Optimization with set relations: Conjugate Duality

Andreas Löhne*

April 30, 2004 (last update: April 4, 2005)

The aim of this paper is to develop a conjugate duality theory, based on set relation approach, for convex set–valued maps. The basic idea is to understand a convex set–valued map as a function with values in the space of closed convex subsets of $\mathbb{R}^p$. The usual inclusion of sets provides a natural ordering relation in this space. Infimum and supremum with respect to this ordering relation can be expressed with the aid of union and intersection. Our main result is a strong duality assertion formulated along the lines of classical duality theorems for extended real–valued convex functions.

Keywords: set–valued optimization; set relations; duality; power structures; embedding of convex sets

Mathematics Subject Classification 2000: 90C48; 52A41; 90C29

1 Introduction

Set–valued optimization problems have been investigated by many authors, see Jahn [9] and the references therein. Set–valued problems naturally occur in vector optimization, for instance, as dual problems, and, of course, vector optimization problems provide a very important special case of set–valued optimization with numerous applications. Set–valued optimization theory is well developed, so there are many papers on optimality conditions, duality theory as well as related topics. For instance, in Hamel et al. [7] a set–valued approach is used to solve the problem of the duality gap in linear vector optimization in the case that the right hand side of the inequality constraints is zero. Given a set–valued objective map $F : X \rightrightarrows Y$ and a set of feasible points $S \subseteq X$, where X and Y are linear spaces and Y is partially ordered by convex pointed cone, in set–valued optimization one deals with minimal points of the set $F(S) := \bigcup_{x \in S} F(x)$, where the minimality notion is understood with respect to the partial ordering in the space Y.

Optimization with set relations provides quite a different approach to set–valued optimization. The main idea is to understand the set–valued objective map $f : X \rightrightarrows Y$ as a function $f : X \to \hat{\mathcal{P}}(Y)$ into the space $\hat{\mathcal{P}}(Y)$ of all subsets of Y. This space is provided with an appropriate ordering relation. Ordering relations on power structures have been investigated, for instance, by Brink [3]. In the special case that Y is a linear space, $K \subseteq Y$ a convex pointed cone (containing zero) and $A, B \subseteq Y$ these relations can be expressed by

$$A \preccurlyeq_K B \;:\Leftrightarrow\; B \subseteq A + K \qquad \text{and} \qquad A \curlyeqprec_K B \;:\Leftrightarrow\; A \subseteq B - K.$$

*Department of Mathematics and Computer Science Martin–Luther–Universität Halle–Wittenberg, 06099 Halle (Saale), Germany, e–mail: loehne@mathematik.uni–halle.de

Kuroiwa [11] formulated corresponding optimization problems. Further references about the origin of these ordering relations can be found in Hamel [6].

In this paper, we investigate convex problems in this context. It turns out that the relations introduced above can be replaced by the usual inclusion of sets in order to describe all the relevant assertions in the paper's framework. This is due to the fact that the relations are not antisymmetric. In order to obtain antisymmetry it is necessary to switch over to equivalence classes. Choosing appropriate representatives of these equivalence classes is the same as using the usual inclusion of sets. A more detailed discussion can be found in [6], [13]. Infimum and supremum with respect to the relation "inclusion of sets" are expressed via union and intersection. We develop a set–valued conjugate duality theory, based on the relation "inclusion of sets", and we proceed completely analogous to the scalar theory of conjugate duality. This work is organized as follows.

In the next section, we investigate the structure of the objective function's image space, namely, the space of closed convex subsets of $\mathbb{R}^p$. We observe that this space is not a linear space. Moreover, it is not possible to embed this space into a linear space. However, as we will see in the third section, certain subsets of this space can be embedded. Essentially, a convex function only attains values in such a subset. This ensures the linear structure of the image space, which is usually needed in duality theory. Furthermore, it is necessary to observe whether infimum and supremum are changed while the embedding procedure. We observe that the infimum is not changed, but the supremum is so. At the first glance, this problem seems to be asymmetric in this sense. With the aid of the concept of oriented sets, due to Rockafellar [18], the symmetry can be re–established. For the details we also refer to [13]. In Section 4, we develop the duality theory. Weak and strong duality assertions are proven. The last section is devoted to some examples.

As for prerequisites, the reader is expected to be familiar with Rockafellar's "Convex Analysis" [18]. Up to a few exceptions, we frequently use the notation therein. The following notations are not in accordance with Rockafellar's book. The symbol $\oplus$ does not mean the direct sum, because it will get an other meaning. If A is a real $m \times n$ matrix, $\operatorname{rg} A := \{Ax \in \mathbb{R}^m |\ x \in \mathbb{R}^n\}$ denotes the range of A and A^T is the transposed matrix. Further, we write $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n |\ \forall i \in \{1, ..., n\} : x_i \geq 0\}$, $\mathbb{R}^n_- := -\mathbb{R}^n_+$ and $\mathbb{R}_+ := \mathbb{R}^1_+$.

The main results of this paper were announced in [12].

2 The structure of the image space

Throughout the paper, Y stands for the Euclidian space $\mathbb{R}^p$, where p is a positive integer. The space of all nonempty closed convex subsets of Y is denoted by $\mathcal{C}(Y)$. For simplicity of notation, we write $\mathcal{C}$ instead of $\mathcal{C}(Y)$. In $\mathcal{C}$ we introduce an addition $\oplus :\ \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a multiplication by nonnegative real numbers $\cdot :\ \mathbb{R}_+ \times \mathcal{C} \to \mathcal{C}$, defined by

$$\forall A, B \in \mathcal{C} :\ A \oplus B := \operatorname{cl}(A + B) = \operatorname{cl}\{a + b |\ a \in A, b \in B\},$$

$$\forall A \in \mathcal{C},\ \alpha \geq 0 :\ \alpha A := \alpha \cdot A := \{\alpha a |\ a \in A\}.$$

Clearly, both operations are well–defined and for all $A, B, C \in \mathcal{C}$ and all $\alpha, \beta \in \mathbb{R}_+$ the following calculus rules hold true:

(C1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$, (C2) $\{0\} \oplus A = A$,
(C3) $A \oplus B = B \oplus A$, (C4) $\alpha\,(\beta\,A) = (\alpha\beta)\,A$,
(C5) $1 \cdot A = A$, (C6) $\alpha\,(A \oplus B) = \alpha\,A \oplus \alpha\,B$,
(C7) $0 \cdot A = \{0\}$, (C8) $\alpha\,A \oplus \beta\,A = (\alpha + \beta)\,A$.

In Hamel [6], a set $(W, \oplus, \cdot)$ is called a *conlinear space* if the axioms (C1)–(C7) are satisfied. In this manner, $\mathcal{C}$ is a *conlinear space* where, additionally, the second distributive law (C8) holds true. For further concepts of this type we refer to [6], [10] and the references therein. It is easy to see that $\mathcal{C}$ is not a linear space, since the axiom of the existence of an inverse element is violated. Moreover, it is *not possible to embed $\mathcal{C}$ into a linear space.* Indeed, assume there is an injective homomorphism j (an embedding) from $\mathcal{C}$ into a linear space L. Given a nonempty closed convex cone $K \subseteq Y$ such that $K \neq \{0\}$ we have $K \in \mathcal{C}$ and $K = K \oplus K$. Hence $j(K) = j(K) + j(K) \neq 0$. Then there must be an inverse element $l \in L$ of $j(K)$, i.e., $j(K) + l = 0$. It follows $0 = j(K) + l = j(K) + l + j(K) = j(K)$, a contradiction.

Although, $\mathcal{C}$ is not a linear space nor can it be embedded into a linear space, its structure is rich enough to define the concept of convexity and that of a cone. A subset $\mathcal{A} \subseteq \mathcal{C}$ is said to be *convex* if $A, B \in \mathcal{A}$ implies that $\lambda A \oplus (1 - \lambda)B \in \mathcal{A}$ for all $\lambda \in [0, 1]$. A subset $\mathcal{A} \subseteq \mathcal{C}$ is said to be a *cone* if $A \in \mathcal{A}$ implies that $\alpha A \in \mathcal{A}$ for all $\alpha > 0$.

Rockafellar [18, Section 39] introduced the concept of *orientation* of convex sets in Y. A convex set $A \subseteq Y$ that is identified with its convex indicator function $\delta(\,\cdot\,|A)$ is said to be *supremum oriented* and a convex set $A \subseteq Y$ that is identified with the concave function $-\delta(\,\cdot\,|A)$ is called *infimum oriented*. This concept plays a crucial role in our theory. Thus we introduce the following notation: The space $\mathcal{C}^\star$ is defined to be the space of all nonempty closed convex subset of Y having supremum orientation. By $\mathcal{C}^\diamond$ we denote the space of all nonempty closed convex subsets of Y having infimum orientation. If not stated otherwise, the orientation is not changed while manipulating sets. For instance, this means $\mathcal{C}^\star$ and $\mathcal{C}^\diamond$ are conlinear spaces, the recession cone of a supremum (infimum) oriented set is supremum (infimum) oriented, and so on.

