On Convergence of Closed Convex Sets

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Abstract

In this paper we introduce a convergence concept for closed convex subsets of a finite dimensional normed vector space. This convergence is called $\mathcal{C}$*-convergence*. It is defined by appropriate notions of upper and lower limits. We compare this convergence with the well-known Painlevé–Kuratowski convergence and with scalar convergence. In fact, we show that a sequence $(A_n)_{n\in\mathbb{N}}$ $\mathcal{C}$-converges to A if and only if the corresponding support functions converge pointwise, except at relative boundary points of the domain of the support function of A, to the support function of A.

1 Introduction

In this paper we introduce a convergence concept in the space of closed convex subsets of a finite dimensional normed vector space X, called $\mathcal{C}$-convergence. We compare this concept with the well-known Painlevé–Kuratowski convergence (shortly PK-convergence), e.g. [6], [2], [9] as well as with scalar convergence (i.e., the pointwise convergence of support functions) of convex sets [13], [10], [11], [12]. This paper is organized as follows. Section 2 is devoted to upper and lower limits in complete lattices. In Section 3 we introduce the $\mathcal{C}$-convergence by upper and lower limits in the complete lattice $(\mathcal{C}, \subset)$ of all closed convex subsets of X. Section 4 is devoted to the relationship between PK-convergence and $\mathcal{C}$-convergence. In Section 5 we investigate the relationship between scalar convergence and $\mathcal{C}$-convergence, which leads to a characterization of $\mathcal{C}$-convergence.

In the sequel X is a finite dimensional real normed vector space whose dimension is $p \geq 1$. Of course, we could identify X with $\mathbb{R}^p$. For nonempty sets $A, B \subset X$ and $\alpha \in \mathbb{R}$ we use the usual Minkowski sum and multiplication:

$$A + B := \{a + b \mid a \in A,\ b \in B\}, \qquad \alpha A := \alpha \cdot A := \{\alpha a \mid a \in A\};$$

moreover, we use the conventions:

$$A + \emptyset := \emptyset + A := \emptyset + \emptyset := \emptyset, \quad 0 \cdot \emptyset := \{0\}, \quad \alpha \cdot \emptyset := \emptyset \text{ if } \alpha \neq 0.$$

We denote by $\mathcal{F} := \mathcal{F}(X)$ the space of closed subsets of X. In $\mathcal{F}$ we introduce an addition $\oplus :\ \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, defined by $A \oplus B := \operatorname{cl}(A + B)$, and a multiplication by real numbers, defined as above. In the sequel $\mathcal{F}$ is equipped with the partial order defined by the usual set inclusion $\subset$. Then $\mathcal{F}$ is a complete lattice; it is easily seen that the supremum and infimum for a nonempty subset $\mathcal{A} \subset \mathcal{F}$, denoted by $\operatorname{SUP}\mathcal{A}$ and $\operatorname{INF}\mathcal{A}$, can be expressed by

$$\operatorname{SUP}\mathcal{A} = \operatorname{cl} \bigcup_{A \in \mathcal{A}} A, \qquad \operatorname{INF}\mathcal{A} = \bigcap_{A \in \mathcal{A}} A. \tag{1}$$

As usual, we set $\operatorname{INF}\emptyset := \operatorname{SUP}\mathcal{F} = X$ and $\operatorname{SUP}\emptyset := \operatorname{INF}\mathcal{F} = \emptyset$.

Let us recall some basic properties of the well-known PK-convergence. Our main reference is the book of Rockafellar and Wets [9]. We use the following notation of [9]:

$$\mathcal{N}_\infty := \{N \subset \mathbb{N} \mid \mathbb{N} \setminus N \text{ finite}\} \quad \text{and} \quad \mathcal{N}_\infty^\# := \{N \subset \mathbb{N} \mid N \text{ infinite}\}. \tag{2}$$

For a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ the *outer* and *inner limits* are the sets

$$\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n := \big\{x \in X \mid \exists N \in \mathcal{N}_\infty^\#,\ \forall n \in N,\ \exists x_n \in A_n : x_n \xrightarrow{N} x\big\},$$
$$\operatorname*{LIMINF}_{n\in\mathbb{N}} A_n := \big\{x \in X \mid \exists N \in \mathcal{N}_\infty,\ \forall n \in N,\ \exists x_n \in A_n : x_n \xrightarrow{N} x\big\},$$

respectively. The limit of a sequence $(A_n)_{n\in\mathbb{N}}$ exists if its outer and inner limits coincide. Then we speak about PK-convergence of $(A_n)_{n\in\mathbb{N}}$ and we write

$$\operatorname*{LIM}_{n\in\mathbb{N}} A_n := \operatorname*{LIMINF}_{n\in\mathbb{N}} A_n = \operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n.$$

As mentioned in [9, page 111], the natural setting for the study of PK-convergence is the space $\mathcal{F}$ of closed subsets of X. In [9], the closedness of the members of the sequence is not supposed a priori. However, as it can be seen in [9, Prop. 4.4], the outer and inner limits only depend on the closure of the sequence's members. In contrast to [9], we use capital letters in the notation of the (outer and inner) limit. This is because the notation with small letters is reserved for the (upper and lower) limit in the space $\mathcal{C}$ of closed convex sets of X to be defined later on. It is an easy task (see [9, Exer. 4.2(b)]) to show that the outer and inner limits of a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ can be expressed by the formulas

$$\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n = \bigcap_{N\in\mathcal{N}_\infty} \operatorname{cl} \bigcup_{n\in N} A_n, \qquad \operatorname*{LIMINF}_{n\in\mathbb{N}} A_n = \bigcap_{N\in\mathcal{N}_\infty^\#} \operatorname{cl} \bigcup_{n\in N} A_n.$$

The expressions above show that in the space $\mathcal{F}$ we have

$$\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n = \operatorname*{INF}_{N\in\mathcal{N}_\infty} \operatorname*{SUP}_{n\in N} A_n, \qquad \operatorname*{LIMINF}_{n\in\mathbb{N}} A_n = \operatorname*{INF}_{N\in\mathcal{N}_\infty^\#} \operatorname*{SUP}_{n\in N} A_n. \tag{3}$$

These formulas are the starting point for defining the upper and lower limits of a sequence in $\mathcal{C}$. First we give these definitions in a general complete lattice.

2 Lower and upper limits in complete lattices

Let $(L, \leq)$ be a complete lattice, that is $(L, \leq)$ is a partially ordered space with the property that every nonempty subset A of L has a greatest lower bound, denoted by $\inf A$, and a smallest upper bound, denoted by $\sup A$. In particular L has a smallest element $l_0 = \inf L$ and a largest element $l_1 = \sup L$. As usual, we consider that $\inf \emptyset := l_1$ and $\sup \emptyset := l_0$. It is obvious that for $A \subset B \subset L$ we have $\inf A \geq \inf B$ and $\sup A \leq \sup B$; moreover, if $A \neq \emptyset$, $\inf A \leq \sup A$. Similarly to the definitions of $\mathcal{N}_\infty$ and $\mathcal{N}_\infty^\#$ in (2), for an infinite subset M of $\mathbb{N}$ we introduce the families

$$\mathcal{N}_\infty(M) := \{P \subset M \mid M \setminus P \text{ finite}\} \quad \text{and} \quad \mathcal{N}_\infty^\#(M) := \{P \subset M \mid P \text{ infinite}\}.$$

It is clear that $\mathcal{N}_\infty(M) \subset \mathcal{N}_\infty^\#(M)$. Moreover, for $M \in \mathcal{N}_\infty^\#$ and $N \in \mathcal{N}_\infty^\#(M)$ one has

$$\mathcal{N}_\infty(N) = \{P \cap N \mid P \in \mathcal{N}_\infty(M)\}, \quad \mathcal{N}_\infty^\#(N) = \{P \in \mathcal{N}_\infty^\#(M) \mid P \subset N\} \subset \mathcal{N}_\infty^\#(M),$$

while for $N \in \mathcal{N}_\infty(M)$ one has

$$\mathcal{N}_\infty^\#(N) = \{P \cap N \mid P \in \mathcal{N}_\infty^\#(M)\}, \quad \mathcal{N}_\infty(N) = \{P \in \mathcal{N}_\infty(M) \mid P \subset N\} \subset \mathcal{N}_\infty(M). \tag{4}$$

Consider an infinite set $M \subset \mathbb{N}$ (that is $M \in \mathcal{N}_\infty^\#$) and $(x_n)_{n\in M} \subset L$. Inspired by the formulas for outer and inner limits of a sequence of elements of $\mathcal{F}$ in (3), we define the upper and lower limits of $(x_n)_{n\in M}$ by

$$\limsup_{n\in M} x_n := \inf_{N\in\mathcal{N}_\infty(M)} \sup_{n\in N} x_n, \quad \liminf_{n\in M} x_n := \inf_{N\in\mathcal{N}_\infty^\#(M)} \sup_{n\in N} x_n.$$

The following result is a collection of several simple, but useful, properties of these limits.

