HIGH DIMENSIONAL INTEGRATION OF SMOOTH FUNCTIONS OVER CUBES

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Abstract. We construct a new algorithm for the numerical integration of functions that are defined on a d-dimensional cube. It is based on the Clenshaw-Curtis rule for $d = 1$ and on Smolyak's construction. This way we make the best use of the smoothness properties of any (nonperiodic) function. We prove error bounds showing that our algorithm is almost optimal (up to logarithmic factors) for different classes of functions with bounded mixed derivative. Numerical results show that the new method is very competitive, in particular for smooth integrands and $d \geq 8$.

1. Introduction

Smolyak (1963) studied tensor product problems and introduced a general construction that leads to almost optimal approximations for any dimension $d > 1$ from optimal approximations for the univariate case $d = 1$. This construction is also known under different names, including "Biermann interpolation", "Boolean method", "discrete blending method", "hyperbolic cross points", and "sparse grid method".

Smolyak's construction has been developed independently in many papers for specific problems. If in the univariate case the operators are polynomial interpolation operators then one obtains discrete versions of polynomial blending interpolation which occur in computer aided design and in the finite element method. See Gordon (1971) and Delvos and Schempp (1989). For $d > 1$ the interpolation points of the Smolyak algorithm form a set of "hyperbolic cross points" or a "sparse grid". These points are also used for spline interpolation and Fourier approximation.

Further results and applications of Smolyak's algorithm can be found in Baszenski, Delvos, and Jester (1992), Bungartz (1992), Griebel, Schneider, and Zenger (1992), Novak and Ritter (1996), Pereverzev (1986), Ritter, Wasilkowski, and Woźniakowski (1995), Temlyakov (1987, 1993, 1994), Wahba (1978), Wasilkowski and Woźniakowski (1995), Woźniakowski (1992), and Zenger (1991).

We are interested, in particular, in numerical integration. This application is also called rth order blending cubature and was already studied in Smolyak (1963). Other papers include Baszenski and Delvos (1993), Bonk (1994), Delvos (1990), Genz (1974), Lyness and Sloan (1995), Paskov (1993), Ritter, Wasilkowski, and Woźniakowski (1993), and Temlyakov (1992). In some of these papers the stress is on the numerical integration of periodic functions. In this case, the multivariate Boolean midpoint rule is very competitive.

In this paper we present a method for high dimensional numerical integration that is almost optimal for different classes of nonperiodic functions. We apply Smolyak's

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construction to the Clenshaw-Curtis rule, see Sections 2 and 3. The respective error bounds are proved in Section 4, which heavily relies on Smolyak's original results and the recent refinements of Wasilkowski and Woźniakowski (1995). The new method is exact for certain "nonclassical" spaces of polynomials, see Section 5. In Section 6 we present numerical results. The testing package of Genz (1984, 1987) is used and, in particular, the results of Sloan and Joe (1994).

In the extensive tests of Sloan and Joe for $d = 5$, 8, and 10 the methods ADAPT and COPY are usually the two most successful ones. The method ADAPT is the famous adaptive method of Van Dooren and De Ridder (1976) in conjunction with the modification of Genz and Malik (1980). Sloan and Joe used the implementation found in the NAG routine D01FCF. The method COPY is an embedded sequence of lattice rules, see Sloan and Joe (1994) for details.

Our tests show that the new method is very good for functions which are at least differentiable and depend on a large number d of variables. For the families 1–4 of Genz the results of the new method were usually not as good as those of ADAPT and COPY for $d = 5$ and comparable for $d = 8$. For $d = 10$ the results of our method were excellent and we expect that the advantage of the new method is even more impressive for $d > 10$. Unfortunately, we could not find many test results for higher dimensions in the literature.

2. Smolyak's Construction

We explain Smolyak's construction not in the most general case but consider, for $d = 1$, always one of the spaces

$$F_1^r = C^r([-1,1]), \qquad r \in \mathbb{N},$$

with the norm

$$\|f\| = \max(\|f\|_\infty, \ldots, \|f^{(r)}\|_\infty).$$

For $d > 1$ we consider the tensor product

$$F_d^r = \{f : [-1,1]^d \to \mathbb{R} \mid D^{\boldsymbol{\alpha}} f \text{ continuous if } \alpha_i \le r \text{ for all } i\}$$

with the norm

$$\|f\| = \max\{\|D^{\boldsymbol{\alpha}} f\|_\infty \mid \boldsymbol{\alpha} \in \mathbb{N}_0^d,\ \alpha_i \le r\}.$$

This means, in particular, that finite linear combinations of functions

$$(f_1 \otimes \cdots \otimes f_d)(x_1, \ldots, x_d) = f_1(x_1) \cdot \cdots \cdot f_d(x_d)$$

with $f_i \in F_1^r$ are dense in F_d^r and

$$\|f_1 \otimes \cdots \otimes f_d\| = \|f_1\| \cdot \cdots \cdot \|f_d\|.$$

We are mainly interested in numerical integration (cubature) of functions $f \in F_d^r$ and define

$$I_d(f) = \int_{[-1,1]^d} f(x)\, dx.$$

We wish to find good approximations to the functional I_d in the multivariate case on the basis of good approximations to I_1 in the univariate case. The algorithm of

Smolyak (1963) provides a general and very efficient construction in such a tensor product situation.

