Discontinuous information in the worst case and randomized settings

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Dedicated to Hans Triebel on the occasion of his 75th birthday

Abstract

We believe that discontinuous linear information is never more powerful than continuous linear information for approximating continuous operators. We prove such a result in the worst case setting. In the randomized setting we consider compact linear operators defined between Hilbert spaces. In this case, the use of discontinuous linear information in the randomized setting cannot be much more powerful than continuous linear information in the worst case setting. These results can be applied when function evaluations are used even if function values are defined only almost everywhere.

1 Introduction

We study the approximation of an operator S defined between normed spaces F and G. The operator S does not have to be linear or continuous. We approximate $S(f)$ by algorithms that use information consisting of finitely many continuous or discontinuous linear functionals $L_i : F \to \mathbb{R}$. The error of such algorithms is defined either in the worst case or randomized setting.

For continuous S, it is hard to imagine that one can learn about $S(f)$ by using discontinuous information. On the other hand, it is well known that the Monte Carlo algorithm works nicely for multivariate integration defined for L_2-functions. This algorithm uses linear functionals given by function evaluations which are indeed discontinuous or even not always well defined. Hence, discontinuous information is actually used in computational practice and seems to be useful, at least in the randomized setting.

This is the subject of this paper. We want to verify the power of *discontinuous* linear information and compare it to the power of *continuous* linear approximation. We study the worst case and randomized settings. This is done by comparing the nth minimal (worst case and randomized) errors which we can achieve by using n discontinuous or continuous linear functionals.

In the worst case setting, we prove that as long as S is a continuous operator (not necessarily linear) then the nth minimal errors are exactly the same for the class $\widetilde{\Lambda}^{\rm all}$ of all discontinuous or continuous linear functionals and the class $\Lambda^{\rm all}$ of all continuous linear functionals, see Theorem 1. This means that the use of discontinuous linear functionals does not help. The situation is quite different if S is discontinuous. We present an easy example of a discontinuous linear functional S for which the nth minimal errors for the class $\widetilde{\Lambda}^{\rm all}$ are zero for all $n \ge 1$, whereas the nth minimal errors for the class $\Lambda^{\rm all}$ are infinity for all $n \ge 1$.

In the randomized setting, we mostly consider compact linear operators S defined between Hilbert spaces F and G. In this case, we know from [14] that the power of continuous linear functionals in the randomized setting is roughly the same as in the worst case setting, see Lemma 2. Here, the word "roughly" means that the nth minimal error in the randomized setting is at least as large as a half of the $(4n-1)$st minimal error in the worst case setting, and obviously it is at most as large as the nth minimal error in the worst case setting. By combining with the result from the worst case setting, we conclude that the power of discontinuous linear functionals in the randomized setting is roughly the same as the power of continuous linear functionals in the worst case setting. On the other hand, if we drop the assumption that S is a compact linear operator between Hilbert spaces then we can construct a problem S which is *not* solvable in the worst case setting and solvable and relatively easy in the randomized setting. Here, not solvable means that the nth minimal errors in the worst case setting do *not* converge to zero, and relatively easy means the nth minimal errors in the randomized setting are of order $n^{-1/2}$.

For many applications the class F consists of functions and we can only use function evaluations for the approximation of S. The class of such evaluations is called *standard* and denoted by $\Lambda^{\rm std}$. These evaluations are always linear but not always continuous. That is, we always have $\Lambda^{\rm std} \subseteq \widetilde{\Lambda}^{\rm all}$, and, depending on the space F, we sometimes have $\Lambda^{\rm std} \subseteq \Lambda^{\rm all}$. In either case, our results apply. In particular, if all function evaluations are discontinuous then they may be useless in the worst case setting since the minimal worst case error of any algorithm that uses n function values is as good as a constant algorithm that uses *no* function values, see Remark 2.

For some applications the space F consists of equivalence classes of functions that are equal almost everywhere. This is the case for $F_1 = L_2(D)$ for some $D \subseteq \mathbb{R}^d$. Then function evaluations are not even well defined. We extend our analysis also to such function

evaluations and show that again the same results as before hold.

2 Worst Case Setting

For arbitrary normed spaces F and G, consider an arbitrary operator $S : F \to G$ that does not have to be linear or continuous. We approximate $S(f)$ for f from the unit ball of F by algorithms that use finitely many linear functionals from $\Lambda^{\rm all}$ or from $\widetilde{\Lambda}^{\rm all}$, respectively. More precisely, we consider algorithms $A_n : F \to G$ given by

$$A_n(f) = \varphi_n(L_1(f), L_2(f), \ldots, L_n(f)), \tag{1}$$

where n is a nonnegative integer, $\varphi_n : \mathbb{R}^n \to G$ is an arbitrary mapping, and $L_j \in \Lambda$, where $\Lambda \in \{\Lambda^{\rm all}, \widetilde{\Lambda}^{\rm all}\}$. Hence, for $\Lambda = \Lambda^{\rm all}$ we only use *continuous* linear functionals, whereas for $\Lambda = \widetilde{\Lambda}^{\rm all}$ we may also use *discontinuous* linear functionals.

The choice of L_j can be *nonadaptive* or *adaptive*. It is nonadaptive if the functionals L_j are the same for all $f \in F$, and it is adaptive if L_j depends on the already computed values $L_1(f), L_2(f), \ldots, L_{j-1}(f)$. That is,

$$N(f) = (L_1(f), L_2(f), \ldots, L_n(f))$$

is the information used by the algorithm A_n and $L_j = L_j(\,\cdot\,; L_1(f), L_2(f), \ldots, L_{j-1}(f)) \in \Lambda$. If the choice of all L_j's is independent of $f \in F$ then N is *nonadaptive* information, otherwise if at least one L_j varies with $f \in F$ then N is *adaptive* information. For $n = 0$, the mapping A_n is a constant element of the space G. More details can be found in e.g., [13, 16, 18]. We define the error of such algorithms by taking the worst case setting, i.e.,

$$e(A_n) = \sup_{\|f\|_F < 1} \big\|S(f) - A_n(f)\big\|_G.$$

Observe that it is enough that the operator S is defined on the open unit ball in F, not necessarily on the whole space F. We take the open unit ball instead of the more standard case of the closed unit ball of F in the definition of the worst case error since this includes also operators with singularities on the boundary of the unit ball. For linear continuous S, or more generally for S uniformly continuous on the closed unit ball of F, this does not make a difference.

