A Dual Variant of Benson's Outer Approximation Algorithm

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Abstract

Geometric duality theory for multiple objective linear programmes is used to derive a dual variant of Benson's outer approximation algorithm to solve multiobjective linear programmes in objective space. We also suggest some improvements of the original version of the algorithm and prove that solving the dual provides a weight set decomposition. We compare both algorithms on small illustrative and on practically relevant examples.

Keywords: Multiobjective optimization, vector optimization, linear programming, duality, objective space, outer approximation.

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1 Introduction

Multiple objective linear programming has been a subject of research since the 1960s. Many algorithms based on extensions of the simplex method to deal with multiple objectives have been published over the years, see for example the brief survey in Ehrgott and Wiecek (2005). Textbooks on multicriteria optimization usually cover multiobjective simplex algorithms (Steuer, 1985; Ehrgott, 2005).

Researchers have noted that the number of efficient basic feasible solutions of multiple objective LP's increases dramatically with the number of objectives and simplex algorithms become rather inefficient. This has motivated research in methods solving the problems in objective or outcome space.

Benson's outer approximation algorithm (Benson, 1998b) is a method to solve a multiple objective linear programme in the outcome space. The motivation for the algorithm is twofold. On the one hand it is in many practical cases not possible for the decision maker to choose a preferred solution from the overwhelming set of all efficient solutions, because the dimension of the decision space is in many problems much bigger than the dimension of the outcome space. On the other hand it seems to be more natural to compare criteria values rather than the decisions leading to them.

We use some new results on duality for multiple objective linear programmes in order to develop a dual variant of Benson's algorithm. Geometric duality (Heyde and Löhne, 2006) defines a dual vector optimization problem which has a completely different outcome set than the primal problem, but it provides a well-defined relationship between the primal and dual outcome set which is easy to handle. The idea of Benson's algorithm can be applied (with some slight modifications) to the dual outcome set. Duality results yield information about the primal outcome set.

Geometric duality theory also yields some new insights into the algorithm. We rediscover well-known principles from scalar duality theory, for instance the embedding of the optimal value between the values of the primal and dual objectives at feasible solutions.

The article is organized as follows. In Section 2 we introduce notation and some basic concepts. In Section 3 we give an introduction to geometric duality. Section 4 is devoted to the original outer approximation algorithm. However, we propose some improvements. Section 5 deals with our dual variant of the algorithm. In Section 6 we prove that the solution of the dual problem provides a weight set decomposition with respect to nondominated extreme points. In Section 7 we present numerical results, comparing the primal and the dual algorithm for several examples.

2 Preliminaries

Throughout the article we use the following notation. The k-th unit vector in $\mathbb{R}^p$ is denoted e^k and a vector of all ones is denoted by e. Given a mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ and a subset $\mathcal{X} \subseteq \mathbb{R}^n$ we write $f(\mathcal{X}) := \{f(x) : x \in \mathcal{X}\}$.

Let $\mathcal{A} \subseteq \mathbb{R}^p$. We denote the boundary, interior, and relative interior of $\mathcal{A}$ by $\operatorname{bd} \mathcal{A}$, $\operatorname{int} \mathcal{A}$, and $\operatorname{ri} \mathcal{A}$. The convex hull of $\mathcal{A}$ is denoted $\operatorname{conv} \mathcal{A}$.

Let $\mathcal{C} \subseteq \mathbb{R}^p$ be a closed convex cone. An element $y \in \mathcal{A}$ is called $\mathcal{C}$-minimal if $(\{y\} - \mathcal{C} \setminus \{0\}) \cap \mathcal{A} = \emptyset$ and $\mathcal{C}$-maximal if $(\{y\} + \mathcal{C} \setminus \{0\}) \cap \mathcal{A} = \emptyset$. A point $y \in \mathcal{A}$ is called weakly $\mathcal{C}$-minimal (weakly $\mathcal{C}$-maximal) if $(\{y\} - \operatorname{ri} \mathcal{C}) \cap \mathcal{A} = \emptyset$ $((\{y\} + \operatorname{ri} \mathcal{C}) \cap \mathcal{A} = \emptyset)$. We set

$$\begin{aligned} \operatorname{wmin}_{\mathcal{C}} \mathcal{A} &:= \{y \in \mathcal{A} : (\{y\} - \operatorname{ri} \mathcal{C}) \cap \mathcal{A} = \emptyset\} \text{ and} \\ \operatorname{wmax}_{\mathcal{C}} \mathcal{A} &:= \operatorname{wmin}_{(-\mathcal{C})} \mathcal{A}. \end{aligned}$$

In this paper we consider two special ordering cones, namely $\mathcal{C} = \mathbb{R}^p_{\geqq} = \{x \in \mathbb{R}^p : x_k \geqq 0, k = 1, \ldots, p\}$ and

$$\mathcal{C} = \mathcal{K} := \mathbb{R}_{\geqq} e^p = \{y \in \mathbb{R}^p : y_1 = \cdots = y_{p-1} = 0, y_p \geqq 0\}.$$

For the choice $\mathcal{C} = \mathbb{R}^p_{\geqq}$ the set of weakly $\mathbb{R}^p_{\geqq}$-minimal elements of $\mathcal{A}$ (also called the set of weakly nondominated points of $\mathcal{A}$) is given by

$$\operatorname{wmin}_{\mathbb{R}^p_{\geqq}} \mathcal{A} := \left\{ y \in \mathcal{A} : (\{y\} - \operatorname{int} \mathbb{R}^p_{\geqq}) \cap \mathcal{A} = \emptyset \right\}.$$

In case of $\mathcal{C} = \mathcal{K}$ the set of $\mathcal{K}$-maximal elements of $\mathcal{A}$ is given by

$$\max_{\mathcal{K}} \mathcal{A} := \{y \in \mathcal{A} : (\{y\} + \mathcal{K} \setminus \{0\}) \cap \mathcal{A} = \emptyset\}.$$

Note that $\operatorname{ri} \mathcal{K} = \mathcal{K} \setminus \{0\}$ so that weakly $\mathcal{K}$-maximal and $\mathcal{K}$-maximal elements of $\mathcal{A}$ coincide.

Since we will always consider minimization with respect to $\mathbb{R}^p_{\geqq}$ and maximization with respect to $\mathcal{K}$, we sometimes omit the subscripts.

Let us recall some facts concerning the facial structure of polyhedral sets (Webster, 1994). Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a convex set. A convex subset $\mathcal{F} \subseteq \mathcal{A}$ is called a *face* of $\mathcal{A}$ if for all $y^1, y^2 \in \mathcal{A}$ and $\alpha \in (0,1)$ such that $\alpha y^1 + (1-\alpha) y^2 \in \mathcal{F}$ it holds that $y^1, y^2 \in \mathcal{F}$. A face $\mathcal{F}$ of $\mathcal{A}$ is called *proper* if $\emptyset \neq \mathcal{F} \neq \mathcal{A}$. A point $y \in \mathcal{A}$ is called an *extreme point* of $\mathcal{A}$ if $\{y\}$ is a face of $\mathcal{A}$.

A *recession direction* of $\mathcal{A}$ is a vector $d \in \mathbb{R}^p$ such that $y + \alpha d \in \mathcal{A}$ for some $y \in \mathcal{A}$ and all $\alpha \geqq 0$. The *recession cone* (or asymptotic cone) $\mathcal{A}_\infty$ of $\mathcal{A}$ is the set of all recession directions

$$\mathcal{A}_\infty := \{d \in \mathbb{R}^p : y + \alpha d \in \mathcal{A} \text{ for some } y \in \mathcal{A} \text{ for all } \alpha \geqq 0\}.$$

A recession direction $d \neq 0$ is called *extreme* if there are no recession directions $d^1, d^2 \neq 0$ with $d^1 \neq \alpha d^2$ for all $\alpha > 0$ such that $d = \frac{1}{2}(d^1 + d^2)$.

A polyhedral convex set $\mathcal{A}$ is defined by $\{x \in \mathbb{R}^n : Ax \geqq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. A polyhedral set $\mathcal{A}$ has a finite number of faces. A subset $\mathcal{F}$ of $\mathcal{A}$ is a face if and only if there are $\lambda \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that $\mathcal{A} \subseteq \left\{y \in \mathbb{R}^n : \lambda^T y \geqq \gamma\right\}$ and $\mathcal{F} = \left\{y \in \mathbb{R}^n : \lambda^T y = \gamma\right\} \cap \mathcal{A}$. Moreover, $\mathcal{F}$ is a proper face if and only if $\mathcal{H} := \left\{y \in \mathbb{R}^n : \lambda^T y = \gamma\right\}$ is a supporting hyperplane to $\mathcal{A}$ with $\mathcal{F} = \mathcal{A} \cap \mathcal{H}$. We call hyperplane $\mathcal{H} = \left\{y \in \mathbb{R}^p : \lambda^T y = \gamma\right\}$ *supporting* if $\lambda^T y \geqq \gamma$ for all $y \in \mathcal{A}$ and there is some $y^0 \in \mathcal{A}$ such that $\lambda^T y^0 = \gamma$. The proper $(r-1)$-dimensional faces of an r-dimensional polyhedral set $\mathcal{A}$ are called *facets* of $\mathcal{A}$.

A polyhedral convex set $\mathcal{A}$ can be represented by both a finite set of inequalities and the set of all extreme points and extreme directions of $\mathcal{A}$ (Rockafellar, 1972, Theorem 18.5). Let $\mathcal{E} = \{x^1, \ldots, x^r, d^1, \ldots, d^t\}$ be the set of all extreme points and extreme directions of $\mathcal{A}$ then

$$\mathcal{A} = \left\{ y \in \mathbb{R}^p : y = \sum_{i=1}^{r} \alpha_i x^i + \sum_{j=1}^{t} \nu_j d^j \text{ with } \alpha_i \geqq 0, \nu_j \geqq 0, \text{ and } \sum_{i=1}^{r} \alpha_i = 1 \right\}.$$

For a polyhedral convex set $\mathcal{A}$, the extreme points are called vertices. The set of all vertices of a polyhedron $\mathcal{A}$ is denoted by $\operatorname{vert} \mathcal{A}$.