Assume that a sequence of quadrature formulas

$$U^i(f) = \sum_{j=1}^{m_i} f(x_j^i) \cdot a_j^i$$

with $m_i \in \mathbb{N}$ is given. In the multivariate case $d > 1$ we first define

$$(U^{i_1} \otimes \cdots \otimes U^{i_d})(f) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_d=1}^{m_{i_d}} f(x_{j_1}^{i_1}, \ldots, x_{j_d}^{i_d}) \cdot (a_{j_1}^{i_1} \cdot \cdots \cdot a_{j_d}^{i_d}).$$

This is a so-called tensor product algorithm, which has a poor order of convergence and only serves as a building block for the more complicated algorithm of Smolyak. We use the notation $U^0 = 0$ and

$$\Delta^i = U^i - U^{i-1}$$

for $i \in \mathbb{N}$. Moreover we put $|\mathbf{i}| = i_1 + \cdots + i_d$ for $\mathbf{i} \in \mathbb{N}^d$. Now the algorithm of Smolyak is given by

$$A(q,d) = \sum_{|\mathbf{i}| \leq q} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d}) \tag{1}$$

for integers $q \geq d$. For the case $d = 2$ we can also write

$$A(q,2) = \sum_{i_1+i_2=q} (U^{i_1} \otimes U^{i_2}) - \sum_{i_1+i_2=q-1} (U^{i_1} \otimes U^{i_2}).$$

For general d the analogous formula is

$$A(q,d) = \sum_{q-d+1 \leq |\mathbf{i}| \leq q} (-1)^{q-|\mathbf{i}|} \cdot \binom{d-1}{q-|\mathbf{i}|} \cdot (U^{i_1} \otimes \cdots \otimes U^{i_d}),$$

see Wasilkowski and Woźniakowski (1995, Lemma 1) and, similar, Delvos (1982, Theorem 1).

Clearly $A(q,d)$ is a linear functional, and $A(q,d)(f)$ depends on $f \in F_d^r$ only through function values at a finite number of points. To describe these points let

$$X^i = \{x_1^i, \ldots, x_{m_i}^i\} \subset [-1,1]$$

denote the set of points that correspond to U^i. The tensor product algorithm $U^{i_1} \otimes \cdots \otimes U^{i_d}$ is based on the grid $X^{i_1} \times \cdots \times X^{i_d}$, and therefore $A(q,d)(f)$ depends (at most) on the values of f at the union

$$H(q,d) = \bigcup_{q-d+1 \leq |\mathbf{i}| \leq q} (X^{i_1} \times \cdots \times X^{i_d}) \subset [-1,1]^d$$

of grids. Nested sets

$$X^i \subset X^{i+1} \tag{2}$$

yield $H(q,d) \subset H(q+1,d)$ and

$$H(q,d) = \bigcup_{|\mathbf{i}|=q} (X^{i_1} \times \cdots \times X^{i_d}). \tag{3}$$

Therefore nested sets seem to be the most economical choice. The points $x \in H(q,d)$ are called hyperbolic cross points and $H(q,d)$ is also called a sparse grid. For instance, if $X^i = \{0, \pm 1/2^i, \dots, \pm(1-1/2^i), \pm 1\}$ then $H(q,d)$ consists of all dyadic grids with product of mesh size given by $1/2^q$.

3. The Clenshaw-Curtis Rule

We want to construct cubature formulas that use in an optimal way the global smoothness properties of any continuous nonperiodic function on $[-1,1]^d$. To this end we employ Smolyak's algorithm with quadrature formulas U^i that use the smoothness of univariate functions optimally.

For any cubature formula Q we have the error bound

$$|I_d(f) - Q(f)| \le \|I_d - Q\| \cdot \|f\|$$

in terms of the norm on F_d^r, i.e., $\|I_d - Q\|$ denotes the respective operator norm of $I_d - Q$. In the univariate case $d=1$,

$$\lim_{n\to\infty}\Big(n^r \cdot \inf_{Q_n} \|I_1 - Q_n\|\Big) = \beta_r \tag{4}$$

with known constants $\beta_r > 0$ for any $r \in \mathbb{N}$, see Strauß (1979). Here the infimum is over all formulas Q_n which use n function values, and formulas which yield the asymptotic error $\beta_r \cdot n^{-r}$ depend on the smoothness r.

Recall that m_i denotes the number of function values used by U^i, and assume that $m_i < m_{i+1}$. In light of (4) we are interested in formulas U^i with

$$\limsup_{i\to\infty}\Big(m_i^r \cdot \|I_1 - U^i\|\Big) < \infty, \qquad \forall\, r \in \mathbb{N}. \tag{5}$$

This property holds, for instance, for any sequence of interpolatory formulas U^i with positive weights, see Brass (1992).

