlyakov [110] for all $p < \infty$. Hence it took more than 30 years to prove Theorem 6 completely.

For functions in $W_p^{k,\mathrm{mix}}([0,1]^d)$ with compact support in $(0,1)^d$ one can take algorithms of the form

$$A_n(f) = \frac{|\det A|}{a^d} \sum_{m \in \mathbb{Z}^d} f\left(\frac{Am}{a}\right),$$

where A is a suitable matrix that does not depend on k or n, and $a > 0$. Of course the sum is finite since we use only the points $\frac{Am}{a}$ in $(0,1)^d$.

This algorithm is similar to a *lattice rule* but is not quite a lattice rule since the points do not build an integration lattice. The sum of the weights is roughly 1, but not quite. Therefore this algorithm is not really a quasi-Monte Carlo algorithm. The algorithm A_n can be modified to obtain the optimal order of convergence for the whole space $W_p^{k,\mathrm{mix}}([0,1]^d)$. The modified algorithm uses different points x_i but still positive weights a_i. For a tutorial on this algorithm see [117]. Error bounds for Besov spaces are studied in [36]. Triebel-Lizorkin spaces and the case of small smoothness are studied in [118] and [75]. □

For the Besov-Nikolskii classes $S^r_{p,q}B(T^d)$ with $1 \le p,q \le \infty$ and $1/p < r < 2$, the optimal rate is

$$n^{-r} (\log n)^{(d-1)(1-1/q)}$$

and can be obtained constructively with QMC algorithms, see [60]. The lower bound was proved by Triebel [116]. □

The Frolov algorithm can be used as a building block for a randomized algorithm that is *universal* in the sense that it has the optimal order of convergence (in the

randomized setting as well as in the worst case setting) for many different function spaces, see [65]. □

A famous algorithm for *tensor product problems* is the *Smolyak algorithm*, also called *sparse grids algorithm*. We can mention just a few papers and books that deal with this topic: The algorithm was invented by Smolyak [107] and, independently, by several other colleagues and research groups. Several error bounds were proved by Temlyakov [109, 111]; explicit error bounds (without unknown constants) were obtained by Wasilkowski and Woźniakowski [122, 124]. Novak and Ritter [81, 82, 83] studied the particular Clenshaw-Curtis Smolyak algorithm. A survey is Bungartz and Griebel [9] and another one is [90, Chap. 15]. For recent results on the order of convergence see Sickel and T. Ullrich [100, 101] and Dinh Dũng and T. Ullrich [30]. The recent paper [63] contains a tractability result for the Smolyak algorithm applied to very smooth functions. We display only one recent result on the Smolyak algorithm.

Theorem 7 (Sickel and T. Ullrich [101]). *For the classes $W_2^{k,\mathrm{mix}}([0,1]^d)$ one can construct a Smolyak algorithm with the order of the error*

$$n^{-k}(\log n)^{(d-1)(k+1/2)}. \tag{9}$$

Remark 7. a) The bound (9) is valid even for L_2 approximation instead of integration, but it is not known whether this upper bound is optimal for the approximation problem. Using the technique of control variates one can obtain the order

$$n^{-k-1/2}(\log n)^{(d-1)(k+1/2)}$$

for the integration problem in the randomized setting. This algorithm is not often used since it is not easy to implement and its arithmetic cost is rather high. In addition, the rate can be improved by the algorithm of [65] to $n^{-k-1/2}(\log n)^{(d-1)/2}$.

b) It is shown in Dinh Dũng and T. Ullrich [30] that the order (9) can not be improved when restricting to Smolyak grids.

c) We give a short description of the Clenshaw-Curtis Smolyak algorithm for the computation of integrals $\int_{[-1,1]^d} f(x)\,\mathrm{d}x$ that often leads to "almost optimal" error bounds, see [82].