3 Geometric Duality

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $P \in \mathbb{R}^{p \times n}$, $e := (1, ..., 1)^T \in \mathbb{R}^p$. Consider the vector optimization problem

$$\text{(P)} \qquad \operatorname{wmin}_{\mathbb{R}^p_{\geqq}} P(\mathcal{X}), \quad \mathcal{X} := \{x \in \mathbb{R}^n : Ax \geqq b\}.$$

Then the dual problem according to the *geometric duality theory* developed in Heyde and Löhne (2006) is

$$\text{(D)} \qquad \max_{\mathcal{K}} D(\mathcal{U}), \quad \mathcal{U} := \left\{(u, \lambda) \in \mathbb{R}^m \times \mathbb{R}^p : (u, \lambda) \geqq 0, A^T u = P^T \lambda, e^T \lambda = 1\right\},$$

where $\mathcal{K} := \{y \in \mathbb{R}^p : y_1 = y_2 = ... = y_{p-1} = 0, y_p \geqq 0\}$ as defined before and $D : \mathbb{R}^{m+p} \to \mathbb{R}^p$ is given by

$$D(u, \lambda) := \left(\lambda_1, ..., \lambda_{p-1}, b^T u\right)^T = \begin{pmatrix} 0 & I_{p-1} & 0 \\ b^T & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix}.$$

The primal problem (P) consists in finding the weakly nondominated points of $P(\mathcal{X})$, the dual problem consists in finding the $\mathcal{K}$-maximal elements of $D(\mathcal{U})$. We introduce the extended polyhedral image sets $\mathcal{P} := P(\mathcal{X}) + \mathbb{R}^p_{\geqq}$ of problem (P) and $\mathcal{D} := D(\mathcal{U}) - \mathcal{K}$ of problem (D). It is known that the $\mathbb{R}^p_{\geqq}$-minimal (nondominated) points of $\mathcal{P}$ and $P(\mathcal{X})$ as well as the $\mathcal{K}$-maximal elements of $\mathcal{D}$ and $D(\mathcal{U})$ coincide, see Heyde and Löhne (2006). An illustration is given in Example 3.1.

Example 3.1 Consider problem (P) with the data

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 4 \\ 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

The extended outcome sets $\mathcal{P}$ and $\mathcal{D}$ of (P) and (D) are shown in Figures 1 and 2.

The geometric duality theory of Heyde and Löhne (2006) establishes a relationship between the (weakly nondominated) vertices of $\mathcal{P}$ and the $\mathcal{K}$-maximal facets of $\mathcal{D}$ and between the weakly nondominated facets of $\mathcal{P}$ and the $\mathcal{K}$-maximal vertices of $\mathcal{D}$. In this example, the five vertices of $\mathcal{D}$, namely $(0, 0)$, $(\frac{1}{3}, \frac{4}{3})$, $(\frac{1}{2}, \frac{3}{2})$, $(\frac{2}{3}, \frac{4}{3})$, and $(1, 0)$, correspond to the facets of $\mathcal{P}$ given by $y_2 = 0$, $y_1 + 2y_2 = 4$, $y_1 + y_2 = 3$, $2y_1 + y_2 = 4$, $y_1 = 0$. The four vertices of $\mathcal{P}$, namely $(0, 4)$, $(1, 2)$, $(2, 1)$, and $(4, 0)$ correspond to the $\mathcal{K}$-maximal facets of $\mathcal{D}$ given by $4v_1 + v_2 = 4$, $v_1 + v_2 = 2$, $-v_1 + v_2 = 1$, and $-4v_1 + v_2 = 0$, respectively.

Geometric duality is an extension of the well-known duality of polytopes to $\mathcal{P}$ and $\mathcal{D}$. Recall that two polytopes $\mathcal{G}$ and $\mathcal{G}^*$ in $\mathbb{R}^p$ are said to be dual to each other provided there

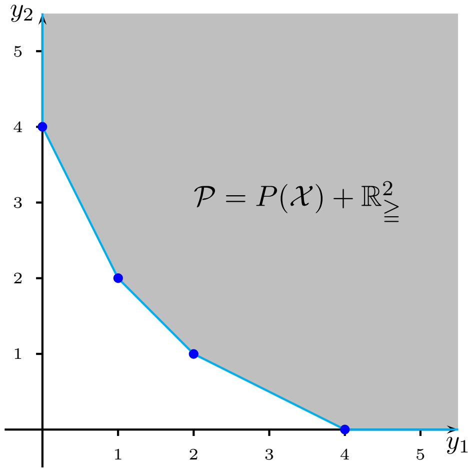

Figure 1: $\mathcal{P}$ in Example 3.1.

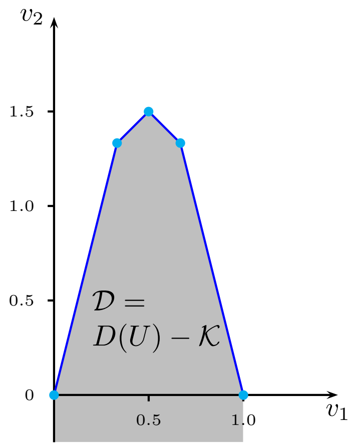

Figure 2: $\mathcal{D}$ in Example 3.1.

exists a one-to-one mapping Ψ between the set of all faces of $\mathcal{G}$ and the set of all faces of $\mathcal{G}^*$ such that Ψ is inclusion-reversing, i.e. faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{G}$ satisfy $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if the faces $\Psi(\mathcal{F}_1)$ and $\Psi(\mathcal{F}_2)$ satisfy $\Psi(\mathcal{F}_1) \supseteq \Psi(\mathcal{F}_2)$ (Grünbaum, 2003). The geometric duality theorem (Heyde and Löhne, 2006, Theorem 1) states that there is a similar duality relationship between $\mathcal{P}$ and $\mathcal{D}$.

To be more precise, we introduce the following notation. We consider the coupling function $\varphi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$, defined by

$$\varphi(y, v) := \sum_{i=1}^{p-1} y_i v_i + y_p \left(1 - \sum_{i=1}^{p-1} v_i\right) - v_p.$$

Note that $\varphi(\cdot, v)$ and $\varphi(y, \cdot)$ are affine. Choosing the values of the primal and dual objective functions for $x \in \mathcal{X}$ and $(u, \lambda) \in \mathcal{U}$ as arguments, we just get

$$\varphi(Px, D(u, \lambda)) = \lambda^T Px - b^T u. \tag{1}$$

Using the coupling function φ, we define the following two set-valued maps

$$\begin{aligned} H : \mathbb{R}^p \rightrightarrows \mathbb{R}^p, \qquad & H(v) := \{y \in \mathbb{R}^p : \varphi(y, v) = 0\}, \\ H^* : \mathbb{R}^p \rightrightarrows \mathbb{R}^p, \qquad & H^*(y) := \{v \in \mathbb{R}^p : \varphi(y, v) = 0\}. \end{aligned}$$

Of course, $H(v)$ and $H^*(y)$ are hyperplanes in $\mathbb{R}^p$ for all $v, y \in \mathbb{R}^p$. Using the notation

$$\begin{aligned} \lambda(v) \quad &:= \quad \left(v_1, \ldots, v_{p-1}, 1 - \sum_{i=1}^{p-1} v_i\right)^T \quad \text{and} \\ \lambda^*(y) \quad &:= \quad \left(y_1 - y_p, \ldots, y_{p-1} - y_p, -1\right)^T \end{aligned}$$

it is easy to see that

$$\begin{aligned} H(v) \quad &= \quad \left\{y \in \mathbb{R}^p : \lambda(v)^T y = v_p\right\} \quad \text{and} \\ H^*(y) \quad &= \quad \left\{v \in \mathbb{R}^p : \lambda^*(y)^T v = -y_p\right\}. \end{aligned}$$

We observe that $\lambda(v) \geqq 0$ if and only if $v_1, \dots, v_{p-1} \geqq 0$ and $\sum_{i=1}^{p-1} v_i \leqq 1$, a fact we will often use.

The map H is now used to define our duality map $\Psi : 2^{\mathbb{R}^p} \to 2^{\mathbb{R}^p}$. Let $\mathcal{F}^* \subseteq \mathbb{R}^p$, then

$$\Psi(\mathcal{F}^*) := \bigcap_{v \in \mathcal{F}^*} H(v) \cap \mathcal{P}.$$

The considerations in the following sections are based on the following geometric duality theorem of Heyde and Löhne (2006).

Theorem 3.2 (Heyde and Löhne (2006)) *Ψ is an inclusion reversing one-to-one map between the set of all proper $\mathcal{K}$-maximal faces of $\mathcal{D}$ and the set of all proper weakly nondominated faces of $\mathcal{P}$ and the inverse map is given by*

$$\Psi^{-1}(\mathcal{F}) = \bigcap_{y \in \mathcal{F}} H^*(y) \cap \mathcal{D}. \tag{2}$$

Moreover, for every proper $\mathcal{K}$-maximal face $\mathcal{F}^*$ of $\mathcal{D}$ it holds $\dim \mathcal{F}^* + \dim \Psi(\mathcal{F}^*) = p - 1$.

We next consider two important consequences.

Corollary 3.3 (Heyde and Löhne (2006)) *The following statements are equivalent*

(i) *v is a $\mathcal{K}$-maximal vertex of $\mathcal{D}$,*

(ii) *$H(v) \cap \mathcal{P}$ is a weakly nondominated $(p-1)$-dimensional facet of $\mathcal{P}$.*

Moreover, if $\mathcal{F}$ is a weakly nondominated $(p-1)$-dimensional facet of $\mathcal{P}$, there is some uniquely defined point $v \in \mathbb{R}^p$ such that $\mathcal{F} = H(v) \cap \mathcal{P}$.

Corollary 3.4 (Heyde and Löhne (2006)) *The following statements are equivalent*

(i) *y is a weakly nondominated vertex of $\mathcal{P}$,*

(ii) *$H^*(y) \cap \mathcal{D}$ is a $\mathcal{K}$-maximal $(p-1)$-dimensional facet of $\mathcal{D}$.*

Moreover, if $\mathcal{F}^*$ is a $\mathcal{K}$-maximal $(p-1)$-dimensional facet of $\mathcal{D}$, there is some uniquely defined point $y \in \mathbb{R}^p$ such that $\mathcal{F}^* = H^*(y) \cap \mathcal{D}$.

The proof of Theorem 3.2 in Heyde and Löhne (2006) is based on the consideration of the following two pairs of dual linear programming problems.

(P$_1(v)$) $$\min_{x \in \mathcal{X}} \lambda(v)^T P x, \quad \mathcal{X} := \{x \in \mathbb{R}^n : Ax \geqq b\}$$

(D$_1(v)$) $$\max_{u \in \mathcal{T}(v)} b^T u, \quad \mathcal{T}(v) := \left\{u \in \mathbb{R}^m : u \geqq 0, A^T u = P^T \lambda(v)\right\}$$

and

(P$_2(y)$) $$\min_{x \in \mathcal{S}(y)} z, \quad \mathcal{S}(y) := \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : Ax \geqq b, Px - ez \leqq y\},$$

$$(\mathrm{D}_2(y)) \qquad \max_{(u,\lambda)\in\mathcal{U}} (b^T u - y^T \lambda), \quad \mathcal{U} := \left\{(u,\lambda) \in \mathbb{R}^m \times \mathbb{R}^p : (u,\lambda) \geqq 0, A^T u = P^T \lambda, e^T \lambda = 1\right\}.$$

Note that Figure 2 illustrates the linear programming duality between $(\mathrm{P}_1(v))$ and $(\mathrm{D}_1(v))$. For any $v \in \mathbb{R}^p$ such that $\lambda(v) \geqq 0$ we have that $x \in \mathcal{X}$ and $u \in \mathcal{T}(v)$ are optimal solutions of $(\mathrm{P}_1(v))$ and $(\mathrm{D}_1(v))$, respectively, if and only if $\varphi(Px, D(u,\lambda(v)) = 0$ in (1). Thus, $(v_1,\dots,v_{p-1},\lambda(v)^T Px) = (v_1,\dots,v_{p-1},b^T u)$ is a boundary point of $D(\mathcal{U})-\mathcal{K}$. Feasible values of $(\mathrm{P}_1(v))$ are "above" that point, feasible values of $(\mathrm{D}_1(v))$ are "below": For feasible $x \in \mathcal{X}$ and $u \in \mathcal{T}(v)$ the value of $\varphi(Px, D(u,\lambda(v))$ measures the duality gap between the two feasible solutions.

