SMOLYAK'S CONSTRUCTION OF CUBATURE FORMULAS OF ARBITRARY TRIGONOMETRIC DEGREE

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ABSTRACT. We study cubature formulas for d-dimensional integrals with a high trigonometric degree. To obtain a trigonometric degree ℓ in dimension d, we need about $d^\ell/\ell!$ function values if d is large. Only a small number of arithmetical operations is needed to construct the cubature formulas using Smolyak's technique. We also compare different methods to obtain formulas with high trigonometric degree.

1. INTRODUCTION

We study cubature formulas

$$Q_n(f) = \sum_{i=1}^{n} a_i \, f(\mathbf{x}_i)$$

for the approximation of d-dimensional integrals

$$I_d(f) = \int_{[0,\,1]^d} f(\mathbf{x}) \, d\mathbf{x}. \tag{1}$$

A trigonometric monomial in the variable $\mathbf{x} = (x_1, x_2, \dots, x_d)$ is a function of the form

$$f : \mathbb{R}^d \to \mathbb{C} : \mathbf{x} \mapsto e^{2\pi\imath\alpha_1 x_1} e^{2\pi\imath\alpha_2 x_2} \cdots e^{2\pi\imath\alpha_d x_d},$$

where $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{Z}$ and $\imath^2 = -1$. The degree of this monomial is $\sum_{j=1}^{d} |\alpha_j|$. The set of all linear combinations of trigonometric monomials of degree at most ℓ is denoted by $\mathbb{T}(\ell, d)$. A cubature formula Q_n has trigonometric degree ℓ if it is exact on $\mathbb{T}(\ell, d)$ but not on $\mathbb{T}(\ell + 1, d)$, i.e.,

$$\forall\, f \in \mathbb{T}(\ell, d) : \qquad Q_n(f) = I_d(f)$$

and

$$\exists\, f \in \mathbb{T}(\ell + 1, d) : \qquad Q_n(f) \neq I_d(f).$$

We define

$$N_{\min}(\ell, d) = \min\{n \in \mathbb{N} : \exists\, Q_n : Q_n = I_d \text{ on } \mathbb{T}(\ell, d)\}$$

to be the minimal number n of knots needed by any cubature formula Q_n of trigonometric degree at least ℓ in dimension d. (We use $\mathbb{N}$ to denote the set of positive integers.)

Date: April 8, 1998.

1991 *Mathematics Subject Classification.* Primary 41A55, 41A63; Secondary 65D32.

The search for $N_{\min}(\ell, d)$ was pioneered by Noskov (1985) and Mysovskikh (1988a). Cools and Sloan (1996) include the known results on lower bounds and minimal cubature formulas. Also the following facts can be found in their paper. The numbers $N_{\min}(\ell, d)$ and corresponding cubature formulas are only known in exceptional cases. More exactly, they are known if $d = 1$ or $d = 2$ or if $\ell \in \{1, 2, 3\}$. Only one more case is known, namely $N_{\min}(5, 3) = 38$. Thus one is interested in upper and lower bounds for this quantity, as well as in "simple" formulas that have a "high" trigonometric degree. It is known that

$$\dim \mathbb{T}(\lfloor \ell/2 \rfloor, d) \leq N_{\min}(\ell, d) \leq \dim \mathbb{T}(\ell, d), \tag{2}$$

where

$$\dim \mathbb{T}(\ell, d) = \sum_{j=0}^{d} \binom{d}{j} \binom{\ell}{j} 2^j.$$

For odd degree ℓ the lower bound in (2) can be improved to

$$\sum_{j=1}^{(\ell+1)/2} \binom{d-1}{j-1} \binom{\frac{\ell+1}{2}}{j} 2^j \leq N_{\min}(\ell, d).$$

For fixed degree ℓ and large dimension d the upper bound in (2) is of the order

$$\frac{2^\ell}{\ell!}\, d^\ell.$$

This upper bound is constructive but, in general, for such Tchakaloff-type upper bounds no construction is known which is feasible for large d. In contrary, the method from Section 5 only uses about

$$\frac{1}{\ell!}\, d^\ell$$

knots to achieve trigonometric degree ℓ, and there is a very simple expression for the corresponding cubature formula. The cost to compute all the knots and weights is proportional to the number of knots, in each dimension d.

