Set-valued Duality Theory for Multiple Objective Linear Programs and Application to Mathematical Finance*

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December 15, 2006

Abstract

We develop a duality theory for multiple objective linear programs which has several advantages in contrast to other theories. For instance, the dual variables are vectors rather than matrices and the dual feasible set is a polyhedron. We use a set-valued dual objective map where its values have a very simple structure, in fact they are hyperplanes. As in other set-valued (but not in vector-valued) approaches, there is no duality gap in the case that the right-hand side of the linear constraints is zero. Moreover, we show that the whole theory can be developed by working in a complete lattice. Thus the duality theory has a high degree of analogy to its classical counterpart. These advantages open the possibility of various applications such as a dual simplex algorithm. Exemplarily, we discuss an application to a Markowitz-type bicriterial portfolio optimization problem where the risk is measured by the Conditional Value at Risk.

1 Introduction

Duality in Multiple Objective Linear Programming has been of interest to researchers for more than 30 years, see e.g. Kornbluth [13], Roedder [20], Isermann [9, 10], Brumelle [1], Jahn [11, 12], Luc [15] and Göpfert and Nehse [4]. Nevertheless the importance in applications is not as high as the importance of duality in scalar optimization (see e.g. the corresponding remark by Göpfert and Nehse [4, page 64]). For instance, no economical interpretation of these vectorial dual problems is known to the authors. Important instruments like a dual simplex algorithm are missing, because the dual variables are matrices (of rank 1) rather than vectors and there is no counterpart to the important fact of the scalar theory that the solutions are attained in vertices of the feasible polyhedron. The latter problem could be partially solved in [6]. The attainment in vertices was shown under additional assumptions, which can be omitted completely in the present approach. Simultaneously, we work with a simpler set-valued objective map in comparison to [5, 6, 14]. The values are hyperplanes, whose parameters depend linearly on the dual variables. Our duality theory provides the theoretical basis for a dual simplex algorithm for multiple objective linear programs. By an example from Mathematical Finance, we show that our duality theory also has practical relevance.

Another item in the present paper is the formulation of the duality results in terms of infimum and supremum with respect to an appropriate complete lattice. The image space of the objective function, which is usually $\mathbb{R}^q$ partially ordered by the ordinary ordering cone $\mathbb{R}^q_+$,

*submitted to *Journal of Optimization Theory and Applications*

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is embedded into a larger space. This larger space is a subset of the power set of $\mathbb{R}^q$, in fact the space of all self-infimal subsets of the power set. The ordering relation induced by the cone $\mathbb{R}^q_+$ is extended appropriately. The lattice structure allows us to carry over many formulations and results from scalar linear programming. For instance, we can answer the question about a natural and expedient concept of the attainment of a solution in multiple objective optimization.

A related approach with vector-valued primal and dual problems, called geometric duality, is developed in [7]. These results are based on duality assertions between the two polyhedral image sets in a similar manner like the classical duality of polytopes.

This paper is organized as follows. In Section 2 we develop our duality theory for multi-objective linear problems. It is our intention to formulate this theory with easy notations and independently from other works. For simplicity, we avoid discussing the theoretical background when we develop the duality theory. But Section 3 is devoted to this topic. We reformulate the duality results in terms of infimum and supremum in the underlying complete lattice and point out the analogies to the classical scalar theory. The last section is devoted to an application of the duality results to a Markowitz-type bicriterial portfolio optimization problem based on the Conditional Value at Risk. We consider the linear approximation of the problem due to Rockafellar and Uryasev [18, 19]. The dual variables and the dual solutions are interpreted by practically relevant quantities.