We assume that for $d = 1$ a sequence of formulas

$$U^i(f) = \sum_{j=1}^{m_i} a_j^i f(x_j^i)$$

is given. In the case of numerical integration the a_j^i are just numbers. The method U^i uses m_i function values and we assume that U^{i+1} has smaller error than U^i and $m_{i+1} > m_i$. Define then, for $d > 1$, the tensor product formulas

$$(U^{i_1} \otimes \cdots \otimes U^{i_d})(f) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_d=1}^{m_{i_d}} a_{j_1}^{i_1} \cdots a_{j_d}^{i_d}\, f(x_{j_1}^{i_1}, \ldots, x_{j_d}^{i_d}).$$

A tensor product formula clearly needs

$$m_{i_1} \cdot m_{i_2} \cdot \cdots \cdot m_{i_d}$$

function values, sampled on a regular grid. The Smolyak formulas $A(q,d)$ are clever linear combinations of tensor product formulas such that

- only tensor products with a relatively small number of knots are used;
- the linear combination is chosen in such a way that an interpolation property for $d = 1$ is preserved for $d > 1$.

The Smolyak formulas are defined by

$$A(q,d) = \sum_{q-d+1 \le |\boldsymbol{i}| \le q} (-1)^{q-|\boldsymbol{i}|} \cdot \binom{d-1}{q-|\boldsymbol{i}|} \cdot (U^{i_1} \otimes \cdots \otimes U^{i_d}),$$

where $q \ge d$. Specifically, we use, for $d > 1$, the Smolyak construction and start, for $d = 1$, with the classical Clenshaw-Curtis formula with

$$m_1 = 1 \quad \text{and} \quad m_i = 2^{i-1} + 1 \ \text{ for } \ i > 1.$$

The Clenshaw-Curtis formulas

$$U^i(f) = \sum_{j=1}^{m_i} a_j^i f(x_j^i)$$

use the knots

$$x_j^i = -\cos \frac{\pi(j-1)}{m_i - 1}, \qquad j = 1, \ldots, m_i$$

(and $x_1^1 = 0$). Hence we use nonequidistant knots. The weights a_j^i are defined in such a way that U^i is exact for all (univariate) polynomials of degree at most m_i. □

It turns out that many tensor product problems are still intractable and suffer from the curse of dimensionality, for a rather exhaustive presentation see [89, 90, 92]. Sloan and Woźniakowski [106] describe a very interesting idea that was further developed in hundreds of papers, the paper [106] is most important and influential. We can describe here only the very beginnings of a long ongoing story; we present just one example instead of the whole theory.

The rough idea is that $f\colon [0,1]^d \to \mathbb{R}$ may depend on many variables, d is large, but some variables or groups of variables are more important than others. Consider, for $d = 1$, the inner product

$$\langle f, g \rangle_{1,\gamma} = \left(\int_0^1 f \, \mathrm{d}x \right) \left(\int_0^1 g \, \mathrm{d}x \right) + \frac{1}{\gamma} \int_0^1 f'(x)\, g'(x) \, \mathrm{d}x,$$

where $\gamma > 0$. If γ is small then f must be "almost constant" if it has small norm. A large γ means that f may have a large variation and still the norm is relatively

small. Now we take tensor products of such spaces and weights $\gamma_1 \geq \gamma_2 \geq \dots$ and consider the complexity of the integration problem for the unit ball F_d with respect to this weighted norm. The kernel K of the tensor product space $H(K)$ is of the form

$$K(x,y) = \prod_{i=1}^{d} K_{\gamma_i}(x_i, y_i),$$

where K_γ is the kernel of the respective space H_γ of univariate functions.

Theorem 8 (Sloan and Woźniakowski [106]). *Assume that $\sum_{i=1}^{\infty} \gamma_i < \infty$. Then the problem is strongly polynomially tractable.*

Remark 8. The paper [106] contains also a lower bound which is valid for all quasi-Monte Carlo methods. The proof of the *upper bound* is very interesting and an excellent example for the *probabilistic method*. Compute the mean of the quadratic worst case error of QMC algorithms over all $(x_1, \dots, x_n) \in [0,1]^{nd}$ and obtain

$$\frac{1}{n}\left(\int_{[0,1]^d} K(x,x)\,\mathrm{d}x - \int_{[0,1]^{2d}} K(x,y)\,\mathrm{d}x\,\mathrm{d}y\right).$$

This expectation is of the form $C_d\, n^{-1}$ and the sequence C_d is bounded if and only if $\sum \gamma_i < \infty$. The *lower bound* in [106] is based on the fact that the kernel K is always non-negative; this leads to lower bounds for QMC algorithms or, more generally, for algorithms with positive weights. □

