bensolve tools

Calculus of Convex Polyhedra

Calculus of Polyhedral Convex Functions

Global Optimization

Vector Linear Programming

for Octave and Matlab

March 14, 2018

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Contents

1. Overview

The package *bensolve tools* contains the following components:

- `polyh` is a class for calculus of convex polyhedra. It is supplemented by the commands `ball`, `bensolvehedron`, `cart`, `chunion`, `cone`, `cube`, `emptyset`, `intsec`, `msum`, `origin`, `point`, `simplex`, `space`, see Section 5 for details.
- `polyf` is a class for calculus of polyhedral convex functions. It is supplemented by the commands `affine`, `fenv`, `finfc`, `fmax`, `fsum`, `gauge`, `indicator`, `maxnorm`, `sumnorm`, `translative`, see Section 6 for explanation.
- `lpsolve` solves linear programs (GLPK interface), see Section 7.
- `pcpsolve` solves polyhedral convex programs, see Section 8.
- `qcsolve` is a solver for global optimization problems with quasi-concave objective function and linear constraints, see Section 9.
- `molpsolve` solves multiple objective linear programs, see Section 10.
- `vlpsolve` solves vector linear programs, see Section 11.

2. Theoretical Background

The package *bensolve tools* is based on the vector linear program solver *bensolve* [10]. As shown in [9], vector linear programming is equivalent to polyhedral projection, which is the basis of all polyhedral calculus tools. The global optimization solver, see Section 9, is based on a modified version of *bensolve* using the theoretical results in [3].

In this exposition, we use several standard concepts from (polyhedral) Convex Analysis without any further explanation. More information can be found in standard books on Convex Analysis, such as [11], [6], [2].

For details about the algorithms used in *bensolve*, see e.g. [1], [4], [7], [5].

3. Citation Policy

Please cite reference [10] if you use *bensolve tools* in scientific papers. In case you use the global optimization solver, please cite reference [3].

4. Installation and Testing

4.1. Installation for Matlab

We assume that Matlab version 2015b or newer is installed.

1. Go to `http://bensolve.org/tools/download.html`.
2. Download and unpack the file `bt-X.Y.zip`

3. Run Matlab and change into the *bensolve tools* directory `.../bt-X.Y`.

Note that *bensolve tools* for Matlab uses a pre-compiled mex-file, see also Section 4.4.

4.2. Installation for Octave

It is necessary to generate a mex-file named `bensolve.mex`. Please follow the instructions depending on your operating system.

Ubuntu

1. Open a terminal (Shift+Ctrl+T)
2. To install Octave (in case Octave is already installed, make sure 'mkoctfile' is available), run:
   `sudo apt-get update`
   `sudo apt-get install liboctave-dev`
3. Go to `http://bensolve.org/tools/download.html`.
4. Download the file `bt-X.Y.tgz`.
5. Change into the folder where `bt-X.Y.tgz` is located by typing into the terminal:
   `cd path_to_your_folder`
6. Unpack the files by typing into the terminal:
   `tar -xz < bt-X.Y.tgz`
7. Change into the generated subdirectory:
   `cd bt-X.Y`
8. Run:
   `mkoctfile --mex src/*.c -lglpk -O3 -o bensolve`

MacOS

The following instructions are based on an installation of GNU Octave using the package manager Homebrew. There are many other possibilities to install Octave, see Section 4.4 for possible problems.

1. Install Homebrew: `https://brew.sh`
2. Install GNU Octave. Open a terminal and enter:
   `brew install octave`
3. Go to `http://bensolve.org/tools/download.html`.
4. Download and unpack `bt-X.Y.zip`
5. Change into the directory `bt-X.Y` by typing into the terminal:
   `cd your_download_location/bt-X.Y`

6. Run Octave by entering:
   `octave`
7. In the octave terminal, run:
   `make_oct`

Windows

1. Go to `https://ftp.gnu.org/gnu/octave/windows/`.
2. Download and run the octave-X.Y.Z-w64-installer.exe to install Octave (we recommend the latest version).
3. Go to `http://bensolve.org/tools/download.html`.
4. Download and unpack `bt-X.Y.zip`
5. Run Octave and move into the *bensolve tools* directory `bt-X.Y`
6. In the Octave terminal, run:
   `make_oct`

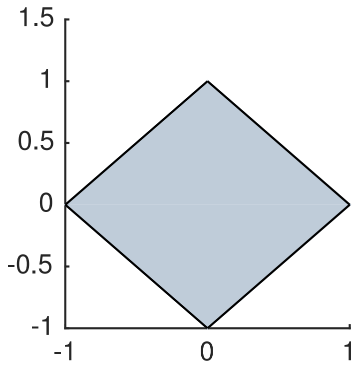
4.3. Test your installation

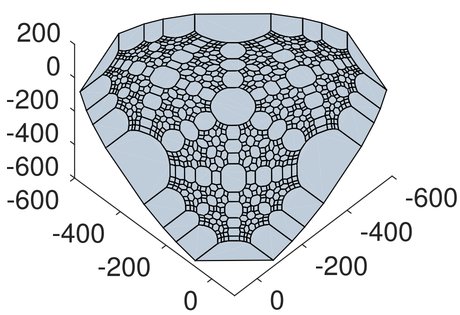
To test your installation, enter

```
P=ball(2);
plot(P);
```

You can also try to generate a more complex image by entering the following commands:

```
P=bensolvehedron(3,3);
Q=cone(3);
R=P+Q;
opt.dirlength=100;
plot(R,opt);
```





4.4. Trouble shooting

If *bensolve tools* does not work correctly, you can try to install a newer version of Matlab or Octave.

Note that *bensolve tools* for Matlab uses a mex-file with the name `bensolve.mexmaci64` (or similar, depending on the operating system). If the pre-compiled mex-file is not working on your operating system, you can try to compile your own mex-file using the source code in the subdirectory `src`. Some hints are given in the file `generate_mex_file_matlab.txt` in the subdirectory `doc`.

If the mex file generation for Octave does not work, we recommend to check whether the GLPK library is installed properly. The usual way to generate a mex file is to run the following in a terminal (exit octave before):

```
cd your_path/bt-X.Y
mkoctfile --mex src/*.c -lglpk -O3 -o bensolve
```

Note that the `O` in `-O3` stands for `Optimization`, 'zero' does not work. Sometimes `glpk.h` is not found. For instance, if it is located in the directory `/usr/include`, enter:

```
mkoctfile --mex src/*.c -I/usr/include -lglpk -O3 -o bensolve
```

Sometimes it can be necessary to specify the glpk library `libglpk.a`. For instance, if it is located in the directory `/usr/lib`, enter:

```
mkoctfile --mex src/*.c -I/usr/include /usr/lib/libglpk.a -O3 -o bensolve
```

4.5. Getting help

Use the `help` command to get help for a class or command (for classes in Matlab also the `doc` command can be useful). For instance:

```
help msum
```

```
 -- P = msum({P1,...,Pn})    Minkowski sum of n polyhedra

    Input:
      {P1,...,Pn}: finitely many polyhedra (cell array of polyh objects)
    Output:
      P: Minkowski sum (polyh object)

    see also: polyh/plus, intsec, chunion, cart
```

5. polyh – Calculus of Convex Polyhedra

`polyh` is a class for computations with convex polyhedra. Throughout this section, a convex polyhedron is called *polyhedron* for short. The most important operations are:

- computing vertices and extreme directions (V-representation)
- computing an inequality representation (H-representation)

- image under linear transformation (in particular projection)
- inverse image under linear transformation
- lineality space, affine hull, dimension, recession cone
- cone generated by a polyhedron
- adjacency list, facet-vertex incidence list
- Minkowski sum of n polyhedra
- intersection of n polyhedra
- closed convex hull of the union of n polyhedra
- cartesian product of n polyhedra
- polar of a polyhedron
- polarcone of a polyhedron
- normal cone of a polyhedron at a point
- comparing polyhedra: subset, proper subset, equality
- plotting 2d and 3d polyhedra

5.1. Representing a convex polyhedron

A convex polyhedron P can be defined in three ways:

- An *H-represenatation*

  $$P = \{x \in \mathbb{R}^n |\ a \le Bx \le b,\ l \le x \le u\}$$

  means that P is given by finitely many linear inequalities. Here B is an $(m \times n)$-matrix. The lower bound vector a has m components being either a real number or $-\infty$. If a is not specified, all of its components are $-\infty$ by default. The remaining bounds have a similar meaning.

- A *V-representation*

  $$P = \{x \in \mathbb{R}^n |\ x = V\lambda + D\mu + L\eta,\ \lambda \ge 0,\ \mu \ge 0,\ e^\top \lambda = 1\}$$

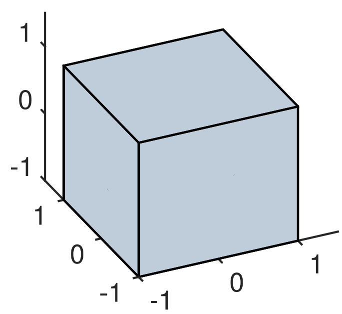
  means that P is given by a generalized convex hull of finitely many points (the columns of the matrix V), finitely many directions (the columns of the matrix D) and finitely many lineality directions (the columns of the matrix L). Here $e = (1, \dots, 1)^\top$ denotes the all-one-vector.

- A *P-representation*
$$P = \{Mx \in \mathbb{R}^q |\ a \le Bx \le b,\ l \le x \le u\}$$
means that another polyhedron $Q = \{x \in \mathbb{R}^n |\ a \le Bx \le b,\ l \le x \le u\}$ is given by an H-representation and P is the image of Q under the linear transformation $x \mapsto Mx$, where M is a $(q \times n)$-matrix. The 'P' stands for "projection" and is motivated by the following reformulation of P:
$$P = \{y \in \mathbb{R}^q |\ \exists x \in \mathbb{R}^n :\ Mx - y = 0,\ a \le Bx \le b,\ l \le x \le u\},$$
which shows that P is a projection of the polyhedron
$$Q = \{((x, y) \in \mathbb{R}^n \times \mathbb{R}^q |\ Mx - y = 0,\ a \le Bx \le b,\ l \le x \le u\}$$
onto the space $\mathbb{R}^q$. Both an H-representation and a V-representation are special cases of a P-representation. The concept of a P-representation is often related to the term "lifting" in the literature.