The four problems above play a key role in the following algorithms.

4 Benson's Outer Approximation Algorithm

We propose an algorithm, which is essentially Benson's algorithm (Benson, 1998b), but involves some slight improvements. We see that it is not necessary to work with bounded simplices as Benson did in the original version. Thus, we compute the nondominated vertices directly and the final step (Benson, 1998b, Theorem 3.2) to check whether a vertex is nondominated or not is superfluous.

In our primal vector optimization problem (P) we assume that the set $\mathcal{P} = P(\mathcal{X}) + \mathbb{R}^p_{\geqq}$ is $\mathbb{R}^p_{\geqq}$-bounded from below, i.e., there exists some $\hat{y} \in \mathbb{R}^p$ such that $\hat{y} \leqq y$ for all $y \in \mathcal{P}$. As a consequence, the ideal point y^I of $\mathcal{P}$ defined by $y^I_k := \min\{y_k : y \in \mathcal{P}\}$ for $i = 1,\dots,p$ exists. Of course, this assumption is weaker than the assumption that $\mathcal{X}$ is a bounded set, which is supposed in Benson (1998b). We also assume that $\mathcal{X}$ is nonempty.

The algorithm first constructs a p-dimensional polyhedral set $\mathcal{S}^0 = y^I + \mathbb{R}^p_{\geqq}$ such that $\mathcal{P} \subseteq \mathcal{S}^0$. In every iteration it chooses an extreme point s^k of $\mathcal{S}^{k-1}$ not contained in $\mathcal{P}$ and constructs a supporting hyperplane to $\mathcal{P}$ by solving a linear programme $(\mathrm{D}_2(y^k))$, where y^k is a boundary point of $\mathcal{P}$ on the line segment connecting s^k with an interior point $\hat{p}$ of $\mathcal{P}$. $\mathcal{S}^k$ is defined by intersecting $\mathcal{S}^{k-1}$ with the halfspace of the hyperplane containing $\mathcal{P}$. The algorithm terminates as soon as no such s^k can be found and $\mathcal{S}^{k-1} = \mathcal{P}$.

For the next result we need Lemma 4.1 from Heyde and Löhne (2006).

Lemma 4.1 *The following three statements are equivalent.*

(i) $y^0 \in \operatorname{wmin}_{\mathbb{R}^p_{\geqq}} \mathcal{P}$.

(ii) *There is some* $x^0 \in \mathbb{R}^n$ *such that* $(x^0, 0)$ *is an optimal solution to* $(\mathrm{P}_2(y^0))$.

(iii) *There is some* $(u^0,\lambda^0) \in \mathcal{U}$ *with* $b^T u^0 = {y^0}^T \lambda^0$ *that is an optimal solution to* $(\mathrm{D}_2(y^0))$.

Proposition 4.2 *Let* $y \in \operatorname{wmin}\mathcal{P}$. *Then there exists an optimal solution of* $(\mathrm{D}_2(y))$ *and for each such solution* $(\bar{u},\bar{\lambda}) \in \mathcal{U}$, $H(D(\bar{u},\bar{\lambda}))$ *is a supporting hyperplane to* $\mathcal{P}$ *with* $y \in H(D(\bar{u},\bar{\lambda}))$.

Proof. By Lemma 4.1 there exists an optimal solution $(\bar{u}, \bar{\lambda})$ of $(\mathrm{D}_2(y))$ such that $b^T\bar{u} = y^T\bar{\lambda}$. Of course, the latter equality is also valid for any other optimal solution of $(\mathrm{D}_2(y))$. For arbitrary $y \in \mathcal{P}$, there exists some $x \in \mathcal{X}$ such that $y \geqq Px$. Hence $(x, 0)$ is feasible for $(\mathrm{P}_2(y))$ and duality between $(\mathrm{P}_2(y))$ and $(\mathrm{D}_2(y))$ implies that $\bar{\lambda}^T y \geqq b^T\bar{u}$. Hence $H(D(\bar{u}, \bar{\lambda})) = \{y \in \mathbb{R}^p : \bar{\lambda}^T y = b^T\bar{u}\}$ is a supporting hyperplane to $\mathcal{P}$. $\square$

We note that Benson (1998b,a) proves similar results to Lemma 4.1 and Proposition 4.2 for his original algorithm.

Proposition 4.3 *Every vertex of $\mathcal{P}$ is nondominated ($\mathbb{R}^p_{\geqq}$-minimal).*

Proof. Let y be a vertex of $\mathcal{P} = P(\mathcal{X}) + \mathbb{R}^p_{\geqq}$ and assume that y is not $\mathbb{R}^p_{\geqq}$-minimal. Hence, there exists some $z \in \left(\{y\} - \mathbb{R}^p_{\geqq} \setminus \{0\}\right) \cap \mathcal{P}$, i.e., $y \in \{z\} + \mathbb{R}^p_{\geqq} \setminus \{0\} \subseteq P(\mathcal{X}) + \mathbb{R}^p_{\geqq} + \left(\mathbb{R}^p_{\geqq} \setminus \{0\}\right) = P(\mathcal{X}) + \mathbb{R}^p_{\geqq} \setminus \{0\}$. Therefore, there is some $\bar{x} \in \mathcal{X}$ and some $\bar{d} \in \mathbb{R}^p_{\geqq} \setminus \{0\}$ such that $y = P\bar{x} + \bar{d} \in \mathcal{P}$. Hence the points $y - \bar{d}$ and $y + \bar{d}$ belong to $\mathcal{P}$ and $y = \frac{1}{2}(y - \bar{d}) + \frac{1}{2}(y + \bar{d})$. This contradicts y being a vertex of $\mathcal{P}$. $\square$

The following Proposition 4.4 shows that we do not need to consider extreme directions but only the vertices (extreme points) in the following algorithm, because the extreme directions are always the same, namely the unit vectors $e^k \in \mathbb{R}^p$.

Proposition 4.4 *Let $y \in \mathbb{R}^p$ and let $\mathcal{S} \subseteq \mathbb{R}^p$ be a polyhedral convex set such that $\mathcal{P} \subseteq \mathcal{S} \subseteq \{y\} + \mathbb{R}^p_{\geqq}$. Letting $\mathcal{E}$ be the set of extreme points of $\mathcal{S}$, we have $\mathcal{S} = \operatorname{conv}(\mathcal{E} + \mathbb{R}^p_{\geqq})$.*

Proof. Since $\mathcal{S}$ and $\mathcal{P}$ are closed and convex, we get $\mathbb{R}^p_{\geqq} \subseteq \mathcal{P}_\infty \subseteq \mathcal{S}_\infty \subseteq \mathbb{R}^p_{\geqq}$, hence $\mathcal{S}_\infty = \mathbb{R}^p_{\geqq}$. Now the conclusion follows from (Rockafellar, 1972, Theorem 18.5 and Theorem 19.5). $\square$

In the following algorithm we construct in iteration k a polyhedron $\mathcal{S}^k$, for which we store both a representation by a finite number of points and a representation by a finite number of inequalities. We cannot always ensure that all the points representing the set $\mathcal{S}^k$ are extreme points, i.e., our set may contain some redundant points. Similarly, it may happen that we have redundant inequalities in the inequality representation of $\mathcal{S}^k$. Therefore, we say that a set $\mathcal{E}$ of finitely many points is a *point representation* of $\mathcal{S}$ if $\mathcal{S} = \operatorname{conv}(\mathcal{E} + \mathbb{R}^p_{\geqq})$. If $\mathcal{E}$ only consists of extreme points of $\mathcal{S}$, we say that $\mathcal{E}$ is *nondegenerate*. Otherwise, $\mathcal{E}$ is called *degenerate*.

Analogously, a system of inequalities is called a *nondegenerate inequality representation* of $\mathcal{S}$ if $\mathcal{S}$ is the solution set of the system and if there are no redundant inequalities. An inequality representation of $\mathcal{S}$ is called *degenerate* if there exist redundant inequalities, i.e., there exists a proper subsystem of inequalities having $\mathcal{S}$ as the solution set.

Algorithm 1.

Initialization ($k = 0$).

(i1) Choose some $\hat{p} \in \operatorname{int} \mathcal{P}$.

(i2) Compute an optimal solution $\bar{u}_i$ and the optimal value y_i^I of ($\mathrm{D}_1(e^i)$), for $i = 1, \dots, p$.

(i3) Set $\mathcal{S}^0 := \{y^I\} + \mathbb{R}^p_{\geqq}$ and $k = 1$.

Iteration steps ($k \geqq 1$).

(k1) If $\operatorname{vert} \mathcal{S}^{k-1} \subseteq \mathcal{P}$ stop, otherwise choose a vertex s^k of $\mathcal{S}^{k-1}$ such that $s^k \notin \mathcal{P}$.

(k2) Compute $\alpha^k \in (0, 1)$ such that $y^k := \alpha^k s^k + (1 - \alpha^k)\hat{p} \in \operatorname{wmin}_{\mathbb{R}^p_{\geqq}} \mathcal{P}$.

(k3) Compute an optimal solution (u^k, λ^k) of ($\mathrm{D}_2(y^k)$).

(k4) Set $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{y \in \mathbb{R}^p : \varphi\big(y, D(u^k, \lambda^k)\big) \geqq 0\}$.

(k5) Set $k := k + 1$ and go to (k1).

Results.

(r1) The set of $\mathbb{R}^p_{\geqq}$-minimal vertices of $\mathcal{P}$ is $\operatorname{vert} \mathcal{S}^{k-1}$. Moreover $\mathcal{S}^{k-1} = \mathcal{P}$.

(r2) The set $\{v \in \mathbb{R}^p : \lambda(v) \geqq 0, \varphi(y, v) \geqq 0 \text{ for all } y \in \operatorname{vert} \mathcal{S}^{k-1}\}$ is defined by a nondegenerate inequality representation of $\mathcal{D}$.