We suggest to use the Clenshaw-Curtis method with a suitable choice of the sequence m_i. See Brass (1977) and Engels (1980) for the following facts. For any choice of $m_i > 1$ let

$$x_j^i = -\cos\frac{\pi \cdot (j-1)}{m_i - 1}, \qquad j = 1, \dots, m_i. \tag{6}$$

The Clenshaw-Curtis formula

$$U^i(f) = \sum_{j=1}^{m_i} f(x_j^i) \cdot a_j^i$$

uses these points x_j^i, and the weights a_j^i are characterized by the demand that U^i is exact for all polynomials of degree less than m_i. Property (5) holds due to the positivity of the weights. In order to obtain nested sets of points, see (2), we choose

$$m_i = 2^{i-1} + 1, \qquad i > 1. \tag{7}$$

It is very important to choose

$$m_1 = 1$$

if we are interested in cubature for relatively large d, because in all other cases the number of points used by $A(q,d)$ increases too fast with d. For instance, this number is m_1^d for $A(d,d)$ and $d\,(m_2-m_1)\,m_1^{d-1}+m_1^d$ for $A(d+1,d)$. It is natural to take $x_1^1=0$, thus

$$U^1(f)=2\cdot f(0).$$

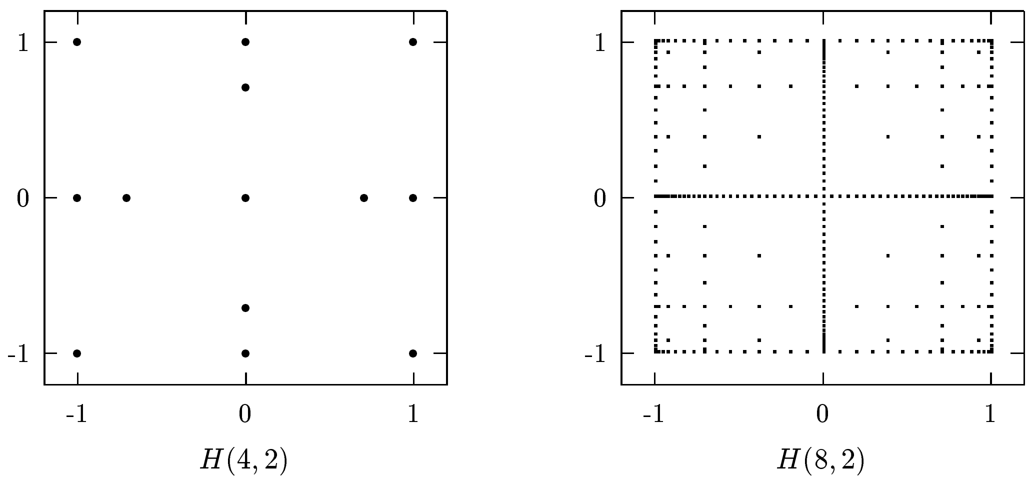
Since m_i is odd for any $i>1$ the respective weights are given by

$$a_j^i=a_{m_i+1-j}^i=\frac{2}{m_i-1}\left(1-\frac{\cos(\pi\cdot(j-1))}{m_i\cdot(m_i-2)}-2\sum_{k=1}^{(m_i-3)/2}\frac{1}{4k^2-1}\cdot\cos\frac{2\pi k\cdot(j-1)}{m_i-1}\right)$$

for $j=2,\dots,m_i-1$. The two missing weights are given by

$$a_1^i=a_{m_i}^i=\frac{1}{m_i\cdot(m_i-2)}.$$

Let us look at two bivariate examples of resulting hyperbolic cross points, see (3).



The cardinalities of $H(4,2)$ and $H(8,2)$ are 13 and 321, respectively.

Although the univariate formulas U^i have only positive weights, the multivariate formulas $A(q,d)$ may also have negative weights. For $d=2$, negative weights already occur in the formula $A(4,2)$ at the knots $\{(0,0),(\pm1,0),(0,\pm1)\}$.

4. Error Bounds

There are general techniques to get error bounds for Smolyak's algorithm for $d>1$ from those for the case $d=1$. For our particular problem we start with the estimate

$$\|I_1-U^i\|\le\gamma_r\cdot2^{-r\cdot i}$$

in the univariate case, see (5) and (7). Explicit values for the constants γ_r can be obtained by known bounds for the respective Peano kernels, see Brass (1993).

In the sequel we also consider tensor products of quadrature formulas and integrals. For a formal definition of tensor products of operators we refer to Delvos and Schempp (1989). In a bivariate example we have

$$(I_1\otimes U^i)(f)=\int_{-1}^{1}\sum_{j=1}^{m_i}f(s,x_j^i)\cdot a_j^i\,ds=\sum_{j=1}^{m_i}\int_{-1}^{1}f(s,x_j^i)\,ds\cdot a_j^i$$

and

$$(I_1 \otimes U^i)(f_1 \otimes f_2) = I_1(f_1) \cdot U^i(f_2).$$

Furthermore,

$$\begin{aligned}
|I_1 \otimes U^i(f)| &\le \|I_1\| \cdot \Big\| \sum_{j=1}^{m_i} f(\cdot, x_j^i) \cdot a_j^i \Big\| \\
&= \|I_1\| \cdot \sup_{0 \le \alpha_1 \le r} \sup_{s \in [-1,1]} |U^i f^{(\alpha_1,0)}(s, \cdot)| \\
&\le \|I_1\| \cdot \sup_{0 \le \alpha_1 \le r} \sup_{s \in [-1,1]} \|U^i\| \cdot \|f^{(\alpha_1,0)}(s, \cdot)\| \\
&= \|I_1\| \cdot \|U^i\| \cdot \|f\|.
\end{aligned}$$

Thus $\|I_1 \otimes U^i\| = \|I_1\| \cdot \|U^i\|$.