5.1.1. H-representation examples

Example 5.1 *Let a cube in $\mathbb{R}^3$ be defined by*
$$P = \{x \in \mathbb{R}^3 |\ -1 \le x_1 \le 1,\ -1 \le x_2 \le 1,\ -1 \le x_3 \le 1\}.$$
A corresponding `polyh` *instance is obtained and plotted by the following commands:*

```
clear rep;
rep.l=[-1;-1;-1];
rep.u=[1;1;1];
P=polyh(rep,'h');
plot(P);
```



`rep` *is a structure to store a representation of the polyhedron. The command* `rep.l=[-1;-1;-1];` *sets lower bounds $x_1 \ge -1$, $x_2 \ge -1$, $x_3 \ge -1$. Likewise,* `rep.u=[1;1;1];` *sets upper bounds $x_1 \le 1$, $x_2 \le 1$, $x_3 \le 1$. The command* `P=polyh(rep,'h');` *defines a* `polyh` *instance. The option* `'h'` *is required to indicate that the polyhedron is given as an H-representation.*

To define a polyhedron by an H-representation, a structure (named `rep` here) with the following fields is used.

```
rep.B
rep.a
rep.b
rep.l
rep.u
```

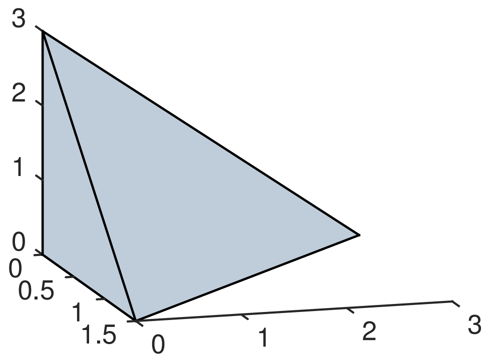
Some fields are optional. Default values for `a`, `l` and `b`, `u` are vectors with entries $-\infty$ and $+\infty$, respectively. At least one of the fields `B`, `l`, `u` is required.

Example 5.2 *Let*

$$P = \{x \in \mathbb{R}^3 |\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ 2x_1 + x_2 + x_3 \leq 3\}.$$

The corresponding `polyh` *instance is defined and plotted by the commands:*

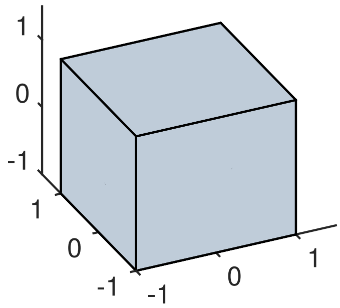
```
clear rep;
rep.B=[2 1 1];
rep.b=3;
rep.l=[0;0;0];
P=polyh(rep,'h');
plot(P);
```



5.1.2. V-representation examples

Example 5.3 *A cube in* $\mathbb{R}^3$*, see Example 5.1, is also given by its eight vertices:* $(0,0,0)^\top$, $(0,0,1)^\top$, $(0,1,0)^\top$, $(0,1,1)^\top$, $(1,0,0)^\top$, $(1,0,1)^\top$, $(1,1,0)^\top$, $(1,1,1)^\top$. *A corresponding* `polyh` *instance is obtained and plotted by*

```
clear rep;
rep.V=[0 0 0 0 1 1 1 1;0 0 1 1 0 0 1 1;0 1 0 1 0 1 0 1];
P=polyh(rep,'v');
plot(P);
```



To define a polyhedron by a V-representation, a structure (named `rep` here) with the following fields must be given.

```
rep.V
rep.D
rep.L
```

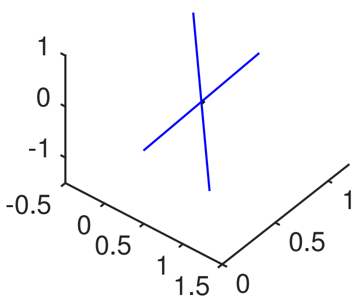
The columns of `rep.V` refer to given points, the columns of `rep.D` refer to given directions and the columns of `rep.L` refer to given lineality directions. At least one point is required to be given.

Example 5.4 *A* `polyh` *instance of the hyperplane*

$$H = \left\{ \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \eta_1 \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \eta_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \middle| \, \eta_1, \eta_2 \in \mathbb{R} \right\}$$

can be generated by the commands:

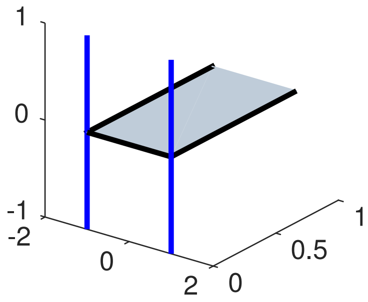
```
clear rep;
rep.V=[3;2;5];
rep.L=[2 1 4;-2 1 0]';
P=polyh(rep,'v');
plot(P);
```



Note that in case of a nontrivial lineality space, we plot only the polyhedron $P \cap U$ (which is a single point in the previous example), where U is a space complementary to the lineality space. Additionaly, lineality directions are drawn by blue lines. This principle becomes more clear by the next example.

Example 5.5 *Plotting the set* $P = \{x \in \mathbb{R}^3 | \; -1 \leq x_1 \leq 1, \; x_2 \geq 0, \; x_3 \in \mathbb{R}\}$*:*

```
P=ball(1):cone(1):space(1);
plot(P);
```

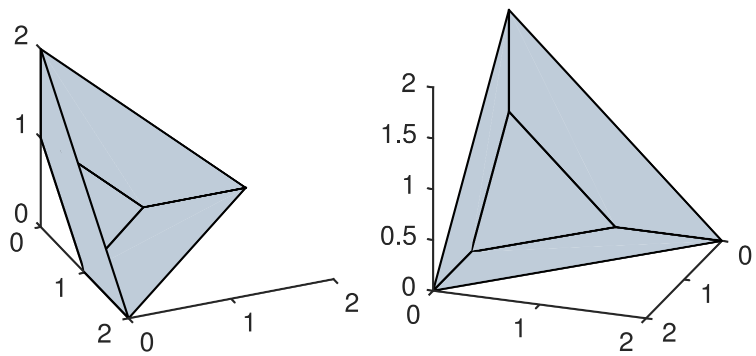


The set P is composed as the cartesian product of three one-dimensional sets: `ball(1)` *defines the interval* $[-1, 1]$, `cone(1)` *defines the nonnegative real numbers* $\mathbb{R}_+$, *and space(1) defines* $\mathbb{R}$.

Example 5.6 *Let the polyhedron P be given as:*

$$P := \text{conv}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} + \text{cone}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}.$$

```
clear rep;
rep.V=eye(3);
rep.D=eye(3);
P=polyh(rep,'v');
plot(P);
```



5.1.3. P-representation examples

Example 5.7 *Let P be the image of the 3-dimensional unit cube mapped onto $\mathbb{R}^2$ by the mapping*

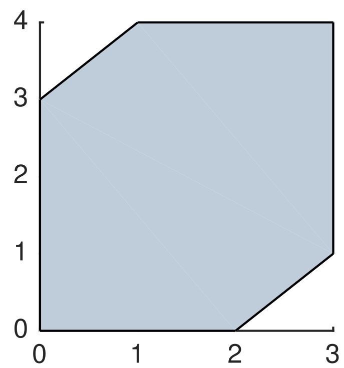
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1 + x_3 \\ 3x_2 + x_3 \end{pmatrix},$$

that is,

$$P = \left\{ \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

A corresponding `polyh` *instance is obtained and plotted by the following commands:*

```
clear rep;
rep.l=zeros(3,1);
rep.u=ones(3,1);
rep.M=[2 0 1;0 3 1];
P=polyh(rep);
plot(P);
```

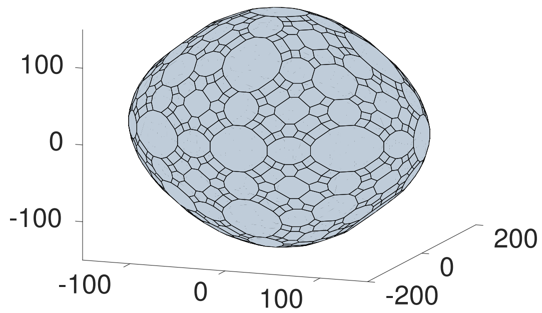


An alternative way to describe this polyhedron is first to define an H-representation of the 3-dimensional unit cube Q and then to compute the image of Q under the linear transformation M by the `im` *command.*

```
clear rep;
rep.l=zeros(3,1);
rep.u=ones(3,1);
Q=polyh(rep,'h');
M=[2 0 1;0 3 1];
P=im(Q,M);
```

Example 5.8 *Another example can be found in the file* `bensolvehedron.m`*. The following command yields the image of a 125-dimensional hypercube with respect to a (3×125)-matrix which consists of all possible column-wise arrangements of the numbers $-2, -1, -0, -1, -2$.*

```
P=bensolvehedron(3,2);
plot(P);
```



5.2. Retrieving representations and evaluation of polyhedra

For a `polyh` instance `P`, an H-, V- or P-representation is returned as a structure, respectively, by the commands:

```
hrep(P)
vrep(P)
prep(P)
```

A `polyh` instance stores (at least) a P-representation of the polyhedron. To obtain an H- or an V-representation, an evaluation of the polyhedron is required. Evaluation can become expensive, in particular, when the dimension increases.