(r3) All $\mathcal{K}$-maximal vertices of $\mathcal{D}$ are contained in the set

$$\mathcal{V} := \left\{ D(\bar{u}^1, e^1), D(\bar{u}^2, e^2), \dots, D(\bar{u}^p, e^p), D(u^1, \lambda^1), \dots, D(u^{k-1}, \lambda^{k-1}) \right\}.$$

(r4) The set $\{y \in \mathbb{R}^p : \varphi(y, v) \geqq 0 \text{ for all } v \in \mathcal{V}\}$ is given by a (possibly degenerate) inequality representation of $\mathcal{P}$.

Details of Algorithm 1.

(i1) It is obvious that $\operatorname{int} \mathcal{P} \neq \emptyset$. For instance, $Px + \alpha e \in \operatorname{int} \mathcal{P}$ for arbitrary $x \in \mathcal{X}$ and $\alpha > 0$.

(i2) Of course, ($\mathrm{D}_1(e^i)$) has an optimal solution because ($\mathrm{P}_1(e^i)$) is bounded.

(i3) From the definition of the ideal point we directly obtain that $\mathcal{S}^0 \supseteq \mathcal{P}$.

(k1) Let $y \in \operatorname{vert} \mathcal{S}^{k-1}$. By computing the optimal value μ of ($\mathrm{P}_2(y)$) or ($\mathrm{D}_2(y)$) it is possible to decide whether $y \in \mathcal{P}$ or not. We have $y \in \mathcal{P}$ if and only if $\mu = 0$.

(k2) Solve the linear programme

$$\alpha^k := \max\{\alpha : x \in \mathcal{X}, \alpha s^k + (1 - \alpha)\hat{p} \geqq Px\}. \tag{3}$$

Of course, for every $\bar{x} \in \mathcal{X}$, $(\bar{x}, 0)$ is feasible for (3). Since $\mathcal{X}$ is nonempty, there exists an optimal solution of (3). From $s^k \notin \mathcal{P}$ and $\hat{p} \in \operatorname{int} \mathcal{P}$ we conclude that $y^k \in \operatorname{bd} \mathcal{P}$ and $\alpha^k \in (0, 1)$. Moreover, we have $\operatorname{bd} \mathcal{P} = \operatorname{wmin}_{\mathbb{R}^p_{\geqq}} \mathcal{P}$ (see e.g. Heyde *et al.* (2006)).

(k3) By Proposition 4.2, there exists an optimal solution.

(k4) By Proposition 4.2, $H(D(\bar{u},\bar{\lambda}))$ is a supporting hyperplane to $\mathcal{P}$ containing y^k. This means, $\varphi\big(y, D(u^k,\lambda^k)\big) \geqq 0$ for all $y \in \mathcal{P}$ and $\varphi\big(y^k, D(u^k,\lambda^k)\big) = 0$. Hence we get $\mathcal{P} \subseteq \mathcal{S}^k \subseteq \mathcal{S}^{k-1}$.

(r1) From (k1) we get $\operatorname{vert} \mathcal{S}^{k-1} \subseteq \mathcal{P}$. By Proposition 4.4 we obtain $\mathcal{S}^{k-1} = \operatorname{conv}(\operatorname{vert} \mathcal{S}^{k-1} + \mathbb{R}^p_{\geqq}) \subseteq \mathcal{P}$. As shown in (k4) we have $\mathcal{P} \subseteq \mathcal{S}^{k-1}$. Together we have $\mathcal{P} = \mathcal{S}^{k-1}$. By Proposition 4.3 the statement follows.

(r2) By Corollary 3.4, $\mathcal{F}^*$ is a $\mathcal{K}$-maximal $(p-1)$-dimensional facet of $\mathcal{D}$ if and only if there exists some $\mathbb{R}^p_{\geqq}$-minimal vertex y of $\mathcal{P}$ such that $\mathcal{F}^* = H^*(y) \cap \mathcal{D}$. Hence a hyperplane $H^*(y)$ supports $\mathcal{D}$ in a facet if and only if y is a $\mathbb{R}^p_{\geqq}$-minimal vertex of $\mathcal{P}$. Of course, the corresponding inequalities are not redundant.

(r3) Let v be a $\mathcal{K}$-maximal vertex of $\mathcal{D}$. By Corollary 3.3, $\mathcal{F} := H(v) \cap \mathcal{P}$ is a $\mathbb{R}^p_{\geqq}$-minimal $(p-1)$-dimensional facet of $\mathcal{P}$. Since $\mathcal{S}^{k-1} = \mathcal{P}$ and by the construction of $\mathcal{S}^{k-1}$, for every weakly $\mathbb{R}^p_{\geqq}$-minimal facet $\mathcal{F}$ of $\mathcal{P}$ there exists some $i \in \{0,\dots,k-1\}$ such that $\mathcal{F} = H(D(u^i,\lambda^i)) \cap \mathcal{P}$. By Corollary 3.3, we get $D(u^i,\lambda^i) = v$.

(r4) This follows from (r3) by the geometric duality theorem.

In Example 4.5 we demonstrate the occurrence of degeneracy.

Example 4.5 Consider problem (P) with the data

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

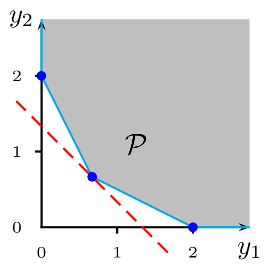

Figure 3: $\mathcal{P}$ and the first supporting hyperplane.

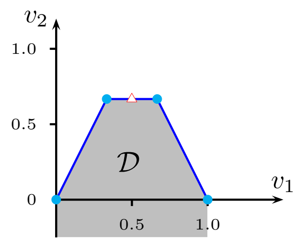

Figure 4: $\mathcal{D}$ and a point in the relative interior of a facet.

We apply Algorithm 1 for the choice $\hat{p} = (1,1)^T$. In the initialization ($k = 0$) we obtain $y^I = (0,0)^T$ and $\mathcal{S}^0 = \mathbb{R}^p_{\geqq}$. In the first iteration ($k = 1$) we get $y^1 = (\frac{2}{3}, \frac{2}{3})^T$. We have to solve $(\mathrm{D}_2(y^1))$. This problem has three optimal extreme point solutions, namely $(u^1,\lambda^1)^T = (0,\frac{1}{3},0,0,0,\frac{1}{3},\frac{2}{3})^T$, $(u^1,\lambda^1)^T = (0,0,\frac{1}{6},0,0,\frac{1}{2},\frac{1}{2})^T$ and $(u^1,\lambda^1)^T = (\frac{1}{3},0,0,0,0,\frac{2}{3},\frac{1}{3})^T$. In the case we choose the second one, we get the redundant inequality $3y_1 + 3y_2 \geqq 4$. The corresponding hyperplane supports $\mathcal{P}$ not in a facet, but just in the vertex y^1, see Figure 3. Also, for the choice $(u^1,\lambda^1)^T = (0,0,\frac{1}{6},0,0,\frac{1}{2},\frac{1}{2})^T$, the point $D(u^1,\lambda^1)$ is not a vertex of $\mathcal{D}$, see Figure 4. This means, Algorithm 1 yields a degenerate inequality representation of $\mathcal{P}$ and a degenerate point representation of $\mathcal{D}$.

Finally we show the finiteness of the modified algorithm in the same way as in the original variant of Benson (1998b).

Theorem 4.6 *The modified outer approximation algorithm is finite.*

Proof. Since $\hat{p} \in \operatorname{int}\mathcal{P}$, the point $y^k \in \mathcal{P}$ computed in iteration k belongs to $\operatorname{int}\mathcal{S}^{k-1}$. We have $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{y \in \mathbb{R}^p : \varphi\big(y, D(u^k,\lambda^k)\big) \geqq 0\}$ and by Proposition 4.2 we know that $\mathcal{F} := \{y \in \mathcal{P} : \varphi\big(y, D(u^k,\lambda^k)\big) = 0\}$ is a face of $\mathcal{P}$ with $y^k \in \mathcal{F}$, where $\mathcal{F} \subseteq \operatorname{bd}\mathcal{S}^k$. This means for the next iteration that $y^{k+1} \notin \mathcal{F}$ (because $y^{k+1} \in \operatorname{int}\mathcal{S}^k$), and therefore y^{k+1} belongs to another face of $\mathcal{P}$. Since $\mathcal{P}$ is polyhedral, it has a finite number of faces, hence the algorithm is finite. $\square$

5 The Dual Variant of Benson's Algorithm

As in the previous section we assume that the primal feasible set $\mathcal{X}$ of problem (P) is nonempty and $\mathcal{P}$ is $\mathbb{R}^p_{\geqq}$-bounded from below.

The dual variant of Benson's algorithm first constructs a p-dimensional polyhedral set $\mathcal{S}^0 = \{v \in \mathbb{R}^p : \lambda(v) \geqq 0, \varphi(Px^0, v) \geqq 0\}$ such that $\mathcal{D} \subseteq \mathcal{S}^0$. Here x^0 is an optimal solution of $(\mathrm{P}_1(\hat{d}))$ for an interior point $\hat{d}$ of $\mathcal{D}$. In every iteration it chooses a vertex s^k of $\mathcal{S}^{k-1}$ not contained in $\mathcal{D}$ and constructs a supporting hyperplane to $\mathcal{D}$ by solving the linear programme $(\mathrm{P}_1(v^k))$, where v^k is a boundary point of $\mathcal{D}$ on the line segment connecting s^k with the interior point $\hat{d}$ of $\mathcal{D}$. $\mathcal{S}^k$ is defined by intersecting $\mathcal{S}^{k-1}$ with the halfspace of the hyperplane containing $\mathcal{D}$ until at termination $\mathcal{S}^{k-1} = \mathcal{D}$.

Proposition 5.1 *Let $\bar{v} \in \max_{\mathcal{K}}\mathcal{D}$, then for every solution $\bar{x}$ of $(\mathrm{P}_1(\bar{v}))$, $H^*(P\bar{x})$ is a supporting hyperplane of $\mathcal{D}$ with $\bar{v} \in H^*(P\bar{x})$.*

Proof. Let $v \in \mathcal{D}$, i.e., there is some u such that $(u, \lambda(v)) \in \mathcal{U}$ and $v_p \leqq b^T u$. From the weak duality between $(\mathrm{P}_1(v))$ and $(\mathrm{D}_1(v))$ we get that $\lambda(v)^T P\bar{x} \geqq b^T u \geqq v_p$, or equivalently, $\varphi(P\bar{x}, v) \geqq 0$. For $\bar{v} \in \max_{\mathcal{K}}\mathcal{D}$ we get similarly even an optimal solution $\bar{u}$ of $(\mathrm{D}_1(\bar{v}))$, and strong duality between $(\mathrm{P}_1(\bar{v}))$ and $(\mathrm{D}_1(\bar{v}))$ implies that $\varphi(P\bar{x}, \bar{v}) = 0$. The result follows from the definition of $H^*(P\bar{x}) = \{v \in \mathbb{R}^p : \varphi(P\bar{x}, v) = 0\}$. $\square$

Proposition 5.2 *Every vertex of $\mathcal{D}$ is $\mathcal{K}$-maximal.*

Proof. Assume there is some vertex $\bar{v} \in \mathcal{D}$ which is not $\mathcal{K}$-maximal. Then there exists some $v \in \bar{v} + \mathcal{K} \cap \mathcal{D}$ with $v \neq \bar{v}$. We get $\bar{v} = \frac{1}{2}v + \frac{1}{2}(\bar{v} - (v - \bar{v}))$, where $v \in \mathcal{D}$ and $(\bar{v} - (v - \bar{v})) \in \mathcal{D}$ are not equal to $\bar{v}$. This contradicts the fact that $\bar{v}$ is a vertex. □

Similarly to Proposition 4.4, we can represent the polyhedra approximating the set $\mathcal{D}$ from outside in the following algorithm by a finite number of (extreme) points, because the (extreme) directions are always the same.