We define the nth minimal errors of approximation of S in the worst case setting as follows.

Definition 1. For $n = 0$ and $n \in \mathbb{N} := \{1, 2, \dots\}$, let

$$e_n^{\text{all}-\text{wor}}(S) = \inf_{A_n \text{ with } L_j \in \Lambda^{\text{all}}} e(A_n)$$

and

$$\widetilde{e}_n^{\text{all}-\text{wor}}(S) = \inf_{A_n \text{ with } L_j \in \widetilde{\Lambda}^{\text{all}}} e(A_n).$$

For $n = 0$, we obtain

$$e_0^{\text{all}-\text{wor}}(S) = \widetilde{e}_0^{\text{all}-\text{wor}}(S) = \inf_{g \in G} \sup_{\|f\|_F < 1} \|S(f) - g\|_G.$$

It is easy to see that the best algorithm is $A_0 = 0$ if we assume that $S(f) = -S(-f)$ for all $\|f\|_F < 1$. Then

$$e_0^{\text{all}-\text{wor}}(S) = \widetilde{e}_0^{\text{all}-\text{wor}}(S) = \sup_{\|f\|_F < 1} \|S(f)\|_G.$$

The error ${e_0}^{\text{all}-\text{wor}}(S)$ is the initial error that can be achieved without computing any linear functional on the elements $f \in F$. Clearly,

$$\widetilde{e}_n^{\text{all}-\text{wor}}(S) \le e_n^{\text{all}-\text{wor}}(S) \quad \text{for all} \quad n \in \mathbb{N}.$$

The sequences $\left\{\widetilde{e}_n^{\text{all}-\text{wor}}(S)\right\}$ and $\left\{e_n^{\text{all}-\text{wor}}(S)\right\}$ are both non-increasing but not necessarily convergent to zero.

We will use the following fact from functional analysis, see, e.g., [1, Ch. 3].

Lemma 1. Assume that F is a normed space and $L \in \widetilde{\Lambda}^{\text{all}}$ is discontinuous. Then for all real α the set $\{f \in F \mid L(f) = \alpha\}$ is dense in F.

We are ready to prove that discontinuous linear functionals do not help for the approximation of continuous operators in the worst case setting.

Theorem 1. Let F and G be normed spaces and let $S : F \to G$ be continuous. Then

$$e_n^{\text{all}-\text{wor}}(S) = \widetilde{e}_n^{\text{all}-\text{wor}}(S).$$

Proof. We may assume that $\dim(F) = \infty$ since otherwise $\widetilde{\Lambda}^{\text{all}} = \Lambda^{\text{all}}$ and there is nothing to prove. Consider arbitrary adaptive information $N = (L_1, L_2, \dots, L_n)$ with

$$L_j = L_j(\cdot; y_1, y_2, \cdots, y_{j-1}) \in \widetilde{\Lambda}^{\text{all}} \quad \text{and} \quad y_i = L_i(f; y_1, y_2, \dots, y_{i-1}) \ \text{ for } \ f \in F.$$

It is well known that the infimum of the worst case errors of algorithms A_n that use information N is given by the radius of information N,

$$r(N) = \sup_{y \in N(F)} \operatorname{rad}(\{S(f) \mid \ N(f) = y, \ \|f\|_F < 1\}), \tag{2}$$

where $\operatorname{rad}(A) = \inf_{g \in G} \sup_{a \in A} \|g - a\|$ denotes the radius of a set $A \subseteq G$, see [18]. Without loss of generality we may assume that the linear functionals $L_1, L_2, \ldots, L_n$ are linearly independent since otherwise the choice of a linear functional, say, L_j that is linearly dependent on $L_1, L_2, \ldots, L_{j-1}$ does not increase our knowledge about the element f. This implies that we may assume that $N(F) = \mathbb{R}^n$.

For $y \in N(F) = \mathbb{R}^n$ and $j = 1, 2, \ldots, n$, define

$$B_k = B_k(y) = \{f \in F \mid \ L_j(f) = y_j \ \text{ for } \ j = 1, 2, \ldots, k\},$$

Then the B_k are affine subspaces of F.

For each affine subspace B of F, we associate the uniquely determined linear subspace $\widetilde{B}$ such that $B = f + \widetilde{B}$ for any $f \in B$. It is easily seen that a linear functional on F is continuous on B if and only if it is continuous on $\widetilde{B}$. In particular,

$$\widetilde{B}_k = \widetilde{B}_k(y) = \ker L_1 \cap \ker L_2 \cap \cdots \cap \ker L_k,$$

and continuity of L_{k+1} on B_k is equivalent to continuity of L_{k+1} on $\widetilde{B}_k$.

We may now further assume for $k = 1, \ldots, n-1$ that the functional L_{k+1} satisfies the following condition:

$$\text{either } L_{k+1} \text{ is continuous on } F \ \text{ or } \ L_{k+1} \text{ is discontinuous on } B_k.$$

Indeed, assume that L_{k+1} is continuous on B_k. Then it is also continuous on $\widetilde{B}_k$. Let L be a continuous linear extension of L_{k+1} on $\widetilde{B}_k$ to the whole space F. Then $L_{k+1} - L$ is a linear functional which is 0 on $\widetilde{B}_k$, so that $\ker(L_{k+1} - L) \supset \widetilde{B}_k$. This implies that $L_{k+1} - L$ is in the span of $L_1, L_2, \ldots, L_k$, and for some numbers a_j we have $L(f) = L_{k+1}(f) + \sum_{j=1}^{k} a_j L_j(f)$ for all f. Hence knowing $L_j(f)$ for $j = 1, 2, \ldots, k$, we know $L(f)$ iff we know $L_{k+1}(f)$. This means that we can replace the functional L_{k+1} in the information N with the continuous functional L without essentially changing the information and without changing its radius.