Example 5.9 *Evaluating the 2-dimensional standard cone*

$$P = \mathbb{R}^2_+ := \{x \in \mathbb{R}^2 |\; x_1 \geq 0,\; x_2 \geq 0\} :$$

```
P=cone(2);
a=iseval(P)
P=eval(P);
b=iseval(P)
```

```
a = 0
b = 1
```

The following commands automatically evaluate a polyhedron if necessary:

```
eval
reinit
hrep
vrep
adj
inc
adj01
inc01
le / <=
ge / >=
eq / ==
ne / ~=
lt / <
gt / >
plot
```

All remaining operations do not require evaluation of the polyhedron. If more than one of the listed functions is used, it is more efficient to store the evaluated polyhedron before calling the commands. For instance

```
P=bensolvehedron(3,2);
P=eval(P);
hrep(P);
vrep(P);
```

is more efficient than

```
P=bensolvehedron(3,2);
hrep(P);
vrep(P);
```

because the first variant requires only one evaluation while the second variant requires two.

A P-representation of a `polyh` instance can be obtained directly by the `prep` command. An evaluation is not necessary.

Example 5.10 *Retrieving a P-representation of the 2-dimensional standard cone:*

```
P=cone(2);
rep=prep(P)
```

```
scalar structure containing the fields:
B = [](0x2)
a = [](0x1)
b = [](0x1)
l =
    0
    0
u =
    Inf
    Inf
```

Example 5.11 *Retrieving an H-representation of the 2-dimensional standard cone:*

```
P=cone(2);
rep=hrep(P)
```

```
ans =
scalar structure containing the fields:

Beq = [](0x2)
beq = [](0x1)
B =
    -1   0
     0  -1
b =
     0
     0
```

Note the `hrep` *returns an H-representation of the form*

$$Bx \leq b, \quad B_{eq}x = b_{eq}.$$

Example 5.12 *Retrieving an V-representation of the 2-dimensional standard cone:*

```
P=cone(2);
rep=vrep(P)
```

```
ans =
scalar structure containing the fields:

L = [](2x0)
V =
    0
    0
D =
    1  0
    0  1
```

5.3. Adjacency lists and incidence lists

After evaluation of a `polyh` instance, adjacency information for vertices and extremal directions is available. The command `adj` returns an adjacency list as a cell array, where indexing starts from 1. Note that the indices of the adjacency list refer to the columns of matrix:

```
[vrep(P).V, vrep(P).D]
```

Moreover, in case of a nontrivial lineality space (that is `vrep(P).L` is nonempty), the adjacency list of $P \cap U$ is returned, where U is a linear space complementary to the lineality space L, that is $L \cap U = \{0\}$, $L + U = \mathbb{R}^n$.

The command `adj01` returns a sparse matrix that contains the same information stored by zero and one entries.

Example 5.13 *The 2-dimensional standard cone has one vertex (index* 1*), which is adjacent to its two extremal directions (indices* 2 *and* 3*):*

```
P=cone(2);
P=eval(P);
a=adj(P)
b=adj01(P)
```

```
a =
{
  [1,1] = 2   3
  [1,2] = 1
  [1,3] = 1
}

b =
Compressed Column Sparse (rows = 3, cols = 3, nnz = 4 [44%])
(2, 1) -> 1
(3, 1) -> 1
(1, 2) -> 1
(1, 3) -> 1
```

After evaluation of a `polyh` instance, "facet-vertex" incidence information is available. Vertices and extremal directions are indexed in the same way as for adjacency lists.

Example 5.14 *The 2-dimensional standard cone shifted by the vector* $(5, 13)^\top$ *has one vertex in* $(5, 13)^\top$*, which is adjacent to its two extremal directions (the two unit vectors):*

```
P=cone(2);
P=P+[5;13];
P=eval(P);
inc(P)
```

```
ans =
{
  [1,1] = 1   3
  [1,2] = 1   2
}
```

The result tells us that the first facet contains "vertex 1" (which is indeed a vertex) and "vertex 3" (which is an extremal direction). Let us verify this information, just for demonstration reasons:

```
...
normalvector=hrep(P).B(1,:)
rhs=hrep(P).b(1)
M=[vrep(P).V,vrep(P).D];
vert1=M(:,1)
vert3=M(:,3)
a=(normalvector * vert1 == rhs)
b=(normalvector * vert3 == 0)
```

```
normalvector =  -1   0
rhs = -5
vert1 =
   5
  13
vert3 =
   0
   1
a = 1
b = 1
```

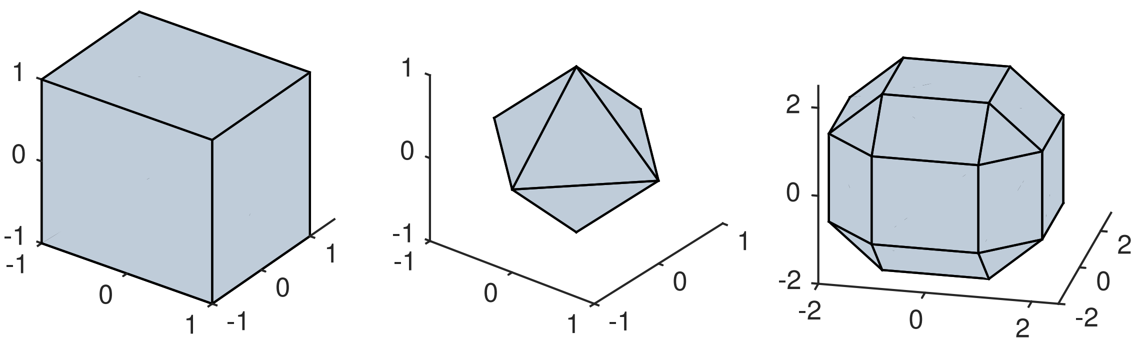
5.4. Calculus examples

The Minkowski sum of two polyhedra $P, Q \subseteq \mathbb{R}^n$ is defined as

$$P + Q = \{x \in \mathbb{R}^n | \; \exists y \in P, \exists z \in Q : \; x = y + z\}.$$

Example 5.15 *The Minkowski sum can be computed by the + operator:*

```
P=cube(3);
Q=ball(3);
R=P+Q;
plot(P);
plot(Q);
plot(R);
```



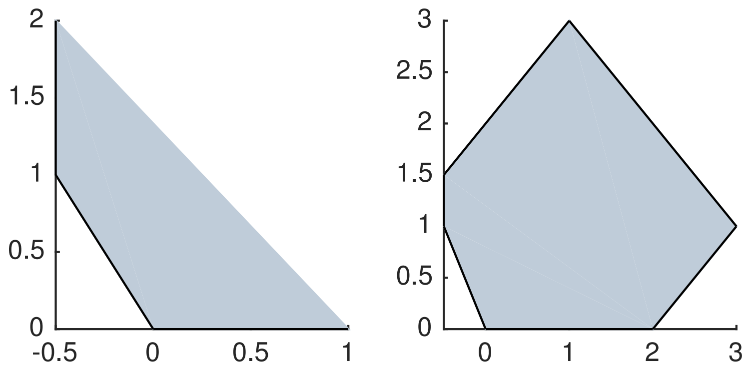
Likewise one can compute

- the *intersection* of two polyhedra by the command: `R=P&Q;`
- the *closed convex hull of the union* of two polyhedra by the command: `R=P|Q;`
- a polyhedron *scaled* by a factor `k`: `R=k*P;`
- a polyhedron *shifted* by a (column) vector `v`: `R=P+v;`

The following example demonstrates how these operations can be combined.

Example 5.16 *In line 3, we compute the closed convex hull of the 2-dimensional standard cone and the point* $(-1/2, 1)^\top$*. In line 6, the result* `R` *is intersected (*`&` *operator) by the sum-norm unit ball (generated by the command* `ball(2)`*), which is scaled by 2 and shifted by the vector* $(1, 1)^\top$*:*

```
1 P=cone(2);
2 Q=point([-1/2;1]);
3 R=Q|P;
4 S=R&(2*ball(2)+[1;1]);
5 plot(R);
6 plot(S);
```

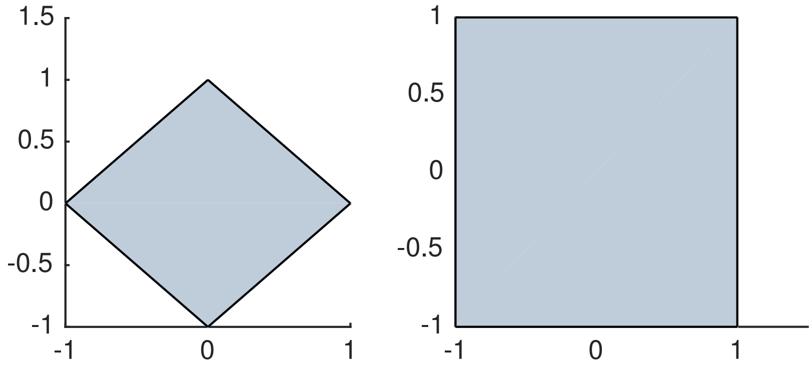


The *polar* of a polyhedron $P \subseteq \mathbb{R}^n$ is defined as

$$P^\circ = \{y \in \mathbb{R}^n | \; \forall x \in P : \; y^\top x \leq 1\}.$$

Example 5.17 *The polar of the 2-dimensional sum-norm unit ball is a square:*

```
P=ball(2);
R=polar(P);
plot(P);
plot(R);
```