For more motivation and information about minimal formulas see Cools (1997). The nonperiodic case (with algebraic polynomials instead of trigonometric polynomials) is studied in Novak and Ritter (1997b), with surprisingly different results. See Section 6.5 for a discussion.

2. The Method

We construct cubature formulas $A(q, d)$ to approximate the integral (1) as follows. First we select quadrature formulas $U^1, U^2, \ldots$ to approximate the one-dimensional integral

$$\int_0^1 f(x)\, dx.$$

Then we use Smolyak's construction to obtain formulas for the d-dimensional integral. This construction has product formulas $U^{i_1} \otimes \cdots \otimes U^{i_d}$ as building blocks. Let n_i denote the number of knots of U^i. Clearly the above product formula needs $n_{i_1} \dots n_{i_d}$ function values, sampled on a grid. The Smolyak formulas $A(q,d)$ are linear combinations of product formulas with the following key properties. Only products with a relatively small number of knots are used and the linear combination is chosen in such a way that an interpolation property for $d = 1$ is preserved for $d > 1$. The formula $A(q,d)$ is defined by

$$A(q,d) = \sum_{q-d+1 \le |\mathbf{i}| \le q} (-1)^{q-|\mathbf{i}|} \cdot \binom{d-1}{q-|\mathbf{i}|} \cdot (U^{i_1} \otimes \cdots \otimes U^{i_d}), \tag{3}$$

where $q \ge d$, $\mathbf{i} \in \mathbb{N}^d$, and $|\mathbf{i}| = i_1 + \cdots + i_d$. Henceforth

$$n = n(q,d)$$

denotes the total number of knots used by $A(q,d)$.

Numerical integration with the Smolyak construction was already studied in Smolyak (1963). Other papers include Baszenski and Delvos (1994), Frank and Heinrich (1996), Genz (1986), Lyness and Sloan (1997), Novak and Ritter (1996, 1997a, 1997b), Temlyakov (1992, 1994), and Wasilkowski and Woźniakowski (1995).

In this paper we always take rectangle rules

$$U^i(f) = \frac{1}{n_i} \sum_{j=0}^{n_i - 1} f(j/n_i) \tag{4}$$

to build the formulas $A(q,d)$. It remains to select a sequence $(n_i)_i$ to specify the formulas $A(q,d)$ completely. We analyze two particular choices, namely,

$$n_i = 2^i \ , \ i \in \mathbb{N}, \tag{5}$$

in Section 4 as well as

$$n_i = 2^{i-1} \ , \ i \in \mathbb{N}, \tag{6}$$

in Section 5.

Error bounds for both methods are known, see Section 6.1. In this paper we study the trigonometric degree of these formulas. For $\ell, d \in \mathbb{N}$ we define

$$N(\ell,d) = \min\{n(q,d) : q \ge d, \ A(q,d) = I_d \text{ on } \mathbb{T}(\ell,d)\}.$$

Hence $N(\ell,d) = n(q,d)$ for the minimal q such that $A(q,d)$ has trigonometric degree at least ℓ. We study the numbers $N(\ell,d)$ for both sequences (5) and (6).

The results from this paper also hold for another choice of univariate formulas, the midpoint rules. The latter are based on knots $j/n_i + 1/(2n_i)$.

3. General Facts

Let $A(q,d)$ denote a cubature formula (3) with univariate formulas U^i of the form (4). Rectangle rules (4) have trigonometric degree $n_i - 1$, and we have

$$N_{\min}(n_i - 1, 1) = n_i.$$

Hence these formulas are optimal for $d = 1$. In the following we put $n_0 = 0$.

Lemma 1. *If $n_i \geq n_{i-1}$ for all $i \in \mathbb{N}$ then*

$$A(q,d)(f) = I_d(f)$$

for all polynomials

$$f \in \sum_{|\mathbf{i}|=q} (\mathbb{T}(n_{i_1} - 1, 1) \otimes \cdots \otimes \mathbb{T}(n_{i_d} - 1, 1)).$$

For a different kind of formulas U^i the proof is given in Novak and Ritter (1996). In that paper the algebraic degree was studied. The proof depends on the fact that the spaces of exactness of U^i are embedded if $n_i \geq n_{i-1}$, and that is true also for $\mathbb{T}(n_i - 1, 1)$, the case we are here investigating.

Now we consider the classical spaces $\mathbb{T}(\ell, d)$, and we require

$$n_{i+1} - n_i \geq n_i - n_{i-1} \tag{7}$$

for all $i \in \mathbb{N}$. This convexity property holds in both cases, (5) and (6).