As already indicated, Sloan and Woźniakowski [106] was continued in many directions. Much more general weights and many different Hilbert spaces were studied. By the probabilistic method one only obtains the *existence* of a good QMC algorithms but, in the meanwhile, there exist many results about the *construction* of good algorithms. In this paper the focus is on the basic complexity results and therefore we simply list a few of the most relevant papers: [7, 11, 24, 28, 29, 54, 55, 56, 66, 67, 68, 70, 93, 94, 103, 104, 105]. See also the books [25, 71, 76, 90] and the excellent survey paper [23]. □

In complexity theory we want to study *optimal* algorithms and it is not clear whether QMC algorithms or quadrature formulas with positive coefficients a_i are optimal. Observe that the Smolyak algorithm uses also negative a_i and it is known that in certain cases positive quadrature formulas are far from optimal; for examples see [85] or [90, Sects. 10.6 and 11.3]. Therefore it is not clear whether the conditions on the weights in Theorem 8 can be relaxed if we allow arbitrary algorithms. The next result shows that this is not the case.

Theorem 9 ([88]). *The integration problem from Theorem 8 is strongly polynomially tractable if and only if $\sum_{i=1}^{\infty} \gamma_i < \infty$.*

Remark 9. Due to the known upper bound of Theorem 8, to prove Theorem 9 it is enough to prove a *lower* bound for arbitrary algorithms. This is done via the technique of *decomposable kernels* that was developed in [88], see also [90, Chap. 11].

We do not describe this technique here and only remark that we need for this technique many non-zero functions f_i in the Hilbert space F_d with disjoint supports. Therefore this technique usually works for functions with finite smoothness, but not for analytic functions. □

Tractability of integration can be proved for many weighted spaces and one may ask whether there are also unweighted spaces where tractability holds as well. A famous example for this are integration problems that are related to the *star discrepancy*.

For $x_1,\dots,x_n \in [0,1]^d$ define the star discrepancy by

$$D_\infty^*(x_1,\dots,x_n) = \sup_{t\in[0,1]^d} \left| t_1\cdots t_d - \frac{1}{n}\sum_{i=1}^n 1_{[0,t)}(x_i)\right|,$$

the respective QMC quadrature formula is $Q_n(f) = \frac{1}{n}\sum_{i=1}^n f(x_i)$.

Consider the Sobolev space

$$F_d = \{f \in W_1^{1,\mathrm{mix}} \mid \|f\| \le 1, f(x) = 0 \text{ if there exists an } i \text{ with } x_i = 1\}$$

with the norm

$$\|f\| := \left\| \frac{\partial^d f}{\partial x_1 \partial x_2 \dots \partial x_d} \right\|_1.$$

Then the Hlawka-Zaremba-equality yields

$$D_\infty^*(x_1,\dots,x_n) = \sup_{f\in F_d} |S_d(f) - Q_n(f)|,$$

hence the star discrepancy is a worst case error bound for integration. We define

$$n(\varepsilon, F_d) = \min\{n \mid \exists x_1,\dots,x_n \text{ with } D_\infty^*(x_1,\dots,x_n) \le \varepsilon\}.$$

The following result shows that this integration problem is polynomially tractable and the complexity is linear in the dimension.

Theorem 10 ([52]).

$$n(\varepsilon, F_d) \le C\, d\, \varepsilon^{-2} \tag{10}$$

and

$$n(1/64, F_d) \ge 0.18\, d.$$

Remark 10. This result was modified and improved in various ways and we mention some important results. Hinrichs [57] proved the lower bound

$$n(\varepsilon, F_d) \ge c\, d\, \varepsilon^{-1} \quad \text{for} \quad \varepsilon \le \varepsilon_0.$$

Aistleitner [1] proved that the constant C in (10) can be taken as 100. Aistleitner and Hofer [2] proved more on upper bounds. Already the proof in [52] showed that

an upper bound $D_\infty^*(x_1,\dots,x_n) \le C\sqrt{\frac{d}{n}}$ holds with high probability if the points $x_1,\dots,x_n$ are taken independently and uniformly distributed. Doerr [31] proved the respective lower bound, hence

$$\mathbb{E}(D_\infty^*(x_1,\dots,x_n)) \asymp \sqrt{\frac{d}{n}} \quad \text{for} \quad n \ge d.$$