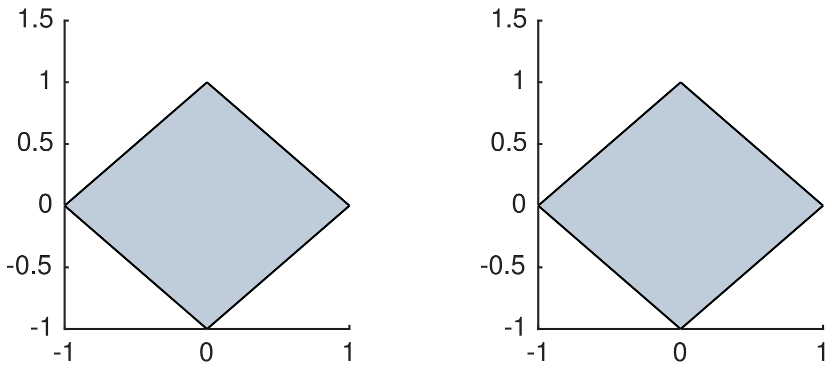


An alternative way to express `R=polar(P)` *is*

```
R=P';
```

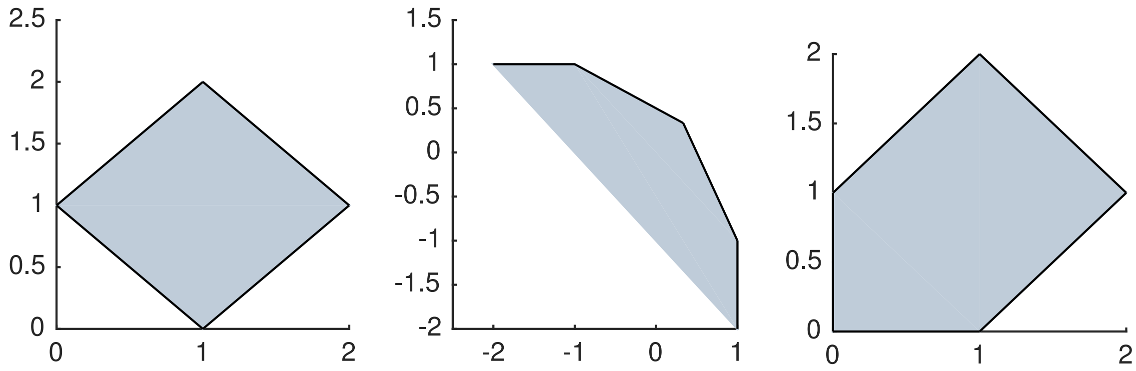
If a polyhedron contains the origin, then the polar of the polar results into the original polyhedron.

```
P=ball(2);
R=P'';
plot(P);
plot(R);
```



Otherwise it is the closed convex hull of the origin and the original polyhedron.

```
P=ball(2)+[1;1];
R=P';
S=R';
plot(P);
plot(R);
plot(S);
```



5.5. Class polyh – list of methods

```
-- polyh : class for calculus of convex polyhedra
```

5.5.1. Initialization and evaluation

```
-- P = polyh(REP,REPTYPE)    constructor

   constructor of polyh class

   Input:
     REP: representation of the polyhedron (struct)
   Optional input:
     REPTYPE: type of representation (char: ’v’, ’h’, ’p’)
            :   reptype ’v’:
            :     V matrix of points
            :     D matrix of directions
            :     L matrix of lineality directions
            :   reptype ’h’
            :     fields B, a, b, l, u according to the H-representation
            :       P = { x | a <= Bx <= b, l <= x <= u}
            :   reptype ’p’ (default)
            :     fields M, B, a, b, l, u according to the P-representation
            :       P = { Mx | a <= Bx <= b, l <= x <= u}
   Output:
     P: polyhedron (polyh object)


-- R = eval(P)    evaluate polyh object

   i.e. compute H- and V-representation, adjacency list, incidence list

   Input:
     P: polyhedron (polyh object)
   Output:
```

```
      R: evaluated polyhedron (polyh object)


 -- R = reinit(P,REPTYPE)    re-initialize polyh object

    re-initialize polyh object using its V- or H-representation
    (can simplify further computations but requires evaluation)

    Input:
      P: polyhedron (polyh object)
    Optional input:
      REPTYPE: type of representation: 'v' (default) or 'h' (char)
    Output:
      R: polyhedron (polyh object)
```

5.5.2. Polyhedral calculus

```
 -- R = plus(P,Q)    sum
    R = P + Q

    compute the Minkowski sum of two polyhedra or
    sum of polyhedron and vector

    Input:
      P, Q: two polyhedra (polyh objects)
          : or one polyhedron and one vector
    Output:
      R: Minkowski sum (polyh object)


 -- R = mtimes(k,P)    scaling
    R = k * P

    scaling of polyhedron

    Input:
      k: scaling factor (number)
      P: polyhedron (polyh objects)
    Output:
      R: scaled polyhedron (polyh object)


 -- R = minus(P,Q)    difference
    R = P - Q

    compute the Minkowski difference of two polyhedra or
```

```
   sum of polyhedron and vector

   Input:
     P, Q: two polyhedra (polyh objects)
         : or one polyhedron and one vector
   Output:
     R: Minkowski difference (polyh object)


-- R = uminus(P)    negative of
   R = -P

   compute negative of polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     R: polyhedron (polyh object)


-- R = and(P,Q)    intersection
   R = P & Q

   compute the intersection of two polyhedra

   Input:
     P, Q: two polyhedra (polyh objects)
   Output:
     R: intersection (polyh object)


-- R = or(P,Q)    closed convex hull of union of two polyhedra
   R = P | Q

   Input:
     P, Q: two polyhedra (polyh objects)
   Output:
     R: closed convex hull of union (polyh object)


-- R = colon(P,Q)    cartesian product of two or three polyhedra
   R = colon(P,Q,S)
   R = P : Q
   R = P : Q : S

   Input:
     P, Q: two polyhedra (polyh objects)
```

```
   Optional input:
     S: third polyhedron (polyh object)
   Output:
     R: cartesian product (polyh object)
```

```
-- R = im(P,M)    image under linear transformation

   Input:
     P: polyhedron (polyh object)
     M: linear transformation (matrix)
   Output:
     R: image M(P) (polyh object)
```

```
-- R = inv(P,M)    inverse image of polyhedron under linear transformation

   Input:
     P: polyhedron (polyh object)
     M: linear transformation (matrix)
   Output:
     R: inverse image {x | Mx in P} (polyh object)
```

5.5.3. Retrieving properties and related objects

```
-- d = sdim(P)    space dimension

   Input:
     P: polyhedron (polyh object)
   Output:
     d: space dimension of P (number)
```

```
-- d = dim(P)    dimension of polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     d: dimension of P (number)
```

```
-- d = ldim(P)    lineality space dimension

   Input:
     P: polyhedron (polyh object)
   Output:
```

```
    d: lineality space dimension of P (number)


-- v = getpoint(P)    point belonging to polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     v: if P is nonempty: point v belonging to polyhedron P (column vector)
      : if P is empty: (spacedim x 0) matrix


-- v = rint(P)    relative interior point

   Input:
     P: polyhedron (polyh object)
   Output:
     v: if P is nonempty: relative interior point v of P (column vector)
      : if P is empty: (spacedim x 0) matrix


-- [R,d] = affine(P)    affine hull

   Input:
     P: polyhedron (polyh object)
   Output:
     R: affine hull of P (polyh object)
     d: dimension of P (number)


-- R = lin(P)    lineality space

   Input:
     P: polyhedron (polyh object)
   Output:
     R: lineality space of P (polyh object)


-- R = polar(P)    polar set of nonempty polyhedron
   R = P'

   Input:
     P: polyhedron (polyh object)
   Output:
     R: polar set of P (polyh object)
```

```
-- R = polarcone(P)    polar cone of nonempty polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     R: polar cone of P (polyh object)
```

```
-- R = conic(P)    closed cone generated by polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     R: closed conic hull of P (polyh object)
```

```
-- R = ncone(P,v)    normal cone of polyhedron P at point v

   Input:
     P: polyhedron (polyh object)
     v: column vector
   Output:
     R: normal cone of P at v (polyh object)
```

```
-- R = recc(P)    reccesion cone of polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     R: recession cone of P (polyh object)
```

5.5.4. Property checking

```
-- flag = isempty(P)    test whether polyhedron is empty

   Input:
     P: polyhedron (polyh object)
   Output:
     flag: nonzero if P is empty (number)
```

```
-- flag = iselem(P,v)    test whether point belongs to polyhedron

   Input:
     P: polyhedron (polyh object)
```

```
    v: point (column vector)
  Output:
    flag: nonzero if v belongs to P (number)
```

```
-- flag = iseval(P)    check whether polyhedron is evaluated

  Input:
    P: polyhedron (polyh object)
  Output:
    flag: nonzero if polyhedron is evaluated (number)
```

```
-- flag = isbounded(P,C,POLARCONE_D,POLARCONE_L)    check boundedness

  test polyhedron P for being bounded (w.r.t. cone C,
  i.e. there is a bounded set B such that P is contained in B+C)

  Input:
    P           : polyhedron (polyh object)
  Optional input:
    C           : cone (polyh object)
    POLARCONE_D : see remark
    POLARCONE_L : see remark
  Output:
    flag: nonzero if P is bounded or C-bounded (number)

  Remark: since the cone C needs to be evaluated, it can be more
    efficient to enter a V-representation of the polar cone (if known)
```

5.5.5. Retrieving representations

```
-- REP = hrep(P)    H-representation

  retrieve H-representation of polyhedron

  Input:
    P: polyhedron (polyh object)
  Output:
    REP: H-representation of P (struct)
```

```
-- REP = vrep(P)    V-representation

  retrieve V-representation of polyhedron
```

```
   Input:
     P: polyhedron (polyh object)
   Output:
     REP: V-representation of P (struct)


-- REP = prep(P)    P-representation

   retrieve P-representation of polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     REP: P-representation of P (struct)
```