Theorem 1. *Assume* (7). *Define $\ell(q,d)$ by*

$$\ell(q,d) = n_{\sigma-1}(d - \tau - 1) + n_\sigma(\tau + 1) - 1$$

where

$$q = \sigma d + \tau$$

for some $\sigma \in \mathbb{N}$ and $\tau \in \{0, \dots, d-1\}$. Then $A(q,d)$ has at least trigonometric degree $\ell(q,d)$.

Proof. The proof is very similar to the proof of Theorem 1 of Novak and Ritter (1997b). In that paper the algebraic degree was studied. Assume that the (algebraic or trigonometric) degree of U^i is m_i. Then, following Novak and Ritter (1997b), the degree of $A(q,d)$ is at least

$$\ell(q,d) = (m_{\sigma-1} + 1)(d - \tau - 1) + (m_\sigma + 1)(\tau + 1) - 1$$

if the convexity assumption

$$m_{i+1} - m_i \geq m_i - m_{i-1}$$

holds for all $i \in \mathbb{N}$, where $m_0 = -1$. The latter condition is satisfied in our case because of (7) and $m_i = n_i - 1$. $\square$

By X^i we denote the set of knots that are used by U^i. Recall that $n_i = \#X^i$, and put $X^0 = \emptyset$. Clearly

$$X^{i-1} \subset X^i \tag{8}$$

for all $i \in \mathbb{N}$ in both cases, (5) and (6). We derive a simple recursion for the number $n(q,d)$ of knots used by $A(q,d)$.

Lemma 2. *Assume* (8). *Then*

$$n(q+1, d+1) = \sum_{s=1}^{q-d+1} n(q+1-s, d) \cdot (n_s - n_{s-1}).$$

Proof. By (3) and (8) the formula $A(q,d)$ is based on the set

$$H(q,d) = \bigcup_{|\mathbf{i}|=q} X^{i_1} \times \cdots \times X^{i_d}$$

of knots. We have

$$\begin{aligned} H(q+1, d+1) &= \bigcup_{s=1}^{q-d+1} \bigcup_{|\mathbf{i}|=q+1-s} X^{i_1} \times \cdots \times X^{i_d} \times (X^s \setminus X^{s-1}) \\ &= \bigcup_{s=1}^{q-d+1} H(q+1-s, d) \times (X^s \setminus X^{s-1}). \end{aligned}$$

Hereby the recursion follows. □

4. The Meritorious Rule

In this section we study the sequence

$$n_i = 2^i \,, \; i \in \mathbb{N},$$

which gives the so-called meritorious rules, see Lyness and Sloan (1997). The notion of merit is based on another order on the set of trigonometric monomials $e^{2\pi\imath\boldsymbol{\alpha}\mathbf{x}} = e^{2\pi\imath\alpha_1 x_1} \dots e^{2\pi\imath\alpha_d x_d}$. Instead of the degree $\sum_{j=1}^{d} |\alpha_j|$ the number $\rho(\boldsymbol{\alpha}) = \prod_{j=1}^{d} \max(1, |\alpha_j|)$ is used, and the merit of a cubature formula Q_n, which is exact for constants, is defined as

$$\min\{\rho(\boldsymbol{\alpha}) : \exists\, \boldsymbol{\alpha} \in \mathbb{Z}^d : Q_n(e^{2\pi\imath\boldsymbol{\alpha}\mathbf{x}}) \neq I_d(e^{2\pi\imath\boldsymbol{\alpha}\mathbf{x}})\}.$$

Theorem 2. *Let* $q = \sigma d + \tau$ *with* $\sigma \in \mathbb{N}$ *and* $\tau \in \{0, \dots, d-1\}$. *Then* $A(q,d)$ *has trigonometric degree*

$$\ell(q,d) = \begin{cases} 2(q-d)+1 & \text{if } q < 3d \\ 2^{\sigma-1}(d+\tau+1)-1 & \text{otherwise.} \end{cases} \tag{9}$$

Proof. Let $\ell = \ell(q,d)$ for arbitrary $q \geq d$. As a consequence of Theorem 1 we see that $A(q,d)$ is exact on $\mathbb{T}(\ell, d)$.