Proposition 5.3 *Let $y \in \mathbb{R}^p$ and let $\mathcal{S} \subseteq \mathbb{R}^p$ be a polyhedral convex set such that $\mathcal{D} \subseteq \mathcal{S} \subseteq \{v \in \mathbb{R}^p : \lambda(v) \geqq 0, \varphi(y, v) \geqq 0\}$. Letting $\mathcal{E}$ be the set of extreme points of $\mathcal{S}$, we have $\mathcal{S} = \operatorname{conv}(\mathcal{E} - \mathcal{K})$.*

Proof. Setting $\mathcal{W} := \{v \in \mathbb{R}^p : \lambda(v) \geqq 0,\ \varphi(y, v) \geqq 0\}$, we have $v \in \mathcal{W}$ if and only if

$$v_1 \geqq 0, \ldots, v_{p-1} \geqq 0, \sum_{i=1}^{p-1} v_i \leqq 1, \lambda^*(y)^T v \geqq -y_p,$$

where the last component of $\lambda^*(y)$ is -1. It follows that $\mathcal{W}_\infty = -\mathcal{K}$. Since $\mathcal{S}$ is closed and convex, we get $-\mathcal{K} \subseteq \mathcal{D}_\infty \subseteq \mathcal{S}_\infty \subseteq -\mathcal{K}$, hence $\mathcal{S}_\infty = -\mathcal{K}$. Now the conclusion follows from (Rockafellar, 1972, Theorem 18.5 and Theorem 19.5.) □

A set $\mathcal{E}$ of finitely many points in $\mathbb{R}^p$ is called a *point representation* of $\mathcal{D}$ if $\mathcal{D} = \operatorname{conv}(\mathcal{E} - \mathcal{K})$. The same notation is used for sets $\mathcal{S}^k$ constructed during the algorithm. Again, we speak about nondegenerate and degenerate point representations depending on whether $\mathcal{E}$ only consists of extreme points of $\mathcal{D}$ or not. With this notation we can say that in the result (r3) of Algorithm 1 we get a (possibly degenerate) point representation of $\mathcal{D}$.

We propose the following algorithm, subsequently called *dual outer approximation algorithm.*

Algorithm 2

Initialization ($k = 0$).

(i1) Choose some $\hat{d} \in \operatorname{int} \mathcal{D}$.

(i2) Compute an optimal solution x^0 of $(\mathrm{P}_1(\hat{d}))$.

(i3) Set $\mathcal{S}^0 := \{v \in \mathbb{R}^p : \lambda(v) \geqq 0, \varphi(Px^0, v) \geqq 0\}$ and $k = 1$.

Iteration steps ($k \geqq 1$).

(k1) If $\operatorname{vert} \mathcal{S}^{k-1} \subseteq \mathcal{D}$ stop, otherwise choose a vertex s^k of $\mathcal{S}^{k-1}$ such that $s^k \notin \mathcal{D}$.

(k2) Compute $\alpha^k \in (0, 1)$ such that $v^k := \alpha^k s^k + (1 - \alpha^k)\hat{d} \in \max_{\mathcal{K}} \mathcal{D}$.

(k3) Compute an optimal solution x^k of $(\mathrm{P}_1(v^k))$.

(k4) Set $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{v \in \mathbb{R}^p : \varphi(Px^k, v) \geqq 0\}$.

(k5) Set $k := k + 1$ and go to (k1)

Results.

(r1) The set of $\mathcal{K}$-maximal vertices of $\mathcal{D}$ is $\operatorname{vert}\mathcal{S}^{k-1}$.

(r2) The set $\left\{y \in \mathbb{R}^p : \varphi(y,v) \geqq 0 \text{ for all } v \in \operatorname{vert}\mathcal{S}^{k-1}\right\}$ is given by a nondegenerate inequality representation of $\mathcal{P}$.

(r3) All $\mathbb{R}^p_{\geqq}$-minimal (nondominated) vertices of $\mathcal{P}$ are contained in the set $\mathcal{Y} := \{Px^0, Px^1, \dots, Px^{k-1}\}$.

(r4) The set $\{v \in \mathbb{R}^p : \lambda(v) \geqq 0, \varphi(y,v) \geqq 0 \text{ for all } y \in \mathcal{Y}\}$ is given by a (possibly degenerate) inequality representation of $\mathcal{D}$.

Details of Algorithm 2.

(i1) We show that $\operatorname{int}\mathcal{D} \neq \emptyset$. Since $\mathcal{X}$ is assumed to be nonempty and $\mathcal{P}$ is $\mathbb{R}^p_{\geqq}$-bounded from below, $(\mathrm{P}_1(v))$ has an optimal solution for every $v \in \mathbb{R}^p$ with $\lambda(v) \geqq 0$. By duality, the same is true for $(\mathrm{D}_1(v))$. Denote by γ^i the optimal value of $(\mathrm{D}_1(e^i))$. Furthermore, set $\gamma = \min\left\{\gamma^i : i \in \{1,\dots,p\}\right\}$. Then γ is a lower bound for the optimal values of the problems $(\mathrm{D}_1(v))$ whenever $\lambda(v) \geqq 0$. From the definition of $\mathcal{D}$ we easily obtain

$$\mathcal{D} = \left\{v \in \mathbb{R}^p : \lambda(v) \geqq 0, A^T u = P^T \lambda(v) \text{ and } v_p \leqq b^T u \text{ for some } u \geqq 0\right\}.$$

Hence

$$\mathcal{D} \supseteq \{v \in \mathbb{R}^p : \lambda(v) \geqq 0, v_p \leqq \gamma\},$$

which shows that $\operatorname{int}\mathcal{D}$ is nonempty. One possible choice for the point $\hat{d} \in \operatorname{int}\mathcal{D}$ is $\hat{d} = \left(\frac{1}{p},\dots,\frac{1}{p},\gamma-1\right)^T$.

(i2) Since $\mathcal{X}$ is assumed to be nonempty, $\mathcal{P}$ is $\mathbb{R}^p_{\geqq}$-bounded from below and $\lambda(\hat{d}) \geqq 0$, $(\mathrm{P}_1(\hat{d}))$ has an optimal solution.

(i3) It holds that $\mathcal{S}^0 \supseteq \mathcal{D}$. It remains to show that

$$\left\{v \in \mathbb{R}^p : A^T u = P^T\lambda(v), v_p \leqq b^T u \text{ for some } u \geqq 0\right\} \subseteq \left\{v \in \mathbb{R}^p : \varphi(Px^0, v) \geqq 0\right\}.$$

This follows from weak duality between $(\mathrm{P}_1(v))$ and $(\mathrm{D}_1(v))$ as in the proof of Proposition 5.1.

(k1) Compute the optimal value μ of $(\mathrm{P}_1(s^k))$ in order to decide whether s^k belongs to $\mathcal{D}$. We have $s^k \in \mathcal{D}$ if and only if $s^k_p \leqq \mu$.

(k2) Solve the linear programme

$$\alpha^k := \max\{\alpha : (u,\lambda) \in \mathcal{U}, \alpha s^k + (1-\alpha)\hat{d} = D(u,\lambda)\}. \tag{4}$$

The existence of an optimal solution of the LP (4) can be shown as follows. If there is some $(\bar{u},\bar{\lambda}) \in \mathcal{U}$ such that $\hat{d} = D(\bar{u},\bar{\lambda})$, then $(\bar{u},\bar{\lambda},0)$ is feasible for problem (4).

Otherwise, we have $\hat{d} \notin D(\mathcal{U})$. Let w^i be the i-th unit vector, but let the last component be replaced by γ^i. It is easy to verify that there is some $\bar{\alpha} \in (0,1)$ such that $\bar{\alpha} s^k + (1-\bar{\alpha})\hat{d} \in \operatorname{conv}\left\{w^1, \dots, w^p\right\} \subseteq D(\mathcal{U})$, i.e., there is some $(\bar{u}, \bar{\lambda}) \in \mathcal{U}$ such that $(\bar{u}, \bar{\lambda}, \bar{\alpha})$ is feasible for problem (4). Furthermore, we have $\bar{\alpha} < 1$ because otherwise we obtain $s^k \in D(\mathcal{U}) \subseteq \mathcal{D}$, a contradiction.

(k3) Since $\mathcal{X}$ is nonempty and $P(\mathcal{X})$ is $\mathbb{R}^p_{\geqq}$-bounded from below, there exists an optimal solution.

(k4) Analogously to (i3) above we get $\mathcal{S}^k \supseteq \mathcal{D}$.

(r1) From (k1) we get $\operatorname{vert} \mathcal{S}^{k-1} \subseteq \mathcal{D}$. By Proposition 5.3 we obtain $\mathcal{S}^{k-1} = \operatorname{conv}(\operatorname{vert} \mathcal{S}^{k-1} - \mathcal{K}) \subseteq \mathcal{D}$. As shown in (k4) we have $\mathcal{D} \subseteq \mathcal{S}^{k-1}$. Together we have $\mathcal{D} = \mathcal{S}^{k-1}$. By Proposition 5.2 the statement follows.

(r2) By Corollary 3.3, $\mathcal{F}$ is a weakly nondominated $(p-1)$-dimensional facet of $\mathcal{P}$ if and only if there exists some $\mathcal{K}$-maximal vertex v of $\mathcal{D}$ such that $\mathcal{F} = H(v) \cap \mathcal{P}$. Hence a hyperplane $H(v)$ supports $\mathcal{P}$ in a facet if and only if v is a $\mathcal{K}$-maximal vertex of $\mathcal{D}$. Thus we have a nondegenerate inequality representation of $\mathcal{P}$.