Now we follow the lines of Wasilkowski and Woźniakowski (1995, Lemma 2). Note that

$$A(q+1, d+1) = \sum_{|\mathbf{i}| \le q} \Big(\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d} \otimes \sum_{k=1}^{q+1-|\mathbf{i}|} \Delta^k \Big) = \sum_{|\mathbf{i}| \le q} \Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d} \otimes U^{q+1-|\mathbf{i}|}$$

due to definition (1). Hence the error of $A(q+1, d+1)$ may be written as

$$I_{d+1} - A(q+1, d+1) = (I_d - A(q, d)) \otimes I_1 + \sum_{|\mathbf{i}| \le q} \Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d} \otimes (I_1 - U^{q+1-|\mathbf{i}|}).$$

Since $\|I_1\| = 2$ and

$$\|S \otimes T\| = \|S\| \cdot \|T\| \tag{8}$$

for the operators involved, we get

$$\|I_{d+1} - A(q+1, d+1)\| \le 2 \cdot \|I_d - A(q, d)\| + \sum_{|\mathbf{i}| \le q} \|\Delta^{i_1}\| \cdot \cdots \cdot \|\Delta^{i_d}\| \cdot \|I_1 - U^{q+1-|\mathbf{i}|}\|.$$

Moreover

$$\|\Delta^{i_k}\| \le \|I_1 - U^{i_k}\| + \|I_1 - U^{i_k - 1}\| \le \gamma_r \cdot 2^{-r \cdot i_k} \cdot (1 + 2^r),$$

and therefore

$$\sum_{|\mathbf{i}| \le q} \|\Delta^{i_1}\| \cdot \cdots \cdot \|\Delta^{i_d}\| \cdot \|I_1 - U^{q+1-|\mathbf{i}|}\| \le \binom{q}{d} \cdot \gamma_r^{d+1} \cdot (1 + 2^r)^d \cdot 2^{-r \cdot (q+1)}.$$

Inductively we obtain the following estimate.

Theorem 1. *Let*

$$\theta_r = \max(2^{r+1}, \gamma_r \cdot (1 + 2^r)).$$

The error of the cubature formula $A(q, d)$ satisfies

$$\|I_d - A(q, d)\| \le \gamma_r \cdot \theta_r^{d-1} \cdot \binom{q}{d-1} \cdot 2^{-r \cdot q}.$$

Now we relate the error of $A(q,d)$ to the number $n(q,d)$ of knots which are used by this cubature formula, i.e., to the cardinality of $H(q,d)$, see (3). Henceforth constants hidden in O-notations depend on r as well as on d. Theorem 1 implies Smolyak's original estimate

$$\|I_d - A(q,d)\| = O(q^{d-1} \cdot 2^{-r\cdot q}).$$

Using $m_i \leq 2^i$ we get

$$n = n(q,d) \leq 2^q \cdot \binom{q-1}{d-1} = O(q^{d-1} \cdot 2^q),$$

and a simple manipulation yields the following bound.

Corollary 1. *Let $n = n(q,d)$ denote the number of knots which are used by $A(q,d)$. Then*

$$\|I_d - A(q,d)\| = O(n^{-r} \cdot (\log n)^{(d-1)\cdot(r+1)}).$$

Remark 1. The crucial property to get the error bounds for $A(q,d)$ is (8), and therefore Theorem 1 and Corollary 1 hold for arbitrary tensor product problems, see Smolyak (1963) and Wasilkowski and Woźniakowski (1995). In the latter paper explicit bounds – with strong emphasis on their dependence on d – are obtained for the error of $A(q,d)$ as well as for the number of points in $H(q,d)$.

Recovery of functions is another interesting application for Smolyak's construction. We look at the particular case of recovering $f \in F_d^r$ in the uniform norm. In this case we define I_d as the embedding

$$I_d(f) = f$$

of F_d^r into $C([-1,1]^d)$. Clearly

$$I_d(f_1 \otimes \cdots \otimes f_d) = I_1(f_1) \otimes \cdots \otimes I_1(f_d).$$

Assume that a sequence of recovering operators

$$U^i(f) = \sum_{j=1}^{m_i} f(x_j^i) \cdot a_j^i$$

with $a_j^i \in C([-1,1])$ is given in the univariate case. The corresponding tensor product operators are

$$(U^{i_1} \otimes \cdots \otimes U^{i_d})(f) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_d=1}^{m_{i_d}} f(x_{j_1}^{i_1}, \ldots, x_{j_d}^{i_d}) \cdot (a_{j_1}^{i_1} \otimes \cdots \otimes a_{j_d}^{i_d})$$

and Smolyak's construction (1) yields an operator $A(q,d)$ mapping F_d^r into $C([-1,1]^d)$. Theorem 1 and Corollary 1 hold analogously, provided that the respective estimate holds for U^i in the univariate case.

Corollary 1 gives the best error bound for Smolyak's construction which holds for arbitrary tensor product problems, see Temlyakov (1987). For some problems, however, this bound is improved in terms of powers of $\log n$, see Temlyakov (1993) and Wasilkowski and Woźniakowski (1995).

Remark 2. The lower bound in (4) clearly extends to the classes F_d^r with $d > 1$. Thus no sequence of cubature formulas yields errors on any class F_d^r which tend to zero faster than n^{-r}. On the other hand our method yields errors of order $n^{-r} \cdot (\log n)^{(d-1)\cdot(r+1)}$ for *all* classes F_d^r. Hence these methods are almost optimal (up to logarithmic factors) on a whole scale of spaces of nonperiodic functions.