2 Duality results

Let us first introduce some notations. For a set $\mathcal{A} \subseteq \mathbb{R}^n$ we denote by $\operatorname{cl}\mathcal{A}$, $\operatorname{int}\mathcal{A}$, $\operatorname{bd}\mathcal{A}$, $\operatorname{ri}\mathcal{A}$ and $\operatorname{rbd}\mathcal{A}$, respectively, the closure, interior, boundary, relative interior and relative boundary of $\mathcal{A}$. Given two vectors $y_1, y_2 \in \mathbb{R}^n$ we write $y_1 \leq y_2$ if $y_2 - y_1 \in \mathbb{R}^n_+ := \{y \in \mathbb{R}^n \mid y_1 \geq 0, \dots, y_n \geq 0\}$ and $y_1 < y_2$ if $y_2 - y_1 \in \operatorname{int}\mathbb{R}^n_+$. We denote by

$$\operatorname{Min}\mathcal{A} := \left\{y \in \mathcal{A} \mid (\{y\} - \operatorname{int}\mathbb{R}^q_+) \cap \mathcal{A} = \emptyset\right\}$$

is the set of *weakly minimal points* of a set $\mathcal{A} \subseteq \mathbb{R}^q$ with respect to $\mathbb{R}^q_+$. The set of weakly maximal points of $\mathcal{A}$ is $\operatorname{Max}\mathcal{A} := -\operatorname{Min}(-\mathcal{A})$.

Let $m, n, q \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}, M \in \mathbb{R}^{q\times n}, b \in \mathbb{R}^m$ be given. We consider the following vector optimization problem

$$\text{(P)} \qquad \operatorname{Min}(M[\mathcal{X}]+\mathbb{R}^q_+), \qquad \mathcal{X} := \{x \in \mathbb{R}^n \mid Ax \geq b\},$$

where

$$M[\mathcal{X}] := \bigcup_{x\in\mathcal{X}} \{Mx\}.$$

A point $x^0 \in \mathcal{X}$ is called a *weakly efficient solution of* (P) iff

$$Mx^0 \in \operatorname{Min} M[\mathcal{X}] + \mathbb{R}^q_+ \quad \text{ or equivalently } \quad Mx^0 \in \operatorname{Min} M[\mathcal{X}].$$

Note the a point x^0 is a weakly efficient solution of (P) if and only if it is a weakly efficient solution of the more common problem

$$\operatorname{Min} M[\mathcal{X}], \qquad \mathcal{X} := \{x \in \mathbb{R}^n \mid Ax \geq b\},$$

even though the set $\operatorname{Min} M[\mathcal{X}]$ and $\operatorname{Min}(M[\mathcal{X}]+\mathbb{R}^q_+)$ can be different. The set $\operatorname{Min}(M[\mathcal{X}]+\mathbb{R}^q_+)$ is closely related to the infimal set of $M[\mathcal{X}]$. The details are discussed in the next section.

Consider the following set-valued dual objective map

$$\mathcal{H} : \mathbb{R}^m \times \mathbb{R}^q \rightrightarrows \mathbb{R}^q, \quad \mathcal{H}(u, c) := \left\{y \in \mathbb{R}^q \mid c^T y = b^T u\right\}.$$

We use the following notation

$$\mathcal{H}[\mathcal{U}] := \bigcup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c) \quad \text{and} \quad k := (1,1,\dots,1)^T \in \mathrm{I\!R}^q.$$

As the dual problem to (P) we consider the problem

$$\text{(D)} \qquad \operatorname{Max} \mathcal{H}[\mathcal{U}], \quad \mathcal{U} := \left\{(u,c) \in \mathrm{I\!R}^m \times \mathrm{I\!R}^q \mid (u,c) \geq 0,\ k^T c = 1,\ A^T u = M^T c\right\}.$$

This means the dual problem consists in determining weakly maximal points of the union of the hyperplanes $\mathcal{H}(u,c)$ defined by the points $(u,c) \in \mathcal{U}$. The new idea in this approach compared to [14] and [6] consists in having a pair (u,c) of dual variables and having hyperplanes as values of the dual objective without making any assumptions on the rank of M.