(r3) Let y be a $\mathbb{R}^p_{\geqq}$-minimal vertex of $\mathcal{P}$. By Corollary 3.4, $\mathcal{F}^* := H^*(y) \cap \mathcal{D}$ is a $\mathcal{K}$-maximal $(p-1)$-dimensional facet of $\mathcal{D}$. Since $\mathcal{S}^{k-1} = \mathcal{D}$ and by the construction of $\mathcal{S}^{k-1}$, for every $\mathcal{K}$-maximal facet $\mathcal{F}^*$ of $\mathcal{D}$ there exists some $i \in \{0, \dots, k-1\}$ such that $\mathcal{F}^* = H^*(Px^i) \cap \mathcal{D}$. By Corollary 3.4, we get $Px^i = y$.

(r4) This follows from (r3) by the geometric duality theorem.

Remark 5.4 If $\hat{d} \in \operatorname{int} \mathcal{D}$ such that $\hat{d}_p \leqq \gamma$, problem (4) is equivalent to the following one having $p+1$ fewer variables and $2p+1$ fewer constraints,

$$\alpha^k := \max\left\{\frac{b^T u - \hat{d}_p}{s^k_p - \hat{d}_p} : u \geqq 0, \tilde{A}u = \tilde{b}\right\}. \tag{5}$$

where

$$\tilde{A} := (s^k_p - \hat{d}_p)A^T + P^T\big(\lambda(\hat{d}) - \lambda(s^k)\big)b^T \text{ and } \tilde{b} := P^T\big(s^k_p \lambda(\hat{d}) - \hat{d}_p \lambda(s^k)\big).$$

This equivalence of problem (4) and problem (5) can be shown in a straightforward way taking into account that $\hat{d}_p \leqq \gamma$ implies $\hat{d}_p < s^k_p$.

Example 5.5 illustrates the occurrence of degenerate representations of $\mathcal{P}$ and $\mathcal{D}$.

Example 5.5 Consider problem (P) with the data

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 7 & 21 & 9 \\ 0 & 0 & -1 \\ -7 & -42 & 3 \\ 1 & 7 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ -1 \\ -39 \\ 6 \end{pmatrix}.$$

We apply Algorithm 2 for the choice $\hat{d} = (\frac{1}{2}, 0)^T$. In the initialization step ($k = 0$) we solve $P_1(\hat{d})$ and obtain the unique optimal solution $x^0 = (0, 1, 1)^T$. Hence

$$\mathcal{S}^0 = \{v \in \mathbb{R}^p : 0 \leqq v_1 \leqq 1,\ v_2 \leqq 1 - v_1\}\,.$$

There is exactly one vertex of $\mathcal{S}^0$, namely $s^1 = (0, 1)^T$, that does not belong to $\mathcal{D}$. Step (k2) yields $v^1 = (\frac{1}{8}, \frac{3}{4})^T$. Note that we have the situation that v^1 is not in the relative interior of a facet of $\mathcal{D}$, because it is a vertex of $\mathcal{D}$. In step (k3) we solve $P_1(v^1)$. We have exactly three extreme point optimal solutions of $P_1(v^1)$, namely $x^1 = (\frac{3}{4}, \frac{3}{4}, 1)^T$, $x^1 = (3, \frac{3}{7}, 0)^T$ and $x^1 = (6, 0, 1)^T$. In case we choose the second one, we get the redundant inequality $-\frac{18}{7}v_1 + v_2 \leqq \frac{3}{7}$. The corresponding hyperplane supports $\mathcal{D}$ not in a facet, but just in the vertex v_1 (see Figure 6). For the choice $x^1 = (3, \frac{3}{7}, 0)^T$, the point Px^1 is not a vertex of $\mathcal{P}$ (see Figure 5). This means, Algorithm 2 yields a degenerate inequality representation of $\mathcal{D}$ and a degenerate point representation of $\mathcal{P}$.

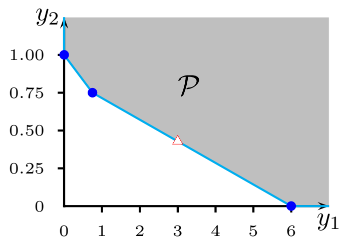

Figure 5: $\mathcal{P}$ and a point in the relative interior of a facet

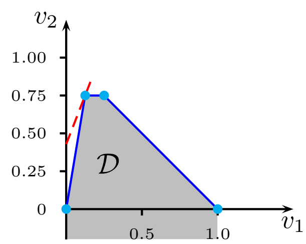

Figure 6: $\mathcal{D}$ and the supporting hyperplane

Finally we show that the algorithm terminates after a finite number of steps.

Theorem 5.6 *The dual outer approximation algorithm is finite.*

Proof. Since $\hat{d} \in \operatorname{int} \mathcal{D}$, the point $v^k \in \mathcal{D}$ computed in iteration k belongs to $\operatorname{int} \mathcal{S}^{k-1}$. We have $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{v \in \mathbb{R}^p : \varphi(Px^k, v) \geqq 0\}$ and, by Proposition 5.1, we know that $\mathcal{F} := \{v \in \mathcal{D} : \varphi(Px^k, v) = 0\}$ is a face of $\mathcal{D}$ with $v^k \in \mathcal{F}$, where $\mathcal{F} \subseteq \operatorname{bd} \mathcal{S}^k$. This means for the next iteration that $v^{k+1} \notin \mathcal{F}$ (because $v^{k+1} \in \operatorname{int} \mathcal{S}^k$), and therefore v^{k+1} belongs to another face of $\mathcal{D}$. Since $\mathcal{D}$ is polyhedral, it has a finite number of faces, hence the algorithm is finite. $\square$

6 Weight Set Decomposition

It is well known that $\mathbb{R}^p_{\geqq}$-minimal points of $\mathcal{P}$ can be characterized by weighted sum scalarization (Isermann, 1974). A point $y \in \mathcal{P}$ is $\mathbb{R}^p_{\geqq}$-minimal if and only if there exists $w \in \mathbb{R}^p_{>} = \{w \in \mathbb{R}^p : w > 0, k = 1, \dots, p\}$ such that $w^T y \leqq w^T y'$ for all $y' \in \mathcal{P}$.

Considering, for fixed $y \in \mathcal{P}$, all w with this property leads to the idea of weight set decomposition, e.g. Benson and Sun (2000). Let $y \in \mathcal{P}$ and define

$$\mathcal{W}(y) := \left\{ w \in \mathbb{R}^p_{\geqq} : w^T y \leqq w^T y' \text{ for all } y' \in \mathcal{P} \right\}.$$

Using the equivalence relation $w^1 \sim w^2$ if and only if $w^1 = \alpha w^2$ for some $\alpha > 0$ it is clear that we can identify $\mathbb{R}^p_{\geqq} \setminus \{0\}$ with $\Lambda = \left\{ \lambda \in \mathbb{R}^p : \lambda \geqq 0, \sum_{k=1}^p \lambda_k = 1 \right\}$ and $\mathcal{W}(y)$ with $\Lambda(y) = \left\{ \lambda \in \Lambda : \lambda^T y \leqq \lambda^T y' \text{ for all } y' \in \mathcal{P} \right\}$.

The following function was already considered in Section 3

$$\lambda : \mathbb{R}^p \to \mathbb{R}^p, \quad \lambda(v) := \left(v_1, \ldots, v_{p-1}, 1 - \sum_{i=1}^{p-1} v_i \right)^T.$$

Proposition 6.1 *Let $\mathcal{P}$ be nonempty and $\mathbb{R}^p_{\geqq}$-bounded below. Let $\{y^1, \ldots, y^q\}$ be the nondominated extreme points of $\mathcal{P}$ and let $\{\mathcal{F}_1^*, \ldots, \mathcal{F}_q^*\}$ be the corresponding $\mathcal{K}$-maximal facets of $\mathcal{D}$ according to the geometric duality theorem. Then for all $i = 1, \ldots, q$ it holds*

$$\Lambda(y^i) = \lambda(\mathcal{F}_i^*) := \{\lambda(v) : v \in \mathcal{F}_i^*\}$$

and $\{\lambda(\mathcal{F}_i^*) : i = 1, \ldots, q\}$ is a weight set decomposition, that is,

$$\Lambda = \bigcup_{i=1}^q \lambda(\mathcal{F}_i^*) \text{ and } \operatorname{ri} \lambda(\mathcal{F}_i^*) \cap \operatorname{ri} \lambda(\mathcal{F}_j^*) = \emptyset \text{ whenever } i \neq j.$$

Proof. Of course, $\lambda(\cdot)$ is a one-to-one map from $\max_{\mathcal{K}} \mathcal{D}$ onto Λ. The inverse map is $v(\lambda') := \lambda^{-1}(\lambda') = (\lambda'_1, \ldots, \lambda'_{p-1}, v_p)^T$ where v_p is the optimal value of the linear programme $(\mathrm{D}_1(\lambda'))$. Moreover, $\lambda(\cdot)$ is affine on convex subsets of $\max_{\mathcal{K}} \mathcal{D}$, in particular on each $\mathcal{K}$-maximal facet of $\mathcal{D}$.

Let $\lambda' \in \Lambda(y^i)$. Determine $v(\lambda')$. By duality between $(\mathrm{P}_1(\lambda'))$ and $(\mathrm{D}_1(\lambda'))$ we get $\varphi(y^i, v(\lambda')) = 0$. Moreover, we have $v(\lambda') \in \mathcal{D}$. Hence $v(\lambda') \in H^*(y^i) \cap \mathcal{D} = \mathcal{F}_i^*$ and so $\lambda' \in \lambda(\mathcal{F}_i^*)$.

Let $\lambda' \in \lambda(\mathcal{F}_i^*)$, i.e., $v(\lambda') \in \mathcal{F}_i^*$. Then $H(v(\lambda'))$ supports $\mathcal{P}$ in y^i. This implies that $\lambda' \in \Lambda(y^i)$.

The second statement follows from the properties of $\lambda(\cdot)$ and the fact that

$$\max_{\mathcal{K}} \mathcal{D} = \bigcup_{i=1}^q \mathcal{F}_i^* \text{ and } \operatorname{ri} \mathcal{F}_i^* \cap \operatorname{ri} \mathcal{F}_j^* = \emptyset \text{ whenever } i \neq j.$$

This completes the proof. $\square$

Proposition 6.1 shows that both Algorithms 1 and 2 can be used to compute a weight set decomposition with respect to the nondominated extreme points of $\mathcal{P}$. This result is very relevant in the context of multiobjective *integer* linear programmes. These are often solved using a two phase algorithm (Ulungu and Teghem, 1995), where the first phase consists in identifying the nondominated extreme points and the second phase finds all other nondominated points. The major problem in Phase 1 is the determination of a weight set decomposition (Przybylski *et al.*, 2007). It can be expected that the algorithms to solve (D) proposed in this paper lead to progress in multiple objective integer linear programming algorithms for problems such as network flow problems, where the single objective counterparts can be solved by linear programming.