It remains to find a polynomial $f \in \mathbb{T}(\ell+1, d)$ such that $A(q,d)(f) \neq I_d(f)$. We first consider the case $q < 2d$ and let

$$f(\mathbf{x}) = \exp(2\pi\imath\, 2x_1) \cdot \cdots \cdot \exp(2\pi\imath\, 2x_r),$$

where $r = q - d + 1$. Then $f \in \mathbb{T}(\ell+1, d)$ and

$$(U^{i_1} \otimes \cdots \otimes U^{i_d})(f) = \begin{cases} 1 & \text{if } i_1 = \cdots = i_r = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $\mathbf{i} \in \mathbb{N}^d$. Put

$$\mathcal{I} = \{\mathbf{i} \in \{1\}^r \times \mathbb{N}^{d-r} : |\mathbf{i}| \leq q\}$$

to obtain

$$A(q,d)(f) = \sum_{\mathbf{i} \in \mathcal{I}} (-1)^{q-|\mathbf{i}|} \cdot \binom{d-1}{q-|\mathbf{i}|}.$$

If $r = d$ then $\mathcal{I} = \{(1,\ldots,1)\}$ and

$$A(2d-1, d)(f) = (-1)^{d-1} \neq 0.$$

If $1 \leq r < d$ then

$$\begin{aligned} A(q,d)(f) &= \sum_{j=d-r}^{q-r} (-1)^{q-r-j} \cdot \binom{d-1}{q-r-j} \cdot \binom{j-1}{d-r-1} \\ &= \sum_{j=0}^{q-d} (-1)^j \cdot \binom{d-1}{j} \cdot \binom{d-j-2}{2d-q-2} \\ &= \sum_{j=0}^{k} (-1)^j \cdot \binom{m+1}{j} \cdot \binom{m-j}{m-k}, \end{aligned}$$

where $k = q - d$ and $m = d - 2$. The identity (27) in Netto (1927, p. 252) yields

$$A(q,d)(f) = (-1)^k.$$

Hence

$$A(q,d)(f) = (-1)^{q-d} \neq I_d(f) = 0$$

Now consider the case $q \geq 2d$. Note that for $2d \leq q < 3d$ both expressions on the right side of (9) are identical. Take $\sigma \geq 2$ and $\tau \in \{0,\ldots,d-1\}$ such that $q = \sigma d + \tau$. Put $r = \tau + 1$ and let

$$f(\mathbf{x}) = \exp(2\pi\imath\, 2^\sigma x_1) \cdot \cdots \cdot \exp(2\pi\imath\, 2^\sigma x_r) \cdot \exp(2\pi\imath\, 2^{\sigma-1} x_{r+1}) \cdot \cdots \cdot \exp(2\pi\imath\, 2^{\sigma-1} x_d).$$

Then $f \in \mathbb{T}(\ell+1, d)$ and

$$(U^{i_1} \otimes \cdots \otimes U^{i_d})(f) = \begin{cases} 1 & \text{if } i_1,\ldots,i_r \leq \sigma \text{ and } i_{r+1},\ldots,i_d \leq \sigma - 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $\mathbf{i} \in \mathbb{N}^d$. Put

$$\mathcal{I} = \{\mathbf{i} \in \{1, \ldots, \sigma\}^r \times \{1, \ldots, \sigma - 1\}^{d-r} : q - d + 1 \leq |\mathbf{i}| \leq q\}$$

to obtain $\mathcal{I} = \{\sigma\}^r \times \{\sigma - 1\}^{d-r}$ and

$$A(q, d)(f) = \sum_{\mathbf{i} \in \mathcal{I}} (-1)^{q-|\mathbf{i}|} \cdot \binom{d-1}{q - |\mathbf{i}|} = (-1)^{d-1} \neq I_d(f) = 0.$$

□

Clearly

$$2^d = n(d, d) \leq N(\ell, d).$$

Hence, for fixed degree ℓ and large dimension d, the numbers $N(\ell, d)$ increase exponentially with d. In the following table we give some explicit values for $N(\ell, d)$ in dimensions $d = 5$, 10, 15, 20 and 25. This table can easily be computed by means of Lemma 2 and Theorem 2.