We now define the information $N^* = (L_1^*, L_2^*, \ldots, L_n^*)$ with adaptively chosen

$$L_j^* = L_j^*(\cdot; y_1^*, y_2^*, \ldots, y_{j-1}^*) \in \Lambda^{\text{all}} \quad \text{and} \quad y_i^* = L_i^*(f; y_1^*, y_2^*, \ldots, y_{i-1}^*) \ \text{ for } \ f \in F$$

such that $r(N^*) \le r(N)$. Since N is arbitrary and N^* consists of continuous linear functionals, this will prove the theorem.

The functionals L_j^* are defined inductively. We define $L_1^* = 0$ if L_1 is discontinuous on F, and otherwise we take $L_1^* = L_1$. Observe that in the case $L_1^* = 0$ the next functional L_2^* cannot be chosen adaptively since $L_1^*(f) = 0$ for all $f \in F$. Therefore, in general, $L_2^* = L_2^*(\cdot, 0)$ will be different from $L_2 = L_2(\cdot; y_1)$, with $y_1 = L_1(f)$, even if L_2 is continuous.

Assume now inductively that for all $j = 1, 2, \ldots, k < n$, we have already defined

$$L_j^* = L_j^*(\cdot; y_1^*, y_2^*, \ldots, y_{j-1}^*) \in \Lambda^{\rm all} \quad \text{with} \quad y_j^* = L_j^*(f; y_1^*, y_2^*, \ldots, y_{j-1}^*) \ \text{ for all } \ f \in F.$$

Let $L_{k+1} = L_{k+1}(\cdot; y_1^*, y_2^*, \ldots, y_k^*) \in \widetilde{\Lambda}^{\rm all}$ be the next linear functional for the original information N. Let $B_k = B_k(y^*)$ be defined as above for $y = y^*$. Define also

$$A_k = A_k(y^*) = \{f \in F \mid \ L_j^*(f) = y_j^* \ \text{ for } \ j = 1, 2, \ldots, k\}.$$

If L_{k+1} is continuous, we set $L_{k+1}^* = L_{k+1}$, if L_{k+1} is discontinuous on B_k, we define $L_{k+1}^* = 0$. In the latter case only $y_{k+1}^* = 0$ needs to be further considered. As shown above, this completes the definition of $N^* = (L_1^*, L_2^*, \ldots, L_n^*)$ that consists of n continuous linear functionals L_j^*.

We now show inductively that $B_j \subseteq A_j$ and B_j is dense in A_j for all $j = 1, 2, \ldots, n$.

Indeed, for $j = 1$ we have $B_1 = A_1$ if L_1 is continuous on F, and

$$B_1 = \{f \in F \mid \ L_1(f) = 0\} \subseteq A_1 = F$$

if L_1 is discontinuous on F. From Lemma 1 we know that B_1 is dense in A_1.

Assume now that $B_j \subseteq A_j$ and B_j is dense in A_j for $j = 1, 2, \ldots, k < n$ with $k \geq 1$. Consider first the case when L_{k+1} is continuous. Then we have $L_{k+1}^* = L_{k+1}$ and

$$\begin{aligned} A_{k+1} &= \{f \in A_k \mid \ L_{k+1}^*(f) = y_{k+1}^*\} \\ B_{k+1} &= \{f \in B_k \mid \ L_{k+1}(f) = y_{k+1}^*\} = \{f \in B_k \mid \ L_{k+1}^*(f) = y_{k+1}^*\}. \end{aligned}$$

Hence

$$A_{k+1} = A_k \cap C_k \quad \text{and} \quad B_{k+1} = B_k \cap C_k \quad \text{with} \quad C_k = \{f \in F \mid \ L_{k+1}^*(f) = y_{k+1}^*\}.$$

Clearly, $B_k \subseteq A_k$ implies that $B_{k+1} \subseteq A_{k+1}$. We need to show that if B_k is dense in A_k then $B_k \cap C_k$ is dense in $A_k \cap C_k$. This is obvious if $L_{k+1}^* = 0$. Assume then that $L_{k+1}^* \neq 0$. Take $f \in A_k \cap C_k$. Then for any positive ε there exists $f_\varepsilon \in B_k$ such that $\|f - f_\varepsilon\| \leq \varepsilon$. For $g_{k+1} \in F$ with $L_j(g_{k+1}) = 0$ for $j = 1, 2, \ldots, k$ and $L_{k+1}^*(g_{k+1}) = 1$, define

$$\tilde{f}_\varepsilon = f_\varepsilon + \left(y_{k+1}^* - L_{k+1}^*(f_\varepsilon)\right) \, g_{k+1}.$$

Then $\tilde{f}_\varepsilon \in B_k \cap C_k$ and since $L^*_{k+1}(f) = y^*_{k+1}$ we have

$$\|f - \tilde{f}_\varepsilon\| \le \|f - f_\varepsilon\| + |L^*_{k+1}(f) - L^*_{k+1}(f_\varepsilon)|\,\|g_{k+1}\| \le \|f - f_\varepsilon\|\left(1 + \|L^*_{k+1}\|\,\|g_{k+1}\|\right).$$

Hence $B_k \cap C_k$ is dense in $A_k \cap C_k$, as needed.