Let $K \subseteq Y$ be a nonempty closed convex cone. The set $\mathcal{C}_K \subseteq \mathcal{C}$ is defined to be the set of all elements $A \in \mathcal{C}$ with $0^+A = K$. If these sets additionally have an orientation, we write $\mathcal{C}_K^\star$ and $\mathcal{C}_K^\diamond$, respectively.

Proposition 2.1 *$\mathcal{C}_K$ is a convex cone in $\mathcal{C}$.*

Proof. Let $A, B \in \mathcal{C}_K$. It remains to show $0^+(A \oplus B) = K$. This is a consequence of [18, Corollary 9.1.1] if we can verify the following condition: If $z_1 \in 0^+A$ and $z_2 \in 0^+B$ are such that $z_1 + z_2 = 0$, then z_1 belongs to the lineality space of A and z_2 belongs to the lineality space of B. Indeed, we have $0^+A = 0^+B = K$ and the lineality spaces of A and B are equal, namely $0^+A \cap (-0^+A) = 0^+B \cap (-0^+B) = K \cap (-K)$. Hence the mentioned condition is satisfied. $\square$

Note that [18, Corollary 9.1.1] also implies that in $\mathcal{C}_K \subseteq \mathcal{C}$ the addition $\oplus$ reduces to the usual Minkowski addition $+$, i.e., the closure operation is superfluous.

The space $\mathcal{C}$ is now equipped with one of the reflexive, transitive and antisymmetric relations $\supseteq$ and $\subseteq$. We establish standard relations, in dependence of the orientation of the members of the space. In fact, let the standard relation be $\supseteq$ in the space $\mathcal{C}^\star$ and $\subseteq$ in $\mathcal{C}^\diamond$. Both these standard relations have the meaning of "less or equal". This identification makes it easier to distinguish between convex and concave functions with values in $\mathcal{C}$ (which is defined below).

We observe the following relation between the conlinear structure and the ordering structure in $(\mathcal{C}, \supseteq)$ and $(\mathcal{C}, \subseteq)$.

$$\forall A, B, C, D \in \mathcal{C},\ \forall \alpha \in \mathbb{R}_+ :\ A \supseteq B,\ C \supseteq D\ \Rightarrow \alpha\,(A \oplus C) \supseteq \alpha\,(B \oplus D).$$

In [6], a conlinear space equipped with a partial ordering and satisfying the latter condition is called an *ordered conlinear space*. This means our spaces $\mathcal{C}^\star$ and $\mathcal{C}^\diamond$ (with its standard relations) are ordered conlinear spaces.

We now repeat some concepts with respect to partially ordered sets, for instance, see [20]. Moreover, we illustrate some crucial facts according to these concepts by simple examples. If $(W, \leq)$ is a partially ordered set, V is a subset of W and the point $w_0 \in W$ satisfies $v \leq w_0$ for all $v \in V$, then w_0 is called an *upper bound* of V. The subset V is now said to be *bounded above*. The definitions of *bounded below* and *lower bound* are analogous. Note that the boundedness depends on the "basis set" W. For instance, letting $W = \mathbb{R}$, ordered by the usual ordering $\leq$, the open interval $V = (0, 1)$ is bounded. If we take instead $W = (0, 1)$, then the same set $V = (0, 1)$ is not bounded. If $w_0 \in W$ is an upper bound of V such that $w_0 \leq \bar{w}$ for any other upper bound $\bar{w} \in W$ of V, then w_0 is called *least upper bound* or *supremum* of V and is denoted by $\sup V$. If V has a supremum then it is uniquely defined. This is an easy consequence of the antisymmetry of the relation $\leq$. The *greatest lower bound* or *infimum* is analogously defined and is denoted by $\inf V$. Supremum and infimum of a set $V \subseteq W$ also depend on the "basis set" W as the following example shows. As above, let $W = \mathbb{R}$ and $V = (0, 1)$. Then, $\sup V = 1$. If we have $W = \{r \in \mathbb{R} |\ r < 1 \vee r \geq 2\}$ instead, $\sup V = 2$. In case of $W = \{r \in \mathbb{R} |\ r < 1 \vee r > 2\}$, the supremum of V does not exist. A partially ordered set W is said to be *order complete* if every subset of W has supremum and infimum. If W is order complete and $V = \emptyset$, then $\sup V = \inf W$ and $\inf V = \sup W$. The set W is called *Dedekind complete* if every nonempty subset of W that is bounded above (bounded below) has a supremum (infimum). Note that for Dedekind completeness an one–sided condition is already sufficient, this means W is Dedekind complete if and only if every nonempty subset of W which is bounded above has a supremum [20, Theorem 1.4]. An element $\bar{w} \in W$ is called the *largest* element of $(W, \leq)$ if $w \leq \bar{w}$ for all $w \in W$. The *smallest* element is defined analogously. If $(W, \leq)$ has a largest (smallest) element, then it is uniquely defined.

Let us apply these concepts to the spaces $\mathcal{C}^\star$ and $\mathcal{C}^\diamond$.

Proposition 2.2 *The spaces $\mathcal{C}^\star$ and $\mathcal{C}^\diamond$ are Dedekind complete and the infimum and supremum can be expressed as follows:*

$$
\begin{array}{llcll}
\text{(i)} & \emptyset \neq \mathcal{A} \subseteq \mathcal{C}^\star \textit{ bounded above} & \Rightarrow & \sup \mathcal{A} & = \bigcap\limits_{A \in \mathcal{A}} A, \\
\text{(ii)} & \emptyset \neq \mathcal{A} \subseteq \mathcal{C}^\star & \Rightarrow & \inf \mathcal{A} & = \operatorname{cl} \operatorname{conv} \bigcup\limits_{A \in \mathcal{A}} A, \\
\text{(iii)} & \emptyset \neq \mathcal{A} \subseteq \mathcal{C}^\diamond & \Rightarrow & \sup \mathcal{A} & = \operatorname{cl} \operatorname{conv} \bigcup\limits_{A \in \mathcal{A}} A, \\
\text{(iv)} & \emptyset \neq \mathcal{A} \subseteq \mathcal{C}^\diamond \textit{ bounded below} & \Rightarrow & \inf \mathcal{A} & = \bigcap\limits_{A \in \mathcal{A}} A.
\end{array}
$$

Proof. (i) Set $S := \bigcap_{A \in \mathcal{A}} A$. Let $\bar{A}$ be an upper bound of $\mathcal{A}$, i.e., $A \supseteq \bar{A}$ for all $A \in \mathcal{A}$. Hence $S \neq \emptyset$. Of course, S is convex and closed. Thus, S belongs to $\mathcal{C}^\star$. For all $A \in \mathcal{A}$ we have $A \supseteq S$, i.e., S is an upper bound of $\mathcal{A}$. Let $\bar{S} \in \mathcal{C}^\star$ be another upper bound of $\mathcal{A}$, i.e., for all $A \in \mathcal{A}$ it holds $A \supseteq \bar{S}$, then it follows $\bigcap_{A \in \mathcal{A}} A \supseteq \bar{S}$, i.e., $S \supseteq \bar{S}$.

(ii) Set $I := \operatorname{cl} \operatorname{conv} \bigcup_{A \in \mathcal{A}} A$. Of course, $I \in \mathcal{C}^\star$. For all $A \in \mathcal{A}$ we have $I \supseteq A$, i.e., I is a lower bound of $\mathcal{A}$. Let $\bar{I} \in \mathcal{C}^\star$ be another lower bound of $\mathcal{A}$, i.e., for all $A \in \mathcal{A}$ it holds $\bar{I} \supseteq A$, then it follows $\bar{I} \supseteq \bigcup_{A \in \mathcal{A}} A$. Since $\bar{I}$ is closed and convex, we obtain $\bar{I} \supseteq I$. The same reasoning applies to (iii) and (iv). $\square$

In many situations, it is convenient to extend a Dedekind complete partially ordered set by a smallest or largest element in order to obtain an order complete set. In $\mathcal{C}^\star$ there already exists the smallest element, namely $Y \in \mathcal{C}^\star$. Therefore, we extend the space $\mathcal{C}^\star$ only by the largest element. Intuitively, we denote this element by $\emptyset$ (and identify it with the empty set). Then, we have $A \supseteq \emptyset$ for all $A \in \mathcal{C}$. This new element is also provided with an orientation, in this case with supremum orientation. Addition and multiplication with this new element are defined by

$$\forall A \in \mathcal{C} \cup \{\emptyset\} : \; A \oplus \emptyset = \emptyset \oplus A = \emptyset, \qquad \forall \alpha > 0 : \; \alpha \cdot \emptyset = \emptyset, \qquad 0 \cdot \emptyset = \{0\}.$$

The resulting order complete conlinear space is denoted by $\hat{\mathcal{C}}^\star$. In the same way the space $\mathcal{C}^\diamond$ is extended by the smallest (infimum oriented) element $\emptyset$ and the resulting order complete conlinear space is denoted by $\hat{\mathcal{C}}^\diamond$.