5.5.6. Retrieving adjacency and incidence lists

```
-- A = adj(P)    adjacency list

   retrieve adjacency list (cell array) of polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     A: adjacency list of P (cell array)

   Remark: In case of a nontrivial lineality space, the list
           corresponds to the intersection of P with
           a complement of the lineality space.


-- M = adj01(P)    adjacency list

   retrieve 0-1 adjacency list of polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     M: adjacency list of P (sparse matrix)

   Remark: In case of a nontrivial lineality space, the list
           corresponds to the intersection of P with
           a complement of the lineality space.


-- A = inc(P)    incidence list
```

```
   retrieve facet-vertex incidence list (cell array) of polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     A: incidence list of P (cell array)

   Remark: In case of a nontrivial lineality space, the list
           corresponds to the intersection of P with
           a complement of the lineality space.


-- M = inc01(P)    incidence list

   retrieve 0-1 incidence list of polyhedron

   Input:
     P: polyhedron (polyh object)
   Output:
     M: incidence list of P (sparse matrix)

   Remark: In case of a nontrivial lineality space, the list
           corresponds to the intersection of P with
           a complement of the lineality space.
```

5.5.7. Comparison of polyhedra

```
-- f = le(P,Q)    subset
   P <= Q

   test: P subset of Q

   Input:
     P, Q: two polyhedra (polyh objects)
   Output:
     f: flag to indicate whether P is subset of Q (number)


-- f = ge(P,Q)    superset
   P >= Q

   test: P superset of Q

   Input:
     P, Q: two polyhedra (polyh objects)
   Output:
```

```
     f: flag to indicate whether P is superset of Q (number)


-- f = eq(P,Q)    equal
   P == Q

   test: P equal to Q

   Input:
     P, Q: two polyhedra (polyh objects)
   Output:
     f: flag to indicate whether P is equal to Q (number)


-- f = ne(P,Q)    unequal
   P ~= Q

   test: P unequal to Q

   Input:
     P, Q: two polyhedra (polyh objects)
   Output:
     f: flag to indicate whether P is unequal to Q (number)


-- f = lt(P,Q)    proper subset
   P < Q

   test: P proper subset of Q

   Input:
     P, Q: two polyhedra (polyh objects)
   Output:
     f: flag to indicate whether P is proper subset of Q (number)


-- f = gt(P,Q) proper superset
   P > Q

   test: P proper superset of Q

   Input:
     P, Q: two polyhedra (polyh objects)
   Output:
     f: flag to indicate whether P is proper superset of Q (number)
```

5.5.8. Plotting of polyhedra

```
-- plot(P,OPT)    plot

   plot polyhedron

   Input:
     P: polyhedron (polyh object)
     OPT: optional options (struct)

   ------------------------------------------------------------------
   option             default value     explanation
   ------------------------------------------------------------------
   color               [3/4 4/5 17/20] color of relative interior
   linealityspacecolor 'blue'          color of lineality directions
   edgewidth           2               edge width
   edgecolor           'black'         edge color
   dirlength           1               lentgh of extremal directions
   graphmode           0               mode for polyf graph plotting
   vertexsize0         4               vertex size for dimension 0
   vertexsize1         4               vertex size for dimension 1
   vertexsize2         0               vertex size for dimension 2
   vertexsize3         0               vertex size for dimension 3
   ------------------------------------------------------------------
```

5.6. Class polyh – list of supplementary functions in alphabetical order

```
-- P = ball(d)    sum-norm unit ball

   Input:
     d: dimension (number)
   Output:
     P: unit ball (polyh object)


-- P = bensolvehedron(d,m)    bensolvehedron

   that is, a projection of a hypercube of dimension (2m+1)^d
   where the columns of the projection matrix consist of all
   possible arrangements of the set {-m,-(m-1),...,-1,0,1,...,m-1,m}

   Input:
     d: dimension
     m: parameter, e.g., 1,2,3,... (number)
   Output:
     P: bensolvehedron (polyh object)
```

```
-- P = cart({P1,...,Pn})    cartesian product of n polyhedra

   Input:
     {P1 ,..., Pn}: finitely many polyhedra (cell array of polyh objects)
   Output:
     P: cartesian product (polyh object)


-- P = chunion({P1,...,Pn})    closed convex hull of union of n polyhedra

   Input:
     {P1,...,Pn}: finitely many polyhedra (cell array of polyh objects)
   Output:
     P: closed convex hull of union (polyh object)


-- P = cone(d)    standard cone

   Input:
     d: dimension
   Output:
     P: standard cone (polyh object)


-- P = cube(d)    max-norm unit ball

   Input:
     d: dimension
   Output:
     P: max-norm unit ball (polyh object)


-- P = emptyset(d)    empty set

   Input:
     d: space dimension
   Output:
     P: empty set (polyh object)


-- P = intsec({P1,...,Pn})    intersection of n polyhedra

   Input:
     {P1,...,Pn}: finitely many polyhedra (cell array of polyh objects)
   Output:
     P: intersection (polyh object)
```

```
-- P = msum({P1,...,Pn})    Minkowski sum of n polyhedra

   Input:
     {P1,...,Pn}: finitely many polyhedra (cell array of polyh objects)
   Output:
     P: Minkowski sum (polyh object)


-- P = origin(d)    origin

   Input:
     d: space dimension
   Output:
     P: origin (polyh object)


-- P = point(v)    point

   Input:
     v: column vector
   Output:
     P: point (polyh object)


-- P = simplex(d)    regular simplex

   Input:
     d: dimension (number)
   Output:
     P: simplex (polyh object)


-- P = space(d)    whole space polyhedron

   Input:
     d: space dimension
   Output:
     P: space (polyh object)
```

6. polyf – Calculus of Polyhedral Convex Functions

`polyf` is a class for computations with polyhedral convex functions. As we only consider convex functions, we say *polyhedral function* for short. The most important operations for polyhedral functions are:

- pointwise maximum of n polyhedral functions

- lower closed convex envelope of n polyhedral functions
- infimal convolution of n polyhedral functions
- pointwise sum of n polyhedral functions
- conjugate of a polyhedral function
- recession function of a polyhedral function
- computation of domain, range, level sets, recession cone of a polyhedral function
- computation of the subdifferential at some point
- test for pointwise ordering and equality of two polyhedral functions
- plot of polyhedral functions with one or two variables

6.1. Representing polyhedral convex functions

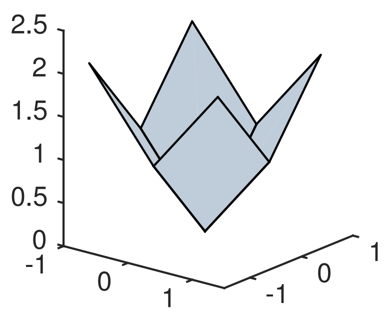
A polyhedral convex function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is represented by its epigraph

$$\operatorname{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} |\ r \geq f(x)\},$$

which is a convex polyhedron. The epigraph of f is stored as a `polyh` object, see Section 5.

Example 6.1 *Consider the sum norm $\|x\|_1 := \sum_{i=1}^n |x_i|$ in $\mathbb{R}^n$. Its epigraph is the cone generated by the set $B \times \{1\}$, where $B := \{x \in \mathbb{R}^n |\ \|x\|_1 \leq 1\}$. To create a* `polyh` *instance of B, we can use the command* `ball`*, see Section 5.*

```
epi=conic(ball(2):point(1));
f=polyf(epi);
plot(f)
```



Alternatively the sum norm can be defined by composition of affine functions:

```
f1=affine([1;0],0);
f2=affine([0;1],0);
f3=affine([-1;0],0);
f4=affine([0;-1],0);
g=fmax(f1,f3) + fmax(f2,f4);
```

The easiest way is to use the `sumnorm` *command:*

```
h=sumnorm(2);
```

All three definitions generate the same function:

```
f==g
g==h
```

```
ans =
     1
ans =
     1
```

The value of f at x, say at $x = (1, 2)^\mathsf{T}$, can be computed as:

```
f=sumnorm(2);
val(f,[1;2])
```

```
ans =
     3
```

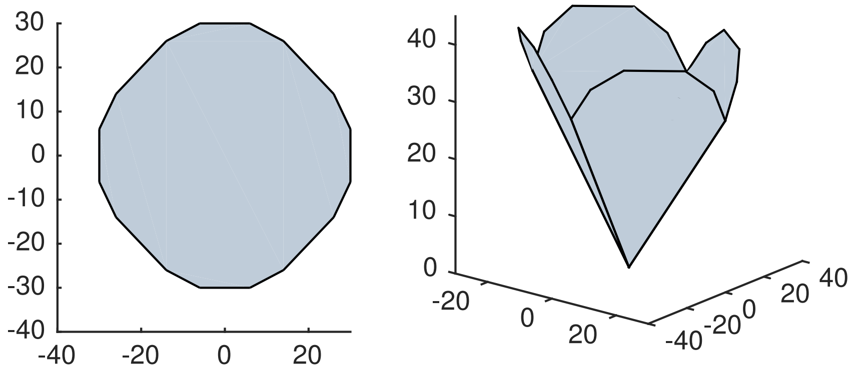
Note that the value of f at x is obtained from the epigraph of f by solving the linear program

$$\min r \quad \text{s.t.} \quad (x, r)^\mathsf{T} \in \operatorname{epi} f.$$

Since $\operatorname{epi} f$ is stored as a P-representation, compare Section 5, the linear program has more than one variable in general.