A point $(u^0,c^0) \in \mathcal{U}$ is called a *weakly efficient solution of* (D) iff

$$\mathcal{H}(u^0,c^0) \cap \operatorname{Max} \mathcal{H}[\mathcal{U}] \neq \emptyset,$$

or equivalently,

$$\exists y^0 \in \mathcal{H}(u^0,c^0) : \forall (u,c) \in \mathcal{U} : \left(\{y^0\} + \operatorname{int} \mathrm{I\!R}^q_+\right) \cap \mathcal{H}(u,c) = \emptyset. \tag{1}$$

The subsequent weakly efficient solutions of (P) or (D) are referred to simply as *solutions* of (P) or (D). The notion of a solution of problem (P) as a feasible point whose image is weakly minimal is common in vector optimization. We adapt this concept for the set-valued dual problem by defining solutions of (D) as feasible points whose image, which is a hyperplane, contains weakly maximal points. Thus the solution concept for the dual problem (D) is different from those in the literature.

In the following we prove weak and strong duality between the two problems directly. In the proofs the following pairs of dual scalar linear optimization problems depending on parameters $c, y \in \mathrm{I\!R}^q$ play an important role.

$$\begin{array}{llll}
(\mathrm{P}_1(c)) & c^T M x \to \min & \text{s.t.} & Ax \geq b, \\
(\mathrm{D}_1(c)) & b^T u \to \max & \text{s.t.} & u \geq 0,\ A^T u = M^T c,
\end{array}$$

$$\begin{array}{llll}
(\mathrm{P}_2(y)) & z \to \min & \text{s.t.} & Ax \geq b,\ Mx - kz \leq y, \\
(\mathrm{D}_2(y)) & b^T u - y^T c \to \max & \text{s.t.} & u, c \geq 0,\ A^T u - M^T c = 0,\ k^T c = 1.
\end{array}$$

The first pair of problems comes from classical linear scalarization and is mainly used for characterizing solutions of (D). The second pair of problems is very useful for characterizing weakly minimal and weakly maximal points in the image space $\mathrm{I\!R}^q$. Similar problems also occur, for instance, in [8]. Note that the problems $(\mathrm{P}_2(y))$ also provide a very common scalarization method in vector optimization, see e.g. [3, 16].

The following notion might also be useful for characterizing solutions of (P) and (D). A pair of points $(x,z) \in \mathrm{I\!R}^n \times \mathrm{I\!R}$ and $(u,c) \in \mathrm{I\!R}^m \times \mathrm{I\!R}^q$ is called *complementary* for the problems $(\mathrm{P}_2(y))$ and $(\mathrm{D}_2(y))$ if $u^T(Ax-b) = 0$ and $c^T(Mx - kz - y) = 0$.

Lemma 1. *If $(x,z) \in \mathrm{I\!R}^n \times \mathrm{I\!R}$ and $(u,c) \in \mathcal{U}$ are complementary points for $(\mathrm{P}_2(y))$ and $(\mathrm{D}_2(y))$ then $z = b^T u - y^T c$.*

Proof. If $(u,c) \in \mathcal{U}$ we have $k^T c = 1$ and $A^T u = M^T c$. Hence $c^T(Mx - kz - y) = 0$ and $u^T(Ax - b) = 0$ imply $z = c^T M x - c^T y = u^T A x - c^T y = u^T b - c^T y$. □

Subsequently, we use the following notation

$$\mathcal{M} := M[\mathcal{X}] + \mathrm{I\!R}^q_+ = \{y \in \mathrm{I\!R}^q \mid \exists x \in \mathcal{X} : Mx \le y\}, \qquad \mathcal{F}(u,c) := \mathcal{H}(u,c) \cap \mathcal{M}.$$

The following lemma can be interpreted as evidence of weak duality. An interpretation of weak duality with the help of set relations is given in the next section.

Lemma 2. *If $(u,c) \in \mathcal{U}$ and $y \in \mathcal{M}$ then $c^T y \ge b^T u$.*

Proof. Since $y \in \mathcal{M}$ there is some $x \in \mathcal{X}$ such that $y \ge Mx$. Hence $(x,0)$ is feasible for $(\mathrm{P}_2(y))$. Duality between $(\mathrm{P}_2(y))$ and $(\mathrm{D}_2(y))$ implies $b^T u - y^T c \le 0$. $\square$

The next lemma states a sufficient optimality condition for (D), which is based on weak duality.