Since the upper bounds are proved with the probabilistic method, we only know the *existence* of points with small star discrepancy. The existence results can be transformed into (more or less explicit) constructions and the problem is, of course, to minimize the computing time as well as the discrepancy. One of the obstacles is that already the computation of the star discrepancy of given points $x_1, x_2, \dots, x_n$ is very difficult. We refer the reader to [19, 26, 27, 32, 33, 34, 35, 42, 59]. □

Recently Dick [20] proved a tractability result for another unweighted space that is defined via an L_1-norm and consists of periodic functions; we denote Fourier coefficients by $\tilde{f}(k)$, where $k \in \mathbb{Z}^d$. Let $0 < \alpha \le 1$ and $1 \le p \le \infty$ and

$$F_{\alpha,p,d} = \left\{ f : [0,1]^d \to \mathbb{R} \mid \sum_{k\in\mathbb{Z}^d} |\tilde{f}(k)| + \sup_{x,h} \frac{|f(x+h)-f(x)|}{\|h\|_p^\alpha} \le 1 \right\}.$$

Dick proved the upper bound

$$e(n, F_{\alpha,p,d}) \le \max\left(\frac{d-1}{\sqrt{n}}, \frac{d^{\alpha/p}}{n^\alpha} \right)$$

for any prime number n. Hence the complexity is at most quadratic in d.

The proof is constructive, a suitable algorithm is the following. Use points $x_k = \left(\left\{ \frac{k^1}{n} \right\}, \left\{ \frac{k^2}{n} \right\}, \dots, \left\{ \frac{k^d}{n} \right\} \right)$, where $k = 0, 1, \dots, n-1$, and take the respective QMC algorithm. □

4 Some Recent Results

We end this survey with two results that were still unpublished at the time of the conference, April 2014. First we return to the classes $C^k([0,1]^d)$, see Theorem 1. We want to be a little more general and consider the computation of

$$S_d(f) = \int_{D_d} f(x)\,\mathrm{d}x \tag{11}$$

up to some error $\varepsilon > 0$, where $D_d \subset \mathbb{R}^d$ has Lebesgue measure 1. The results hold for arbitrary sets D_d, the standard example of course is $D_d = [0,1]^d$. For convenience we consider functions $f\colon \mathbb{R}^d \to \mathbb{R}$. This makes the function class a bit smaller and the result a bit stronger, since our emphasis is on lower bounds.

It has *not* been known if the curse of dimensionality is present for probably the most natural class which is the unit ball of r times continuously differentiable functions,

$$F_d^k = \{f \in C^k(\mathbb{R}^d) \mid \|D^\alpha f\|_\infty \le 1 \quad \text{for all} \quad |\alpha| \le k\},$$

where $k \in \mathbb{N}$.

Theorem 11 ([61]). *The curse of dimensionality holds for the classes F_d^k with the* super-exponential *lower bound*

$$n(\varepsilon, F_d^k) \ge c_k\,(1-\varepsilon)\,d^{d/(2k+3)} \quad \textit{for all } d \in \mathbb{N} \textit{ and } \varepsilon \in (0,1),$$

where $c_k > 0$ depends only on k.

Remark 11. In [61, 62] we also prove that the curse of dimensionality holds for even smaller classes of functions F_d for which the norms of arbitrary directional derivatives are bounded proportionally to $1/\sqrt{d}$.

We start with the fooling function

$$f_0(x) = \min\left\{1, \frac{1}{\delta\sqrt{d}}\,\mathrm{dist}(x, \mathscr{P}_\delta)\right\} \quad \text{for all} \quad x \in \mathbb{R}^d,$$

where

$$\mathscr{P}_\delta = \bigcup_{i=1}^n B_\delta^d(x_i)$$

and $B_\delta^d(x_i)$ is the ball with center x_i and radius $\delta\sqrt{d}$. The function f_0 is Lipschitz. By a suitable smoothing via convolution we construct a smooth fooling function $f_k \in F_d$ with $f_k|_{\mathscr{P}_0} = 0$.