Proposition 2.1 *Let $M \in \mathcal{N}_\infty^\#$ and consider the sequence $(x_n)_{n\in\mathbb{N}} \subset L$.*
(i) $\liminf_{n\in M} x_n \le \limsup_{n\in M} x_n$.
(ii) *Assume that $N \in \mathcal{N}_\infty^\#(M)$; then*

$$\limsup_{n\in N} x_n \le \limsup_{n\in M} x_n \quad \textit{and} \quad \liminf_{n\in N} x_n \ge \liminf_{n\in M} x_n.$$

(iii) *Assume that $N \in \mathcal{N}_\infty(M)$; then*

$$\limsup_{n\in N} x_n = \limsup_{n\in M} x_n \quad \textit{and} \quad \liminf_{n\in N} x_n = \liminf_{n\in M} x_n.$$

(iv) *Let $(y_n)_{n\in\mathbb{N}} \subset L$ be another sequence and let $N \in \mathcal{N}_\infty(M)$ be such that $x_n \le y_n$ for every $n \in N$, then*

$$\limsup_{n\in M} x_n \le \limsup_{n\in M} y_n \quad \textit{and} \quad \liminf_{n\in M} x_n \le \liminf_{n\in M} y_n.$$

(v) *Let be given a family of sequences $(z_n^i)_{n\in\mathbb{N}} \subset L$, where i belongs to a nonempty set I. Then*

$$\limsup_{n\in M} \inf_{i\in I} z_n^i \le \inf_{i\in I} \limsup_{n\in M} z_n^i, \quad \limsup_{n\in M} \sup_{i\in I} z_n^i \ge \sup_{i\in I} \limsup_{n\in M} z_n^i,$$
$$\liminf_{n\in M} \inf_{i\in I} z_n^i \le \inf_{i\in I} \liminf_{n\in M} z_n^i, \quad \liminf_{n\in M} \sup_{i\in I} z_n^i \ge \sup_{i\in I} \liminf_{n\in M} z_n^i.$$

Proof. (i) Since $\mathcal{N}_\infty(M) \subset \mathcal{N}_\infty^\#(M)$, it is clear that $\liminf_{n\in M} x_n \le \limsup_{n\in M} x_n$.
(ii) Let $N \in \mathcal{N}_\infty^\#(M)$. Since $\mathcal{N}_\infty(N) = \{P \cap N \mid P \in \mathcal{N}_\infty(M)\}$, we have

$$\limsup_{n\in N} x_n = \inf_{P\in\mathcal{N}_\infty(N)} \sup_{n\in P} x_n = \inf_{P\in\mathcal{N}_\infty(M)} \sup_{n\in P\cap N} x_n \le \inf_{P\in\mathcal{N}_\infty(M)} \sup_{n\in P} x_n = \limsup_{n\in M} x_n.$$

Since $\mathcal{N}_\infty^\#(N) \subset \mathcal{N}_\infty^\#(M)$, we have

$$\liminf_{n\in N} x_n = \inf_{P\in\mathcal{N}_\infty^\#(N)} \sup_{n\in P} x_n \ge \inf_{P\in\mathcal{N}_\infty^\#(M)} \sup_{n\in P} x_n = \liminf_{n\in M} x_n.$$

(iii) Let $N \in \mathcal{N}_\infty(M)$. Since $N \in \mathcal{N}_\infty^\#(M)$, taking into account (ii), we have only to show that $\limsup_{n\in N} x_n \ge \limsup_{n\in M} x_n$ and $\liminf_{n\in N} x_n \le \liminf_{n\in M} x_n$. Indeed, since $\mathcal{N}_\infty(N) \subset \mathcal{N}_\infty(M)$ we have

$$\limsup_{n\in N} x_n = \inf_{P\in\mathcal{N}_\infty(N)} \sup_{n\in P} x_n \ge \inf_{P\in\mathcal{N}_\infty(M)} \sup_{n\in P} x_n = \limsup_{n\in M} x_n;$$

using (4) we deduce that

$$\liminf_{n\in N} x_n = \inf_{P\in\mathcal{N}_\infty^\#(N)} \sup_{n\in P} x_n = \inf_{P\in\mathcal{N}_\infty^\#(M)} \sup_{n\in P\cap N} x_n \leq \inf_{P\in\mathcal{N}_\infty^\#(M)} \sup_{n\in P} x_n = \liminf_{n\in M} x_n.$$

(iv) By (iii) we may assume that $N = M$. Then the conclusion is immediate.

(v) Let $u_n := \inf_{i\in I} z_n^i$ and $v_n := \sup_{i\in I} z_n^i$. Then $u_n \leq z_n^i \leq v_n$ for every $n \in M$ and $i \in I$. The conclusion follows from (iv). □

Of course, when for $(x_n)_{n\in\mathbb{N}} \subset L$ and $M \in \mathcal{N}_\infty^\#$ we have $x := \liminf_{n\in M} x_n = \limsup_{n\in M} x_n$ we say that $(x_n)_{n\in M}$ is convergent to x and x is denoted by $\lim_{n\in M} x_n$.

Corollary 2.2 *Let $M \in \mathcal{N}_\infty^\#$ and $(x_n)_{n\in\mathbb{N}} \subset L$. If $x = \lim_{n\in M} x_n$ and $N \in \mathcal{N}_\infty^\#(M)$, then $x = \lim_{n\in N} x_n$.*

An important particular case is that of monotone sequences.

Proposition 2.3 *Let $(x_n)_{n\in\mathbb{N}} \subset L$ be a monotone sequence and $N \in \mathcal{N}_\infty^\#$.*

(i) *If $(x_n)_{n\in\mathbb{N}}$ is increasing, that is $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, then $\lim_{n\in N} x_n = \sup_{n\in\mathbb{N}} x_n$.*

(ii) *If $(x_n)_{n\in\mathbb{N}}$ is decreasing, that is $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, then $\lim_{n\in N} x_n = \inf_{n\in\mathbb{N}} x_n$.*

Proof. We only prove (ii) (because the proof of (i) is even easier). For $k \in \mathbb{N}$ define the set $P_k := \{n \in N \mid n \geq k\} \in \mathcal{N}_\infty(N)$. Since $\sup_{n\in P_k} x_n \leq x_k$, we have

$$\limsup_{n\in N} x_n = \inf_{P\in\mathcal{N}_\infty(N)} \sup_{n\in P} x_n \leq \sup_{n\in P_k} x_n \leq x_k$$

for all $k \in \mathbb{N}$. Hence $\limsup_{n\in N} x_n \leq \inf_{k\in\mathbb{N}} x_k \leq \liminf_{n\in N} x_n$. □

Let $(L', \leq)$ be another complete lattice. Then $(L \times L', \leq)$ is a complete lattice when the order $\leq$ on $L \times L'$ is defined coordinate-wise: $(x, x') \leq (y, y')$ iff $x \leq y$ and $x' \leq y'$. Moreover, for $A \subset L \times L'$ and $\mathrm{Pr}_L, \mathrm{Pr}_{L'}$ being the projections of $L \times L'$ onto L and L', respectively, we have $\sup A = (\sup \mathrm{Pr}_L(A), \sup \mathrm{Pr}_{L'}(A))$ and $\inf A = (\inf \mathrm{Pr}_L(A), \inf \mathrm{Pr}_{L'}(A))$. Using this property we get immediately the next result.

Proposition 2.4 *Let $(x_n)_{n\in\mathbb{N}} \subset L$, $x' \in L'$ and $M \in \mathcal{N}_\infty^\#$. Then*

$$\liminf_{n\in M}(x_n, x') = \big(\liminf_{n\in M} x_n, x'\big) \quad \textit{and} \quad \limsup_{n\in M}(x_n, x') = \big(\limsup_{n\in M} x_n, x'\big).$$

Remark 2.5 It is easy to see that for a sequence $(x_n)_{n\in\mathbb{N}} \subset L$, it holds $\limsup_{n\in\mathbb{N}} x_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} x_n$, that is $\limsup_{n\in\mathbb{N}} x_n$ is the usual limit superior of $(x_n)_{n\in\mathbb{N}}$. On the other hand, we always have $\liminf_{n\in\mathbb{N}} x_n \geq \sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n$. In general, this inequality is strict. For this take the lattice $(\mathcal{F}, \subset)$ and the sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n := [(n+2)^{-1}, (n+1)^{-1}]$. Then $\liminf_{n\in\mathbb{N}} x_n = \{0\}$ and $\sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n = \emptyset$.

When $(L, \leq)$ is the lattice $(\mathcal{F}, \subset)$, Propositions 2.1, 2.3 and Corollary 2.2 provide well-known properties of the Painlevé–Kuratowski outer and inner limits.