Consider now the case when L_{k+1} is discontinuous on B_k. Then $L^*_{k+1} = 0$ and $y^*_{k+1} = 0$. We now have

$$A_{k+1} = A_k \quad \text{and} \quad B_{k+1} = B_k \cap C_k \quad \text{with} \quad C_k = \{f \in F \mid \ L_{k+1}(f) = 0\}.$$

Clearly $B_{k+1} \subseteq B_k \subseteq A_k = A_{k+1}$. Since L_{k+1} is discontinuous on B_k it is also discontinuous on F. Then Lemma 1 says that C_k is a linear subspace which is dense in F. We want to show that $B_k \cap C_k$ is dense in $A_k = A_{k+1}$. Similarly as before, we take $f \in A_k$. For any positive ε we can find $f_\varepsilon \in B_k$ such that $\|f - f_\varepsilon\| \le \varepsilon$. If $L_{k+1}(f_\varepsilon) = 0$ then $f_\varepsilon \in B_k \cap C_k$ and we are done. Assume then that $L_{k+1}(f_\varepsilon) \neq 0$. We now choose

$$g_{k+1} \in \widetilde{B}_k = \ker L_1 \cap \ker L_2 \cdots \cap \ker L_k \quad \text{with} \quad L_{k+1}(g_{k+1}) = 1.$$

Since L_{k+1} is discontinuous on B_k, it is also discontinuous on $\widetilde{B}_k$, so the element g_{k+1} can be of an arbitrary small norm. We choose a nonzero g_{k+1} such that $\|g_{k+1}\| \le \varepsilon/|L_{k+1}(f_\varepsilon)|$. Then

$$\tilde{f}_\varepsilon = f_\varepsilon - L_{k+1}(f_\varepsilon)\, g_{k+1}$$

belongs to $B_k \cap C_k$ and

$$\|f - \tilde{f}_\varepsilon\| \le \|f - f_\varepsilon\| + |L_{k+1}(f_\varepsilon)|\,\|g_{k+1}\| \le 2\varepsilon.$$

Hence, $B_k \cap C_k$ is dense in A_k, as needed. This completes the proof that $B_n = N^{-1}(y^*)$ is dense in $A_n = (N^*)^{-1}(y^*)$.

Due to continuity of S, the set $B(y^*) := \{S(f) \in G \mid N(f) = y^*,\ \|f\|_F < 1\}$ is dense in $A(y^*) := \{S(f) \in G \mid N^*(f) = y^*,\ \|f\|_F < 1\}$ and therefore $\mathrm{rad}(A(y^*)) = \mathrm{rad}(B(y^*))$. This holds for all $y^* \in N^*(F)$ and therefore $r(N^*) \le r(N)$, as needed. □

The assumption on continuity of S in Theorem 1 is needed. Indeed, assume that $\dim(F) = \infty$. Then there are discontinuous linear functionals $L : F \to \mathbb{R}$. Define $S = L$. Note that now $e_0^{\mathrm{all-wor}}(S) = \widetilde{e}_0^{\mathrm{all-wor}}(S) = \infty$.

Clearly, the worst case error $A_1(f) = L(f) = S(f)$ is zero. Therefore $\widetilde{e}_n^{\mathrm{all-wor}}(S) = 0$ for all $n \ge 1$. On the other hand, we know that adaption does not help for linear functionals as proved by Bakhvalov. Furthermore, if we use n linear continuous nonadaptive functionals L_j then Smolyak's theorem tells us that the best φ in (1) is linear, i.e., there are some real

numbers a_j for which the algorithm $A_n(f) = \sum_{j=1}^n a_j L_j(f)$ minimizes the worst case error among all algorithms that use $N = (L_1, L_2, \ldots, L_n)$. The results of Bakhvalov and Smolyak can be found in [13, 16, 17, 18]. However, $S(f) - A_n(f)$ is still a discontinuous linear functional and therefore its worst case error is infinite. Since this holds for all continuous N, we have $e_n^{\text{all-wor}}(S) = \infty$. Hence,

$$\widetilde{e}_n^{\text{all-wor}}(S) = 0 < e_n^{\text{all-wor}}(S) = \infty \quad \text{for all} \quad n \in \mathbb{N}.$$

Although Theorem 1 deals with adaptive information, it is known that adaptive information does *not* help for many problems. This holds for linear operators S defined over Hilbert spaces F or if S is a linear functional, whereas for linear operators defined over arbitrary normed spaces adaption may help at most by a factor of two. The reader may find a survey of such results in Chapter 4 of [16].

Remark 1. Seeing the proof of Theorem 1, we may think that the use of discontinuous linear functionals is useless for approximating continuous S. More precisely, assume that we use nonadaptive $N = (L_1, L_2, \ldots, L_n)$ for which *all* L_j's are discontinuous linear functionals. What is the radius of N? Is it the same as the radius of zero information? This is not true. We now show that we can achieve the radius of nonadaptive information consisting of $n-1$ *continuous* linear functionals. Indeed, let $N^* = (L_2^*, L_3^*, \ldots, L_n^*)$ be nonadaptive continuous information, $L_j^* \in \Lambda^{\text{all}}$. Take $N = (L_1, L_1 + L_2^*, \ldots, L_1 + L_n^*)$ with a discontinuous linear functional L_1. Then all $L_1 + L_j^*$'s are discontinuous. However, if we compute $y_1 = L_1(f)$ and $y_j = L_1(f) + L_j^*(f)$ then we also know $L_j^*(f) = y_j - y_1$ for $j = 2, 3, \ldots, n$. Hence, we know $N^*(f)$ and therefore $r(N) \le r(N^*)$, as claimed.

It is interesting to see what happens if we apply the proof of Theorem 1 to N. Since L_1 is discontinuous we obtain $L_1^* = 0$. However, $L_1 + L_2^*$ on $B_1 = \{f \in F \mid \ L_1(f) = y_1\}$ is $y_1 + L_1^*$ and therefore it is *continuous*. Then we replace L_2 by L_2^*. Similarly, all L_j with $j \ge 2$ will be replaced by L_j^*. The proof of Theorem 1 shows that $r(N) = r(N^*)$.