A `polyf` instance with one ore two variables can be plotted. By default, the plotting region is the interval $[-1, 1]$ and the square $[-1, 1] \times [-1, 1]$, respectively. Other plotting regions are possible:

```
f=sumnorm(2)
R=bensolvehedron(2,2);
plot(R);
plot(f,R);
```

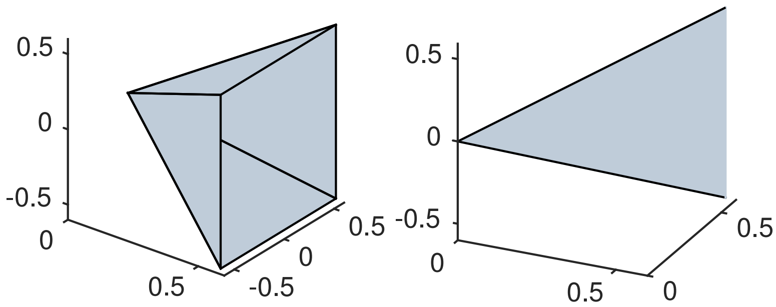


6.2. Calculus examples

Example 6.2 *Subdifferential: We compute the subdifferential of the indicator function of the sum-norm unit ball B at $x = (1, 0, 0)^\mathsf{T}$ and, which is known to be the same, the normal cone of B at x. Then we compute the subdifferential at $x = (1/2, 1/2, 0)^\mathsf{T}$:*

```
B=ball(3);
f=indicator(B);
P=subdiff(f,[1;0;0]);
Q=ncone(B,[1;0;0]);
P==Q
plot(P);
R=subdiff(f,[1/2;1/2;0]);
plot(R);
```

```
ans =
     1
```

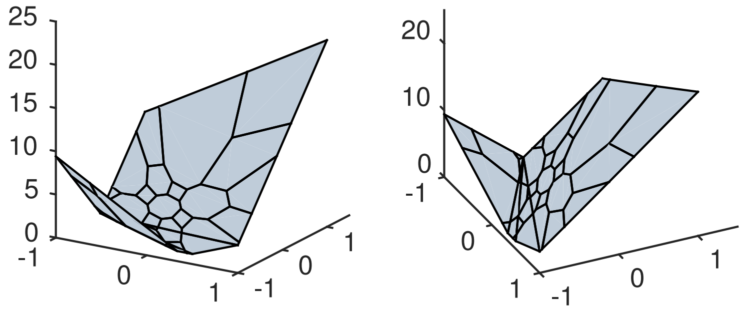


Example 6.3 *Assume we want to compose a polyhedral function of the form*

$$f(x) = \sum_{i=1}^{k} d_i(A^i x + b^i)$$

where $d_i : \mathbb{R}^m \to (-\infty, \infty]$ is a gauge function of a polytope $B_i \in \mathbb{R}^m$ with $0 \in B_i$, $A_i \in \mathbb{R}^{m \times n}$, $b^i \in \mathbb{R}^m$, for $i = 1, \dots, k$. To illustrate the idea, we use a small sample instance with $m = 3$, $n = 2$, $k = 2$:

```
A1=[1 2;2 3;3 4]; b1=[1;2;3];
A2=[3 2;1 4;3 2]; b2=[2;1;0];
B1=ball(3);
B2=bensolvehedron(3,1);
f=precomp(gauge(B1),A1,b1)+precomp(gauge(B2),A2,b2);
plot(f);
```



6.3. Class polyf – list of methods

```
-- polyh : class for calculus of polyhedral convex functions

   Public properties:
     nvar      : number of variables
```

6.3.1. Initialization and evaluation

```
-- f = polyf(P)    constructor

   constructor of polyf class

   Input:
     P: epigraph of polyhedral convex function (polyh object)
   Output:
     f: polyhedral convex function (polyf object)


-- h = eval(f)    evaluate polyf object

   i.e. evaluate epigraph

   Input:
     f: function (polyf object)
   Output:
     h: evaluated function (polyf object)


-- h = reinit(f,REPTYPE)    re-initialize function

   re-initialize polyf object using the V- or H-representation of the epigraph
   (can simplify further computations but requires evaluation)
```

```
Input:
  f: function (polyf object)
Optional input:
  REPTYPE: type of representation: 'v' (default) or 'h' (char)
Output:
  h: function (polyh object)
```

6.3.2. Basic operations and retrieving related objects

```
-- y = val(f,x)    value y=f(x)

   Input:
     f: function (polyf object)
     x: argument (column vector)
   Output:
     y: value y=f(x) (number)

   Remark: command requires to solve one linear program


-- P = epi(f)    epigraph

   epigraph of function f

   Input:
     f: function (polyf object)
   Output:
     P: epigraph of f (polyh object)


-- P = dom(f)    domain

   domain of function f

   Input:
     f: function (polyf object)
   Output:
     P: domain of f (polyh object)


-- P = level(f,a)    sublevel set

   sublevel set of function of f w.r.t. level a

   Input:
     f: function (polyf object)
```

```
    a: level (number)
  Output:
    P: sublevel set (polyh object)


-- P = recc(f)    recession cone of polyf object

  i.e. the recession cone of nonempty sublevel sets

  Input:
    f: function (polyf object)
  Output:
    P: recession cone of f (polyh object)


-- P = subdiff(f,x)    subdifferential

  subdifferential of proper function f at argument x

  Input:
    f: function (polyf object)
    x: argument (column vector)
  Output:
    P: subdifferential (polyh object)
```

6.3.3. Property checking

```
-- flag = iseval(f)    check whether function is evaluated

  Input:
    f: function (polyf objects)
  Output:
    flag: nonzero if function is evaluated (number)


-- flag = isimproper(f)    check whether function is improper

  Input:
    f: function (polyf objects)
  Output:
    flag:
       0 = proper
       1 = improper with nonempty domain
       2 = improper with empty domain
```

6.3.4. Calculus and composition of polyhedral convex functions

```
-- h = fmax(f,g)    pointwise maximum of two functions
   h = f & g

   Input:
     f,g: two functions (polyf objects)
   Output:
     h: pointwise maximum function (polyf object)


-- h = finfc(f,g)    infimal convolution of two functions

   Input:
     f,g: two functions (polyf objects)
   Output:
     h: infimal convolution (polyf object)


-- h = fenv(f,g)    lower closed convex envelope of two functions
   h = f | g

   Input:
     f,g: two functions (polyf objects)
   Output:
     h: lower closed convex envelope (polyf object)


-- h = fsum(f,g)    sum of two functions
   h = f + g

   Input:
     f,g: two functions (polyf objects)
   Output:
     h: sum (polyf object)


-- h = precomp(f,M,v)    pre-composition with affine transformation

   h(x) = f(M x + v)

   Input:
     f: function (polyf object)
     M: matrix
     v: column vector
   Output:
     h: function (polyf object)
```

```
-- h = conj(f)    conjugate function

   Input:
     f: function (polyf object)
   Output:
     h: conjugate function (polyf object)


-- h = recf(f)    recession function

   recession function of f

   Input:
     f: function (polyf object)
   Output:
     h: recession function of f (polyf object)
```

6.3.5. Comparing polyhedral convex functions

```
-- flag = le(f,g)    <= for two functions
   f <= g

   Input:
     f,g: two functions (polyf objects)
   Output:
     flag: nonzero if f(x)<=g(x) for all x (number)


-- flag = ge(f,g)    >= for two functions
   f >= g

   Input:
     f,g: two functions (polyf objects)
   Output:
     flag: nonzero if f(x)>=g(x) for all x (number)


-- flag = eq(f,g)    == for two functions
   f == g

   Input:
     f,g: two functions (polyf objects)
   Output:
     flag: nonzero if f(x)==g(x) for all x (number)
```

```
-- flag = ne(f,g)    ~= for two functions
   f ~= g

   Input:
     f,g: two functions (polyf objects)
   Output:
     flag: nonzero if f(x)~=g(x) for some x (number)


-- flag = lt(f,g)    (<= and ~=)  for two functions
   f < g

   Input:
     f,g: two functions (polyf objects)
   Output:
     flag: nonzero if f(x)<=g(x) for all x and f(x)<g(x) for some x (number)


-- flag = gt(f,g)    (>= and ~=)  for two functions
   f > g

   Input:
     f,g: two functions (polyf objects)
   Output:
     flag: nonzero if f(x)>=g(x) for all x and f(x)>g(x) for some x (number)
```

6.3.6. Plotting polyhedral convex functions

```
-- plot(f,P,OPT)    plot function

   Input:
     f: function (polyf object)
     P: bounded region where f is plotted (poly object)
     OPT: optional options (struct)

   ------------------------------------------------------------------
   option              default value    explanation
   ------------------------------------------------------------------
   color                [3/4 4/5 17/20] color of relative interior
   linealityspacecolor 'blue'           color of lineality directions
   edgewidth            2               edge width
   edgecolor            'black'         edge color
   dirlength            0               lentgh of extremal directions
   graphmode            1               mode for polyf graph plotting
   vertexsize0          4               vertex size for dimension 0
   vertexsize1          4               vertex size for dimension 1
```

```
vertexsize2        0              vertex size for dimension 2
vertexsize3        0              vertex size for dimension 3
------------------------------------------------------------------
```

6.4. Class polyf – list of supplementary functions in alphabetical order

```
-- f = affine(a,b)    affine function

   f(x)=a^T x + b

   Input:
     a: column vector a
     b: number b
   Output:
     f: affine function (polyf object)


-- f = fenv({f1,...,fn})    lower closed convex envelope

   lower closed convex envelope of n polyhedral functions

   corresponds to convex hull of the union of epigraphs

   Input:
     {f1,...,fn}: n polyhedral convex functions with same number
                  of variables (cell array of polyf objects)
   Output:
     f: lower closed convex envelope (polyf object)


-- f = finfc({f1,...,fn})    infimal convolution

   infimal convolution of n polyhedral functions

   corresponds to Minkowski sum of epigraphs

   Input:
     {f1,...,fn}: polyhedral convex functions with same number
                  of variables (cell array of polyf objects)
   Output:
     f: infimal convolution (polyf object)


-- f = fmax({f1,...,fn})    pointwise maximum

   pointwise maximum of n polyhedral functions
```

```
   corresponds to intersection of epigraphs

   Input:
     {f1,...,fn}: polyhedral convex functions with same number
                  of variables (cell array of polyf objects)
   Output:
     f: pointwise maximum (polyf object)


-- f = fsum({f1,...,fn})    pointwise sum of n polyhedral functions

   Input:
     {f1,...,fn}: n polyhedral convex functions with same number
                  of variables (cell array of polyf objects)
   Output:
     f: pointwise sum (polyf object)


-- f = gauge(P)    gauge function

   gauge function f of a polyhedral set P,
   where zero must be contained in P

   f(x) = inf{r > 0 | x \in r P}

   Input:
     P: polyhedon (polyh object)
   Output:
     f: gauge function (polyf object)


-- f = indicator(P)    indicator function

   indicator function f of a polyhedral set P

   Input:
     P: polyhedon (polyh object)
   Output:
     f: indicator function (polyf object)


-- f = maxnorm(n)    maximum norm

   Input:
     n: number of variables
   Output:
```

```
      f: maximum norm (polyf object)


 -- f = sumnorm(n)    sum norm

    Input:
      n: number of variables
    Output:
      f: sum norm (polyf object)


 -- f = translative(P,C,k)    translative function

    translative function of polyhedron P, polyhedral cone C, vector k in C

      f(x)=inf{r : x + r*k in P+C}

    Input:
      P: polyhedron (polyh object)
      C: polyhedral cone (polyh object)
      k: column vector
    Output:
      f: translative function (polyf object)
```