3 $\mathcal{C}$-Convergence

In the sequel we shall mainly deal with the space $\mathcal{C} := \mathcal{C}(X)$ of closed convex subsets of X; we also set $\mathcal{C}_0 := \mathcal{C} \setminus \{\emptyset\}$. The set $\mathcal{C}$ is equipped with the same operations and the same order relation like $\mathcal{F}$. Of course, $\mathcal{C}$ is also a complete lattice. It is easy to see that the supremum and the infimum of the nonempty subset $\mathcal{A} \subset \mathcal{C}$, denoted by $\sup \mathcal{A}$ and $\inf \mathcal{A}$, are expressed by

$$\sup \mathcal{A} = \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A}} A, \qquad \inf \mathcal{A} = \bigcap_{A \in \mathcal{A}} A. \tag{5}$$

As usual we set $\inf \emptyset := \sup \mathcal{C} = X$ and $\sup \emptyset := \inf \mathcal{C} = \emptyset$. For nonempty subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ (or $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$) we set

$$\mathcal{A} + \mathcal{B} := \{A \oplus B \mid A \in \mathcal{A}, B \in \mathcal{B}\}, \quad \mathcal{A} + \emptyset := \emptyset + \mathcal{A} := \emptyset + \emptyset := \emptyset.$$

It is easily seen that for arbitrary sets $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ and $\bar{\mathcal{A}}, \bar{\mathcal{B}} \subset \mathcal{C}$ the following statements hold:

$$\operatorname{INF}(\mathcal{A} + \mathcal{B}) \supset \operatorname{INF} \mathcal{A} \oplus \operatorname{INF} \mathcal{B} \qquad \text{and} \qquad \inf(\bar{\mathcal{A}} + \bar{\mathcal{B}}) \supset \inf \bar{\mathcal{A}} \oplus \inf \bar{\mathcal{B}}, \tag{6}$$

$$\operatorname{SUP}(\mathcal{A} + \mathcal{B}) = \operatorname{SUP} \mathcal{A} \oplus \operatorname{SUP} \mathcal{B} \qquad \text{and} \qquad \sup(\bar{\mathcal{A}} + \bar{\mathcal{B}}) = \sup \bar{\mathcal{A}} \oplus \sup \bar{\mathcal{B}}. \tag{7}$$

As in the preceding section, the upper and lower limits of a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ are defined, respectively, by

$$\limsup_{n\in\mathbb{N}} A_n := \inf_{N \in \mathcal{N}_\infty} \sup_{n \in N} A_n \quad \text{and} \quad \liminf_{n\in\mathbb{N}} A_n := \inf_{N \in \mathcal{N}_\infty^\#} \sup_{n \in N} A_n.$$

The upper and lower limits are used to introduce a convergence concept in $\mathcal{C}$. So, we say that a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ is $\mathcal{C}$-*convergent* to some $A \in \mathcal{C}$ if

$$A = \limsup_{n\in\mathbb{N}} A_n = \liminf_{n\in\mathbb{N}} A_n.$$

In this case the limit A is denoted by $\lim_{n\in\mathbb{N}} A_n$ and we write $A_n \to A$ or $A_n \xrightarrow{\mathcal{C}} A$. Similarly, if $M \in \mathcal{N}_\infty^\#$, the upper and lower limits of $(A_n)_{n\in M} \subset \mathcal{C}$ are defined by

$$\limsup_{n\in M} A_n := \inf_{N \in \mathcal{N}_\infty(M)} \sup_{n \in N} A_n, \qquad \liminf_{n\in M} A_n := \inf_{N \in \mathcal{N}_\infty^\#(M)} \sup_{n \in N} A_n;$$

we say that $(A_n)_{n\in M}$ $\mathcal{C}$-converges to $A \in \mathcal{C}$ when $A = \limsup_{n\in M} A_n = \liminf_{n\in M} A_n$ and we write $A = \lim_{n\in M} A_n$. Proposition 2.1 and Corollary 2.2 apply immediately to sequences $(A_n) \subset \mathcal{C}$. In the next examples we show that the inequalities in the assertions of Proposition 2.1(v) can be strict in the case of $(\mathcal{C}, \subset)$.

Example 3.1 Let $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R})$, $A_n = \{n\}$, $B_n = \{-n\}$. Then $\lim_{n\in\mathbb{N}} A_n = \lim_{n\in\mathbb{N}} B_n = \emptyset$, hence $\sup\{\lim_{n\in\mathbb{N}} A_n, \lim_{n\in\mathbb{N}} B_n\} = \emptyset$. But, $\sup\{A_n, B_n\} = [-n, n]$ and consequently, $\lim_{n\in\mathbb{N}} \sup\{A_n, B_n\} = \mathbb{R}$.

Note further that the sum of limits could be different from the limit of the sum.

Example 3.2 Let the sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ as in Example 3.1. Then we have $\emptyset = \lim_{n\in\mathbb{N}} A_n \oplus \lim_{n\in\mathbb{N}} B_n \neq \lim_{n\in\mathbb{N}}(A_n \oplus B_n) = \{0\}$.

In the next example we show that the sum of two $\mathcal{C}$-convergent sequences in $\mathcal{C}$ is not necessarily $\mathcal{C}$-convergent.

Example 3.3 Let the sequences $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^2)$ be defined by

$$A_n := \begin{cases} \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = nx_1,\ x_2 \geq 0\} & \text{if } n \text{ is odd,} \\ \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 \geq 0\} & \text{if } n \text{ is even,} \end{cases}$$

$$B_n := \begin{cases} \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = -nx_1,\ x_2 \leq 0\} & \text{if } n \text{ is odd,} \\ \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 \leq 0\} & \text{if } n \text{ is even.} \end{cases}$$

Then, $(A_n)_{n\in\mathbb{N}}$ $\mathcal{C}$-converges to $A = \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$ and $(B_n)_{n\in\mathbb{N}}$ $\mathcal{C}$-converges to $B = \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$. But, the subsequence $(A_{2n} \oplus B_{2n})_{n\in\mathbb{N}}$ $\mathcal{C}$-converges to $\{(0, x_2) \mid x_2 \in \mathbb{R}\}$ and $(A_{2n+1} \oplus B_{2n+1})_{n\in\mathbb{N}}$ $\mathcal{C}$-converges to $\{(x_1, x_2) \mid x_1 \geq 0,\ x_2 \in \mathbb{R}\}$, and so $(A_n \oplus B_n)_{n\in\mathbb{N}}$ is not $\mathcal{C}$-convergent.

Proposition 3.4 *Consider the sequence $(x_n)_{n\in\mathbb{N}} \subset X$.*

(i) *If $x_n \to x$ for some $x \in X$ then $\{x_n\} \xrightarrow{\mathcal{C}} \{x\}$.*

(ii) *If $\|x_n\| \to \infty$ then $\liminf_{n\in\mathbb{N}} \{x_n\} = \emptyset$; moreover, if $\|x_n\|^{-1} x_n \to u$ for some $u \in X$ then $\lim_{n\in\mathbb{N}}\{x_n\} = \emptyset$.*

(iii) *Assume that $\{x_n\} \xrightarrow{\mathcal{C}} A \in \mathcal{C}$. Then either $A = \emptyset$ and $\|x_n\| \to \infty$, or $A = \{x\}$ and $x_n \to x$ for some $x \in X$.*

Proof. (i) follows from Proposition 5.1 below.

(ii) Let $\|x_n\| \to \infty$. First we assume that $\|x_n\|^{-1}x_n \to u$ for some $u \in X$. Since $\left\langle \|x_n\|^{-1} x_n, u\right\rangle \to 1$, we have $\langle x_n, u\rangle \to \infty$. Fix some $r > 0$; then there exists $N \in \mathcal{N}_\infty$ such that $\langle x_n, u\rangle \geq r$ for every $n \in N$, and so $\sup_{n\in N}\{x_n\} \subset \{x \mid \langle x, u\rangle \geq r\}$. It follows that

$$\limsup_{n\in\mathbb{N}}\{x_n\} = \inf_{N\in\mathcal{N}_\infty} \sup_{n\in N}\{x_n\} \subset \bigcap_{r>0}\{x \mid \langle x, u\rangle \geq r\} = \emptyset,$$

and so $\lim_{n\in\mathbb{N}}\{x_n\} = \emptyset$. Assume now only that $\|x_n\| \to \infty$. Then there exists $N \in \mathcal{N}_\infty^\#$ such that $\|x_n\|^{-1} x_n \xrightarrow{N} v \in X$. The argument above and Proposition 2.1(ii) imply that $\liminf_{n\in\mathbb{N}}\{x_n\} \subset \liminf_{n\in N}\{x_n\} = \lim_{n\in N}\{x_n\} = \emptyset$.

(iii) Assume first that $A = \emptyset$; then $\|x_n\| \to \infty$. Otherwise, there exist $N \in \mathcal{N}_\infty^\#$ and $x \in X$ such that $x = \lim_{n\in N} x_n$. By (i) we get $\lim_{n\in N}\{x_n\} = \{x\}$, contradicting the fact that $\lim_{n\in\mathbb{N}}\{x_n\} = \emptyset$ and Corollary 2.2. Assume now that $A \neq \emptyset$. If $(x_n)_{n\in\mathbb{N}}$ is unbounded then there exist $N \in \mathcal{N}_\infty^\#$ and $v \in X$ such that $\|x_n\| \xrightarrow{N} \infty$ and $v = \lim_{n\in N} \|x_n\|^{-1} x_n$. Then, from (ii), we have $\lim_{n\in N}\{x_n\} = \emptyset$, contradicting the fact that $\lim_{n\in\mathbb{N}}\{x_n\} = A \neq \emptyset$ and Corollary 2.2. Hence $(x_n)_{n\in\mathbb{N}}$ is bounded. Assuming that there are two subsequences with distinct limits, by Corollary 2.2, we obtain again a contradiction. $\square$

Without requiring that $\left(\|x_n\|^{-1} x_n\right)$ is convergent in Proposition 3.4(ii) it is not possible to obtain $\lim_{n\in\mathbb{N}}\{x_n\} = \emptyset$.