In every order complete ordered conlinear space $(W, \oplus, \cdot, \leq)$ it is evident that

$$\inf(\mathcal{A} + \mathcal{B}) \geq \inf \mathcal{A} \oplus \inf \mathcal{B} \qquad \text{and} \qquad \sup(\mathcal{A} + \mathcal{B}) \leq \sup \mathcal{A} \oplus \sup \mathcal{B}, \tag{1}$$

where $\mathcal{A} + \mathcal{B} := \{A \oplus B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. In general, (1) does not hold with equality.

Example 2.3 Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $\mathbb{B} := \{x \in \mathbb{R}^2 \mid \; \|x\| \leq 1\}$, $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}^\star$, $\mathcal{A} := \{\{(0,1)^T\} + K, \{(1,0)^T\} + K\}$, $\mathcal{B} := \{\mathbb{B}\}$. Then, $\sup \mathcal{A} \oplus \sup \mathcal{B} = \big((\{(0,1)^T\} + K) \cap (\{(1,0)^T\} + K) \big) + \mathbb{B} = \{(1,1)^T\} + K + \mathbb{B}$. However, $\sup(\mathcal{A} + \mathcal{B}) = \big(\{(0,1)^T\} + K + \mathbb{B}\big) \cap \big(\{(1,0)^T\} + K + \mathbb{B}\big) = K$. Hence, $\sup(\mathcal{A} + \mathcal{B}) \neq \sup \mathcal{A} \oplus \sup \mathcal{B}$.

However, for the infimum in $\mathcal{C}^\star$ and the supremum in $\mathcal{C}^\diamond$, (1) even holds with equality.

Proposition 2.4 *For nonempty sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}^\star$ and $\bar{\mathcal{A}}, \bar{\mathcal{B}} \subseteq \mathcal{C}^\diamond$ it holds*

$$\inf(\mathcal{A}+\mathcal{B}) = \inf \mathcal{A} \oplus \inf \mathcal{B} \qquad \textit{and} \qquad \sup(\bar{\mathcal{A}}+\bar{\mathcal{B}}) = \sup \bar{\mathcal{A}} \oplus \sup \bar{\mathcal{B}}.$$

Proof. For nonempty subsets $A, B \subseteq Y$ it holds $\operatorname{conv} A + \operatorname{conv} B = \operatorname{conv}(A+B)$, for instance, see [16]. Furthermore, it is easy to check that $\operatorname{cl}(\operatorname{cl} A + \operatorname{cl} B) = \operatorname{cl}(A+B)$. Hence, we conclude $\operatorname{cl}(\operatorname{cl}\operatorname{conv} A + \operatorname{cl}\operatorname{conv} B) = \operatorname{cl}\operatorname{conv}(A+B)$. This yields

$$\begin{aligned}
\inf(\mathcal{A}+\mathcal{B}) &\overset{\text{Prop. 2.2}}{=} \operatorname{cl}\operatorname{conv} \bigcup_{C \in \mathcal{A}+\mathcal{B}} C \;=\; \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A},\, B \in \mathcal{B}} \{A \oplus B\} \\
&\supseteq \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A},\, B \in \mathcal{B}} \{A + B\} \;=\; \operatorname{cl}\operatorname{conv} \left(\bigcup_{A \in \mathcal{A}} A + \bigcup_{B \in \mathcal{B}} B \right) \\
&= \operatorname{cl} \left(\operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A}} A + \operatorname{cl}\operatorname{conv} \bigcup_{B \in \mathcal{B}} B \right) \overset{\text{Prop. 2.2}}{=} \inf \mathcal{A} \oplus \inf \mathcal{B}.
\end{aligned}$$

By (1) (or directly) we deduce equality. The second part is completely the same. $\square$

The following definitions of convex and concave functions are clear having in mind that both the standard relation $\supseteq$ in $\mathcal{C}^\star$ and $\subseteq$ in $\mathcal{C}^\diamond$ have the meaning of "less or equal". Let X be a linear space and $S \subseteq X$ convex. A function $f : S \to \hat{\mathcal{C}}^\star$ is said to be *convex* if

$$\forall \lambda \in [0,1],\, \forall x, u \in S: \quad f(\lambda \cdot x + (1-\lambda) \cdot u) \supseteq \lambda\, f(x) \oplus (1-\lambda)\, f(u). \tag{2}$$

However, if a function $f : S \to \hat{\mathcal{C}}^\diamond$ satisfies (2) it is said to be *concave*. Analogously, a function $f : S \to \hat{\mathcal{C}}^\star$ is *concave* and a function $f : S \to \hat{\mathcal{C}}^\diamond$ is *convex* if the following dual condition is satisfied:

$$\forall \lambda \in [0,1],\, \forall x, u \in S: \quad f(\lambda \cdot x + (1-\lambda) \cdot u) \subseteq \lambda\, f(x) \oplus (1-\lambda)\, f(u).$$

At the first glance, a convex function $f : S \to \hat{\mathcal{C}}^\star$ does not subsume the important case of an extended real–valued convex function. In Example 4.5 that follows, however, we see that extended real–valued problems are equivalent to special set–valued problems.

Up to now, all the operations used did not influence the orientation of a set. We want to express the change of the orientation of a set as follows: Given an oriented set A we denote by $\boxplus A$ the same set, but with opposite orientation. As usual, the negative of a convex set A is defined by

$$-A := \{y \in Y |\; -y \in A\}.$$

By convention, if A is an oriented set, this operation does not manipulate the orientation of A. In contrast to this, we introduce a second concept of a negative of a convex set which does so. Given an oriented set A we define $\boxminus A$ being the set $-A$, but with the opposite orientation. Instead of two signs, we now have four signs, namely $+, -, \boxplus, \boxminus$. Obviously, the following assertions hold true:

$$\begin{aligned} A &= \boxplus \boxplus A = \boxminus \boxminus A, & -A &= \boxplus \boxminus A = \boxminus \boxplus A, \\ \boxplus A &= + \boxplus A = - \boxminus A, & \boxminus A &= + \boxminus A = - \boxplus A. \end{aligned}$$

Clearly, an expression is independent of the order of the signs. Note that $\boxplus$ and $\boxminus$ are signs but not operations. This means, adding elements $A \in \mathcal{C}^\star$ with elements $B \in \mathcal{C}^\diamond$ is not allowed. Nevertheless, we write $A \boxplus B := A + (\boxplus B)$ and $A \boxminus B := A + (\boxminus B)$, if these expressions are defined, i.e., A and B are contrarily oriented. If $\emptyset^\star$ is the largest element in $\hat{\mathcal{C}}^\star$ and $\emptyset^\diamond$ is the smallest element in $\mathcal{C}^\diamond$, then let us use the convention $\boxminus \emptyset^\diamond = \emptyset^\star$. Thus we obtain the well–known convexity–concavity dualism also for convex functions with values in $\hat{\mathcal{C}}$. A function $f : S \to \hat{\mathcal{C}}^\star$ is convex (concave) if and only if $\boxminus f : S \to \hat{\mathcal{C}}^\diamond$ is concave (convex). For a given set $\mathcal{A} \subseteq \hat{\mathcal{C}}^\star$ and using the notation $\boxminus \mathcal{A} := \{\boxminus A |\ A \in \mathcal{A}\}$, it can be easily shown (Proposition 2.2) that

$$\boxminus \inf \mathcal{A} = \sup \boxminus \mathcal{A} \quad \text{and} \quad \boxminus \sup \mathcal{A} = \inf \boxminus \mathcal{A}. \tag{3}$$

A further motivation for the usage of the sign $\boxminus$ will be given in the next section.

3 Embedding subsets of $\mathcal{C}$ into a linear space

Embedding of spaces of convex sets into linear spaces was investigated by Rådström [17]. For further results in this field, compare [1] and the references therein.

The aim of this section is to embed the convex cone $\mathcal{C}_K \subseteq \mathcal{C}$ into a partially ordered linear space. In dependence of the orientation of the members of $\mathcal{C}_K$ we use different embedding maps. This procedure allows us to re–interpret the inverse element of the embedding map's image of a member of $\mathcal{C}_K$ as an element of $\mathcal{C}_{-K}$ having the opposite orientation.

The following lemma is an important tool in the proof of the duality theorem in the next section. It is a refinement of [18, Theorem 13.1]. Denoting the polar cone of a cone $K \subseteq Y$ by K°, it is shown that only the set $\operatorname{ri}(0^+A)^\circ := \operatorname{ri}((0^+A)^\circ)$ (instead of the whole space $Y^* = \mathbb{R}^p$) is essential for the description of a nonempty closed convex set via its support function.