Important elements of the proof are volume estimates (in the spirit of Elekes [38] and Dyer, Füredi and McDiarmid [37]), since we need that the volume of a neighborhood of the convex hull of n arbitrary points is exponentially small in d. □

Also classes of C^∞-functions were studied recently. We still do not know whether the integration problem suffers from the curse of dimensionality for the classes

$$F_d = \{f : [0,1]^d \to \mathbb{R} \mid \|D^\alpha f\|_\infty \le 1 \text{ for all } \alpha \in \mathbb{N}_0^d\},$$

this is Open Problem 2 from [89]. We know from Vybíral [120] and [62] that the curse is present for somewhat larger spaces and that a weak tractability holds for smaller classes; this can be proved with the Smolyak algorithm, see [63]. □

We now consider univariate oscillatory integrals for the standard Sobolev spaces H^s of periodic and non-periodic functions with an arbitrary integer $s \ge 1$. We study the approximate computation of Fourier coefficients

$$I_k(f) = \int_0^1 f(x)\,\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x, \qquad \mathrm{i} = \sqrt{-1},$$

where $k \in \mathbb{Z}$ and $f \in H^s$.

There are several recent papers about the approximate computation of highly oscillatory univariate integrals with the weight $\exp(2\pi \mathrm{i} kx)$, where $x \in [0,1]$ and k is an integer (or $k \in \mathbb{R}$) which is assumed to be large in the absolute sense, see Huybrechs and Olver [64] for a survey.

We study the Sobolev space H^s for a finite $s \in \mathbb{N}$, i.e.,

$$H^s = \{f : [0,1] \to \mathbb{C} \mid f^{(s-1)} \text{ is abs. cont., } f^{(s)} \in L_2\} \tag{12}$$

with the inner product

$$\begin{aligned} \langle f,g\rangle_s &= \sum_{\ell=0}^{s-1} \int_0^1 f^{(\ell)}(x)\,\mathrm{d}x \int_0^1 \overline{g^{(\ell)}(x)}\,\mathrm{d}x + \int_0^1 f^{(s)}(x)\,\overline{g^{(s)}(x)}\,\mathrm{d}x \\ &= \sum_{\ell=0}^{s-1} \langle f^{(\ell)},1\rangle_0\,\overline{\langle g^{(\ell)},1\rangle_0} + \langle f^{(s)},g^{(s)}\rangle_0, \end{aligned} \tag{13}$$

where $\langle f,g\rangle_0 = \int_0^1 f(x)\,\overline{g(x)}\,\mathrm{d}x$, and norm $\|f\|_{H^s} = \langle f,f\rangle_s^{1/2}$.

For the periodic case, an algorithm that uses n function values at equally spaced points is nearly optimal, and its worst case error is bounded by $C_s(n+|k|)^{-s}$ with C_s exponentially small in s. For the non-periodic case, we first compute successive derivatives up to order $s-1$ at the end-points $x=0$ and $x=1$. These derivatives values are used to periodize the function and this allows us to obtain similar error bounds like for the periodic case. Asymptotically in n, the worst case error of the algorithm is of order n^{-s} independently of k for both periodic and non-periodic cases.

Theorem 12 ([87]). *Consider the integration problem I_k defined over the space H^s of non-periodic functions with $s \in \mathbb{N}$. Then*

$$\frac{c_s}{(n+|k|)^s} \;\le\; e(n,k,H^s) \;\le\; \left(\frac{3}{2\pi}\right)^s \frac{2}{(n+|k|-2s+1)^s},$$

for all $k \in \mathbb{Z}$ and $n \ge 2s$.

Remark 12. The minimal errors $e(n,k,H^s)$ for the non-periodic case have a peculiar property for $s \ge 2$ and large k. Namely, for $n=0$ we obtain the initial error which is of order $|k|^{-1}$, whereas for $n \ge 2s$ it becomes of order $|k|^{-s}$. Hence, the dependence on $|k|^{-1}$ is short-lived and disappears quite quickly. For instance, take $s=2$. Then $e(n,k,H^s)$ is of order $|k|^{-1}$ only for $n=0$ and maybe for $n=1,2,3$, and then becomes of order $|k|^{-2}$. □

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