For periodic functions, cubature formulas are known which are asymptotically optimal on a whole scale of spaces, see Temlyakov (1992). Periodization is a standard technique to derive error bounds for related spaces of nonperiodic functions. In particular this approach leads to cubature formulas $Q_{n,r}$ which even satisfy

$$\|I_d - Q_{n,r}\| = O(n^{-r} \cdot (\log n)^{(d-1)/2})$$

on the class F_d^r.

However, periodization takes into account the smoothness r in order to preserve the order of the error, and therefore the resulting formulas $Q_{n,r}$ depend on r. Moreover, computational problems may occur and the exactness of the cubature formulas may be spoiled. We refer to Niederreiter and Sloan (1993) and Sloan and Joe (1994) for a detailed discussion of this topic.

Remark 3. Property (5) is the essential requirement for the U^i in the univariate case. It also holds for the Gauß formulas. These formulas have a higher polynomial exactness than the Clenshaw-Curtis formulas and yield methods $A(q,d)$ with a higher degree of exactness also for $d > 1$, cf. Theorem 2.

We prefer the Clenshaw-Curtis formulas mainly because of (2), which has two important consequences. The number of knots in $H(q,d)$ is reduced because different grids have many common points. Furthermore, weights of different sign at common points partially cancel. Hence we get better stability properties.

Property (5) is also satisfied for certain extrapolation methods, such as the Romberg method. The application of Smolyak's algorithm to a method of this kind is studied in Bonk (1994). However, the Clenshaw-Curtis method has a much higher degree of exactness and also leads to better numerical results.

Remark 4. Our method is easily modified to work for weighted integrals

$$I_d(f) = \int_{[-1,1]^d} f(x) \cdot \rho(x)\, dx$$

where $\rho = \omega_1 \otimes \cdots \otimes \omega_d$ is a tensor product. We only need to redefine the weights in the cubature formulas $A(q,d)$.

In the univariate case we consider generalized Clenshaw-Curtis formulas

$$U^{i,\omega}(f) = \sum_{j=1}^{m_i} f(x_j^i) \cdot a_j^{i,\omega}$$

which use the Clenshaw-Curtis points x_j^i, see (6), and are exact for all polynomials of degree less than m_i. Sloan and Smith (1978) show that

$$\lim_{i\to\infty} \sum_{j=1}^{m_i} |a_j^{i,\omega}| = \int_{-1}^{1} |\omega(x)|\, dx$$

if $\omega \in L_p([-1,1])$ for some $p > 1$. In particular, the functionals $U^{i,\omega}$ are uniformly bounded on $C([-1,1])$. As in the case $\omega \equiv 1$, the error bound (5) follows for $U^i = U^{i,\omega}$ and $I_1(f) = \int_{-1}^{1} f(x) \cdot \omega(x)\,dx$.

Again we choose $m_1 = 1$ and $m_i = 2^{i-1}+1$ for $i > 1$ and apply Smolyak's construction (1) with $\Delta^{i_k} = U^{i,\omega_k} - U^{i-1,\omega_k}$. Theorem 1, with different constants, and Corollary 1 still hold for the resulting cubature formula $A(q,d)$ if $\omega_k \in L_p([-1,1])$ for some $p > 1$ and all $k = 1, \dots, d$. Observe that the set $H(q,d)$ of knots is not changed.

Remark 5. There is another important modification of our method which yields different sets of knots. We only indicate the basic idea. Assume that the smoothness of f varies with the coordinates. For instance, this may be due to a transformation of a rectangular domain with different sidelengths onto the cube $[-1,1]^d$. Then it makes sense to increase the number of knots in different coordinates with different speed by letting m_i depend on the coordinates.

5. Polynomial Exactness

Let $\mathbb{P}_k$ denote the space of polynomials in one variable of degree at most k. The Clenshaw-Curtis formula U^i is exact on

$$V^i = \mathbb{P}_{m_i}$$

since m_i is odd. In the multivariate case our method is exact on certain "nonclassical" spaces of polynomials.

Theorem 2. *The cubature formula $A(q,d)$ is exact on*

$$\sum_{|\mathbf{i}|=q} (V^{i_1} \otimes \cdots \otimes V^{i_d}).$$

Proof. Our proof is via induction over d. For $d = 1$ we have $A(q,1) = U^q$ and the statement follows. For the multivariate case we use the fact that $A(q,d+1)$ can be written in terms of the $A(\ell,d)$, we have

$$A(q,d+1) = \sum_{\ell=d}^{q-1} A(\ell,d) \otimes (U^{q-\ell} - U^{q-\ell-1})$$

due to definition (1).

The induction step is as follows. Assume that

$$f = f_{i_1} \otimes \cdots \otimes f_{i_{d+1}}, \qquad f_{i_k} \in V^{i_k},$$

with $|\mathbf{i}| = q$ and $i_1 + \cdots + i_d = m$ whence $m + i_{d+1} = q$. Then

$$A(q,d+1)(f) = \sum_{\ell=d}^{q-1} A(\ell,d)(f_{i_1} \otimes \cdots \otimes f_{i_d}) \cdot (U^{q-\ell} - U^{q-\ell-1})(f_{i_{d+1}}).$$