Lemma 3.1 *Let A be a nonempty closed convex subset of Y. Then*

$$A = \bigcap_{y^* \in \operatorname{ri}(0^+A)^\circ} \{y \in Y |\ \langle y^*, y \rangle \le \delta^* (y^* |\, A)\}.$$

Proof. As a consequence of [18, Theorem 14.2] we have $\operatorname{cl} \operatorname{dom} \delta^*(\,\cdot\,|\,A\,) = (0^+A)^\circ$ (compare [8, Theorem 2.2.4], too). Together with [18, Theorem 6.3] this yields

$$\operatorname{ri}(0^+A)^\circ = \operatorname{ri} \operatorname{cl} \operatorname{dom} \delta^*(\,\cdot\,|\,A\,) \subseteq \operatorname{dom} \delta^*(\,\cdot\,|\,A\,) \subseteq (0^+A)^\circ. \tag{4}$$

By [18, Theorem 13.1] and (4) we obtain

$$A = \bigcap_{y^* \in Y^*} \big\{y \in Y |\ \langle y^*, y \rangle \le \delta^* (y^* |\, A)\big\} = \bigcap_{y^* \in (0^+A)^\circ} \big\{y \in Y |\ \langle y^*, y \rangle \le \delta^* (y^* |\, A)\big\}.$$

It remains to show

$$Y_1 := \bigcap_{y^* \in (0^+A)^\circ} \left\{y \in Y \mid \langle y^*, y\rangle \le \delta^*\left(y^* \mid A\right)\right\} = \bigcap_{y^* \in \operatorname{ri}(0^+A)^\circ} \left\{y \in Y \mid \langle y^*, y\rangle \le \delta^*\left(y^* \mid A\right)\right\} =: Y_2.$$

The inclusion $Y_1 \subseteq Y_2$ is obvious. In order to show $Y_2 \subseteq Y_1$ let $y \in Y_2$ be arbitrarily chosen. It holds $\langle y^*, y\rangle \le \delta^*\left(y^* \mid A\right)$ for all $y^* \in \operatorname{ri}(0^+A)^\circ$. Let $\bar{y}^* \in (0^+A)^\circ$ and $y^* \in \operatorname{ri}(0^+A)^\circ$, then $\lambda\bar{y}^* + (1-\lambda)y^* \in \operatorname{ri}(0^+A)^\circ$ for all $\lambda \in [0,1)$ (compare [18, Theorem 6.1]). It follows

$$\langle \lambda\bar{y}^* + (1-\lambda)y^*, y\rangle \le \delta^*\left(\lambda\bar{y}^* + (1-\lambda)y^* \mid A\right) \le \lambda\,\delta^*\left(\bar{y}^* \mid A\right) + (1-\lambda)\delta^*\left(y^* \mid A\right).$$

By virtue of (4), we deduce that $\delta^*\left(y^* \mid A\right) < +\infty$. Letting $\lambda \to 1$ we obtain $\langle \bar{y}^*, y\rangle \le \delta^*\left(\bar{y}^* \mid A\right)$, i.e., $y \in Y_1$. $\square$

With the aid of the preceeding lemma we are able to give an equivalent characterization of the ordered conlinear spaces $\mathcal{C}_K^\star$ and $\mathcal{C}_K^\diamond$ (the axioms (C1) – (C8) are satisfied, if we replace $\{0\}$ by K, i.e., K is the neutral element). This can be used to embed the spaces $\mathcal{C}_K^\star$ and $\mathcal{C}_K^\diamond$ into a linear space. This gives a further motivation of oriented sets and the usage of the sign $\boxminus$ introduced above and sheds new light on the structure of the objective function's image space $\mathcal{C}$.

Let $\Gamma_K^\star$ be the space of all positively homogeneous concave functions from $\operatorname{ri} K^\circ$ into $\mathbb{R}$ and let $\Gamma_K^\diamond$ be the space of all positively homogeneous convex functions from $\operatorname{ri} K^\circ$ into $\mathbb{R}$. The spaces $\Gamma_K^\star$ and $\Gamma_K^\diamond$ are conlinear spaces, with respect to the addition and the multiplication by nonnegative real numbers, defined pointwise using the corresponding operations in $\mathbb{R}$. Moreover, $\Gamma_K^\star$ and $\Gamma_K^\diamond$, equipped with the ordering relation $\le$, defined pointwise using the usual $\le$ relation in $\mathbb{R}$, are ordered conlinear spaces.

Corollary 3.2 *Let $K \subseteq Y$ be a nonempty closed convex cone. Then, the following assertions hold true:*

(i) *There exists a bijective map $j^\star : \mathcal{C}_K^\star \to \Gamma_K^\star$ such that for all $A, B \in \mathcal{C}_K^\star$ and all positive real numbers $\alpha > 0$:*

(a) $j^\star(A+B) = j^\star(A) + j^\star(B)$, (b) $j^\star(\alpha A) = \alpha j^\star(A)$,

(c) $j^\star(K) = 0_{\Gamma_K^\star}$, (d) $A \supseteq B \;\Leftrightarrow\; j^\star(A) \le j^\star(B)$.

(ii) *There exists a bijective map $j^\diamond : \mathcal{C}_{-K}^\diamond \to \Gamma_K^\diamond$ such that for all $A, B \in \mathcal{C}_{-K}^\diamond$ and all positive real numbers $\alpha > 0$:*

(a) $j^\diamond(A+B) = j^\diamond(A) + j^\diamond(B)$, (b) $j^\diamond(\alpha A) = \alpha j^\diamond(A)$,

(c) $j^\diamond(-K) = 0_{\Gamma_K^\diamond}$, (d) $A \subseteq B \;\Leftrightarrow\; j^\diamond(A) \le j^\diamond(B)$.

Proof. (i) Consider the map $j^\star$ which assigns every $A \in \mathcal{C}^\star_K$ the negative support function of the set $A \subseteq Y$, restricted to the set $\operatorname{ri} K^\circ \subseteq Y^*$. More precisely, $\gamma_A = j^\star(A)$ is defined by $\gamma_A : \operatorname{ri} K^\circ \to \mathbb{R} \cup \{\pm\infty\}$, $\gamma_A(y^*) := -\delta^*(y^* | \, A\,)$. Of course, $j^\star$ is a function. Moreover, $j^\star$ is a function from $\mathcal{C}^\star_K$ into $\Gamma^\star_K$. Indeed, let $A \in \mathcal{C}^\star_K$. Since A is nonempty, we have $\delta^*(y^* | \, A\,) > -\infty$ for all $y^* \in Y^*$. With the aid of (4) we obtain $\delta^*(y^* | \, A\,) < +\infty$ for all $y^* \in \operatorname{ri} K^\circ$. Hence $\gamma_A = j^\star(A)$ only attains values in $\mathbb{R}$. Since support functions are sublinear and $\operatorname{ri} K^\circ$ is a convex cone, $\gamma_A = j^\star(A)$ is positively homogeneous and concave on $\operatorname{ri} K^\circ$. Hence $j^\star(A) \in \Gamma^\star_K$.

$j^\star : \mathcal{C}^\star_K \to \Gamma^\star_K$ is injective. Indeed, let $A, B \in \mathcal{C}^\star_K$ such that $j^\star(A) = j^\star(B)$. Note that A and B are nonempty, closed, convex and $0^+A = 0^+B = K$. Lemma 3.1 yields $A = B$.

$j^\star : \mathcal{C}^\star_K \to \Gamma^\star_K$ is surjective. Indeed, given an element $\gamma \in \Gamma^\star_K$, define a function $d : Y^* \to \mathbb{R} \cup \{+\infty\}$ by

$$d(y^*) := \begin{cases} -\gamma(y^*) & \text{if} \quad y^* \in \operatorname{ri} K^\circ \\ +\infty & \text{else.} \end{cases}$$

Then d is convex, positively homogeneous, not identically $+\infty$ and $d > -\infty$. With the aid of [18, Corollary 13.2.1] we conclude that $\operatorname{cl} d$ is the support function of the nonempty closed convex set

$$A_\gamma := \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in Y | \; \langle y^*, y\rangle \leq d(y^*)\} = \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in Y | \; \langle y^*, y\rangle \leq -\gamma(y^*)\}\,.$$

Applying [18, Corollary 8.3.3], taking into account the considerations in [18, page 62] and applying Lemma 3.1 we obtain

$$0^+A_\gamma = \bigcap_{y^* \in \operatorname{ri} K^\circ} 0^+ \{y \in Y | \; \langle y^*, y\rangle \leq d(y^*)\} = \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in Y | \; \langle y^*, y\rangle \leq 0\} = K.$$

By definition, we have $j^\star(A_\gamma)(y^*) = -\operatorname{cl} d(y^*)$ for all $y^* \in \operatorname{ri} K^\circ$. With the aid of [18, Theorem 7.4] we have $\operatorname{cl} d(y^*) = d(y^*)$ for all $y^* \in \operatorname{ri} K^\circ$. Hence $j^\star(A_\gamma) = \gamma$.

(i)(a) and (i)(b) follow from elementary properties of the supremum in $\mathbb{R}$, compare [18, page 113], too. (i)(c) follows from the definition of the polar cone and of the support function. (i)(d) is a consequence of [18, Corollary 13.1.1].