Remark 2. Assume now that F is a space of functions $f : D \to \mathbb{R}$. We consider the class Λ^{std} of all function evaluations given by the linear functionals $L_x(f) = f(x)$ for $f \in F$. Let $e_n^{\text{std-wor}}(S)$ denote the nth minimal worst case errors of algorithm A_n with $L_j \in \Lambda^{\text{std}}$, i.e., algorithms that use at most n function evaluations.

Assume that L_x is *discontinuous* on

$$F \cap \ker L_{x_1} \cap \ker L_{x_2} \cap \ldots \cap \ker L_{x_n}$$

for all $x \in D \setminus \{x_1, x_2, \ldots, x_n\}$. For instance $F = L_2(D) \cap C(D)$ equipped with the $L_2(D)$ norm is such an example.

Then it is easy to check that the proof of Theorem 1 yields $L_j^* = 0$ for all j. That is,

$$e_n^{\mathrm{std-wor}}(S) = e_0^{\mathrm{std-wor}}(S) = e_0^{\mathrm{all-wor}}(S).$$

Hence, in this case the use of function evaluations is completely useless.

However, similarly as in Remark 1, one can construct examples where all function evaluations are *discontinuous* on F but may be *continuous* on $F \cap \ker L_{x_1} \cap \ker L_{x_2} \cap \ldots \cap \ker L_{x_n}$, and still they are useful. Indeed, consider

$$F = C([0,1]) \cap \left\{ f \;\middle|\; \int_0^1 f(x)\,\mathrm{d}x = \frac{f(0)+f(1)}{2} \right\}$$

equipped with the L_2 norm. Then the integration problem $S(f) = \int_0^1 f(x)\,\mathrm{d}x$ is continuous and all function evaluations are discontinuous on F. Nevertheless, $L_1(f) = f(1) = S(f)/2$ on $F \cap \ker L_0$ is continuous, and we can compute $S(f)$ exactly using two function values of f.

Remark 3. For a continuous linear S and a Banach space F, Theorem 1 can be proved modulo a factor of $\frac{1}{2}$ by using the known relations between the Gelfand numbers $c_n(S)$ and the minimal errors $e_n^{\mathrm{all-wor}}(S)$, see [16] for a survey of related results. In particular, we use that

$$c_n(S) \le e_n^{\mathrm{all-wor}}(S) \le 2c_n(S),$$

for any linear and continuous operator S, see [18, Section 5.4 of Chapter 4].

It is known, see [2, Prop. 2.7.5], that the Gelfand numbers $c_n(S)$ are local in the sense that

$$c_n(S) = \sup_M c_n(S|_M)$$

where the supremum is taken over all finite dimensional subspaces M and $S|_M$ is the restriction of S to M.

Altogether, we obtain the following inequality:

$$\begin{aligned} e_n^{\mathrm{all-wor}}(S) &\le 2c_n(S) = 2\sup_M c_n(S|_M) \le 2\sup_M e_n^{\mathrm{all-wor}}(S|_M) = 2\sup_M \widetilde{e}_n^{\mathrm{all-wor}}(S|_M) \\ &\le 2\widetilde{e}_n^{\mathrm{all-wor}}(S), \end{aligned}$$

i.e.

$$e_n^{\mathrm{all-wor}}(S) \le 2 \cdot \widetilde{e}_n^{\mathrm{all-wor}}(S).$$

We return to general continuous not necessarily linear operators S. We conjecture that the worst case error $e_n^{\mathrm{all-wor}}(S)$ is itself a local quantity at least for compact operators S, i.e.,

$$e_n^{\mathrm{all-wor}}(S) = \sup_M e_n^{\mathrm{all-wor}}(S|_M), \tag{3}$$

where again the supremum is taken over all finite dimensional subspaces M of F, and $S|_M$ is the restriction of S to M. This would give another and a much shorter proof of Theorem 1. Indeed, (3) implies

$$e_n^{\mathrm{all-wor}}(S) = \sup_M e_n^{\mathrm{all-wor}}(S|_M) = \sup_M \widetilde{e}_n^{\mathrm{all-wor}}(S|_M) \le \widetilde{e}_n^{\mathrm{all-wor}}(S),$$

as claimed.

Although we do not know if (3) holds, we easily conclude from the local property of the Gelfand numbers that at least the weak local property holds for a continuous linear S and a Banach space F, namely

$$e_n^{\mathrm{all-wor}}(S) \le c_n(S) = \sup_M c_N(S|_M) \le 2 \sup_M e_n^{\mathrm{all-wor}}(S|_M).$$

It is also known that the approximation numbers $a_n(S)$ are local as long as S is a compact operator or G is a dual space, see [2, Prop. 2.7.1 and 2.7.3]. That translates into the fact that for linear nonadaptive algorithms in the worst case setting, the use of discontinuous functionals does not help. A weak form (with a factor 5) is true for arbitrary G and S, see [2, Prop. 2.7.4].

Remark 4. Heinrich [3] proves relations between linear n-widths and approximation numbers and shows that they coincide for compact and absolutely convex subsets of a normed space. There is also an example showing that, in general, for relatively compact absolutely convex sets equality does not hold. The spirit of these results is similar, but there is a difference as can be seen from Proposition 1.3 of that paper: The aim is to compare general or continuous linear information applied to $g = S(f) \in G$, while we compare general or continuous linear information applied to $f \in F$.

3 Randomized Setting

We now deal with randomized algorithms. We consider, as in [16, Theorem 4.42], only measurable algorithms. Hence we use the following definitions.

A randomized algorithm A is a pair consisting of a probability space (Ω, Σ, μ) and a family $(N_\omega, \varphi_\omega)_{\omega\in\Omega}$ of mappings such that the following holds:

1. For each fixed $\omega \in \Omega$, the mapping $A_\omega = \varphi_\omega \circ N_\omega$ is a deterministic algorithm defined as before, based on adaptive information N_ω consisting of linear functionals from a class Λ.

2. Let $n(f, \omega)$ be the cardinality of the information N_ω for $f \in F$. We assume that the function n is measurable.