Example 3.5 Let $(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2$ be defined by $x_{2n} := (n, 0)$, $x_{2n+1} := (-n, 0)$, $z_{2n} := (n, n^{1/2})$, $z_{2n+1} := (-n, n^{1/2})$ for $n \in \mathbb{N}$. Then $\limsup_{n\in\mathbb{N}}\{x_n\} = \mathbb{R} \times \{0\}$ and $\limsup_{n\in\mathbb{N}}\{z_n\} = \emptyset$.

The following characterization of the upper limit is useful to show further properties of the upper and lower limits. For simplicity of notation we denote the set $\{m, m+1, \ldots, k\} \subset \mathbb{N}$ ($m, k \in \mathbb{N}$, $m \leq k$) by $\overline{m,k}$. Further we set $\Delta_p := \big\{\lambda \in [0,1]^p \mid \sum_{i \in \overline{0,p-1}} \lambda_n^i = 1\big\}$.

Proposition 3.6 *Consider a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. Then $x \in \limsup_{n\in\mathbb{N}} A_n$ if and only if the following assertion holds:*

$$\begin{aligned} &\exists\, (\lambda_n)_{n\in\mathbb{N}} \subset \Delta_{p+1},\ \exists\, (k_n)_{n\in\mathbb{N}} \subset \mathbb{N}^{p+1},\ \exists\, (z_n)_{n\in\mathbb{N}} \subset X^{p+1},\ \forall n \in \mathbb{N},\ \forall j \in \overline{0,p}\ : \\ &\quad k_n^j \geq n,\ z_n^j \in A_{k_n^j},\quad x = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i. \end{aligned}$$

Proof. For the necessity part, let $x \in A := \limsup_{n\in\mathbb{N}} A_n$. It is easy to see that $A = \bigcap_{n\in\mathbb{N}} \operatorname{cl} V_n$, where $V_n := \operatorname{conv} Z_n$ with $Z_n := \bigcup_{k\geq n} A_k$. Hence $x \in \operatorname{cl} V_n$ for all $n \in \mathbb{N}$. This yields

$$\forall n \in \mathbb{N},\ \forall \varepsilon > 0,\ \exists v_{n,\varepsilon} \in V_n\ :\ \|x - v_{n,\varepsilon}\| < \varepsilon.$$

Choosing $\varepsilon := 1/(n+1)$ and setting $x_n := v_{n,1/(n+1)}$, we obtain a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converging to x. Since $x_n \in \operatorname{conv} Z_n$ with $Z_n \subset X$, by the Carathéodory theorem, for every $j \in \overline{0,p}$ there exist $\lambda_n^j \in [0,1]$ and $z_n^j \in Z_n$ such that $\sum_{i\in\overline{0,p}} \lambda_n^i = 1$ and $x_n = \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i$. Since $z_n^j \in Z_n = \bigcup_{k\geq n} A_k$, there exists $k_n^j \geq n$ such that $z_n^j \in A_{k_n^j}$.

For the sufficiency part observe that for an arbitrary $m \in \mathbb{N}$ we have

$$x = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i = \lim_{n\geq m} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i.$$

Hence $x \in \operatorname{cl}\operatorname{conv} \bigcup_{k\geq m} A_k$ for all $m \in \mathbb{N}$. This yields $x \in \limsup A_n$. □

Remark 3.7 Of course, the previous proposition remains true if one replaces $\overline{0,p}$ by $\overline{0,q}$ with $q \geq p$.

4 PK-convergence versus $\mathcal{C}$-convergence

The following examples show that (in case of existence) the limit with respect to PK-convergence can be different from the limit with respect to $\mathcal{C}$-convergence. It can be seen that neither $\mathcal{C}$-convergence implies PK-convergence nor vice versa.

Example 4.1 PK-convergence does not coincide with $\mathcal{C}$-convergence:

(i) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^2)$ be defined by $A_n := \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq n x_1\big\}$. By an easy calculation it can be seen that

$$\big\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1\big\} = \operatorname*{LIM}_{n\in\mathbb{N}} A_n \neq \lim_{n\in\mathbb{N}} A_n = \mathbb{R}^2.$$

(ii) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^2)$ be defined by

$$A_n := \begin{cases} \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq n x_1\big\} & \text{if } n \text{ is odd}, \\ \mathbb{R}^2 & \text{if } n \text{ is even}. \end{cases}$$

In view of (i), it can be easily seen that $\lim_{n\in\mathbb{N}} A_n = \mathbb{R}^2$, but $\operatorname{LIM}_{n\in\mathbb{N}} A_n$ does not exist. In fact, we have $\operatorname{LIMSUP}_{n\in\mathbb{N}} A_n = \mathbb{R}^2$, but $\operatorname{LIMINF}_{n\in\mathbb{N}} A_n = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1\big\}$.

(iii) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^2)$ be defined by

$$A_n := \begin{cases} \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \le nx_1\} & \text{if} \quad n \text{ is odd,} \\ \{(x_1,x_2) \in \mathbb{R}^2 \mid 0 \le x_1\} & \text{if} \quad n \text{ is even.} \end{cases}$$

By (i), it can be easily seen that $\mathrm{LIM}_{n\in\mathbb{N}} A_n = \{(x_1,x_2) \in \mathbb{R}^2 \mid 0 \le x_1\}$, but $\lim_{n\in\mathbb{N}} A_n$ does not exist. In fact, we have $\liminf_{n\in\mathbb{N}} A_n = \{(x_1,x_2) \in \mathbb{R}^2 \mid 0 \le x_1\}$ and $\limsup_{n\in\mathbb{N}} A_n = \mathbb{R}^2$.

Of course, we have the following relationships between the outer and inner limits, and the upper and lower limits of a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C} \subset \mathcal{F}$:

$$\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n \subset \limsup_{n\in\mathbb{N}} A_n, \qquad \operatorname*{LIMINF}_{n\in\mathbb{N}} A_n \subset \liminf_{n\in\mathbb{N}} A_n.$$

In this section, we are looking for conditions which ensure the opposite inclusions. We start with a technical assertion, which is used several times in the sequel. Before stating it we recall that the *recession cone* of $A \in \mathcal{C}_0$ is the set $0^+A := \{u \in X \mid a + tu \in A \ \forall a \in A, \ \forall t \ge 0\}$; moreover, $0^+\emptyset := \{0\}$. Furthermore, the *lineality space* of $A \in \mathcal{C}$ is the linear space $L(A) := 0^+A \cap (-0^+A)$. The representation of $A \in \mathcal{C}$ in the following lemma was observed in [14, p. 268] in the particular case where A is a closed convex cone and Y is its lineality space.

Lemma 4.2 *Let $A \in \mathcal{C}_0$ and let $Y \subset 0^+A$ be a linear space. If $Z \subset X$ is a linear space such that $X = Y + Z$ and $Y \cap Z = \{0\}$ (that is X is the direct sum of Y and Z) then $A = Y + (A \cap Z)$ and $0^+(A \cap Z) = Z \cap 0^+A$; in particular, if Y is the lineality space of A then $\{0\}$ is the lineality space of $A \cap Z$.*

Proof. We have $Y + (A \cap Z) \subset Y + A = A$. Conversely, let $x \in A$. Then $x = y + z$ with $y \in Y$ and $z \in Z$. Since $A + Y = A$, we obtain $z = x + (-y) \in A$, and so $x \in Y + (A \cap Z)$. Moreover, $0^+(A \cap Z) = 0^+A \cap 0^+(Z) = 0^+A \cap Z$. □

Another useful auxiliary result is the following.

Lemma 4.3 *Let Y, Z be two finite dimensional normed vector spaces, $B \in \mathcal{C}(Y)$ and $(C_i)_{i\in I} \subset \mathcal{C}(Z)$. Then*

$$\operatorname*{SUP}_{i\in I} B \times C_i = B \times \operatorname*{SUP}_{i\in I} C_i, \quad \operatorname*{INF}_{i\in I} B \times C_i = B \times \operatorname*{INF}_{i\in I} C_i,$$
$$\sup_{i\in I} B \times C_i = B \times \sup_{i\in I} C_i, \quad \inf_{i\in I} B \times C_i = B \times \inf_{i\in I} C_i.$$

Moreover, if $N \in \mathcal{N}_\infty^\#$ and $(D_n)_{n\in N} \subset \mathcal{C}(Z)$, then

$$\operatorname*{LIMSUP}_{n\in N} B \times D_n = B \times \operatorname*{LIMSUP}_{n\in N} D_n, \quad \operatorname*{LIMINF}_{n\in N} B \times D_n = B \times \operatorname*{LIMINF}_{n\in N} D_n,$$
$$\limsup_{n\in N} B \times D_n = B \times \limsup_{n\in N} D_n, \quad \liminf_{n\in N} B \times D_n = B \times \liminf_{n\in N} D_n.$$

Proof. It is sufficient to observe that for $E \subset Y$ and $F \subset Z$ we have $\mathrm{cl}(E \times F) = \mathrm{cl}\, E \times \mathrm{cl}\, F$ and $\mathrm{conv}(E \times F) = \mathrm{conv}\, E \times \mathrm{conv}\, F$. □

The next two theorems provide sufficient conditions for the coincidence of PK-convergence and $\mathcal{C}$-convergence. The statements refer to the class $\mathcal{C}_K$ of those sets $A \in \mathcal{C}_0$ with $0^+A = K$, where $K \subset X$ is a fixed closed convex cone. For the special case $K = \{0\}$, the statement of the next theorem can be found in [1, Lemma 1.1.9].