(ii) Define $j^\diamond(A) := -j^\star(\boxminus A)$ and use (i). $\square$

Let Γ_K be the space of all positively homogeneous (and not necessarily convex or concave) functions $\gamma : \operatorname{ri} K^\circ \to \mathbb{R}$. Let Γ_K be equipped with an addition and a multiplication by scalars, defined pointwise using the corresponding operations in $\mathbb{R}$, and with an ordering relation $\leq$, defined pointwise using the usual $\leq$ relation in $\mathbb{R}$. Then, the space Γ_K is a partially ordered linear space. Corollary 3.2 yields that $(\mathcal{C}^\star_K, \supseteq)$ and $(\mathcal{C}^\diamond_{-K}, \subseteq)$ are isomorphic to convex cones in the partially ordered linear space $(\Gamma_K, \leq)$. Let $j^\star : \mathcal{C}^\star_K \to \Gamma_K$ be the injective homomorphism which embeds $\mathcal{C}^\star_K$ into Γ_K and let $j^\diamond : \mathcal{C}^\diamond_{-K} \to \Gamma_K$ be the injective homomorphism which embeds $\mathcal{C}^\diamond_{-K}$ into Γ_K. Then, it easily follows that

$$\forall A \in \mathcal{C}^\star_K : \;\; j^\star(A) + j^\diamond(\boxminus A) = 0, \qquad \forall A \in \mathcal{C}^\diamond_{-K} : \;\; j^\diamond(A) + j^\star(\boxminus A) = 0. \tag{5}$$

In this sense, $\boxminus A$ can be regarded as the "inverse element" of a nonempty closed convex set A. However, this does not imply that $\mathcal{C}^\star_K \cup \mathcal{C}^\diamond_{-K}$ is a linear space, because it is not a conlinear space.

The possibility of embedding seems to be the natural background of the duality theory, although the embedding is not explicitely used in the proof of the duality theorem in the next section. It can be shown (see [14], [13]) that a convex function $f : X \to \hat{\mathcal{C}}^\star$ essentially (i.e., on the interior of its domain or, under the additional assumption of lower semi–continuity [13], everywhere on its domain) attains values in $\mathcal{C}^\star_K$. This means, these values can be embedded into a linear space.

The next proposition tells us what happens to the infimum and supremum while the embedding procedure. A more detailed discussion in terms of the isomorphisms $j^\star$ and $j^\diamond$ can be found in [13].

Proposition 3.3 *For an arbitrary index set I, let be given a set $\mathcal{A} := \{A_i \in \hat{\mathcal{C}}^\star |\ i \in I\} \subseteq \hat{\mathcal{C}}^\star$. Then it holds*

$$\forall y^* \in Y^* :\ -\delta^*\big(y^*|\inf_{i\in I} A_i\big) = \inf_{i\in I}\big\{-\delta^*\big(y^*|A_i\big)\big\},$$

$$\forall y^* \in Y^* :\ -\delta^*\big(y^*|\sup_{i\in I} A_i\big) \geq \sup_{i\in I}\big\{-\delta^*\big(y^*|A_i\big)\big\}.$$

Proof. Without loss of generality we can assume $\mathcal{A} \subseteq \mathcal{C}^\star$. We have $\inf_{i\in I} A_i = \operatorname{cl}\operatorname{conv}\bigcup_{i\in I} A_i$. Hence, the first assertion follows from the first part of [18, Corollary 16.5.1].

Taking into account that $\sup_{i\in I} A_i = \bigcap_{i\in I} A_i$, the second part of [18, Corollary 16.5.1] yields $\delta^*\big(\cdot\,|\sup_{i\in I} A_i\big) = \operatorname{cl}\operatorname{conv}\big\{\delta^*\big(\cdot\,|A_i\big)\,|\, i \in I\big\}$, where the convex hull of a collection of functions is defined as the convex hull of the pointwise infimum of the collection, i.e., $\operatorname{cl}\operatorname{conv}\big\{\delta^*\big(\cdot\,|A_i\big)\,|\, i \in I\big\} = \operatorname{cl}\operatorname{conv}\inf_{i\in I}\delta^*\big(\cdot\,|A_i\big)$, compare [18, page 37]. It follows that $\delta^*\big(\cdot\,|\sup_{i\in I} A_i\big) \leq \inf_{i\in I}\delta^*\big(\cdot\,|A_i\big)$, which proves the second assertion. □

4 Conjugate duality

In this section, the conjugate duality theory for optimization problems based on set relations is developed. We start with two auxiliary assertions, which will be used in the proof of the duality theorem.

Lemma 4.1 *Let $A \subseteq Y$ be a nonempty closed convex set and $K \subseteq Y$ be a nonempty closed convex cone. Then*

$$\big(\forall y^* \in \operatorname{ri} K^\circ :\ \delta^*(y^*|\,A) < +\infty\big) \qquad \Rightarrow \qquad K \supseteq 0^+A.$$

Proof. From (4) we conclude that $\operatorname{ri} K^\circ \subseteq \operatorname{dom}\delta^*(\cdot\,|\,A) \subseteq (0^+A)^\circ$. With the aid of [18, Theorem 6.3] we obtain $K^\circ \subseteq (0^+A)^\circ$. It easily follows that $K^{\circ\circ} \supseteq (0^+A)^{\circ\circ}$. The bipolar theorem [18, Theorem 14.1] yields $K \supseteq 0^+A$. □

Lemma 4.2 *Let $K \subseteq Y$ be a nonempty closed convex cone which is not a linear subspace of Y. Then it holds*

$$(y \in \operatorname{ri} K \wedge y^* \in \operatorname{ri} K^\circ) \;\Rightarrow\; \langle y^*, y\rangle < 0.$$

Proof. Assume the contrary. By the definition of the polar cone this means there exists $\bar{y} \in \operatorname{ri} K$ and $\bar{y}^* \in \operatorname{ri} K^\circ$ such that $\langle \bar{y}^*, \bar{y}\rangle = 0$. We show that

$$\forall y^* \in K^\circ: \quad \langle y^*, \bar{y}\rangle = 0. \tag{6}$$

Assume that (6) is not true. Then, there is some $\tilde{y}^* \in K^\circ$ such that $\langle \tilde{y}^*, \bar{y}\rangle < 0$. Since $\bar{y}^* \in \operatorname{ri} K^\circ$ there exists some $\mu > 1$ such that $\hat{y}^* := \mu \bar{y}^* + (1-\mu)\tilde{y}^* \in K^\circ$. Hence $\langle \hat{y}^*, \bar{y}\rangle > 0$, which contradicts $\hat{y}^* \in K^\circ$.

With the aid of (6) and the bipolar theorem [18, Theorem 14.1] we obtain $-\bar{y} \in K^{\circ\circ} = K$. Since K is a convex cone and $\bar{y} \in K \cap (-K)$ we have $K = K + \{-\bar{y}\}$. From $\bar{y} \in \operatorname{ri} K$ we conclude $0 \in \operatorname{ri} K + \{-\bar{y}\} = \operatorname{ri}(K + \{-\bar{y}\}) = \operatorname{ri} K$. This implies $\operatorname{lin} K = \operatorname{aff} K = K$, i.e., K is a linear subspace of Y, a contradiction. □

Let $X := \mathbb{R}^n$, $U := \mathbb{R}^m$ (hence $X^* = \mathbb{R}^n$, $U^* = \mathbb{R}^m$) and let $f : X \to \hat{\mathcal{C}}^\star$ be a function. Recall that both the standard relation $\supseteq$ in $\hat{\mathcal{C}}^\star$ the standard relation $\subseteq$ in $\hat{\mathcal{C}}^\diamond$ have the meaning of "less or equal". Therefore, we write $\leq$ instead of $\supseteq$ and $\subseteq$ in order to emphasize the analogy of our statements to the well–known scalar case. So, the interpretation of $\leq$ depends on the orientation of the sets being compared.

Let $c \in Y$ and let $\{c\}$ be infimum oriented. The function $f_c^* : X^* \to \hat{\mathcal{C}}^\diamond$, defined by

$$f_c^*(x^*) := \sup_{x \in X} \left\{ \langle x^*, x\rangle \cdot \{c\} \boxminus f(x) \right\},$$

is said to be the *conjugate of f* with respect to c. From (1) we conclude that f_c^* is convex (even if f is not). As an easy consequence of the definition of f_c^*, we obtain the Fenchel–Young inequality

$$\forall x \in X,\, x^* \in X^*,\, c \in Y: \quad f_c^*(x^*) \geq \langle x^*, x\rangle \cdot \{c\} \boxminus f(x). \tag{7}$$

For functions $f : X \to \hat{\mathcal{C}}^\star$ and $h : X \to \hat{\mathcal{C}}^\star$ it is evident that

$$\big(\forall x \in X: \; f(x) \leq h(x)\big) \quad \Rightarrow \quad \big(\forall x^* \in X^*, \forall c \in Y: \; f_c^*(x^*) \geq h_c^*(x^*)\big).$$

It follows the main result of this paper, a duality theorem for functions with values in the space of closed convex subsets of $Y = \mathbb{R}^p$.