TABLE 1

ℓ	$N(\ell, 5)$	$N(\ell, 10)$	$N(\ell, 15)$	$N(\ell, 20)$	$N(\ell, 25)$
5	832	77 824	4 947 968	263 192 576	$1.262 \cdot 10^{10}$
7	3 072	425 984	35 586 048	$2.339 \cdot 10^{9}$	$1.334 \cdot 10^{11}$
9	10 272	2 013 184	214 990 848	$1.715 \cdot 10^{10}$	$1.148 \cdot 10^{12}$
11	32 064	8 579 072	$1.147 \cdot 10^9$	$1.095 \cdot 10^{11}$	$8.515 \cdot 10^{12}$
13	95 104	33 820 672	$5.568 \cdot 10^9$	$6.281 \cdot 10^{11}$	$5.627 \cdot 10^{13}$

5. A Modified Rule with Polynomial Bounds

We want to modify the method from Section 4 such that $N(\ell, d)$ increases for fixed degree ℓ like a polynomial in the dimension d. Since $n(d, d) = n_1^d$ we require

$$n_1 = 1. \tag{10}$$

It turns out that $X^2 \setminus X^1$ should also be small in order to obtain a slowly growing sequence $n(k + d, d)$ in d. Hence we require

$$\#(X^2 \setminus X^1) = 1 \tag{11}$$

for $j = 1, \ldots, d$. Therefore we use the modified sequence

$$n_i = 2^{i-1}\ ,\ i \in \mathbb{N},$$

in this section.

Theorem 3. *Let $q = \sigma d + \tau$ with $\sigma \in \mathbb{N}$ and $\tau \in \{0, \ldots, d-1\}$. Then $A(q,d)$ has trigonometric degree*

$$\ell(q,d) = \begin{cases} q - d & \text{if } q < 3d \\ 2^{\sigma-2}(d+\tau+1) - 1 & \text{otherwise.} \end{cases} \tag{12}$$

Proof. Let $\ell = \ell(q,d)$ for arbitrary $q \geq d$. As a consequence of Theorem 1 we see that $A(q,d)$ is exact on $\mathbb{T}(\ell,d)$.

Let $q = \sigma d + \tau$ with $\sigma \in \mathbb{N}$ and $\tau \in \{0, \ldots, d-1\}$. Put $r = \tau + 1$ and let

$$f(\mathbf{x}) = \exp(2\pi\imath\, x_1) \cdot \cdots \cdot \exp(2\pi\imath\, x_r)$$

if $q < 2d$ and

$$f(\mathbf{x}) = \exp(2\pi\imath\, 2^{\sigma-1}x_1) \cdot \cdots \cdot \exp(2\pi\imath\, 2^{\sigma-1}x_r) \cdot \exp(2\pi\imath\, 2^{\sigma-2}x_{r+1}) \cdot \cdots \cdot \exp(2\pi\imath\, 2^{\sigma-2}x_d)$$

if $q \geq 2d$. Then $f \in \mathbb{T}(\ell+1, d)$ and similar to the proof of Theorem 2 we obtain $A(q,d)(f) \neq I_d(f)$. □

Now we derive an estimate on the number $n(q,d)$ of knots used by $A(q,d)$. We use $\approx$ to denote the strong equivalence of sequences, i.e.,

$$v_n \approx w_n \qquad \text{iff} \qquad \lim_{n\to\infty} v_n/w_n = 1.$$

Lemma 3. *Assume* (10) *and* (11). *For $d \to \infty$ and fixed ℓ*

$$n(\ell+d, d) \approx \frac{1}{\ell!} d^\ell.$$

Proof. This is very similar to Lemma 2 of Novak and Ritter (1997b). □

Corollary 1. *Let $\ell < 2d$. Then $A(\ell+d,d)$ has degree ℓ and the corresponding number of knots satisfies*

$$N(\ell,d) = n(\ell+d,d) \approx \frac{1}{\ell!} d^\ell.$$

Here, as in Lemma 3, ℓ is fixed and d tends to infinity.

The upper bound (2) yields

$$N_{\min}(\ell,d) \leq \dim \mathbb{T}(\ell,d) = \sum_{j=0}^{d} \binom{d}{j} \binom{\ell}{j} 2^j \approx \frac{2^\ell}{\ell!} d^\ell.$$

We conclude that the polynomial dependence on d for the modified method is the same as in the upper bound (2) but better by a constant of 2^ℓ.

We use Lemma 2 and Theorem 3 to compute the following table. The table gives explicit values for $N(\ell,d)$ in the same range as in Table 1.