Theorem 4.4 *Let $K \subset X$ be a closed convex cone and let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}_K$ be such that* $\sup_{n\in\mathbb{N}} A_n \in \mathcal{C}_K$. *Then,*

$$\limsup_{n\in\mathbb{N}} A_n = \operatorname{cl}\operatorname{conv}\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n.$$

Proof. Since $\mathrm{LIMSUP}_{n\in\mathbb{N}} A_n \subset \limsup_{n\in\mathbb{N}} A_n$, it remains to prove the inclusion $A := \limsup_{n\in\mathbb{N}} A_n \subset \operatorname{cl}\operatorname{conv}\mathrm{LIMSUP}_{n\in\mathbb{N}} A_n$.

(a) Assume that K is pointed, that is $K \cap -K = \{0\}$. Take $x \in A$. By Proposition 3.6 we have

$$\begin{gathered}\exists\,(\lambda_n)_{n\in\mathbb{N}} \subset \Delta_{p+1},\ \exists\,(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}^{p+1},\ \exists\,(z_n)_{n\in\mathbb{N}} \subset X^{p+1},\ \forall n \in \mathbb{N},\ \forall j \in \overline{0,p}\ : \\ k_n^j \geq n,\ z_n^j \in A_{k_n^j},\quad x = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i.\end{gathered}$$

Set $v_n := \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i$. Without loss of generality we assume that

$$\forall n \in \mathbb{N}\ :\ \left\|\lambda_n^0 z_n^0\right\| \leq \left\|\lambda_n^1 z_n^1\right\| \leq ... \leq \|\lambda_n^p z_n^p\| \neq 0. \tag{8}$$

There exists $N \in \mathcal{N}_\infty^\#$ such that

$$\forall j \in \overline{0,p}\ :\ \lambda_n^j \xrightarrow{n\in N} \lambda^j \in [0,1],\ \|\lambda_n^p z_n^p\|^{-1} \lambda_n^j z_n^j \xrightarrow{n\in N} y^j \in X. \tag{9}$$

Assume that the sequence $(\lambda_n^p z_n^p)_{n\in N}$ is unbounded. It follows that there exists $N' \in \mathcal{N}_\infty^\#(N)$ such that $\|\lambda_n^p z_n^p\| \xrightarrow{n\in N'} \infty$, whence $\lambda_n^j / \|\lambda_n^p z_n^p\| \xrightarrow{n\in N'} 0$ for all $j \in \overline{0,p}$. By the characterization of recession cones in [8, Th. 8.2] applied to the set $\sup_{n\in\mathbb{N}} A_n$, we deduce that $y^j \in K$ for all $j \in \overline{0,p}$. Passing to the limit in the relation

$$\|\lambda_n^p z_n^p\|^{-1} v_n = \sum_{j\in\overline{0,p}} \|\lambda_n^p z_n^p\|^{-1} \lambda_n^j z_n^j$$

we obtain $0 = \sum_{j=0}^p y^j$. Thus we get $y^p \in K \cap -K = \{0\}$, a contradiction (because $\|y^p\| = 1$). Hence the sequence $(\lambda_n^j z_n^j)_{n\in N}$ is bounded for each $j \in \overline{0,p}$. It follows that there exists $N' \in \mathcal{N}_\infty^\#(N)$ such that $\lambda_n^j z_n^j \xrightarrow{n\in N'} v^j \in X$ for every $j \in \overline{0,p}$. Let $q \in \overline{0,p}$ be such that $\lambda^j \neq 0$ for $j \in \overline{0,q}$ and $\lambda^j = 0$ for $j \in \overline{q+1,p}$. If $\lambda^j = 0$, as above, $v^j \in K$. Fix $j \in \overline{0,q}$. Then $\lambda^j \neq 0$, and so $z_n^j \xrightarrow{n\in N'} z^j := (\lambda^j)^{-1} v^j$. Since $z_n^j \in \cup_{k\geq n} A_k$, we obtain $z^j \in \mathrm{LIMSUP}_{n\in N'} A_n \subset \mathrm{LIMSUP}_{n\in\mathbb{N}} A_n$. Setting $k := \sum_{j=q+1}^p v^j$ ($k := 0$ if $q = p$), we obtain

$$x = k + \sum_{j\in\overline{0,q}} \lambda^j z^j \in K + \operatorname{conv}\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n \subset K + \operatorname{cl}\operatorname{conv}\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n = \operatorname{cl}\operatorname{conv}\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n.$$

(b) We now turn to the general case, i.e., the lineality space $Y := K \cap -K$ is not necessarily $\{0\}$. Take a linear subspace $Z \subset X$ such that $X = Y + Z$ and $Y \cap Z = \{0\}$. Of course, we can identify X with $Y \times Z$, and so, by Lemma 4.2, every set $A \in \mathcal{C}_K$ is identified with $Y \times B$, where $B := A \cap Z$. In particular, we have $A_n = Y \times B_n$ with $B_n \in \mathcal{C}_{\{0\}}(Z)$. By Lemma 4.3 it follows that

$$\operatorname*{LIMSUP}_{n\in\mathbb{N}} A_n = Y \times \operatorname*{LIMSUP}_{n\in\mathbb{N}} B_n, \quad \limsup_{n\in\mathbb{N}} A_n = Y \times \limsup_{n\in\mathbb{N}} B_n.$$

Moreover, $\sup_{n\in\mathbb{N}} B_n \in \mathcal{C}_{\{0\}}(Z)$. The conclusion now follows from (a). □

An analogous result for the lower and inner limits can be obtained even under weaker assumptions. The result is proved using the previous theorem.

Theorem 4.5 *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ be a sequence such that for all $N \in \mathcal{N}_\infty^\#$ there exist some $N' \in \mathcal{N}_\infty^\#(N)$ and some closed convex cone $K \subset X$ such that $(A_n)_{n\in N'} \subset \mathcal{C}_K$ and $\sup_{n\in N'} A_n \in \mathcal{C}_K$. Then*

$$\liminf_{n\in\mathbb{N}} A_n = \operatorname*{LIMINF}_{n\in\mathbb{N}} A_n.$$

Proof. Clearly, we have $A := \liminf_{n\in\mathbb{N}} A_n \supset \mathrm{LIMINF}_{n\in\mathbb{N}} A_n$. To show the opposite inclusion fix $N \in \mathcal{N}_\infty^\#$. By the cluster point description of outer limits [9, Prop. 4.19] there exists some $N' \in \mathcal{N}_\infty^\#(N)$ such that $(A_n)_{n\in N'}$ is PK-convergent. By assumption, there exist $N'' \in \mathcal{N}_\infty^\#(N')$ and a closed convex cone K such that $A_n \in \mathcal{C}_K$ for $n \in N''$ and $\sup_{n\in N''} A_n \in \mathcal{C}_K$. Of course, $(A_n)_{n\in N''}$ is PK-convergent, too. It follows that $\mathrm{LIM}_{n\in N''} A_n = \mathrm{LIMINF}_{n\in N''} A_n = \mathrm{LIMSUP}_{n\in N''} A_n$; moreover, $\mathrm{LIMSUP}_{n\in N''} A_n$ is closed and convex, because $\mathrm{LIMINF}_{n\in N''} A_n$ is so (A_n being convex for every n). Using also Proposition 2.1(ii), (i) and Theorem 4.4 we deduce that

$$\begin{aligned} A = \liminf_{n\in\mathbb{N}} A_n \subset \liminf_{n\in N''} A_n \subset \limsup_{n\in N''} A_n = \operatorname{cl}\operatorname{conv} \operatorname*{LIMSUP}_{n\in N''} A_n \\ = \operatorname*{LIMSUP}_{n\in N''} A_n \subset \operatorname{cl} \bigcup_{n\in N''} A_n \subset \operatorname{cl} \bigcup_{n\in N} A_n. \end{aligned}$$

Since $N \in \mathcal{N}_\infty^\#$ was chosen arbitrarily, it follows that $A \subset \mathrm{LIMINF}_{n\in\mathbb{N}} A_n$. $\square$

An immediate consequence of Theorems 4.4 and 4.5 is the next result.