Theorem 4.3 (Duality theorem) *For given functions $f: X \to \hat{\mathcal{C}}^\star$ and $g: U \to \hat{\mathcal{C}}^\star$, a linear map $A: X \to U$ and a vector $c \in Y$, let*

$$p : X \to \hat{\mathcal{C}}^\star \quad \textit{and} \quad d_c : U^* \to \hat{\mathcal{C}}^\star$$

be defined, respectively, by

$$p(x) := f(x) \oplus g(Ax) \quad \textit{and} \quad d_c(u^*) := \boxminus\big(f_c^*(A^T u^*) \oplus g_c^*(-u^*)\big).$$

These functions satisfy the weak duality inequality

$$D_c := \sup_{u^* \in U^*} d_c(u^*) \leq \inf_{x \in X} p(x) =: P. \tag{8}$$

Furthermore, let f and g be convex, let

$$0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f) \tag{9}$$

and, in dependence of $K := 0^+P$, let the element $c \in Y$ be chosen as follows:

(i) *$c \in \operatorname{ri} K \cup (-\operatorname{ri} K)$, if $K \subsetneq Y$ is not a linear subspace of Y or $K = Y$,*

(ii) *$c \in Y \setminus K$, if $K \subsetneq Y$ is a linear subspace of Y.*

Then, we have strong duality, i.e., $D_c = P$.

Proof. Let $x \in X$, $u^* \in U^*$ and $c \in Y$ be arbitrarily given and let $u := Ax$. With the aid of the Fenchel–Young inequality (7) we obtain the weak duality inequality (8) as follows:

$$\begin{aligned} d_c(u^*) &= \boxminus\big(f_c^*(A^T u^*) \oplus g_c^*(-u^*)\big) \\ &\leq \big(f(x) \boxminus \langle A^T u^*, x\rangle \cdot \{c\}\big) \oplus \big(g(Ax) \boxminus \langle -u^*, Ax\rangle \cdot \{c\}\big) \\ &= \big(f(x) \oplus g(Ax)\big) \boxminus \langle u^*, Ax\rangle \cdot \{c\} \boxplus \langle u^*, Ax\rangle \cdot \{c\} \;=\; p(x). \end{aligned}$$

The proof of the strong duality assertion is organized as follows. We start with case (i). Then we show that case (ii) is a consequence of case (i).

(i) In case of $K = Y$ there is nothing to prove because the strong duality immediately follows from the weak duality assertion. Therefore, let $K \subsetneq Y$ be not a linear subspace of Y. It is easy to verify that $D_c = D_{-c}$. Hence it suffices to consider the case $c \in \operatorname{ri} K$.

With the aid of Proposition 3.3 it follows that

$$\forall y^* \in Y^*: \; -\delta^*(y^*|P) = -\delta^*(y^*| \inf_{x \in X} p(x)) = \inf_{x \in X} \big\{-\delta^*\big(y^*|\, p(x)\big)\big\}.$$

By the extended real–valued functions $f_{y^*} : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ and $g_{y^*} : U \to \mathbb{R} \cup \{-\infty, +\infty\}$ being defined, respectively, by $f_{y^*}(x) := -\delta^*\big(y^*|\, f(x)\big)$ and $g_{y^*}(u) := -\delta^*\big(y^*|\, g(u)\big)$ this can be rewritten as a collection of scalar optimization problems

$$\forall y^* \in Y^*: \; -\delta^*(y^*|P) = \inf_{x \in X} \big\{f_{y^*}(x) + g_{y^*}(Ax)\big\}. \tag{10}$$

The convexity of f and g implies the convexity of f_{y^*} and g_{y^*}, respectively. Clearly, we have $\operatorname{dom} f = \operatorname{dom} f_{y^*}$ and $\operatorname{dom} g = \operatorname{dom} g_{y^*}$. Hence, (9) implies that $0 \in \operatorname{ri}(\operatorname{dom} g_{y^*} - A \operatorname{dom} f_{y^*})$. A scalar duality theorem, for instance [2, Theorem 3.3.5], now yields that

$$\forall y^* \in Y^*: \; -\delta^*(y^*|P) = \sup_{u^* \in U^*} \left\{-f_{y^*}^*(A^T u^*) - g_{y^*}^*(-u^*)\right\}. \tag{11}$$

Let $y^* \in \operatorname{ri} K^\circ$ be arbitrarily given. Since $c \in \operatorname{ri} K$, Lemma 4.2 yields that $\langle y^*, c\rangle < 0$. Hence there exists $\alpha_{y^*} > 0$ such that $\langle \alpha_{y^*} y^*, c\rangle = -1$. This can be rewritten as

$$\forall t \in \mathbb{R}: \;\; -\delta^*\big(\alpha_{y^*} y^*|\, \{t \cdot c\}\big) = -\langle \alpha_{y^*} y^*, t \cdot c\rangle = t. \tag{12}$$

For $\alpha := \alpha_{y^*} > 0$ we have

$$\begin{aligned}
\alpha \cdot \big(-\delta^*(y^*|P)\big) &= -\delta^*(\alpha y^*|P) \overset{(11)}{=} \sup_{u^*\in U^*}\left\{-f^*_{\alpha y^*}(A^T u^*) - g^*_{\alpha y^*}(-u^*)\right\} \\
&= \sup_{u^*\in U^*}\left\{\inf_{x\in X}\left\{-\left\langle A^T u^*, x\right\rangle + f_{\alpha y^*}(x)\right\} + \inf_{u\in U}\left\{\langle u^*, u\rangle + g_{\alpha y^*}(u)\right\}\right\} \\
&\overset{(12)}{=} \sup_{u^*\in U^*}\Big\{\inf_{x\in X}\left\{-\delta^*\big(\alpha y^*|\boxplus(-\left\langle A^T u^*, x\right\rangle\cdot\{c\})\big) - \delta^*\big(\alpha y^*|f(x)\big)\right\} \\
&\qquad + \inf_{u\in U}\left\{-\delta^*\big(\alpha y^*|\boxplus(\langle u^*, u\rangle\cdot\{c\})\big) - \delta^*\big(\alpha y^*|g(u)\big)\right\}\Big\} \\
&= \sup_{u^*\in U^*}\Big\{\inf_{x\in X}\left\{-\delta^*\big(\alpha y^*|\boxminus\left\langle A^T u^*, x\right\rangle\{c\} + f(x)\big)\right\} \\
&\qquad + \inf_{u\in U}\left\{-\delta^*\big(\alpha y^*|\boxminus\langle -u^*, u\rangle\{c\} + g(u)\big)\right\}\Big\} \\
&\overset{\text{Prop. 3.3}}{=} \sup_{u^*\in U^*}\Big\{-\delta^*\big(\alpha y^*|\inf_{x\in X}\{\boxminus\left\langle A^T u^*, x\right\rangle\{c\} + f(x)\}\big) \\
&\qquad -\delta^*\big(\alpha y^*|\inf_{u\in U}\{\boxminus\langle -u^*, u\rangle\{c\} + g(u)\}\big)\Big\} \\
&\overset{(3)}{=} \sup_{u^*\in U^*}\Big\{-\delta^*\big(\alpha y^*|\boxminus\sup_{x\in X}\{\left\langle A^T u^*, x\right\rangle\{c\}\boxminus f(x)\}\big) \\
&\qquad -\delta^*\big(\alpha y^*|\boxminus\sup_{u\in U}\{\langle -u^*, u\rangle\{c\}\boxminus g(u)\}\big)\Big\} \\
&= \sup_{u^*\in U^*}\Big\{-\delta^*\Big(\alpha y^*\Big|\boxminus\sup_{x\in X}\{\left\langle A^T u^*, x\right\rangle\{c\}\boxminus f(x)\} \\
&\qquad \oplus \boxminus\sup_{u\in U}\{\langle -u^*, u\rangle\{c\}\boxminus g(u)\}\Big)\Big\} \\
&\overset{\text{Prop. 3.3}}{\leq} -\delta^*\Big(\alpha y^*\Big|\sup_{u^*\in U^*}\Big\{\boxminus\sup_{x\in X}\{\left\langle A^T u^*, x\right\rangle\{c\}\boxminus f(x)\} \\
&\qquad \oplus \boxminus\sup_{u\in U}\{\langle -u^*, u\rangle\{c\}\boxminus g(u)\}\Big\}\Big) \\
&= -\delta^*\Big(\alpha y^*\Big|\sup_{u^*\in U^*}\left\{\boxminus f^*_c(A^T u^*)\oplus\boxminus g^*_c(-u^*)\right\}\Big) \\
&= -\delta^*(\alpha y^*|D_c) = \alpha\cdot\big(-\delta^*(y^*|D_c)\big).
\end{aligned}$$

Since $y^* \in \operatorname{ri} K^\circ$ was arbitrarily chosen and taking into account (4), we obtain

$$\forall y^* \in \operatorname{ri} K^\circ : \ \delta^*(y^*|D_c) \leq \delta^*(y^*|P) < \infty.$$

Lemma 4.1 and the weak duality inequality yield $0^+ D_c = K$. By Lemma 3.1 we obtain $P \supseteq D_c$. Finally, the weak duality inequality yields $P = D_c$.