TABLE 2

ℓ	$N(\ell,5)$	$N(\ell,10)$	$N(\ell,15)$	$N(\ell,20)$	$N(\ell,25)$
5	1 002	8 378	35 004	104 380	253 756
7	8 472	122 468	765 314	3 158 460	10 105 856
9	62 912	1 462 563	13 049 304	72 053 110	295 574 206
11	165 504	15 157 188	186 519 138	$1.343 \cdot 10^{9}$	$6.921 \cdot 10^{9}$
13	427 264	141 264 528	$2.333 \cdot 10^9$	$2.146 \cdot 10^{10}$	$1.367 \cdot 10^{11}$

6. DISCUSSION

6.1. **Error bounds.** It is known that both sequences (5) and (6) lead to cubature formulas $A(q,d)$ that are universal, i.e., they are almost optimal for many different classes of periodic functions. If f has bounded derivatives up to order r then

$$|I_d(f) - Q_n(f)| = \mathcal{O}\Big(n^{-r/d}(\log n)^{(d-1)\,r/d}\Big). \tag{13}$$

If f has a bounded mixed derivative $f^{(r,\dots,r)}$ then

$$|I_d(f) - Q_n(f)| = \mathcal{O}\Big(n^{-r}(\log n)^{(d-1)\,(r+1)}\Big). \tag{14}$$

These two estimates are optimal – up to logarithmic factors – in both smoothness scales and hold for all $r \in \mathbb{N}$. The bound (14) follows from general estimates in Smolyak (1963), see also Wasilkowski and Woźniakowski (1995) and Novak and Ritter (1996). See Temlyakov (1994, Section IV.6) for the estimate (13). The Korobov classes are studied in Baszenski and Delvos (1994), and a result similar to (14) is obtained.

6.2. **Comparison between meritorious and modified rules.** Let $N_{\mathrm{mod}}(\ell,d)$ denote the number of points needed by a modified rule to obtain trigonometric degree ℓ in dimension d. In the sequel we compare some other constructions of cubature formulas with the modified rules. To this end we compute ratios

$$r(\ell,d) = \frac{N(\ell,d)}{N_{\mathrm{mod}}(\ell,d)},$$

where $N(\ell,d)$ denotes the number of points used by a respective construction to obtain trigonometric degree at least ℓ in dimension d.

We begin with a comparison between meritorious and modified rules. The recursion from Lemma 2 is valid in both cases but with a different sequence n_i. From this follows immediately that

$$n_{\mathrm{mer}}(q,d) = 2^d\, n_{\mathrm{mod}}(q,d). \tag{15}$$

Here $n_{\mathrm{mer}}(q,d)$ and $n_{\mathrm{mod}}(q,d)$ denote the number of points used by the meritorious and modified rules, respectively. In Lyness and Sloan (1997) it is shown that

$$n_{\mathrm{mer}}(q,d) = 2^{q-d+1}\, p_{d-1}(q-d+1) \tag{16}$$

where p_{d-1} is a polynomial of degree $d-1$.

From Theorems 2 and 3 follows that for a given d and $\ell \geq 2d+1$, a meritorious rule $A(q,d)$ will have the same odd degree ℓ as a modified rule $A(q+d,d)$. Hence

$$r(\ell,d) = \frac{n_{\mathrm{mer}}(q,d)}{n_{\mathrm{mod}}(q+d,d)} = \frac{2^{q-d+1} p_{d-1}(q-d+1)}{2^{q-d+1} p_{d-1}(q+1)}.$$

Here q of course depends on ℓ but is the same for both methods. Consequently

$$\lim_{\ell\to\infty} r(\ell,d) = 1$$

for every dimension d.

We use $\asymp$ to denote the weak equivalence of sequences, i.e.,

$$v_n \asymp w_n \qquad \text{iff} \qquad c_1\, v_n \leq w_n \leq c_2 w_n, \quad \forall n \in \mathbb{N},$$

with two constants $c_i > 0$. Let $d \in \mathbb{N}$ be fixed. From Theorem 3 we have $\ell(q,d) \asymp 2^{q/d}$, and hence $\log \ell(q,d) \asymp q$. Using (15) and (16) we get

$$n_{\mathrm{mod}}(q,d) \asymp 2^q q^{d-1} \asymp (\ell(q,d))^d (\log \ell(q,d))^{d-1}.$$