Corollary 4.6 *Let $K \subset X$ be a closed convex cone and let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}_K$ be a sequence with $\sup_{n\in\mathbb{N}} A_n \in \mathcal{C}_K$. Then, $(A_n)_{n\in\mathbb{N}}$ is $\mathcal{C}$-convergent if and only if $(A_n)_{n\in\mathbb{N}}$ is PK-convergent. In case of convergence both limits coincide.*

5 Scalar convergence versus $\mathcal{C}$-convergence

Let us recall another convergence for sequences of closed convex sets. For this recall that the support function of the subset A of X is the function

$$\sigma_A : X^* \to \overline{\mathbb{R}}, \quad \sigma_A(x^*) := \sup\{\langle x, x^*\rangle \mid x \in A\},$$

where X^* is the (topological) dual of X and $\langle x, x^*\rangle := x^*(x)$ for $x \in X$ and $x^* \in X^*$ (and $\sup\emptyset = -\infty$). We say that the sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ *scalar converges* (or simply S-converges) to $A \in \mathcal{C}$ and we write $A_n \xrightarrow{S} A$ or $A = S\text{-}\lim_{n\in\mathbb{N}} A_n$ if

$$\sigma_{A_n}(x^*) \to \sigma_A(x^*) \quad \forall x^* \in X^*.$$

This convergence was introduced by Wijsman [13] and studied by several authors (see Salinetti and Wets [11], De Blasi and Myjak [3], Sonntag and Zălinescu [12], and Beer [2] for an overview and further references). Note that $\sigma_A = \sigma_{\operatorname{cl}\operatorname{conv} A}$ and $\sigma_A(0) = 0$ for every nonempty set A; this shows that the natural framework for this convergence is the class $\mathcal{C}_0$ of nonempty closed convex subsets of X. In this section we investigate the relationship between scalar convergence and $\mathcal{C}$-convergence. We start with an extension of a result of Sonntag and Zălinescu [12].

Proposition 5.1 *Assume that $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}_0$ is S-convergent to $A \in \mathcal{C}_0$. Then $A_n \xrightarrow{\mathcal{C}} A$.*

Proof. In [12, Prop. 1] it is shown that $A = \limsup_{n\in\mathbb{N}} A_n$. In order to show that $A = \liminf_{n\in\mathbb{N}} A_n$, fix some $N \in \mathcal{N}_\infty^\#$. Of course, $A = S\text{-}\lim_{n\in N} A_n$, and so $A = \limsup_{n\in N} A_n \subset \sup_{n\in N} A_n$. Since $N \in \mathcal{N}_\infty^\#$ is arbitrary, we obtain $A \subset \liminf_{n\in\mathbb{N}} A_n$. Hence $A_n \xrightarrow{\mathcal{C}} A$. $\square$

It is easy to see that the reverse implication is not true in $\mathcal{C}$. For this just take $A_n := \{n\} \subset \mathbb{R}$; then $A_n \xrightarrow{\mathcal{C}} \emptyset$, but (A_n) does not S-converge (in fact if $A_n \xrightarrow{S} \emptyset$ then $\{n \mid A_n = \emptyset\} \in \mathcal{N}_\infty$). Even for $A, A_n \in \mathcal{C}_0$ $(n \in \mathbb{N})$ the reverse implication is not true.

Example 5.2 Consider the sets $A := \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\} \in \mathcal{C}_0(\mathbb{R}^2)$ and $A_n := \{(x_1, nx_1) \in \mathbb{R}^2 \mid x_1 \geq 0\} \in \mathcal{C}_0(\mathbb{R}^2)$ for $n \in \mathbb{N}$. It is easy to see that $A_n \xrightarrow{\mathcal{C}} A$. However $\sigma_{A_n}(1,0) = \infty \to \infty \neq 0 = \sigma_A(1,0)$, whence (A_n) does not S-converge to A.

An easy calculation shows that in the previous example the domain $\operatorname{dom}\sigma_A$ of σ_A is the set $\{(u_1,u_2) \in \mathbb{R}^2 \mid u_2 \leq 0\}$ and $\sigma_{A_n}(u_1,u_2) \to \sigma_A(u_1,u_2)$ for every $(u_1,u_2) \in \mathbb{R}^2 \setminus \operatorname{bd}\operatorname{dom}\sigma_A$. In fact this characterizes the $\mathcal{C}$-convergence of (A_n) to A. Below we will show that a sequence $(A_n)_{n\in\mathbb{N}}$ $\mathcal{C}$-converges to $A \in \mathcal{C}$ if and only if $(\sigma_{A_n})_{n\in\mathbb{N}}$ converges pointwise to σ_A excepting the relative boundary points of $\operatorname{dom}\sigma_A$. We start with some auxiliary assertions.

The next result, related to proper lower semicontinuous convex functions, will be useful. For a nonempty convex set $A \subset X$ we set $\operatorname{rbd} A := \operatorname{cl} A \setminus \operatorname{ri} A$, the relative boundary of A.

Proposition 5.3 *Let $f, g : X \to \overline{\mathbb{R}}$ be two proper lower semicontinuous convex functions.*

(i) *If $f(x) \leq g(x)$ for every $x \in \operatorname{ri}\operatorname{dom} g$ then $f \leq g$.*

(ii) *If $\operatorname{dom} g \cap \operatorname{ri}\operatorname{dom} f \neq \emptyset$ and $f(x) \leq g(x)$ for every $x \in X \setminus \operatorname{rbd}\operatorname{dom} f$ then $f \leq g$.*

(iii) *If $\operatorname{aff}\operatorname{dom} f \subset \operatorname{aff}\operatorname{dom} g$ (in particular if $\operatorname{int}\operatorname{dom} g \neq \emptyset$) and $f(x) \leq g(x)$ for every $x \in X \setminus \operatorname{rbd}\operatorname{dom} f$ then $f \leq g$.*

(iv) *If $f(x) = g(x)$ for every $x \in X \setminus (\operatorname{rbd}\operatorname{dom} f \cup \operatorname{rbd}\operatorname{dom} g)$. Then $f = g$.*

(v) *For every $x^* \in X^*$ one has $f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x) \mid x \in \operatorname{ri}\operatorname{dom} f\}$.*

Proof. Note that in (i), (ii) and (iii) we have to prove $f(x) \leq g(x)$ for $x \in \operatorname{dom} g$.

(i) Fix some $x_0 \in \operatorname{ri}\operatorname{dom} g$ and take $x \in \operatorname{dom} g$. Then $]x, x_0] := \{(1-\lambda)x + \lambda x_0 \mid \lambda \in (0,1]\} \subset \operatorname{ri}\operatorname{dom} g$, and so $f((1-\lambda)x + \lambda x_0) \leq g((1-\lambda)x + \lambda x_0)$ for every $\lambda \in (0,1]$. Since the restrictions of g to the segment $[x_0, x]$ is continuous (see [15, Prop. 2.1.6]) and f is lsc, we obtain $f(x) \leq g(x)$ (taking the limit for $\lambda \downarrow 0$).

(ii) If $x \notin \operatorname{cl}\operatorname{dom} f$ then $x \in X \setminus \operatorname{rbd}\operatorname{dom} f$, and so $\infty = f(x) \leq g(x)$. Hence $\operatorname{dom} g \subset \operatorname{cl}\operatorname{dom} f$. Fix $x_0 \in \operatorname{dom} g \cap \operatorname{ri}\operatorname{dom} f$ and take $x \in \operatorname{dom} g$. Then $[x_0, x] \subset \operatorname{dom} g$ and $]x, x_0] \subset \operatorname{ri}\operatorname{dom} f$. As in (i) we obtain $f(x) \leq g(x)$.

(iii) As in (ii) we have $\operatorname{dom} g \subset \operatorname{cl}\operatorname{dom} f$, whence $\operatorname{aff}\operatorname{dom} g \subset \operatorname{aff}\operatorname{cl}\operatorname{dom} f = \operatorname{aff}\operatorname{dom} f$. Hence $\operatorname{aff}\operatorname{dom} f = \operatorname{aff}\operatorname{dom} g$. Doing a translation, we may assume that $X_0 := \operatorname{aff}\operatorname{dom} g$ is a linear space. Since outside X_0, f and g coincide, we may assume that $X_0 = X$, and so $\operatorname{int}\operatorname{dom} g \neq \emptyset$. From the inclusion $\operatorname{dom} g \subset \operatorname{cl}\operatorname{dom} f$ we obtain $\operatorname{int}\operatorname{dom} g \subset \operatorname{int}\operatorname{cl}\operatorname{dom} f = \operatorname{int}\operatorname{dom} f$, and so $\operatorname{dom} g \cap \operatorname{ri}\operatorname{dom} f \neq \emptyset$. The conclusion follows from (ii).

(iv) First observe that $\operatorname{ri}\operatorname{dom} f \subset \operatorname{cl}\operatorname{dom} g$. Indeed, if $x \in \operatorname{ri}\operatorname{dom} f$ and $x \notin \operatorname{cl}\operatorname{dom} g$ then $x \in X \setminus (\operatorname{rbd}\operatorname{dom} f \cup \operatorname{rbd}\operatorname{dom} g)$, and so $f(x) = g(x) < \infty$. It follows that $x \in \operatorname{dom} g \subset \operatorname{cl}\operatorname{dom} g$, a contradiction. Similarly, we have $\operatorname{ri}\operatorname{dom} g \subset \operatorname{cl}\operatorname{dom} f$. Since $\operatorname{dom} f$ and $\operatorname{dom} g$ are convex subsets of a finite dimensional space, we obtain $\operatorname{ri}\operatorname{dom} f = \operatorname{ri}\operatorname{dom} g$ and $\operatorname{cl}\operatorname{dom} f = \operatorname{cl}\operatorname{dom} g$, and so $\operatorname{rbd}\operatorname{dom} f = \operatorname{rbd}\operatorname{dom} g$. The conclusion follows using (i) or (ii).