(ii) Let $K \subsetneq Y$ be a linear subspace of Y and let $c \in Y \setminus K$. Consider the set $B := \boxplus\mathbb{R}_+\{c\} \subseteq \mathcal{C}^\star$. We define a new objective function by $\tilde{p} : X \to \hat{\mathcal{C}}^\star$, $\tilde{p}(x) := p(x) \oplus B = f(x) \oplus (g(Ax) \oplus B)$. By Proposition 2.4, we have $\tilde{P} := \inf_{x\in X}\tilde{p}(x) = \big(\inf_{x\in X} p(x)\big) \oplus B =$

$P \oplus B$. With the aid of [18, Corollary 9.1.2] we conclude that $\tilde{P} = P \oplus B = P + B$ and $\tilde{K} := 0^+\tilde{P} = 0^+P + B = K + B$. Clearly, $\tilde{K}$ is not a linear space and $c \in \operatorname{ri} \tilde{K}$. It is an easy task to show that $\tilde{g} : U \to \hat{\mathcal{C}}^\star$, $\tilde{g}(\cdot) := g(\cdot) \oplus B$ is convex and (9) remains true for the new problem, hence, we have strong duality by part (i) of this theorem. For the conjugate $\tilde{g}^*_c : U^* \to \hat{\mathcal{C}}^\diamond$ of $\tilde{g}$ it holds

$$\begin{aligned} \tilde{g}^*_c(u^*) &= \sup_{u \in U} \{\langle u, u^* \rangle \{c\} \boxminus (g(u) \oplus B)\} \\ &= \sup_{u \in U} \big\{ \big(\langle u, u^* \rangle \{c\} \boxminus g(u) \big) \oplus \boxminus B \big\} \overset{\text{Prop. 2.4}}{=} g^*_c(u^*) \oplus \boxminus B. \end{aligned}$$

Hence, the dual objective function $\tilde{d} : U^* \to \hat{\mathcal{C}}^\star$ for the problem $\inf_{x \in X} \tilde{p}(x)$ is given by

$$\tilde{d}_c(u^*) = \boxminus f^*_c(A^T u^*) \oplus \boxminus g^*_c(-u^*) \oplus B = d_c(u^*) \oplus B.$$

Since $0 \in B$ we deduce that $\tilde{d}_c \leq d_c$, hence $\tilde{D}_c := \sup_{u^* \in U^*} \tilde{d}_c(u^*) \leq D_c$. The strong duality assertion for the problem $\inf_{x \in X} \tilde{p}(x)$ yields $P + B = \tilde{P} = \tilde{D}_c \leq D_c$. Analogously (replace c by $-c$) it follows $P - B \leq D_{-c} = D_c$. Hence, $(P + B) \cap (P - B) \leq D_c$.

We next show that $P \leq (P+B) \cap (P-B)$. By the definition of the space $\mathcal{C}$, P is a convex subset of Y. Let $y \in (P+B) \cap (P-B)$ be given. This means $y = p_1 + r_1 c = p_2 - r_2 c$ for some elements $p_1, p_2 \in P$ and real numbers $r_1, r_2 \geq 0$. If $r_1 + r_2 = 0$ there is nothing to prove. For $r_1 + r_2 > 0$ it follows

$$y = \frac{r_2}{r_1 + r_2}(p_1 + r_1 c) + \frac{r_1}{r_1 + r_2}(p_2 - r_2 c) = \frac{r_2}{r_1 + r_2} p_1 + \frac{r_1}{r_1 + r_2} p_2 \in P.$$

Hence, $P \supseteq (P + B) \cap (P - B)$, this means $P \leq (P + B) \cap (P - B)$.

Together we have shown that $P \leq D_c$. Taking into account the weak duality assertion we obtain $P = D_c$. □

We next express the preceeding theorem by conventional notations. Although the analogy to the scalar theory is more difficult to see, this form is more convenient for applications. Let $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set–valued map. As usual, the set $\operatorname{gr} f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p |\ y \in f(x)\}$ is called the *graph* of f. We say f has closed (convex) values if $f(x) \subseteq \mathbb{R}^p$ is closed (convex) for all $x \in \mathbb{R}^n$. Clearly, if f has a closed (convex) graph, then f has closed (convex) values. The opposite implication is not true, in general. The map f has closed values and a convex graph if and only if f can be interpreted as a convex function $f : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$.

Corollary 4.4 *For given set–valued maps $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $g : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$, a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ and a vector $c \in \mathbb{R}^p$, we have*

$$\bigcup_{x \in \mathbb{R}^n} \{f(x) + g(Ax)\} \subseteq \bigcap_{u^* \in \mathbb{R}^m} \left\{ \bigcup_{x \in \mathbb{R}^n} \{f(x) - \langle A^T u^*, x \rangle \{c\}\} + \bigcup_{u \in \mathbb{R}^m} \{g(u) + \langle u^*, u \rangle \{c\}\} \right\}.$$

If, furthermore, f and g have convex graphs and closed values and satisfy the condition $0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f)$ and, in dependence of $K := 0^+(\operatorname{cl}\operatorname{conv} \bigcup_{x \in \mathbb{R}^n} \big(f(x) + g(Ax)\big)$, the vector $c \in \mathbb{R}^p$ is chosen as in Theorem 4.3, we have strong duality, i.e.,

$$\operatorname{cl} \bigcup_{x \in \mathbb{R}^n} \{f(x) + g(Ax)\} = \bigcap_{u^* \in \mathbb{R}^m} \operatorname{cl} \left(\bigcup_{x \in \mathbb{R}^n} \{f(x) - \langle A^T u^*, x \rangle \{c\}\} + \bigcup_{u \in \mathbb{R}^m} \{g(u) + \langle u^*, u \rangle \{c\}\} \right).$$

Proof. For all $u^* \in \mathbb{R}^m$, we have

$$\begin{aligned}\bigcup_{x\in\mathbb{R}^n} \{f(x)+g(Ax)\} &= \bigcup_{x\in\mathbb{R}^n} \left\{f(x) - \left\langle A^T u^*, x\right\rangle \{c\} + g(Ax) + \langle u^*, Ax\rangle \{c\}\right\} \\ &\subseteq \bigcup_{x\in\mathbb{R}^n} \left\{f(x) - \left\langle A^T u^*, x\right\rangle \{c\}\right\} + \bigcup_{u\in\mathbb{R}^m} \{g(u) + \langle u^*, u\rangle \{c\}\}.\end{aligned}$$

Taking the intersection over all $u^* \in \mathbb{R}^m$ we obtain the weak duality inclusion.

Let f and g have convex graphs and closed values. This means that f and g can be interpreted as convex functions $f : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$ and $g : \mathbb{R}^m \to \hat{\mathcal{C}}^\star$. The expression $\inf_{x\in\mathbb{R}^n} \{f(x) \oplus g(Ax)\}$ in Theorem 4.3 has the meaning of $\operatorname{cl}\operatorname{conv} \bigcup_{x\in\mathbb{R}^n} \operatorname{cl}\big(f(x)+g(Ax)\big)$. We next show that this expression can be simplified in the present case, namely

$$\inf_{x\in\mathbb{R}^n} \{f(x) \oplus g(Ax)\} = \operatorname{cl} \bigcup_{x\in\mathbb{R}^n} \{f(x)+g(Ax)\}. \tag{13}$$

It is easy to show that the set $\bigcup_{x\in\mathbb{R}^n} \operatorname{cl}\big(f(x)+g(Ax)\big)$ is convex. Moreover, one can easily verify that $\operatorname{cl} \bigcup_{x\in\mathbb{R}^n} \operatorname{cl} p(x) = \operatorname{cl} \bigcup_{x\in\mathbb{R}^n} p(x)$, where $p(x) = f(x)+g(Ax)$. Together we obtain (13). By analogous arguments the right–hand side of the strong duality equality equals the dual value D_c in Theorem 4.3. □

In the next example we show that the duality theorem for extended real–valued convex functions (which was used in the proof) follows from the set–valued duality theorem by a simple reformulation of the problem.

Example 4.5 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be convex functions satisfying the condition $0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f)$. A given extended real–valued problem $P = \inf_{x\in\mathbb{R}^n}\big\{f(x)+g(Ax)\big\}$ can be rewritten as $P = \inf \bigcup_{x\in\mathbb{R}^n}\big\{\{f(x)\} + \{g(Ax)\} + \mathbb{R}_+\big\}$. Consider the inner (set–valued) problem. It is easy to see that (9) is satisfied for this problem. If P is finite, we have $K = \mathbb{R}^+$. Hence the choice $c = 1$ is possible in order to obtain strong duality by Corollary 4.4. By some simple calculations (using Corollary 4.4) we obtain $P = \inf \bigcup_{x\in\mathbb{R}^n}\big\{\{f(x)\} + \{g(Ax)\} + \mathbb{R}_+\big\} = \inf \bigcap_{u^*\in\mathbb{R}^m}\big\{\{-f^*(A^T u^*)\} - \{g^*(-u^*)\} + \mathbb{R}_+\big\}$, where f^* and g^* are the classical conjugate functions of f and g. It is an easy task to show that $\inf \bigcap_{u^*\in\mathbb{R}^m}\big\{\{-f^*(A^T u^*)\} - \{g^*(-u^*)\} + \mathbb{R}_+\big\} = \sup_{u^*\in\mathbb{R}^m}\big\{-f^*(A^T u^*) - g^*(-u^*)\big\}$. The latter expression is exactly the classical dual problem for the extended real–valued problem.