Given $\ell \in \mathbb{N}$, choose $q(\ell)$ minimal such that $\ell(q(\ell),d) \geq \ell$. Then $\ell(q(\ell),d) \asymp \ell$, and therefore

$$N_{\mathrm{mod}}(\ell,d) = n_{\mathrm{mod}}(q(\ell),d) \asymp \ell^d (\log \ell)^{d-1}. \tag{17}$$

For degree $\ell = 2k+1$ or $\ell = 2k$ with $k < d$ we obtain

$$r(\ell,d) = \frac{n_{\mathrm{mer}}(k+d,d)}{n_{\mathrm{mod}}(\ell+d,d)} = \frac{2^d\, n_{\mathrm{mod}}(k+d,d)}{n_{\mathrm{mod}}(\ell+d,d)}.$$

For fixed k and large d, Lemma 3 yields

$$r(\ell,d) \approx \frac{\ell!}{k!}\, \frac{2^d}{d^{\ell-k}}.$$

Hence we obtain

$$\lim_{d\to\infty} r(\ell,d) = \infty$$

for every degree ℓ.

Although in this paper we focus on cubature formulas for large dimensions d, in comparisons we consider values of ℓ and d which make sense in practical computation. The corresponding number of points must not be too large, say, must not exceed 10^9. Thus, if $N(\ell,d)$ and $N_{\mathrm{mod}}(\ell,d)$ are greater than 10^9, instead of the ratio $r(\ell,d)$ a '–' is shown in the following tables.

For the meritorious rule some values of $r(\ell,d)$ are listed in the next table. Based on further computations we conjecture that the modified rule is better whenever $d > \ell$.

TABLE 3

ℓ	$r(\ell,5)$	$r(\ell,10)$	$r(\ell,15)$	$r(\ell,20)$	$r(\ell,25)$
5	0.83	9.3	141	2 521	49 718
7	0.36	3.5	46	741	13 201
9	0.16	1.4	16	238	3 885
11	0.19	0.57	6	–	–
13	0.22	0.24	–	–	–

6.3. **Comparison with minimal formulas.** Minimal formulas are known for degree $\ell \in \{1,2,3\}$ as well as for dimension $d \le 2$. See Cools and Sloan (1996) for results and references; see also Section 6.4. We have

$$N_{\min}(1,d) = 2, \tag{18}$$

$$N_{\min}(2,d) = 2d+1, \tag{19}$$

and

$$N_{\min}(3,d) = 4d \tag{20}$$

for every d, while

$$N_{\mathrm{mod}}(1,d) = d+1,$$
$$N_{\mathrm{mod}}(2,d) = \frac{1}{2}\,d^2 + \frac{5}{2}\,d + 1,$$

and

$$N_{\mathrm{mod}}(3,d) = \frac{1}{6}\,d^3 + 2\,d^2 + \frac{29}{6}\,d + 1$$

holds for the modified rule. Moreover,

$$N_{\min}(2k+1,2) = 2\,(k+1)^2 \tag{21}$$

for every $k \ge 0$. One has, for instance, $N_{\min}(11,2) = 72$ while $N_{\mathrm{mod}}(11,2) = 256$. A meritorious rule needs 192 points for trigonometric degree 11 if $d = 2$.

6.4. **Comparison with copy rules.** For dimensions $d > 2$, all the known minimal formulas are lattice rules. The copy construction is a simple way to derive a sequence of lattice rules from a basic lattice rule, see, e.g., Niederreiter (1992) and Sloan and Joe (1994).

Consider a basic lattice rule in dimension d that uses t_0 knots and has trigonometric degree ℓ_0. Let $m \in \mathbb{N}$. The corresponding m-copy rule is constructed by subdividing $[0,1]^d$ into m^d congruent subcubes and scaling the basic rule onto these subcubes. The m-copy rule uses $t_0 m^d$ knots, and its trigonometric degree is given by $(\ell_0+1)m-1$. Therefore we have

$$N(\ell,d) = t_0 \lceil (\ell+1)/(\ell_0+1) \rceil^d$$

for the copy construction. For fixed degree ℓ the number $N(\ell, d)$ increases exponentially in d. In particular,

$$\lim_{d\to\infty} r(\ell, d) = \infty.$$

On the other hand, for fixed dimension d the number $N(\ell, d)$ is of order ℓ^d, while $N_{\mathrm{mod}}(\ell, d)$ is of order $\ell^d(\log \ell)^{d-1}$, see (17). Therefore

$$\lim_{\ell\to\infty} r(\ell, d) = 0.$$

The simplest copy rules are product rectangle rules. In this case $t_0 = 1$ and $\ell_0 = 0$, and therefore $N(\ell, d) = (\ell + 1)^d$.