3. The mapping $(f, \omega) \mapsto \varphi_\omega(N_\omega(f)) \in G$ is measurable.

Let A be a randomized algorithm. Then the cardinality of A is defined as

$$n(A) = \sup_{\|f\|_F \le 1} \int_\Omega n(f, \omega) \, \mathrm{d}\mu(\omega),$$

whereas the error of A in the randomized setting is

$$e^{\mathrm{ran}}(A) = \sup_{\|f\|_F \le 1} \left(\int_\Omega \|S(f) - \varphi_\omega(N_\omega(f))\|^2 \, \mathrm{d}\mu(\omega) \right)^{1/2}.$$

For $n \in \mathbb{N}$, we define the nth minimal error of S in the randomized setting by

$$e_n^{\mathrm{all-ran}}(S) = \inf\{e^{\mathrm{ran}}(A) \, : \, n(A) \le n\},$$

if A uses linear functionals from $\Lambda = \Lambda^{\mathrm{all}}$. Similarly we define $\widetilde{e}_n^{\,\mathrm{all-ran}}(S)$ with $\Lambda = \widetilde{\Lambda}^{\mathrm{all}}$.

Obviously, we can interpret deterministic algorithms as randomized with a singleton Ω. That is why the nth minimal errors in the randomized setting cannot be larger than the nth minima errors in the worst case setting,

$$\widetilde{e}_n^{\,\mathrm{all-ran}}(S) \le \widetilde{e}_n^{\,\mathrm{all-wor}}(S) \quad \text{and} \quad e_n^{\mathrm{all-ran}}(S) \le e_n^{\mathrm{all-wor}}(S).$$

Basically, there exists only one proof technique to obtain lower bounds for randomized algorithms, and this technique goes back to Bakhvalov, see Section 4.3.3 of [16]. The main point is to observe that the errors in the randomized setting cannot be smaller than the errors in the average case setting for an arbitrary probability measure on F.

If we restrict ourselves to measurable randomized algorithms, then

$$e_n^{\mathrm{all-ran}}(S) \ge \tfrac{\sqrt{2}}{2} \, e^{\mathrm{avg}}(2n, \varrho), \tag{4}$$

where $e^{\mathrm{avg}}(2n, \varrho)$ denotes the $2n$-th minimal average case error of deterministic algorithms that use at most $2n$ linear functionals from Λ^{all} and ϱ is an arbitrary (Borel) probability measure on F, for more details see Lemma 4.37 and Remark 4.41 of [16]. The next result is identical with Theorem 4.42 from [16], see also [14]. It is stated here with a proof since we need a small modification of this result.

Lemma 2. Assume that $S : F \to G$ is a compact linear operator between Hilbert spaces F and G. Then

$$\tfrac{1}{2}\, e_{4n-1}^{\mathrm{all-wor}}(S) \le e_n^{\mathrm{all-ran}}(S).$$

Proof. We know that $S(e_i) = \sigma_i \tilde{e}_i$ with orthonormal $\{e_i\}$ in F and $\{\tilde{e}_i\}$ in G. Here, $\{\sigma_i\}$ is a sequence of non-increasing singular values σ_i of S and $\lim_i \sigma_i = 0$. We also know that $e_n^{\mathrm{all-wor}}(S) = \sigma_{n+1}$. For $m > n$, consider the normed $(m-1)$-dimensional Lebesgue measure ϱ_m on the unit sphere $E_m = \{\sum_{i=1}^m \alpha_i e_i \,:\, \alpha_i \in \mathbb{R},\ \sum_{i=1}^m \alpha_i^2 = 1\}$. Then

$$A_n^*\left(\sum_{i=1}^{\infty} \alpha_i e_i\right) = \sum_{i=1}^{n} \sigma_i \alpha_i \tilde{e}_i$$

is the optimal algorithm using continuous linear information of cardinality n. This is true for the worst case setting, with error σ_{n+1}, as well as for the average case setting with respect to ϱ_m. Hence

$$e^{\mathrm{avg}}(n, \varrho_m)^2 = \int_{E_m} \sum_{i=n+1}^{m} \sigma_i^2 \alpha_i^2 \, d\varrho_m(\alpha).$$

Since $\int_{E_m} \alpha_i^2 \, d\varrho_m(\alpha) = 1/m$ we obtain

$$e^{\mathrm{avg}}(n, \varrho_m)^2 = \frac{1}{m} \sum_{i=n+1}^{m} \sigma_i^2.$$

If we put $m = 2n$ then we obtain

$$e^{\mathrm{avg}}(n, \varrho_{2n}) \ge \tfrac{1}{2}\sqrt{2}\, \sigma_{2n}.$$

Together with (4), we obtain

$$e_n^{\mathrm{all-ran}}(S) \ge \tfrac{1}{2}\sqrt{2}\, e^{\mathrm{avg}}(2n, \varrho_{4n}) \ge \tfrac{1}{2}\, \sigma_{4n} = \tfrac{1}{2}\, e_{4n-1}^{\mathrm{all-wor}}(S).$$

□

We stress that in the proof we only use finite dimensional subspaces of F and linear functionals on such subspaces. Of course, for finite dimensional spaces, we have $\widetilde{\Lambda}^{\mathrm{all}} = \Lambda^{\mathrm{all}}$ and therefore we obtain the following result.

Theorem 2. Assume that $S : F \to G$ is a compact operator between Hilbert spaces. Then

$$\tfrac{1}{2}\, e_{4n-1}^{\mathrm{all-wor}}(S) \le \widetilde{e}_n^{\mathrm{all-ran}}(S) \le e_n^{\mathrm{all-wor}}(S).$$

In this sense, randomization as well as allowing discontinuous linear functionals from $\widetilde{\Lambda}^{\rm all}$ does not (essentially) help for the approximation of compact operators between Hilbert spaces.