7 Numerical Results

In this section, we solve several multiple objective linear programmes by both the primal and the dual outer approximation algorithm. We start with some small examples in order to illustrate the relationship between the primal outcome set $\mathcal{P}$ and the dual outcome set $\mathcal{D}$. Then we address some larger problems taken from real world applications. In each example, the primal problem is solved by (our slightly modified) primal outer approximation algorithm and the dual problem is solved by the dual variant of the algorithm. We show the primal and dual sets $\mathcal{P}$ and $\mathcal{D}$ and list the vertices and the facets for some of the smaller examples. We also compare the computation time of solving the primal and the dual problem. As seen in the considerations above it is sufficient to solve one problem to obtain the outcome set of both the primal and dual problems.

Both algorithms were implemented in Matlab 7.1(R14) using CPLEX 10.0 as LP solver and the tests were run on a dual processor CPU with 1.8 GHz and 1 GB RAM. We used the dual simplex method to solve the LPs. At step (k4), the method of Chen and Hansen (1991) for on-line vertex enumeration by adjacency lists was used to calculate a vertex representation from the inequality representation of $\mathcal{S}^k$.

As Benson's algorithm and its dual variant have steps of the same type, the number of facets of the primal, respectively dual, outcome set seems to correlate with the computation time. We observe in each of our examples that the dual variant is faster if the dual outcome set has fewer facets than the primal. Otherwise the primal method is faster. This means that it depends on the structure of the problem whether the primal or the dual algorithm is the better choice.

Example 7.1 In this example we consider the LP relaxation of an assignment problem with three objectives. The cost matrices of the three objectives are

$$\begin{pmatrix} 3 & 6 & 4 & 5 \\ 2 & 3 & 5 & 4 \\ 3 & 5 & 4 & 2 \\ 4 & 5 & 3 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 5 & 4 \\ 5 & 3 & 4 & 3 \\ 5 & 2 & 6 & 4 \\ 4 & 5 & 2 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 4 & 2 \\ 4 & 2 & 4 & 6 \\ 4 & 2 & 6 & 3 \\ 2 & 4 & 5 & 3 \end{pmatrix}.$$

Figures 7 and 8 show the weakly nondominated set of $\mathcal{P}$ and the $\mathcal{K}$-maximal subset of $\mathcal{D}$. The four nondominated vertices of $\mathcal{P}$ are $(11, 11, 14)$, $(19, 14, 10)$, $(15, 9, 17)$, and $(13, 16, 11)$. They correspond to the $\mathcal{K}$-maximal facets of $\mathcal{D}$ given by $3v_1+3v_2+v_3 = 14$, $-9v_1-4v_2+v_3 = 10$, $2v_1 + 8v_2 + v_3 = 17$, and $-2v_1 - 5v_2 + v_3 = 11$, respectively. $\mathcal{D}$ has nine vertices which we list along with the corresponding facets of $\mathcal{P}$ in Table 1.

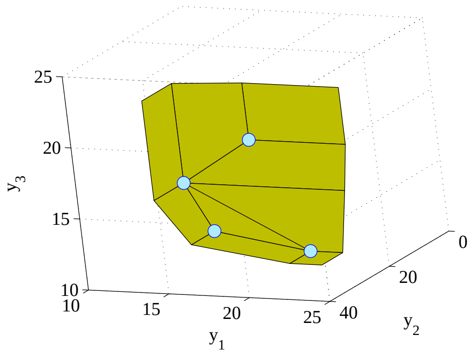

Figure 7: $\mathrm{wmin}_{\mathbb{R}^3_{\geqq}} \mathcal{P}$ in Example 7.1.

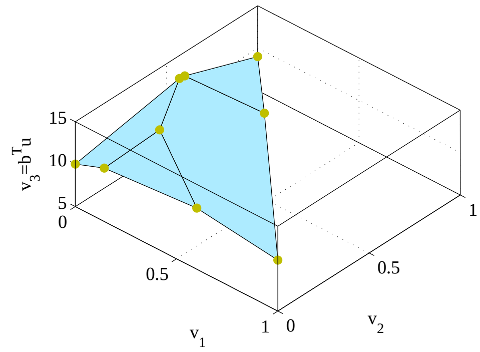

Figure 8: $\max_{\mathcal{K}} \mathcal{D}$ in Example 7.1.

Vertices of $\mathcal{D}$			Facets of $\mathcal{P}$
v_1	v_2	v_3	
1	0	11	$y_1 = 11$
0	1	9	$y_2 = 9$
0	0	10	$y_3 = 10$
$\frac{1}{3}$	$\frac{2}{3}$	11	$\frac{1}{3}y_1 + \frac{2}{3}y_2 = 11$
$\frac{3}{5}$	0	$12\frac{1}{5}$	$\frac{3}{5}y_1 + \frac{2}{5}y_3 = 12\frac{1}{5}$
0	$\frac{4}{7}$	$12\frac{2}{7}$	$\frac{4}{7}y_2 + \frac{3}{7}y_3 = 12\frac{2}{7}$
$\frac{1}{7}$	0	$11\frac{2}{7}$	$\frac{1}{7}y_1 + \frac{6}{7}y_3 = 11\frac{2}{7}$
0	$\frac{3}{5}$	$12\frac{1}{5}$	$\frac{3}{5}y_2 + \frac{2}{5}y_3 = 12\frac{1}{5}$
$\frac{11}{61}$	$\frac{16}{61}$	$12\frac{41}{61}$	$\frac{11}{61}y_1 + \frac{16}{61}y_2 + \frac{34}{61}y_3 = 12\frac{41}{61}$

Table 1: Example 7.1: Vertices of $\mathcal{D}$ and corresponding facets of $\mathcal{P}$.

Example 7.2 The next small example has again three objectives. The data are

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -3 & -1 \\ -3 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ b = \begin{pmatrix} -5 \\ -9 \\ -16 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Figures 9 and 10 show the weakly nondominated subset of $\mathcal{P}$ and the $\mathcal{K}$-maxiaml subset of $\mathcal{D}$. Seven vertices of $\mathcal{P}$ and their corresponding $\mathcal{K}$-maximal facets of $\mathcal{D}$ are shown in Table 2. $\mathcal{D}$ has nine vertices we list these vertices and their corresponding facets of $\mathcal{P}$ in Table 3.

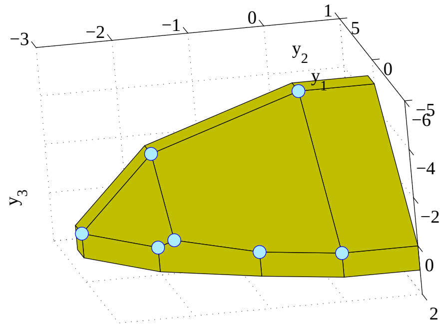

Figure 9: $\mathrm{wmin}_{\mathbb{R}^3_{\geqq}} \mathcal{P}$ in Example 7.2.

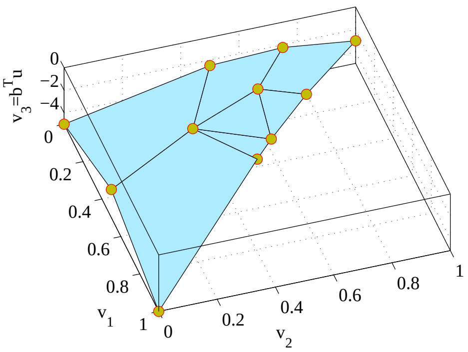

Figure 10: $\max_{\mathcal{K}} \mathcal{D}$ in Example 7.2.

Vertices of $\mathcal{P}$			$\mathcal{K}$-maximal Facets of $\mathcal{D}$
v_1	v_2	v_3	
-5	0	0	$5v_1 + v_3 = 0$
0	-3	0	$3v_2 + v_3 = 0$
0	0	-5	$-5v_1 - 5v_2 + v_3 = -5$
$-2\frac{2}{5}$	$-2\frac{1}{5}$	0	$2\frac{2}{5}v_1 + 2\frac{1}{5}v_2 + v_3 = 0$
0	-2	-3	$-3v_1 - v_2 + v_3 = -3$
-4	-1	0	$4v_1 + v_2 + v_3 = 0$
$-2\frac{2}{3}$	-2	$-\frac{1}{3}$	$2\frac{1}{3}v_1 + 1\frac{2}{3}v_2 + v_3 = -\frac{1}{3}$

Table 2: Example 7.2: Vertices of $\mathcal{P}$ and corresponding $\mathcal{K}$-maximal facets of $\mathcal{D}$.

Vertices of $\mathcal{D}$			Facets of $\mathcal{P}$
v_1	v_2	v_3	
$\frac{1}{3}$	$\frac{1}{3}$	$-1\frac{2}{3}$	$\frac{1}{3}y_1 + \frac{1}{3}y_2 + \frac{1}{3}y_3 = -1\frac{2}{3}$
1	0	-5	$y_1 = -5$
0	1	-3	$y_2 = -3$
0	0	-5	$y_3 = -5$
$\frac{1}{2}$	$\frac{1}{2}$	$-2\frac{1}{2}$	$\frac{1}{2}y_1 + \frac{1}{2}y_2 = -2\frac{1}{2}$
$\frac{1}{2}$	0	$-2\frac{1}{2}$	$\frac{1}{2}y_1 + \frac{1}{2}y_3 = -2\frac{1}{2}$
0	$\frac{3}{4}$	$-2\frac{1}{4}$	$\frac{3}{4}y_2 + \frac{1}{4}y_3 = -2\frac{1}{4}$
0	$\frac{1}{2}$	$-2\frac{1}{2}$	$\frac{1}{2}y_2 + \frac{1}{2}y_3 = -2\frac{1}{2}$
$\frac{1}{4}$	$\frac{3}{4}$	$-2\frac{1}{4}$	$\frac{1}{4}y_1 + \frac{3}{4}y_2 = -2\frac{1}{4}$
$\frac{3}{7}$	$\frac{4}{7}$	$-2\frac{2}{7}$	$\frac{3}{7}y_1 + \frac{4}{7}y_2 = -2\frac{2}{7}$
$\frac{1}{5}$	$\frac{3}{5}$	$-1\frac{4}{5}$	$\frac{1}{5}y_1 + \frac{3}{5}y_2 + \frac{1}{5}y_3 = -1\frac{4}{5}$

Table 3: Example 7.2: Vertices of $\mathcal{D}$ and corresponding facets of $\mathcal{P}$.