7. lpsolve – Solving Linear Programs

`lpsolve` can be used to solve linear programs. It is based on the GNU linear programming kit (GLPK). The input data consist of the objective function c, which is a column vector, the feasible set S, which is a `polyh` object, and an optimization direction.

```
 -- [optval,sol_p,sol_d,status] = lpsolve(c,S,optdir)    solve linear program

    min c^T y  s.t.  y in S

    where S is given by a P-representation:

    S = {Mx : l <= x <= u, a <= Bx <= b}

    Input:
      c        objective function (column vector)
      S        feasible set (polyh object)
      optdir   'min' (default) or 'max'
    Output:
      optval    optimal value of the problem (number)
      sol_p     primal solution (column vector)
      sol_d     dual solution (column vector)
```

```
    status    solution status (string)

  Remark:
    the dual solution refers to the P-representation of S and consists of
    row variables:     the first m entries
    column variables:  the last n entries
    where [m,n] = size(S.prep.B)
```

Example 7.1 *Consider the minimization problem with $c^{\mathsf{T}} = (0,\dots,0,1,\dots,1)$ where the feasible set is the sum-norm unit ball.*

```
n=5;
k=2;
c=[zeros(n-k,1);ones(k,1)];
S=ball(n);
[optval,sol_p]=lpsolve(c,S)
```

```
optval = -1
sol_p =
   0
   0
   0
   0
  -1
```

The following code computes the set of all optimal solutions:

```
...
rep.B=c';
rep.b=optval;
H=polyh(rep,'h');
P=S&H;
vrep(P)
```

An alternative variant using the `polyf` *class is:*

```
...
P=S&level(affine(c,0),optval);
vrep(P)
```

```
ans =
  scalar structure containing the fields:
    L = [](5x0)
    V =
       0   0
       0   0
       0   0
      -1   0
       0  -1
    D = [](5x0)
```

Note that the last command (computation of a V-representation) requires evaluation and is therefore expensive in larger example. Less expensive is, for instance, the computation of the dimension of the set of all solutions:

```
...
dim(P)
```

```
ans =  1
```

8. pcpsolve – Solving Polyhedral Convex Programs

The command `pcpsolve` provides a convenient way to solve polyhedral convex programs. Internally, the polyhedral convex program is reformulated as a linear program and solved by GLPK.

```
 -- [optval, sol]=pcpsolve(f,S)    solve polyhedral convex program

    minimize f(x)  s.t.  x in S

    where S is given by a P-representation:

    S = {Mx : a <= Bx <= b , l <= x <= u}

    Input:
      f     polyhedral convex objective function (polyf object)
      S     feasible set (polyh object)
    Output:
      optval: optimal value
      sol: an optimal solution (column vector)
```

Example 8.1 *Locational analysis: Given five points in the plane:* $a^{(1)} = (1,4)^\mathsf{T}$, $a^{(2)} = (2,2)^\mathsf{T}$, $a^{(3)} = (3,3)^\mathsf{T}$, $a^{(4)} = (1,2)^\mathsf{T}$, $a^{(5)} = (6,5)^\mathsf{T}$, *we are looking for* $x \in \mathbb{R}^2$ *minimizing the function*

$$\sum_{i=1}^{5} \|x - a^{(i)}\|_1$$

subject to the constraint $x_1 + x_2 \leq 2$*:*

```
A=[1 2 3 1 6;4 2 3 2 5];
m=size(A,2);
C=cell(m,1);
for i=1:m
  C{i,1}=precomp(sumnorm(2),eye(2),-A(:,i));
end
S=level(affine([1;1],0),2);
[optval,sol]=pcpsolve(fsum(C),S)
```

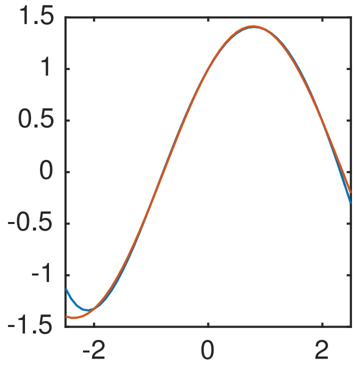
```
optval =  19
sol =
   0
   2
```

Example 8.2 *Chebyshev Approximation: The function* $f(x) = cos(x) + sin(x)$ *is approximated by a polynomial* $g_a(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ *of degree 4 at a finite set* $X = [-2, -1, 0, 1, 2]$ *by solving the polyhedral convex optimization problem*

$$\min_{a \in \mathbb{R}^5} \max_{x \in X} |f(x) - g_a(x)|.$$

```
X=-2:1:2;
m=length(X);
f=@(x)cos(x)+sin(x);
g=@(x)([x.^4;x.^3;x.^2;x.^1;x.^0]);
C=cell(2*m,1);
for i=1:m
  C{i,1}  = affine( g(X(i)),-f(X(i)));
  C{i+m,1}= affine(-g(X(i)), f(X(i)));
end
[approx_error,a]=pcpsolve(fmax(C),space(5))
h=@(x)(a'*g(x));
x=-2.5:.1:2.5;
plot(x,h(x),x,f(x));
```

```
approx_error = 0
a =
   0.035220
  -0.128941
  -0.494918
   0.970412
   1.000000
```



9. qcsolve – Global Optimization Solver

The command `qcsolve` provides a solver for a class of global optimization problems with quasi-concave objective function. For more information about the theoretical background, the reader is referred to [3].

```
 -- [fmin,x] = qcsolve (S,fname,P,args)    solve quasi-concave program

    solve quasi-concave global optimization problem

    minimze f(Px) s.t. x in S

     where

     -- f is a function from R^p to [-inf,Inf) which is quasi-concave
        (i.e. f has convex super-level sets)
     -- P is a p times n matrix
     -- S is a polyhedral feasible set in R^n such that P[S] is bounded

    If f is monotone with respect to a polyhedral pointed convex cone C
    on dom f = {y | f(y) > -Inf}, then C can be used as optional input argument.
    Specifying such a cone can speed up the algorithm. Moreover, boundedness
    of P[S] can be weakened to C-boundedness of P[S]. C-monotonicity is not
    checked by the program and has to be ensured by the user.

    Input:
      S:      feasible set S (polyh object)
      fname:  name of the objective function (string)
              for requirements of the function itself, see below.
      P:      matrix
      args:   optional arguments:
        args.C:             monotonicity cone C (polyh object)
        args.opt.display    flag to display solution
    Output:
      fmin:  optimal value
```

```
  x:     optimal solution

Remark: The objective function f is required to be given as
        Matlab/Octave function. A single argument of f is a
        column vector. It is important to guarantee, that
        multiple arguments are possible: If the input for f is a
        matrix X the output of f is expected to be a row vector
        the entries of which are the functions values of the
        corresponding columns of X.
```

Example 9.1 *Concave quadratic programming: Let $Q \in \mathbb{R}^{n\times n}$ be a positive semi-definite symmetric matrix. Then, the function $g\colon \mathbb{R}^n \to \mathbb{R}$ with $g(x) = -x^{\mathsf{T}}Qx$ is concave, hence quasi-concave. In order to minimize g under linear constraints, Q can be factorized, i.e. $Q = P^{\mathsf{T}}P$ for some matrix $P \in \mathbb{R}^{p\times n}$. We obtain an appropriated problem instance*

$$\min f(Px) \quad \text{s.t.} \quad x \in S$$

for `qcsolve` *with $f : \mathbb{R}^p \to \mathbb{R}$ with $f(y) := -y^{\mathsf{T}}y$.*

Let us solve this global optimization problem for the choice (compare [8, 3])

$$P_{ij} = \lfloor p \cdot \sin((j-1)\cdot p + i)\rfloor,$$

where $\lfloor x\rfloor := \max\{z \in \mathbb{Z} \mid z \le x\}$ and $S = \{x \in \mathbb{R}^n \mid -e \le x \le e\}$.

First, we need to store the objective function f in a file, say in `f.m`*:*

```
function y=f(x)
  y=-sum(x.^2);
end
```

To illustrate the preceding remark about multiple arguments of f, consider:

```
X=[1 2 3;3 4 5]
Y=f(X)
```

```
X =
    1     2     3
    3     4     5
Y =
  -10   -20   -34
```

The correct result is obtained with our function, whereas `y=-X'*X` *yields a wrong result in case* `X` *has more than one column.*

Next we generate the matrix P and the feasible region S and call `qcsolve`*:*

```
n=1000; p=6;
P=floor(p*sin(reshape(1:p*n,p,n)));
S=cube(n);
qcsolve(S,'f',P)
```

In order to run the dual algorithm described in [3], an option for bensolve *must be set:*

```
set_bensolve_option('a','dual');
```

See Section 12 for more details.

10. molpsolve – Multiple Objective Linear Programming Solver

The command `molpsolve` can be used to solve a multiple objective linear program of the form

$$\min Px \quad \text{s.t.} \quad x \in S. \tag{MOLP}$$

where the feasible set S is a polyhedron given as a `polyh` object, see Section 5, and P is a $q \times n$ matrix. For more information see e.g. [10].