The lattice rule with knots $(0, \dots, 0)$ and $(1/2, \dots, 1/2)$, which is probably first mentioned as such in Mysovskikh (1988b), has trigonometric degree one. By (18) it is a minimal formula. The corresponding copy rules are sometimes called body-centered cubic rules, and they yield

$$N(2k + 1, d) = 2(k + 1)^d.$$

These copy rules are minimal formulas for dimension $d = 2$, see (21). This result is probably first obtained by Noskov (1985). Moreover, the rules are better than the modified rules for small dimensions. Specifically, $r(\ell, d) < 1$ for $d \le 6$. The following table shows superiority of the modified rule for larger dimensions.

TABLE 4

ℓ	$r(\ell,5)$	$r(\ell,10)$	$r(\ell,15)$	$r(\ell,20)$	$r(\ell,25)$
5	0.49	14.1	820	66 809	$6.6 \cdot 10^6$
7	0.24	17.1	2 806	696 233	$2.2 \cdot 10^8$
9	0.10	13.3	4 677	2 647 143	$2.0 \cdot 10^9$
11	0.09	8.0	5 042	–	–
13	0.08	4.0	–	–	–

The lattice rule whose lattice is generated by

$$\left(\frac{1}{4d}, \frac{3}{4d}, \dots, \frac{2d-1}{4d}\right)$$

has trigonometric degree 3 and uses $4d$ knots. It is a minimal formula due to (20). This result is due to Noskov (1988). The corresponding copy rules yield

$$N(4k + 1, d) = N(4k + 3, d) = 4d(k + 1)^d.$$

For $\ell = 4k + 3$ with $k \in \mathbb{N}$ these copy rules are better than the modified rules in moderate dimensions. Specifically, $r(\ell, d) < 1$ for $d \le 12$ and $\ell = 4k + 3$. The following table shows superiority of the modified rule for large dimensions. To further illustrate this fact consider the case $\ell = 7$ and $d = 40$. Then $N_{\mathrm{mod}}(\ell, d) = 1.3 \cdot 10^8$ while $N(\ell, d) = 1.7 \cdot 10^{14}$ for the copy rule.

TABLE 5

ℓ	$r(\ell,5)$	$r(\ell,10)$	$r(\ell,15)$	$r(\ell,20)$	$r(\ell,25)$
5	.64	4.89	56.2	804	13223
7	.08	.33	2.6	27	332
9	.08	1.61	66.0	3871	286658
11	.03	.16	4.6	–	–
13	.05	.30	–	–	–

Note that the degree of m-copy rules increases proportional to m with slope $\ell_0 + 1$. This causes $N(\ell, d)$ to be constant on certain ranges of ℓ if $\ell_0 > 0$. Theorems 2 and 3 show that the same phenomenon is present for the modified and meritorious rules. Since the plateaus of $N(\ell, d)$ and of $N_{\mathrm{mod}}(\ell, d)$ appear in different ranges, one obtains the irregular behavior illustrated in Table 5.

6.5. **Are the periodic and nonperiodic cases really different?** A comparison between the periodic and the nonperiodic case, i.e., with algebraic instead of trigonometric degree, reveals the following facts. In both cases

- the bounds (2) hold with the respective spaces of polynomials;
- for degree ℓ and large dimension d, the upper bound is of order d^ℓ while the lower bound is of order d^k where $k = \lfloor \ell/2 \rfloor$.

What we achieve so far by constructive methods, however, is different. In the nonperiodic case, see Novak and Ritter (1997b), one has methods with about d^k knots, which also is the order of the lower bound. In the periodic case, studied in this paper, we only obtained methods with about d^ℓ knots, which is the order of the upper bound. Looking at (18) and (20), we conjecture that about d^k knots are always sufficient also in the periodic case.

It is not clear to us whether the methods of ideal theory, as presented in Temirgaliev (1991), can be developed to a point where similar (or even better) results can be proven.

Acknowledgments. The formulas on which Table 1 is immediately based were previously known to James Lyness. We thank him for making available to us his independent (and different) derivation and for his version of this table.

One of the authors (E.N.) was supported by a Heisenberg scholarship of the DFG.

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