Remark 5. Assume that $S : F \to G$ is linear and F and G are normed spaces. We do not know whether

$$\lim_{n\to\infty} e_n^{\rm all-ran}(S) = 0$$

implies that S is compact. It is shown in [10] that the embedding $I : \ell_1 \to \ell_\infty$ is a universal non-compact operator in the sense that it factors through any non-compact linear bounded operator $S : F \to G$ between Banach spaces F and G:

$$\begin{array}{ccc} F & \xrightarrow{S} & G \\ {\scriptstyle V}\uparrow & & \downarrow{\scriptstyle U} \\ \ell_1 & \xrightarrow{I} & \ell_\infty \, . \end{array}$$

Here, $I = USV$ for some linear bounded operators U and V. It follows that

$$e_n^{\rm all-ran}(I) = e_n^{\rm all-ran}(USV) \le \|U\|\, e_n^{\rm all-ran}(S)\, \|V\|.$$

Thus it is sufficient to decide whether

$$\lim_{n\to\infty} e_n^{\rm all-ran}(I) > 0 \quad \text{or} \quad \lim_{n\to\infty} e_n^{\rm all-ran}(I) = 0.$$

Remark 6. Also in the randomized setting it would be very interesting to know whether the error $e_n^{\rm all-ran}(S)$ is a local quantity, i.e. whether

$$e_n^{\rm all-ran}(S) = \sup_M e_n^{\rm all-ran}(S|_M),$$

or at least

$$e_n^{\rm all-ran}(S) \le c \sup_M e_n^{\rm all-ran}(S|_M)$$

for some constant c independent of n. This would lead to an analogue of Theorem 1 in the randomized setting.

Remark 7. It is interesting to mention that there is a continuous *nonlinear* operator $S : F \to G$ for normed spaces F and G which is solvable in the randomized setting but not in the worst case setting, i.e.,

$$\lim_n \widetilde{e}_n^{\rm all-wor}(S) = \lim_n e_n^{\rm all-wor}(S) > 0 \qquad \text{and} \qquad \lim_n e_n^{\rm all-ran}(S) = 0.$$

Obviously, the first equality above follows from Theorem 1. Indeed, let

$$S : F := \ell_1 \to G := \mathbb{R}, \qquad S(x) = \|x\|_2^2 \quad \text{for all} \quad x \in F.$$

Clearly, S is a continuous nonlinear functional. Note that the constant algorithm $\frac{1}{2}$ has the worst case error $\frac{1}{2}$ since we have $S(x) \in [0,1]$ for x from the unit ball of ℓ_1. Let

$$c := \lim_{n\to\infty} e_n^{\text{all}-\text{wor}}(S).$$

Then $c \le \frac{1}{2}$. We now show that $c > 0$. We will use the known result of Kashin about the Gelfand width $c_n(B_1^m, \ell_2^m)$ for the unit ball B_1^m of $x \in \mathbb{R}^m$ with $\|x\|_1 \le 1$, and with the error measured in the ℓ_2 norm. Namely, there are two positive numbers $c_{1,2}$ and $C_{1,2}$ such that for all $n < m$ we have

$$c_{1,2}\,\min\left(1, \frac{\ln(m/n)+1}{n}\right)^{1/2} \le c_n(B_1^m, \ell_2^m) \le C_{1,2}\,\min\left(1, \frac{\ln(m/n)+1}{n}\right)^{1/2}.$$

This means that $\lim_{m\to\infty} c_n(B_1^m, \ell_2) \ge c_{1,2}$. Using the definition of the Gelfand width, we conclude that

$$\lim_{n\to\infty} \inf_{L_1,L_2,\ldots,L_n \in \widetilde{\Lambda}^{\text{all}}} \sup_{x\in\ell_1,\ L_j(x)=0,\, j=1,2,\ldots,n,\ \|x\|_1\le 1} \|x\|_2 \ge c_{1,2}.$$

Take now arbitrary adaptive $N = (L_1, L_2, \ldots, L_n)$ with linear functionals L_i. Then there exists x in the unit ball of ℓ_1 such that $L_j(x) = 0$ for $j = 1, 2, \ldots, n$ and $\|x\|_2 \ge c_{1,2}/2$. Let $\alpha \in [0,1]$. Then $\alpha\, x$ belongs to the unit ball of ℓ_1 and $N(\alpha\, x) = 0$. For an arbitrary algorithm $A = \varphi_n(N(\cdot))$ we have $A_n(\alpha\, x) = \varphi_n(0)$ and $S(\alpha\, x) - \varphi_n(0) = \alpha\|x\|_2^2 - \varphi_n(0)$. Therefore

$$e(A_n) \ge \max_{\alpha\in[0,1]} |\alpha\|x\|_2^2 - \varphi_n(0)| \ge \tfrac{1}{2}\|x\|_2^2 \ge \tfrac{1}{4}c_{1,2}.$$

Hence $e_n^{\text{all}-\text{wor}}(S) \ge \frac{1}{4}c_{1,2}$ for all n. This proves that $c > 0$, as claimed.

In the randomized setting, consider the (random) linear functional

$$L(x) = \sum_{k=1}^{\infty} (\pm)\, x_k$$

with random and independent signs of probability $\frac{1}{2}$. The variance of the random variable $L(x)$ is $\sigma^2(L(x)) = \|x\|_2^2$ and it can be easily estimated with independent copies L_i of L. It is well known that the "empirical variance"

$$A_n(x) = \frac{1}{n-1}\sum_{i=1}^{n}\left(L_i - \sum_{k=1}^{n} L_k\right)^2$$

with independent copies L_i of L has expectation $S(x) = \|x\|_2^2$ and variance

$$\sigma^2(A_n(x)) \le \frac{1}{n} \cdot \|x\|_4^4 \le \frac{1}{n} \cdot \|x\|_1^4.$$

Therefore

$$e^{\rm ran}(A_n) = \sup_{\|x\|_{\ell_1}<1} \sigma(A_n(x)) \le \frac{1}{\sqrt{n}},$$

and $\lim_n e_n^{\rm all-ran}(S) = 0$, as claimed.