Optimization problems with set relations naturally occur in vector optimization (for instance, see [15], [5]) and in set–valued optimization in the sense of [9]. Let $p : X \to Y \cup \{+\infty\}$ or $p : X \rightrightarrows Y$, and let $C \subseteq Y$ be a closed convex and pointed ($C \cap -C = \{0\}$) cone. Usually one asks for the set $\operatorname{Eff}[P; C]$ of efficient points of the set $P := \bigcup_{x\in X} p(x)$ with respect to C. If the assumptions of Theorem 4.3 are satisfied, and if we can ensure that $\bigcup_{x\in X} p(x)$ is closed (by additional assumptions), this problem can be equivalently expressed by

$$\operatorname{Eff}\Big[\bigcup_{x\in X} p(x); C\Big] = \operatorname{Eff}\Big[\bigcap_{u^*\in U^*} d_c(u^*); C\Big],$$

For the details we refer to [13]. There, we also prove a biconjugation theorem as well as a strong duality assertion for the Lagrange duality approach. Furthermore, the relationship to the usual duality in set–valued and vector–valued optimization (e.g., [4], [15], [19], [5], [9]) is investigated.

5 Some special cases

We consider the problem of minimizing a convex function $f : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$ with respect to a nonempty closed convex set $S \subseteq \mathbb{R}^n$. This problem can be formulated as

$$\inf_{x \in \mathbb{R}^n} \big\{ f(x) + g(Ax) \big\},$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, and $g : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$ is the "set–valued indicator function", defined by

$$g(x) := \begin{cases} \{0\} & \text{if} \quad x \in S \\ \emptyset & \text{else.} \end{cases}$$

The conjugate function $g_c^* : \mathbb{R}^n \to \hat{\mathcal{C}}^\diamond$ of g can be understood as the "set–valued support function" of the set S. It is given by

$$g_c^*(x^*) = \operatorname{cl} \bigcup_{x \in S} \big\{ \langle x^*, x \rangle \{c\} \big\} = \big[-\delta^*(x^*, -S), \delta^*(x^*, S) \big] \cdot \{c\} \,,$$

where, by convention,

$$\forall \alpha \in \mathbb{R} : \quad [-\infty, \alpha] \{c\} := (-\infty, \alpha] \{c\} = \bigcup_{\lambda \le \alpha} \big\{ \lambda \{c\} \big\},$$

$$\forall \alpha \in \mathbb{R} : \quad [\alpha, +\infty] \{c\} := [\alpha, +\infty) \{c\} = \bigcup_{\lambda \ge \alpha} \big\{ \lambda \{c\} \big\},$$

$$[-\infty, +\infty] \{c\} := (-\infty, +\infty) \{c\} = \mathbb{R} \cdot \{c\} = \bigcup_{\lambda \in \mathbb{R}} \big\{ \lambda \{c\} \big\},$$

$$[+\infty, -\infty] \{c\} := \emptyset \quad \text{(this case occurs if } S = \emptyset).$$

As a special case, let us consider $f(x) = \{M \cdot x\}$ where M is a real $p \times n$ matrix. An easy computation shows that

$$f_c^*(x^*) = \bigcup_{x \in \mathbb{R}^n} \big\{ (c \cdot (x^*)^T - M) \cdot x \big\} = \big(c \cdot (x^*)^T - M \big) \cdot \mathbb{R}^n.$$

In the special case of $Y = \mathbb{R}^n$ and $M := I$ being the $n \times n$ unit matrix, Corollary 4.4 yields the following dual description of a nonempty closed convex set $S \subseteq \mathbb{R}^n$. In contrast to the usual dual description $S = \bigcap_{x^* \in \mathbb{R}^n} \{x \in \mathbb{R}^n | \; \langle x^*, x \rangle \le \delta^*(x^*|S)\}$ we have a different parameterization in the following formula, i.e., the same x^* may generate different sets. This parameterization depends on the choice of c. If 0^+S is not a linear subspace of $\mathbb{R}^n$, for all $c \in \mathrm{ri}\,(0^+S) \cup (-\mathrm{ri}\,(0^+S))$ it holds

$$S = \bigcap_{x^* \in \mathbb{R}^n} \Big((I - c \cdot (x^*)^T) \cdot \mathbb{R}^n + \big[-\delta^*(x^*, -S), \delta^*(x^*, S)\big] \cdot \{c\} \Big). \tag{14}$$

Moreover, if 0^+S is a linear subspace of $\mathbb{R}^n$, (14) is valid for all $c \in \mathbb{R}^n \setminus 0^+S$. Note that in (14) the constraint qualification (9) is superfluous, see Remark 5.1 below.

We next turn to the case of linear inequality constraints. Let A be a real $m \times n$ matrix and $b \in \mathbb{R}^m$ a given vector. We write $u \le v$ if $v - u \in \mathbb{R}^m_+$. Consider the problem

$$\inf_{x \in S} \{Mx\}, \quad S = \{x \in X |\ Ax \ge b\}. \tag{15}$$

Since S is polyhedral, we have $\inf_{x \in S} \{Mx\} = \bigcup_{x \in S} \{Mx\} =: M \cdot S$. We introduce $g : \mathbb{R}^m \to \hat{\mathcal{C}}^\star$ as

$$g(u) := \begin{cases} \{0\} & \text{if} \quad u \ge b \\ \emptyset & \text{else.} \end{cases}$$

Thus, (15) can be rewritten as $\inf_{x \in \mathbb{R}^n} \big\{ f(x) + g(Ax) \big\}$. A simple calculation yields

$$\begin{aligned} g^*_c(u^*) &= \bigcup_{u \ge b} \{\langle u^*, u\rangle \{c\}\} = \bigcup_{u \ge 0} \{\langle u^*, u + b\rangle \{c\}\} \\ &= \big\{c \cdot (u^*)^T \cdot b\big\} + \bigcup_{u \ge 0} \{\langle u^*, u\rangle \{c\}\} = \big\{c \cdot (u^*)^T \cdot b\big\} + h_c(u^*), \end{aligned}$$

where

$$h_c(u^*) := \bigcup_{u \ge 0} \{\langle u^*, u\rangle \{c\}\} = \begin{cases} \mathbb{R}_- \cdot \{c\} & \text{if} \quad u^* \in \mathbb{R}^m_- \setminus \{0\} \\ \mathbb{R}_+ \cdot \{c\} & \text{if} \quad u^* \in \mathbb{R}^m_+ \setminus \{0\} \\ \{0\} & \text{if} \quad u^* = 0 \\ \mathbb{R} \cdot \{c\} & \text{else} \end{cases}$$

is the "set–valued support function" of $\mathbb{R}^m_+$. Note that $h(u^*) = -h(-u^*)$, hence, the dual objective function is given by

$$d_c(u^*) = (M - c \cdot (A^T \cdot u^*)^T) \cdot \mathbb{R}^n + \big\{c \cdot (u^*)^T \cdot b\big\} + h(u^*).$$

By Theorem 4.3 we obtain the following strong duality assertion. Let $K := 0^+(M \cdot S)$. If there exists some $x \in \mathbb{R}^n$ such that $Ax \ge b$, then, respectively, for all $c \in \mathrm{ri}\,K \cup (-\mathrm{ri}\,K)$ if K is not a linear space and for all $c \in Y \setminus K$ if K is a linear subspace of Y it is true that

$$M \cdot S = \bigcap_{u^* \in \mathbb{R}^m} \Big\{ \big(M - c \cdot (A^T \cdot u^*)^T\big) \cdot \mathbb{R}^n + \big\{c \cdot (u^*)^T \cdot b\big\} + h(u^*) \Big\}. \tag{16}$$

Remark 5.1 In Theorem 4.3 (duality theorem) we suppose the constraint qualification (9). In the proof of this theorem we use this condition in order to obtain the corresponding condition for the scalar problems (10). If all these problems are polyhedral, (9) can be replaced by $\mathrm{dom}\, g \cap A\mathrm{dom}\, f \neq \emptyset$, compare e.g. [2, Corollary 5.1.9]. Hence, in (14) and (16) the constraint qualification reduces to $S \neq \emptyset$.

Acknowledgments. This paper is part of the author's Ph.D. thesis, written under the supervision of Christiane Tammer and Andreas H. Hamel at Martin–Luther–Universität Halle–Wittenberg. The author wishes to express his deepest gratitude. Furthermore, the author is greatly indebted to C. Zălinescu and both referees for their useful hints.

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