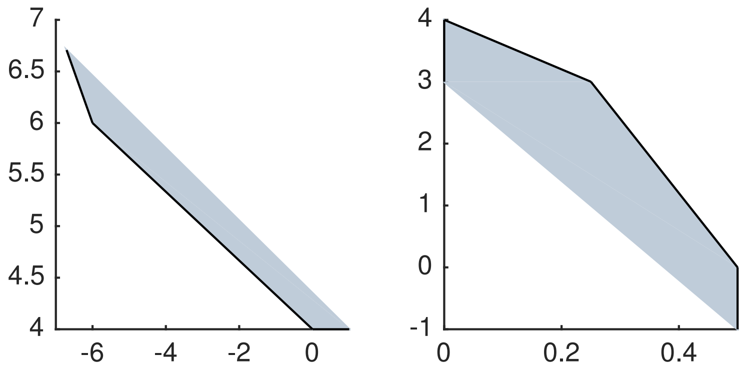
```
 -- [img_p,img_d,sol_p,sol_d]=molpsolve(P,S,optdir)    solve MOLP

    solve multiple objective linear program

    minimize Px  s.t  x in S

    where S is given by a P-representation:

    S = {Mx : l <= x <= u, a <= Bx <= b}

    Input:
      P:  objective matrix
      S:  feasible set (polyh object)
      optdir:  'min' (default) or 'max'
    Output:
      img_p:  extended image of the primal problem (polyh object)
      img_d:  extended image of the dual problem (polyh object)
      sol_p:  primal solution (matrix)
      sol_d:  dual solution (matrix)

    Remark:
      the dual solution refers to the P-representation of S and consists of
      row variables:     the first m entries
      column variables:  the next n entries
      weight variables:  the last q entries
      where [m,n] = size(S.prep.B) and q is the number of objectives
      see Section 1.7 at http://bensolve.org/files/manual.pdf
```

Example 10.1 *Consider the following MOLP with two objectives:*

$$\min \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} \quad s.t. \quad 6 \le 2x_1 + x_2,\ 6 \le x_1 + 2x_2,\ x_1 \ge 0,\ x_2 \ge 0$$

```
clear('rep');
rep.B=[2 1;1 2];
rep.a=[6;6];
rep.l=[0;0];
S=polyh(rep,'h');
P=[1 -1; 1 1];
[PP,DD,sol]=molpsolve(P,S);
plot(PP)
plot(DD)
```

The picture shows the upper image of (MOLP) *and the lower image of its dual problem.*




```
sol =
  0  2  0  0
  6  2  1  0
```

To interpret the solution we need to know the number of vertices of the V-representation of `PP`*:*

```
size(vrep(PP).V,2)
```

```
ans =  2
```

There are two vertices, hence the first two columns of `sol` *are points and the third column is a direction. Zero directions are not part of the solution of* (MOLP)*. Here the fourth column of* `sol` *corresponds to*

```
[vrep(PP).V, vrep(PP).D](:,4)
```

```
ans =
   1
   0
```

The vector $(1,0)^\mathsf{T}$ belongs to the ordering cone $\mathbb{R}^2_+$. Therefore it is not a minimal direction and does not belong to a solution, see e.g. [10] for further explanation.

11. vlpsolve – Vector Linear Programming Solver

The command `vlpsolve` provides a convenient way to use the VLP solver *bensolve*.

```
 -- [img_p,img_d,c,sol_p,sol_d]=vlpsolve(P,S,optdir,C,c)    solve VLP

    solve vector linear program

    minimize Px  s.t  x in S  w.r.t  ordering cone C

    where S is given by a P-representation:

    S = {Mx : l <= x <= u, a <= Bx <= b}

    Input:
      P        objective matrix
      S        feasible set (polyh object)
      optdir   'min' (default) or 'max'
      C        ordering cone (polyh object)
      c        duality parameter vector
    Output:
      img_p:    extended image of the primal problem (polyh object)
      img_d:    extended image of the dual problem (polyh object)
      c_ret:    duality parameter corresponding to dual solution
      sol_p:    primal solution (matrix)
      sol_d:    dual solution (matrix)

    Remark:
      the dual solution refers to the P-representation of S and consists of
      row variables:     the first m entries
      column variables:  the next n entries
      weight variables:  the last q entries
      where [m,n] = size(S.prep.B) and q is the number of objectives
      see Section 1.7 at http://bensolve.org/files/manual.pdf
```

Example 11.1 *Let the following code define the feasible set S of a vector linear program:*

```
clear('rep');
rep.B=[ones(1,3);1 2 2;2 2 1;2 1 2];
rep.a=[1;3/2;3/2;3/2];
rep.l=[0;0;0];
S=polyh(rep,'h');
```

The objective matrix of the vector linear program is:

```
P=[1 0 1; 1 1 0; 0 1 1];
```

Let the ordering cone C be generated by the vectors

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \begin{pmatrix}-1\\0\\2\end{pmatrix} \quad \begin{pmatrix}0\\-1\\2\end{pmatrix},$$

which can be entered as:

```
clear ('rep');
rep.V=[0 0 0]';
rep.D=[1 0 0; 0 1 0; -1 0 2; 0 -1 2]';
C=polyh(rep,'v');
```

It is possible to specify a geometric duality parameter vector c. It must belong in the interior of C and must have a nonzero last component. If c is not specified, it is computed by the solver.

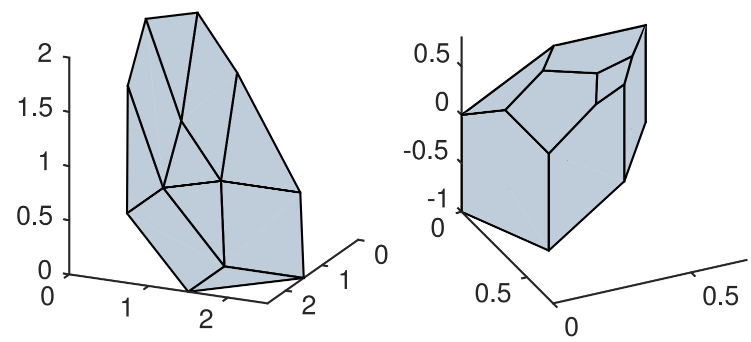
```
c=[2;2;2];
```

Now the vector linear programming solver is called.

```
[img_p,img_d,c,sol_p]=vlpsolve(P,S,'min',C,c);
```

We plot the upper image of the primal problem and the lower image of the dual problem:

```
plot(img_p);
plot(img_d);
```



Now we display a V-representation of the upper image of the primal problem:

```
vrep(img_p)
```

```
V =
   1.5   0.5   0.5   1
   1.5   1     0.5   0.5
   0     0.5   1     0.5
D =
   0     1    -0.5   0
   1     0     0    -0.5
   0     0     1     1
L = [](3x0)
```

The duality parameter vector is returned by the solver. Displaying the returned vector shows that c is scaled by the solver. The dual lower image is defined with respected to this scaled vector c.

```
c =
   1
   1
   1
```

Now a solution of the vector linear program is displayed:

```
sol_p
```

```
sol_p =
   1.5   0.5   0     0.5   0   0   0   0
   0     0.5   0.5   0     0   0   0   0
   0     0     0.5   0.5   0   0   0   0
```

A solution has to be interpreted together with the V-representation of the upper image of the primal problem. The upper image has four vertices, hence the first four columns of the solution are points and the remaining columns are directions. The V-representation of the optimal solution has four extremal directions, however all four belong to the ordering cone. Thus the last four zero columns are just place holders, they do not belong to the solution.

Further details can be found in [10] and in the *bensolve* reference manual[1].

12. Bensolve options

bensolve tools is based on the VLP solver *bensolve*. The command `qcsolve` is based on a modified variant of *bensolve*. An option for *bensolve* can be set globally using the command `set_bensolve_option`.

```
-- set_bensolve_option(fn,val)    set options for bensolve

   Input:
     fn: fieldname (string)
     val: value (of different type)
   No input:
     reset of options

   --------------------------------------------------------------------------
   fn   val              explanation
   --------------------------------------------------------------------------
   'b'  0 1              assume VLP to be bounded (can be faster)
   'g'  0 1              enable global optimization mode (for internal use)
```

[1] http://bensolve.org/files/manual.pdf

```
's'  0 1              enable outut of solutions (pre-image information)
'k'  * (see below)    simplex type in phase 0 of Benson's algorithm
'L'  * (see below)    simplex type in phase 1 of Benson's algorithm
'l'  * (see below)    simplex type in phase 2 of Benson's algorithm
'm'  '0' '1' '2' '3'  display less or more messages
'M'  '0' '1' '2' '3'  display less or more messages of internal lp solver
'A'  'primal' 'dual'  type of Benson algorithm in phase 1
'a'  'primal' 'dual'  type of Benson algorithm in phase 2
'E'  e.g. '1e-6'      epsilon for Benson algorithm in phase 1
'e'  e.g. '1e-6'      epsilon for Benson algorithm in phase 2
--------------------------------------------------------------------------

* 'primal_simplex' 'dual_simplex' 'dual_primal_simplex'
```

Example 12.1 *Increasing the message level of* bensolve*:*

```
set_bensolve_option('m','3');
A=eval(ball(2));
```

Choosing the dual variant of Benson's algorithm:

```
set_bensolve_option('a','dual');
A=eval(ball(2));
```

Reset to default options:

```
set_bensolve_option();
```

References

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[5] A. H. Hamel, A. Löhne, and B. Rudloff. A Benson type algorithm for linear vector optimization and applications. *Journal of Global Optimization*, 59(4):811–836, 2014. see also http://arxiv.org/abs/1302.2415.

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A. APPENDIX: GNU GENERAL PUBLIC LICENSE

Preamble

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