Example 7.3 In this example the primal solves faster than the dual. The data are

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 3 \\ 4 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The weakly $\mathbb{R}^3_{\geqq}$-minimal set of $\mathcal{P}$ and the $\mathcal{K}$-maximal set of $\mathcal{D}$ are shown in Figures 11 and 12, respectively. $\mathcal{P}$ has seven vertices, they are $(0,0,3)$, $(2,0,1)$, $(0,2,1)$, $(0,4,0)$, $(4,0,0)$, $(1,2,0)$ and $(2,1,0)$. The corresponding $\mathcal{K}$-maximal facets of $\mathcal{D}$ are $3v_1+3v_2+v_3=3$, $v_1-v_2+v_3=1$, $-v_1+v_2+v_3=1$, $-4v_2+v_3=0$, $-4v_1+v_3=0$, $-v_1-2v_2+v_3=0$, and $-2v_1-v_2+v_3=0$, respectively. The six vertices $(1,0,0)$, $(0,1,0)$, $(0,0,0)$, $(\frac{1}{3},\frac{1}{3},1)$, $(\frac{2}{5},\frac{1}{5},\frac{4}{5})$ and $(\frac{1}{5},\frac{2}{5},\frac{4}{5})$ of $\mathcal{D}$ correspond to the facets $y_1=0$, $y_2=0$, $y_3=0$, $\frac{1}{3}y_1+\frac{1}{3}y_2+\frac{1}{3}y_3=1$, $\frac{2}{5}y_1+\frac{1}{5}y_2+\frac{2}{5}y_3=\frac{4}{5}$, and $\frac{1}{5}y_1+\frac{2}{5}y_2+\frac{2}{5}y_3=\frac{4}{5}$ of $\mathcal{P}$.

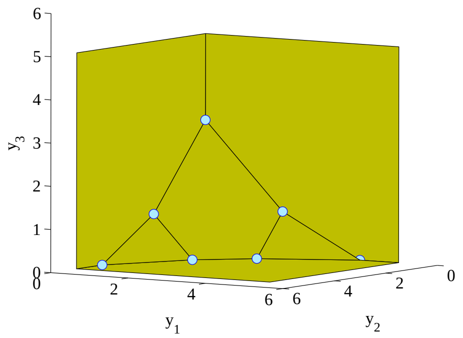

Figure 11: $\mathrm{wmin}_{\mathbb{R}^3_{\geqq}}\mathcal{P}$ in Example 7.3.

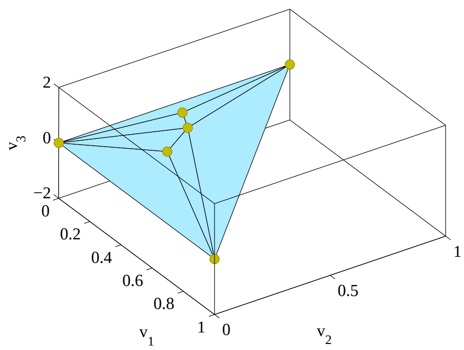

Figure 12: $\max_{\mathcal{K}}\mathcal{D}$ in Example 7.3.

Example 7.4 This problem is a portfolio selection problem (example 2050 in Steuer (1989)) with three objectives, 21 variables and 45 constraints. $\mathcal{P}$ has 52 nondominated extreme points. $\mathcal{D}$ has 99 extreme points and 52 facets, see Figures 13 and 14.

Example 7.5 The problem of intensity optimization in radiotherapy treatment planning can be formulated as a multiobjective linear programme (Shao and Ehrgott, 2006). We use one of the examples from Shao and Ehrgott (2006), an acoustic neuroma. The problem has three objectives, 597 variables and 1664 constraints. $\mathcal{P}$ (see Figure 15) has 55 vertices and 85 facets, $\mathcal{D}$ (see Figure 16) has 85 vertices and 55 facets.

Example 7.6 Our second radiotherapy treatment planning example concerns a prostate case. In this three-objective problem $\mathcal{P}$ has 3165 nondominated extreme points and 3280 facets. $\mathcal{P}$ and $\mathcal{D}$ are shown in Figures 17 and 18, respectively.

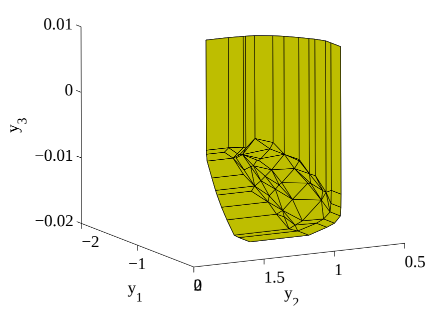

Figure 13: $\mathrm{wmin}_{\mathbb{R}^3_{\geqq}} \mathcal{P}$ in Example 7.4.

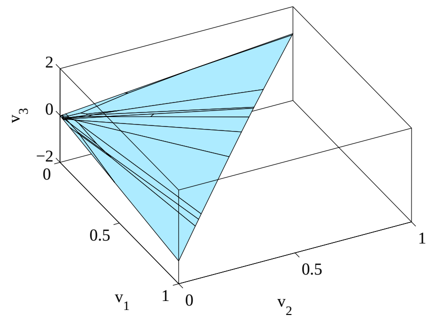

Figure 14: $\max_{\mathcal{K}} \mathcal{D}$ in Example 7.4.

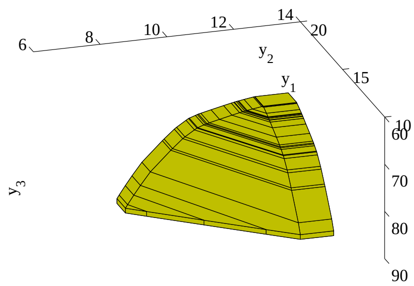

Figure 15: $\mathrm{wmin}_{\mathbb{R}^3_{\geqq}} \mathcal{P}$ in Example 7.5.

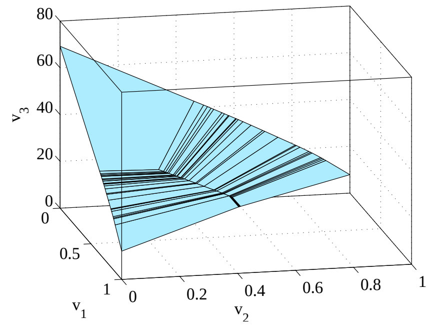

Figure 16: $\max_{\mathcal{K}} \mathcal{D}$ in Example 7.5.

Finally, we compare the computation times of our examples (see Table 4). We only give the number of vertices and the number of facets of $\mathcal{P}$, because the number of facets and the number of vertices of $\mathcal{D}$ correspond to them by geometric duality theory.

We see that the dual variant of the algorithm may have a computational speed advantage. It can be regarded as an alternative method which is preferable depending on the structure of the problem.

References

Benson, H. (1998a). Further analysis of an outcome set-based algorithm for multiple-objective linear programming. *Journal of Optimization Theory and Applications*, **97**(1), 1–10.

Benson, H. (1998b). An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem. *Journal of Global Optimization*, **13**, 1–24.

Benson, H. and Sun, E. (2000). Outcome space partition of the weight set in multiobjective linear programming. *Journal of Optimization Theory and Applications*, **105**(1), 17–36.

Chen, P. and Hansen, P. (1991). On-line and off-line vertex enumeration by adjacency lists. *Operations Research Letters*, **10**, 403–409.

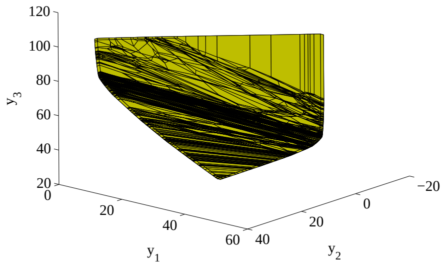

Figure 17: $\text{wmin}_{\mathbb{R}^3_{\geqq}} \mathcal{P}$ in Example 7.6.

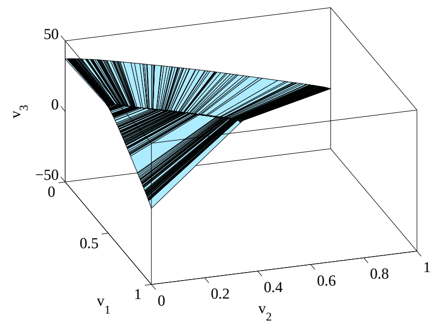

Figure 18: $\max_{\mathcal{K}} \mathcal{D}$ in Example 7.6.

		Size of A		$\mathcal{P}$		CPU time (seconds)	
Example	p	m	n	Vertices	Facets	primal	dual
3.1	2	5	2	4	5	0.0310	0.0310
7.1	3	16	16	4	9	0.0620	0.0470
7.2	3	6	3	7	11	0.0940	0.0470
7.3	3	6	3	7	6	0.0460	0.0630
7.4	3	45	21	52	99	0.7660	0.5310
7.5	3	1664	597	55	85	13.9840	8.8640
7.6	3	2142	824	3165	3280	995.0500	792.3900

Table 4: Computation times for the examples.

Ehrgott, M. (2005). *Multicriteria Optimization*. Springer-Verlag, Berlin, 2nd edition.

Ehrgott, M. and Wiecek, M. (2005). Multiobjective programming. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multicriteria Decision Analysis: State of the Art Surveys*, pages 667–722. Springer Science + Business Media, New York.

Grünbaum, B. (2003). *Convex Polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition. Prepared by V. Kaibel, V. Klee, and G.M. Ziegler.

Heyde, F. and Löhne, A. (2006). Geometric duality in multi-objective linear programming. Reports on Optimization and Stochastics 06-15, Department of Mathematics and Computer Science, Martin-Luther-University Halle-Wittenberg. Submitted to SIAM Journal on Optimization.

Heyde, F., Löhne, A., and Tammer, C. (2006). Set-valued duality theory for multiple objective linear programs and application to mathematical finance. Reports on optimization and stochastics, Department of Mathematics and Computer Science, Martin-Luther-University Halle-Wittenberg. Submitted to Journal of Optimization Theory and Applications.

Isermann, H. (1974). Proper efficiency and the linear vector maximum problem. *Operations Resaerch*, **22**, 189–191.

Przybylski, A., Gandibleux, X., and Ehrgott, M. (2007). Recursive algorithms for finding all nondominated extreme points in the outcome set of a multiobjective integer program. *INFORMS Journal on Computing*, **Submitted for publication**.

Rockafellar, R. (1972). *Convex Analysis*. Princeton University Press, Princeton.

Shao, L. and Ehrgott, M. (2006). Approximately solving multiobjective linear programmes in objective space and an application in radiotherapy treatment planning. Report 646, Department of Engineering Science, The University of Auckland. Submitted to Mathematical Methods of Operations Research.

Steuer, R. (1985). *Multiple Criteria Optimization: Theory, Computation and Application*. John Wiley & Sons, New York, NY.

Steuer, R. (1989). Adbase multiple objective linear programming package. Technical report, University of Georgia, Athens, Georgia.

Ulungu, E. and Teghem, J. (1995). The two phases method: An efficient procedure to solve bi-objective combinatorial optimization problems. *Foundations of Computing and Decision Sciences*, **20**, 149–165.

Webster, R. (1994). *Convexity*. Oxford Science Publications, Oxford, Oxford University Press.