Since

$$U^{q-\ell}(f_{i_{d+1}}) = U^{q-\ell-1}(f_{i_{d+1}}) = I_1(f_{i_{d+1}}), \qquad d \le \ell \le m-1,$$

and

$$A(\ell,d)(f_{i_1} \otimes \cdots \otimes f_{i_d}) = I_d(f_{i_1} \otimes \cdots \otimes f_{i_d}), \qquad m \le \ell \le q-1,$$

we obtain

$$\begin{aligned} A(q,d+1)(f) &= \sum_{\ell=m}^{q-1} I_d(f_{i_1}\otimes\cdots\otimes f_{i_d})\cdot(U^{q-\ell}-U^{q-\ell-1})(f_{i_{d+1}}) \\ &= I_d(f_{i_1}\otimes\cdots\otimes f_{i_d})\cdot U^{q-m}(f_{i_{d+1}}) = I_{d+1}(f) \end{aligned}$$

as claimed. □

It follows, for instance, that $A(d+2,d)$ is exact for the polynomials

$$x_j^4,\ x_j^2,\ 1,\ x_j^2 x_k^2$$

with $1 \le j,k \le d$ as well as for all monomials having at least one odd exponent. Thus $A(d+2,d)$ is exact for all polynomials of degree at most 5.

Remark 6. Theorem 2 holds for general tensor product problems if the spaces

$$V^i = \{f \in F_1^r \mid I_1(f) = U^i(f)\}$$

of exactness for the univariate problem are nested, i.e., $V^i \subset V^{i+1}$. In the case of polynomial interpolation this fact is shown in Delvos (1982).

Remark 7. For trigonometric polynomials, the space

$$\operatorname{span}\{\exp(i\langle \mathbf{k},\cdot\rangle) \mid \mathbf{k}\in\mathbb{Z}^d,\ \prod_{j=1}^{d}(1+|k_j|)\le K\} \tag{9}$$

is an analogon to the space of exactness of $A(q,d)$. Here $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$. The frequencies $\mathbf{k}$ which appear in (9) form a hyperbolic cross in $\mathbb{Z}^d$. These spaces of trigonometric polynomials were introduced by Babenko (1960). Further results and applications can be found in Frank, Heinrich, and Pereverzev (1995), Temlyakov (1994), and Tikhomirov (1990).

6. Numerical Examples

Achim Steinbauer (1995) implemented our method which is denoted by NEW in this section. To compare NEW with different existing methods, computations were made with NEW that are analogous to those of Sloan and Joe (1994). These authors compared four different methods for functions of $d = 5$, 8, and 10 variables using the testing package of Genz (1984, 1987). The two most successful methods are generally ADAPT, the adaptive method of Van Dooren and De Ridder (1976), and COPY, an embedded sequence of lattice rules. The two other methods are an adaptive Monte Carlo method found in the NAG routine D01GBF and certain rank-1 lattice rules.

The testing package of Genz is based on a collection of six families of integrands $f_1,\dots,f_6$ which are defined on $[0,1]^d$. The formulas $A(q,d)$ are modified accordingly. Each of these families is given a name or attribute as follows:

1. OSCILLATORY: $$f_1(x) = \cos\Big(2\pi w_1 + \sum_{i=1}^{d} c_i x_i\Big),$$
2. PRODUCT PEAK: $$f_2(x) = \prod_{i=1}^{d} \Big(c_i^{-2} + (x_i - w_i)^2\Big)^{-1},$$
3. CORNER PEAK: $$f_3(x) = \Big(1 + \sum_{i=1}^{d} c_i x_i\Big)^{-(d+1)},$$
4. GAUSSIAN: $$f_4(x) = \exp\Big(-\sum_{i=1}^{d} c_i^2\,(x_i - w_i)^2\Big),$$
5. CONTINUOUS: $$f_5(x) = \exp\Big(-\sum_{i=1}^{d} c_i |x_i - w_i|\Big),$$
6. DISCONTINUOUS: $$f_6(x) = \begin{cases} 0, & \text{if } x_1 > w_1 \text{ or } x_2 > w_2, \\ \exp\left(\sum_{i=1}^{d} c_i x_i\right), & \text{otherwise.} \end{cases}$$

Different test integrals were obtained by varying the parameters $c = (c_1, \ldots, c_d)$ and $w = (w_1, \ldots, w_d)$. The parameter w acts as a shift parameter, and the difficulty of the integrand is monotonically increasing with the $c_i > 0$. For $d = 10$ we require

$$\sum_{i=1}^{10} c_i = b_j \tag{10}$$

where b_j depends on the family f_j and is given by

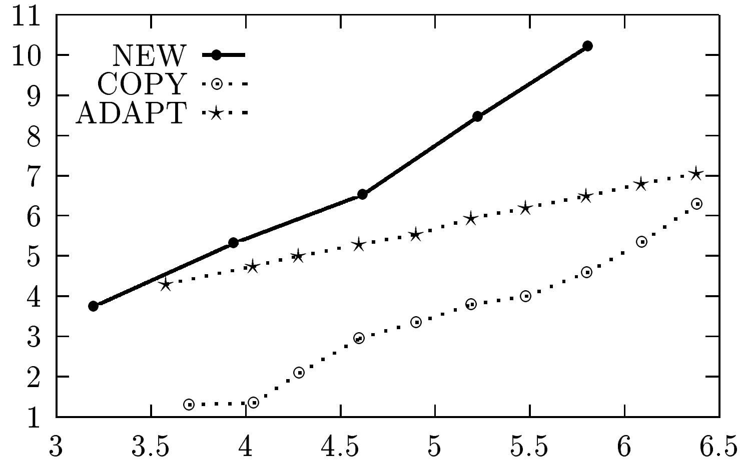
j	1	2	3	4	5	6
b_j	9.0	7.25	1.85	7.03	2.04	4.3.