4 Function Values

So far we did not assume that F is a space of functions and we only compared continuous linear information with arbitrary linear information. Now we assume that F is a normed space of functions $f : D \to \mathbb{R}$ for some nonempty set D.

Let $L_x(f) = f(x)$ for all $f \in F$ and $x \in D$. Since $f(x)$ is well defined for all $f \in F$ and $x \in D$, the functionals L_x's are linear but *not* necessarily continuous. This class of information is called *standard* and denoted by $\Lambda^{\rm std}$. Obviously $\Lambda^{\rm std} \subseteq \widetilde{\Lambda}^{\rm all}$, however, $\Lambda^{\rm std} \subseteq \Lambda^{\rm all}$ only if all L_x's are continuous. Still Theorems 1 and 2 apply in this case.

We now consider a more general case when F is a space of equivalence classes of functions $f : D \to \mathbb{R}$. A major example is $F = L_2(D)$. Then L_x is not even well defined for $f \in F$. On the other hand, we know that for $F = L_2(D)$ the functional $L_x(\tilde{f}) = \tilde{f}(x)$ is well defined for each $\tilde{f} \in f$ and all x from D. Here $\tilde{f} \in f$ means that the well defined function $\tilde{f}$ is in the equivalence class f. This type of information is successfully used for multivariate integration by the standard Monte Carlo algorithm M_n in the randomized setting. Here for $D = [0,1]^d$, we have $M_n(f) = n^{-1} \sum_{j=1}^n f(x_j)$ for independent and uniformly distributed points x_j. Then

$$M_n(\tilde{f}_1) = M_n(\tilde{f}_2) \qquad a.s.$$

if $\tilde{f}_1, \tilde{f}_2 \in f$ and one usually uses only algorithms with this property. We would like to extend the analysis presented in the previous section also for the case when the elements of F are equivalence classes of functions.

We argue as follows. As in Section 3, we assume that $S : F \to G$ is a compact linear operator between the Hilbert spaces F and G. We know that then

$$S(e_i) = \sigma_i e_i'$$

with a non-increasing sequence of singular values σ_i of S, $\lim_i \sigma_i = 0$, with orthonormal $\{e_i\}$ in F and orthonormal $\{e_i'\}$ in G. We also know that $e_n^{\mathrm{all-wor}}(S) = \sigma_{n+1}$. Then

$$A_n^*\left(\sum_{i=1}^{\infty} \alpha_i e_i\right) = \sum_{i=1}^{n} \sigma_i \alpha_i e_i'$$

is an optimal algorithm using continuous linear information of cardinality n. This is also true if we replace F by the $n+k$ dimensional space $V_{n+k} = \mathrm{span}(e_1, e_2, \dots e_{n+k})$. Hence

$$\widetilde{e}_n^{\,\mathrm{all-wor}}(V_{n+k}) = e_n^{\mathrm{all-wor}}(V_{n+k}) = \sigma_{n+1} = e_n^{\mathrm{all-wor}}(S) \quad \text{for all} \quad k \geq 1.$$

Suppose that the functions $\tilde{e}_i$ are elements of the equivalence classes e_i, for all $i = 1, 2, \dots, n+k$. Then we have functions in $\widetilde{V}_{n+k} = \mathrm{span}(\tilde{e}_1, \tilde{e}_2, \dots \tilde{e}_{n+k})$ that are well defined everywhere. With this assumption we only make the oracle more powerful, i.e., the lower bound is even stronger. In this sense we can think of Λ^{std} as a subset of $\widetilde{\Lambda}^{\mathrm{all}}$. By $e_n^{\,\mathrm{std-wor}}(S)$ we denote the n minimal worst case errors of algorithms that use at most n function values for approximating S over $\widetilde{V}_{n+k}$. We obtain the following corollary.

Corollary 1. Assume that $S : F \to G$ is a compact linear operator between Hilbert spaces F and G. Then

$$e_n^{\,\mathrm{std-wor}}(S) \geq e_n^{\mathrm{all-wor}}(S) \quad \text{for all} \quad n \in \mathbb{N}.$$

The same can be said for randomized algorithms. In this case we take $k \geq 3n$, update the definition of $e_n^{\,\mathrm{std-ran}}(S)$ and obtain the following corollary.

Corollary 2. Assume that $S : F \to G$ is a compact linear operator between Hilbert spaces F and G. Then

$$e_n^{\mathrm{std-ran}}(S) \geq \tfrac{1}{2}\, e_{4n-1}^{\mathrm{all-wor}}(S).$$

Remark 8. Note that we did *not* compare $e_n^{\mathrm{std-ran}}(S)$ with $e_n^{\mathrm{std-wor}}(S)$. We stress that randomization may help a lot for the class Λ^{std}. This holds if function evaluations are continuous and also if function evaluations are not continuous. Examples for both cases can be found in Chapter 17 of [17].

A major example is the embedding of a function space into another (larger) function space. The literature is very rich, see, e.g., [4, 5, 6, 7, 8, 11, 12, 15, 16, 19, 20]. One can study the classes Λ^{all}, $\widetilde{\Lambda}^{\mathrm{all}}$ as well as Λ^{std} in the worst case setting and in the randomized setting. In the randomized setting we do *not* know whether Λ^{all} and $\widetilde{\Lambda}^{\mathrm{all}}$ always lead to the same results since in Theorem 2 we assume that both F and G are Hilbert spaces. It is open what happens for general normed spaces F and G and if an analogue of Theorem 1 for the worst case setting also holds in the randomized setting.

We also add that the lower bounds in the randomized setting of Heinrich and Mathé for specific spaces F and G are valid not only for $\Lambda^{\rm all}$ but also for $\widetilde{\Lambda}^{\rm all}$, see [6, 7, 8, 11]. The reason is similar as above in the proof of Theorem 2. Namely, finite dimensional subspaces of F can be used for the lower bounds and here all linear functionals are continuous.

Added in proof: Stefan Heinrich [9] recently proved that (3) is true for compact operators, but not in general.

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