(v) Let $x^* \in X^*$. Of course,

$$f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x) \mid x \in X\} \geq \sup\{\langle x, x^*\rangle - f(x) \mid x \in \operatorname{ri}\operatorname{dom} f\} := \gamma.$$

If $\gamma = \infty$ there is nothing to prove. Let $\gamma \in \mathbb{R}$ and take $g(x) := \langle x, x^*\rangle - \gamma$. Then $g(x) \leq f(x)$ for every $x \in \operatorname{ri}\operatorname{dom} f$. Using (i) we obtain $g \leq f$, and so $f^*(x^*) \leq \gamma$. □

The next result is a refinement of some well-known assertions of Convex Analysis.

Lemma 5.4 *Let $A, B \in \mathcal{C}_0$. Then*

$$\operatorname{ri}(0^+A)^\circ \subset \operatorname{dom}\sigma_A \subset (0^+A)^\circ, \qquad \operatorname{rbd}(0^+A)^\circ = \operatorname{rbd}\operatorname{dom}\sigma_A, \tag{10}$$

$$A = \bigcap_{x^* \in \operatorname{ri}(0^+A)^\circ} \{x \in X \mid \langle x, x^*\rangle \leq \sigma_A(x^*)\}, \tag{11}$$

$$A \subset B \;\Leftrightarrow\; \forall x^* \in \operatorname{ri}(0^+B)^\circ :\; \sigma_A(x^*) \leq \sigma_B(x^*). \tag{12}$$

Moreover, if $L(B) \subset L(A)$, then

$$A \subset B \;\Leftrightarrow\; \forall x^* \in X \setminus \operatorname{rbd}(0^+A)^\circ :\; \sigma_A(x^*) \leq \sigma_B(x^*). \tag{13}$$

Proof. As a consequence of [8, Th. 14.2] we have $\operatorname{cl}\operatorname{dom}\sigma_A = (0^+A)^\circ$ (compare [5, Th. 2.2.4], too). Together with [8, Th. 6.3] this yields $\operatorname{ri}(0^+A)^\circ = \operatorname{ri}\operatorname{cl}\operatorname{dom}\sigma_A = \operatorname{ri}\operatorname{dom}\sigma_A \subset \operatorname{dom}\sigma_A \subset (0^+A)^\circ$. Therefore, (10) holds. Since $A \in \mathcal{C}_0$, we have $\iota_A = (\iota_A)^{**} = (\sigma_A)^*$ (where $\iota_A(x) := 0$ if $x \in A$ and $\iota_A(x) := \infty$ if $x \notin A$). Using Proposition 5.3(v) we obtain (11) as follows:

$$\begin{aligned} x \in A \Leftrightarrow \iota_A(x) &= \sup\{\langle x, x^*\rangle - \sigma_A(x^*) \mid x^* \in \operatorname{ri}\operatorname{dom}\sigma_A\} = 0 \\ &\Leftrightarrow\; \forall x^* \in \operatorname{ri}\operatorname{dom}\sigma_A = \operatorname{ri}(0^+A)^\circ :\; \langle x, x^*\rangle \leq \sigma_A(x^*). \end{aligned}$$

Since $A \subset B \Leftrightarrow \sigma_A \leq \sigma_B$ and $\operatorname{ri}\sigma_B = \operatorname{ri}(0^+B)^\circ$, (12) is an immediate consequence of Proposition 5.3(i). Furthermore, (13) follows from Proposition 5.3(iii) if we succeed to prove that $\operatorname{aff}\operatorname{dom}\sigma_A \subset \operatorname{aff}\operatorname{dom}\sigma_B$ whenever $L(B) \subset L(A)$. For this recall that for closed convex cones $P, Q \subset X$ we have $(P \cap Q)^\circ = \operatorname{cl}(P^\circ + Q^\circ)$. It follows that $(P \cap -P)^\circ = \operatorname{cl}(P^\circ - P^\circ) = \operatorname{cl}\operatorname{aff} P^\circ = \operatorname{aff} P^\circ$ (because every linear subspace of a finite dimensional normed vector space is closed). Therefore $L(B) \subset L(A)$ implies that (in fact is equivalent to) $\operatorname{aff}\operatorname{dom}\sigma_A \subset \operatorname{aff}\operatorname{dom}\sigma_B$. □

Note that (13) is not sufficient for $A \subset B$ without the assumption $L(B) \subset L(A)$. Indeed, consider $A = \mathbb{R}^2_+ := \{x \in \mathbb{R}^2 \,|\, x_1, x_2 \geq 0\}$ and $B = \{x \in \mathbb{R}^2 \,|\, x_1 \geq 1\}$. Then we have $(0^+B)^\circ \subset \operatorname{rbd}(0^+A)^\circ$ and, by (10), the righthand side of (13) is satisfied. But $A \not\subset B$.

The next easy result is an immediate consequence of [8, Cor. 16.5.1].

Proposition 5.5 *Let $(A_i)_{i \in I} \subset \mathcal{C}$ and set $B := \inf_{i \in I} A_i$ and $C := \sup_{i \in I} A_i$. Then $\sigma_B \leq \inf_{i \in I}\sigma_{A_i}$ and $\sigma_C = \sup_{i \in I}\sigma_{A_i}$.*

Proposition 5.6 *Let $K \subset X$ be a pointed closed convex cone. Then $\langle x, x^*\rangle < 0$ for all $x^* \in \operatorname{ri} K^\circ$ and $x \in K \setminus \{0\}$.*

Proof. Let $x^* \in \operatorname{ri} K^\circ$ and $x \in K \setminus \{0\}$. Since K is pointed, we have $\operatorname{int} K^\circ \neq \emptyset$, and so $x^* \in \operatorname{int} K^\circ$. Let $\overline{x}^* \in X^*$ be such that $\langle x, \overline{x}^* \rangle = 1$. There exists some $\varepsilon > 0$ such that $x^* + \varepsilon \overline{x}^* \in K^\circ$. Then $\langle x, x^* \rangle + \varepsilon = \langle x, x^* + \varepsilon \overline{x}^* \rangle \leq 0$, whence $\langle x, x^* \rangle < 0$. $\square$

Proposition 5.7 *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ be such that $\limsup_{n\in\mathbb{N}} A_n \neq \emptyset$ and let $(x_n)_{n\in\mathbb{N}} \subset X \setminus \{0\}$ be a sequence such that $x_n \in A_n$ for all $n \in \mathbb{N}$, $\|x_n\| \to \infty$ and $x_n / \|x_n\| \to u$. Then $u \in 0^+ \limsup_{n\in\mathbb{N}} A_n$.*

Proof. We show that $x + tu \in A := \limsup_{n\in\mathbb{N}} A_n$ for every $x \in A$ and $t > 0$. Fix $x \in A$ and $t > 0$. Then, by Proposition 3.6,

$$\begin{aligned} &\exists\, (\lambda_n)_{n\in\mathbb{N}} \subset \Delta_{p+1},\ \exists\, (k_n)_{n\in\mathbb{N}} \subset \mathbb{N}^{p+1},\ \exists\, (z_n)_{n\in\mathbb{N}} \subset X^{p+1},\ \forall n \in \mathbb{N},\ \forall j \in \overline{0,p}\ : \\ &\qquad k_n^j \geq n,\ z_n^j \in A_{k_n^j},\quad x = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i. \end{aligned}$$

Hence

$$x + tu = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i + \lim_{n\in\mathbb{N}} t \frac{x_n}{\|x_n\|}.$$

Setting $\lambda_n^{p+1} := t \|x_n\|^{-1}$, $z_n^{p+1} := x_n$, $k_n^{p+1} := n$ and $\tilde{\lambda}_n^j := \lambda_n^j \big(1 + \lambda_n^{p+1}\big)^{-1}$ for $j \in \overline{0,p+1}$, we obtain

$$x + tu = \lim_{n\in\mathbb{N}} \left(1 + \lambda_n^{p+1}\right) \sum_{j\in\overline{0,p+1}} \tilde{\lambda}_n^j z_n^j = \lim_{n\in\mathbb{N}} \sum_{j\in\overline{0,p+1}} \tilde{\lambda}_n^j z_n^j.$$

By Proposition 3.6 and taking into account Remark 3.7 we obtain $x + tu \in A$. $\square$

Lemma 5.8 *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ be such that $A := \limsup_{n\in\mathbb{N}} A_n \neq \emptyset$. Then*

$$\forall x^* \in \operatorname{ri}(0^+A)^\circ :\ \limsup_{n\in\mathbb{N}} \sigma_{A_n}(x^*) = \sigma_A(x^*).$$

Proof. Let $K := 0^+A$. From Proposition 5.5 we easily deduce $\sigma_A \leq \limsup_{n\in\mathbb{N}} \sigma_{A_n}$. It remains to show that $\limsup_{n\in\mathbb{N}} \sigma_{A_n}(x^*) \leq \sigma_A(x^*)$ for all $x^* \in \operatorname{ri} K^\circ$.

(a) We first prove the case where K is pointed. Assume the assertion is not true. This means there exists some $x^* \in \operatorname{ri} K^\circ$ such that $\limsup_{n\in\mathbb{N}} \sigma_{A_n}(x^*) > \sigma_A(x^*)$. Hence, there are $\varepsilon > 0$ and $N \in \mathcal{N}_\infty^\#$ such that $\sigma_{A_n}(x^*) > \sigma_A(x^*) + \varepsilon$ for all $n \in N$. It follows that

$$\forall n \in N,\ \exists x_n \in A_n\ :\ \langle x_n, x^* \rangle > \sigma_A(x^*) + \varepsilon. \tag{14}$$

We distinguish between two cases.