We add that the values b_j correspond to the level of difficulty $L = 1$ for the families PRODUCT PEAK, GAUSSIAN and DISCONTINUOUS, and to the level $L = 2$ for OSCILLATORY, CORNER PEAK, and CONTINUOUS, see Sloan and Joe (1994).

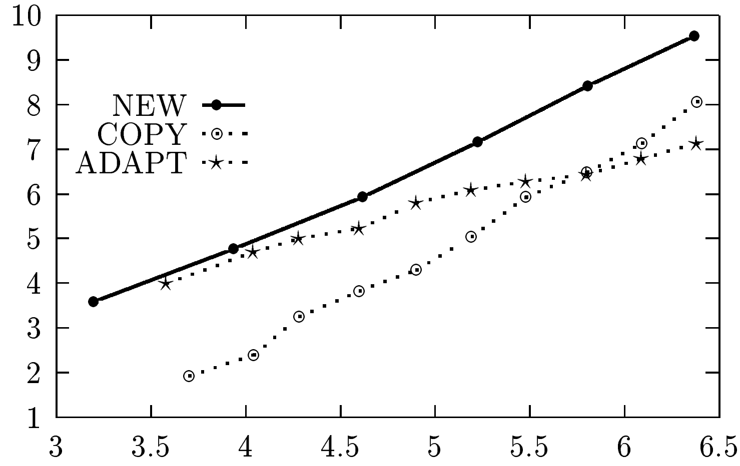
Twenty vectors w and c' were generated independently and uniformly distributed in $[0, 1]^d$. Then c' was renormalized to satisfy (10). This way we get 20 integrands for each test family in dimension $d = 10$. The integrals were computed with $A(q, 10)$ with $q = 13, \ldots, 18$. The corresponding number n of knots is 1581, 8801, 41265, 171425, 652065, and 2320385. For any n and any family f_j, we use the median of the actual number

$$-\log_{10} \frac{|I_d(f_j) - A(q,d)(f_j)|}{|I_d(f_j)|}$$

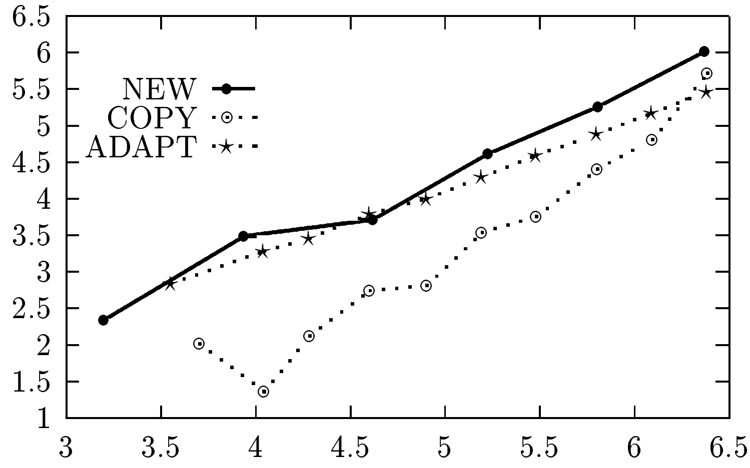
of correct digits as the measure for the accuracy of NEW. Each of the figures shows the dependence of the median on $\log_{10} n$. To allow a comparison with the methods ADAPT and COPY we also show the respective results of Sloan and Joe (1994).



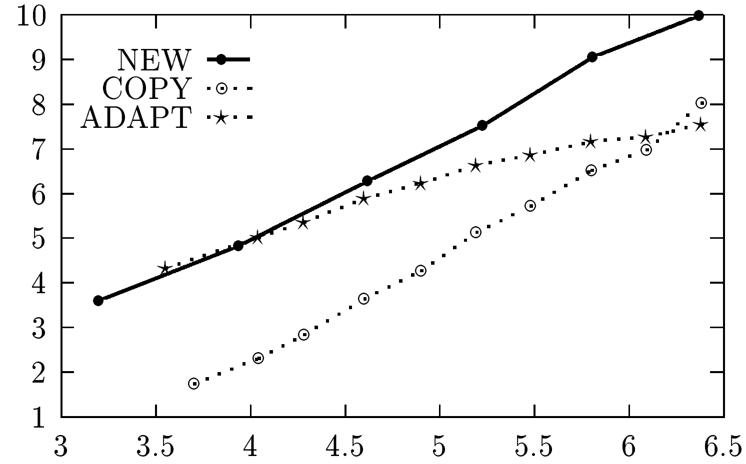
Median number of correct digits for OSCILLATORY with $d = 10$



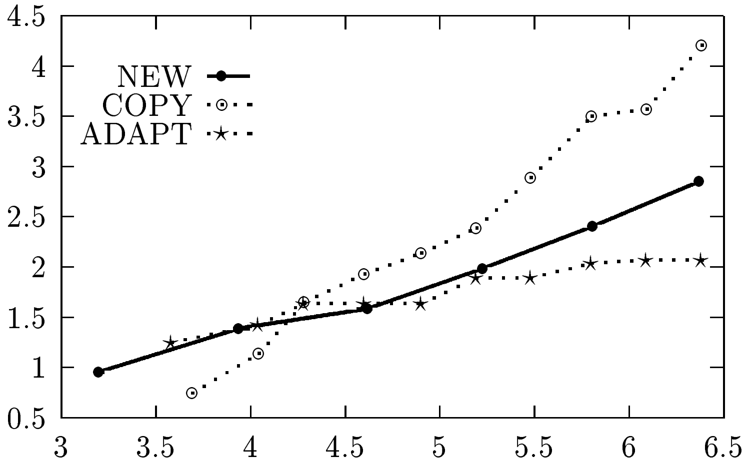
Median number of correct digits for PRODUCT PEAK with $d = 10$