(i) There exists some $N' \in \mathcal{N}_\infty^\#(N)$ such that $x_n \to x \in X$. Then $x \in \operatorname{LIMSUP}_{n\in\mathbb{N}} A_n \subset \limsup_{n\in\mathbb{N}} A_n = A$, which contradicts (14).

(ii) Otherwise there is some $N' \in \mathcal{N}_\infty^\#(N)$ such that $\|x_n\| \neq 0$ for $n \in N'$, $\|x_n\| \xrightarrow{N'} \infty$ and $\|x_n\|^{-1} x_n \xrightarrow{N'} u \in X$. From Proposition 5.7 we deduce that $u \in K = 0^+A$. Proposition 5.6 yields $\langle u, x^* \rangle < 0$. On the other hand, dividing both sides of the inequality in (14) by $\|x_n\|$ for $n \in N'$ and taking the limit we obtain $\langle u, x^* \rangle \geq 0$, a contradiction.

(b) Let $Y := K \cap -K \neq \{0\}$. First observe that $A = \limsup_{n\in\mathbb{N}} A_n'$, where $A_n' := A_n \oplus Y \in \mathcal{C}$. For this, set $A_N := \sup_{n\in N} A_n = \operatorname{cl}\operatorname{conv}\left(\cup_{n\in N} A_n\right)$ for every $N \in \mathcal{N}_\infty$. Since $A \subset A_N$ we have $Y \subset 0^+A \subset 0^+A_N$, and so $A_N = Y + A_N$. It follows that for every $n \in N$ we have $A_n \subset A'_n \subset A_N \oplus Y = A_N$, whence $A_N = \sup_{n\in N} A_n \subset \sup_{n\in N} A'_n \subset A_N$. Hence $A = \limsup_{n\in\mathbb{N}} A'_n$. Moreover, $\sigma_{A'_n} = \sigma_{A_n} + \sigma_Y$ if $A_n \neq \emptyset$ and $\sigma_{A'_n} = \sigma_{A_n}$ if $A_n = \emptyset$, and so $\sigma_{A'_n}(x^*) = \sigma_{A_n}(x^*)$ for every $x^* \in K^\circ$ (because $K^\circ \subset Y^\circ = Y^\perp$, and so $\sigma_Y(x^*) = 0$ for every $x^* \in K^\circ$). These arguments show that, without loss of generality, we may assume that $A_n + Y = A_n$ for every $n \in \mathbb{N}$. Consider a linear space $Z \subset X$ such that $X = Y + Z$ and $Y \cap Z = \{0\}$. Then $A_n = Y + (Z \cap A_n)$. Setting $B_n := Z \cap A_n \in \mathcal{C}(Z)$ and identifying X with $Y \times Z$, we have $A_n = Y \times B_n$ for every $n \in \mathbb{N}$. Taking $B := \limsup_{n\in\mathbb{N}} B_n$, by Lemma 4.3 we have $A = Y \times B$ and $K = 0^+A = Y \times 0^+B$, whence $K^\circ = \{0\} \times (0^+B)^\circ$. Since $Y = K \cap -K$, it follows that 0^+B is pointed. Moreover, $\operatorname{ri} K^\circ = \{0\} \times \operatorname{ri}(0^+B)^\circ$ and $\sigma_A(y^*, z^*) = \sigma_B(z^*)$ for $y^* = 0$, $\sigma_A(y^*, z^*) = \infty$ for $y^* \neq 0$ (and similarly for A, B replaced by A_n, B_n, respectively). The conclusion follows applying (a). □

We now state the main result of this paper, a characterization of $\mathcal{C}$-convergence.

Theorem 5.9 *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $A \in \mathcal{C}_0$; the following statements are equivalent:*
(i) $A_n \xrightarrow{\mathcal{C}} A$,
(ii) $\forall x^* \in X^* \setminus \operatorname{rbd}(0^+A)^\circ : \lim_{n\in\mathbb{N}} \sigma_{A_n}(x^*) = \sigma_A(x^*)$.

Proof. (i) ⇒ (ii). We have $A = \limsup_{n\in\mathbb{N}} A_n = \liminf_{n\in\mathbb{N}} A_n$. From Proposition 5.5 we easily obtain $\sigma_A \leq \liminf_{n\in\mathbb{N}} \sigma_{A_n}$, and so

$$\sigma_A \leq \liminf_{n\in\mathbb{N}} \sigma_{A_n} \leq \limsup_{n\in\mathbb{N}} \sigma_{A_n}. \tag{15}$$

Since $\operatorname{dom} \sigma_A \subset (0^+A)^\circ$ (see (10)), it follows that $\lim_{n\in\mathbb{N}} \sigma_{A_n}(x^*) = \sigma_A(x^*) = \infty$ for every $x^* \in X^* \setminus (0^+A)^\circ$. It remains to show that $\lim_{n\in\mathbb{N}} \sigma_{A_n}(x^*) = \sigma_A(x^*)$ for every $x^* \in \operatorname{ri}(0^+A)^\circ$. This follows from Lemma 5.8 and (15).

(ii) ⇒ (i). First note that $A_n \in \mathcal{C}_0$ for $n \in N_0$ for some $N_0 \in \mathcal{N}_\infty$ and so we may assume that $A_n \in \mathcal{C}_0$ for every n. Moreover, $\operatorname{rbd} \operatorname{dom} \sigma_A = \operatorname{rbd}(0^+A)^\circ$.

Let us prove that $A \subset \liminf_{n\in\mathbb{N}} A_n$. First we prove that $A \subset B := \limsup_{n\in\mathbb{N}} A_n$. Assuming that this is done, for $N \in \mathcal{N}_\infty^\#$ we have $\lim_{n\in N} \sigma_{A_n}(x^*) = \sigma_A(x^*)$ for every $x^* \in X^* \setminus \operatorname{rbd}(0^+A)^\circ$, whence $A \subset \limsup_{n\in N} A_n \subset \sup_{n\in N} A_n$. It follows that $A \subset \inf_{N\in\mathcal{N}_\infty^\#} \sup_{n\in N} A_n = \liminf_{n\in\mathbb{N}} A_n$.

In order to show that $A \subset \limsup_{n\in\mathbb{N}} A_n$, let $B_n := \sup_{k\geq n} A_k$. Since $B_{n+1} \subset B_n$, we have $\sigma_{B_{n+1}} \leq \sigma_{B_n}$ for every $n \in \mathbb{N}$. It follows that

$$\limsup_{n\in\mathbb{N}} \sigma_{A_n}(x^*) = \inf_{n\in\mathbb{N}} \sup_{k\geq n} \sigma_{A_k}(x^*) = \inf_{n\in\mathbb{N}} \sigma_{B_n}(x^*) = \lim_{n\in\mathbb{N}} \sigma_{B_n}(x^*)$$

and so

$$\forall n \in \mathbb{N},\ \forall x^* \in X^* \setminus \operatorname{rbd} \operatorname{dom} \sigma_A : \ \sigma_A(x^*) \leq \sigma_{B_n}(x^*)\ . \tag{16}$$

On the other hand, because $L(B_{n+1}) \subset L(B_n)$ for every n, there exists some n_0 such that $L(B_n) = Y$ for $n \geq n_0$. We show that $Y \subset L(A)$. Indeed, take $x^* \in X^* \setminus Y^\perp$; then $x^* \notin (0^+B_n)^\circ$, and so $\sigma_{B_n}(x^*) = \infty$ for every n. If $x^* \notin \operatorname{rbd} \operatorname{dom} \sigma_A$, by hypothesis, we obtain $\sigma_A(x^*) = \infty$. Hence $x^* \in X^* \setminus \operatorname{ri} \operatorname{dom} \sigma_A$. It follows that $\operatorname{ri} \operatorname{dom} \sigma_A \subset Y^\perp$, whence $(0^+A)^\circ = \operatorname{cl} \operatorname{dom} \sigma_A \subset Y^\perp$. Thus we obtain $Y \subset 0^+A$, hence $Y \subset L(A)$. It follows that $L(B_n) \subset L(A)$ for every $n \geq n_0$. Taking into account (16), from (13) we conclude that $A \subset B_n$ for every $n \geq n_0$, and so $A \subset B_n$ for every n. This proves that $A \subset \limsup A_n = B$. Hence, as observed above, $A \subset \liminf_{n\in\mathbb{N}} A_n$.

But $B = \inf_{n\in\mathbb{N}} B_n$, and so, by Proposition 5.5, we have

$$\sigma_B \leq \inf_{n\in\mathbb{N}} \sigma_{B_n} = \inf_{b\in\mathbb{N}} \sup_{k\geq n} \sigma_{A_k} = \limsup_{n\in\mathbb{N}} \sigma_{A_n}.$$

It follows that $\sigma_B(x^*) \leq \sigma_A(x^*)$ for every $x^* \in X^* \setminus \operatorname{rbd}(0^+A)^\circ$. Using (12) we obtain $B \subset A$. Hence $A = \lim A_n$. □

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