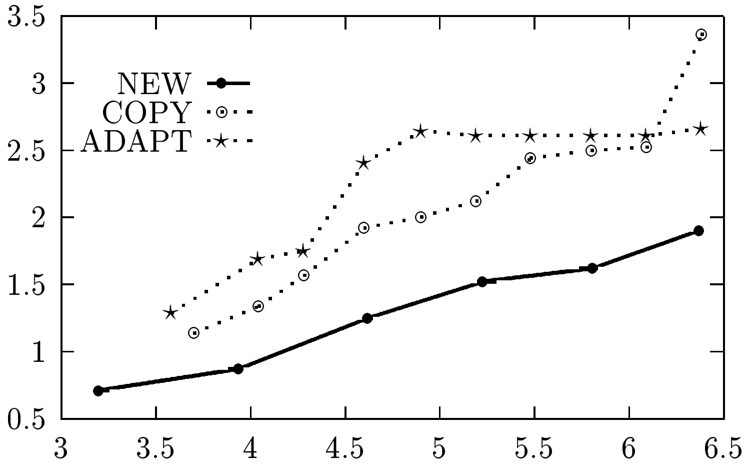
Median number of correct digits for CORNER PEAK with $d = 10$



Median number of correct digits for GAUSSIAN with $d = 10$

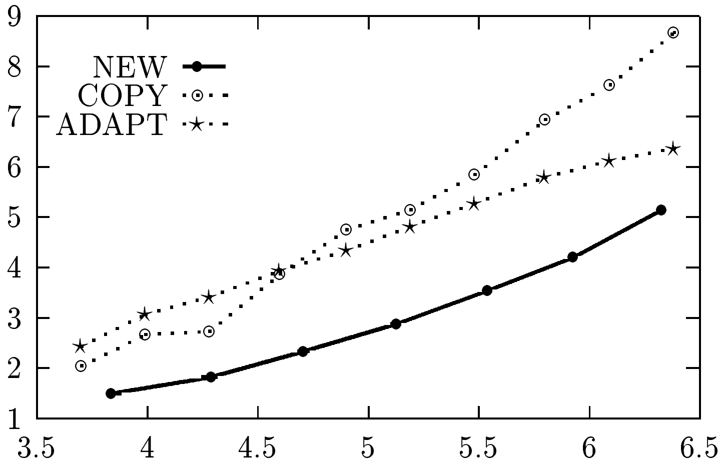


Median number of correct digits for CONTINUOUS with $d = 10$

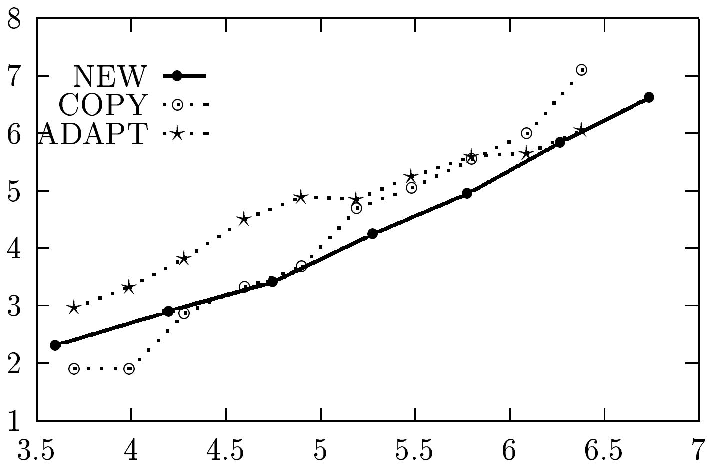


Median number of correct digits for DISCONTINUOUS with $d = 10$

For the family PRODUCT PEAK we also show the results in dimensions $d = 5$ and 8. Here the respective constant b_j is chosen as 29 for $d = 5$ and 14.32 for $d = 8$, which corresponds to the same level $L = 1$ of difficulty as for $d = 10$.



Median number of correct digits for PRODUCT PEAK with $d = 5$



Median number of correct digits for PRODUCT PEAK with $d = 8$

Furthermore, we have also tested NEW with the other examples from Sloan and Joe (1994) and with the examples from Bonk (1994). The results of our computations can be summarized as follows:

- The new method leads to good results if the integrand is smooth and the dimension d is high.

More precisely we made the following observations:

- For certain very smooth functions the results are excellent in any dimension. This can be seen with $f(x) = \exp(\sum_{i=1}^{d} x_i)$ on $[-h, h]^d$ with a relatively small h, see also Bonk (1994).

- For the families 1–4 of Genz the results of NEW were usually not as good as those of ADAPT and COPY for $d = 5$ and comparable for $d = 8$. For $d = 10$ the results of NEW were excellent and we expect that the advantage of the new method is even more impressive for $d > 10$.
- For the family 6 of discontinuous functions the results were poor in any dimension $d \leq 10$.

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