Lemma 3. *If $(u^0,c^0) \in \mathcal{U}$ and $y^0 \in \mathcal{F}(u^0,c^0)$ then $y^0 \in \operatorname{Max}\mathcal{H}[\mathcal{U}]$.*

Proof. Let $(u^0,c^0) \in \mathcal{U}$ and $y^0 \in \mathcal{F}(u^0,c^0)$. Therefore we have $y^0 \in \mathcal{H}[\mathcal{U}]$. We show that $\left(\{y^0\} + \operatorname{int}\mathrm{I\!R}^q_+\right) \cap \mathcal{H}(u,c) = \emptyset$ for all $(u,c) \in \mathcal{U}$. Assume on the contrary that there are $(u,c) \in \mathcal{U}$ and $y \in \mathcal{H}(u,c)$ with $y > y^0$. Since $c \ge 0, c \ne 0$ this implies $c^T y > c^T y^0 \ge b^T u = c^T y$, a contradiction. $\square$

The following theorem provides different characterizations of (weakly efficient) solutions of (D).

Theorem 4. *Let $(u^0,c^0) \in \mathcal{U}$. Then the following statements are equivalent.*

(i) *(u^0,c^0) is a solution of* (D)*,*

(ii) *u^0 solves $(\mathrm{D}_1(c^0))$,*

(iii) *there exists some $x^0 \in \mathcal{X}$ with ${c^0}^T M x^0 = b^T u^0$,*

(iv) *$\mathcal{F}(u^0,c^0)$ is nonempty.*

Proof. (i)⇒(ii). Assume u^0 does not solve $(\mathrm{D}_1(c^0))$. Then there is some $u \in \mathrm{I\!R}^m$ such that $(u,c^0) \in \mathcal{U}$ and $b^T u > b^T u^0$. But for each $y \in \mathcal{H}(u^0,c^0)$ we get

$$y + k(b^T u - b^T u^0) \in \left(\{y\} + \operatorname{int}\mathrm{I\!R}^q_+\right) \cap \mathcal{H}(u,c^0)$$

contradicting (1), i.e., (u^0,c^0) being a solution of (D).

(ii)⇒(iii). If u^0 solves $(\mathrm{D}_1(c^0))$ then by duality between the problems $(\mathrm{P}_1(c^0))$ and $(\mathrm{D}_1(c^0))$ there is some $x^0 \in \mathcal{X}$ such that ${c^0}^T M x^0 = b^T u^0$.

(iii)⇒(iv). If (iii) holds then $Mx^0 \in \mathcal{H}(u^0,c^0)$. Since $Mx^0 \in \mathcal{M}$ we have $Mx^0 \in \mathcal{F}(u^0,c^0)$.

(iv)⇒(i). By Lemma 3. $\square$

We continue with a strong duality theorem in the sense that the set of weakly minimal points for (P) and the set of weakly maximal points for (D) coincide.

Theorem 5. *The following four statements are equivalent.*

(i) *$y^0 \in \operatorname{Min}\mathcal{M}$,*

(ii) *there is some $x^0 \in \mathrm{I\!R}^n$ such that $(x^0,0)$ solves $(\mathrm{P}_2(y^0))$,*

(iii) *there is some $(u^0,c^0) \in \mathcal{U}$ with $b^T u^0 = {y^0}^T c^0$ solving $(\mathrm{D}_2(y^0))$,*

(iv) *$y^0 \in \operatorname{Max}\mathcal{H}[\mathcal{U}]$.*

Proof. (ii)⇒(i). If $(x^0, 0)$ solves $(\mathrm{P}_2(y^0))$ then $x^0 \in \mathcal{X}$ and $Mx^0 \leq y^0$ hence $y^0 \in \mathcal{M}$. Assume that there is some $y \in \mathcal{M}$ (i.e., there is some $x \in \mathcal{X}$ with $Mx \leq y$) with $y < y^0$. Then there is some $z < 0$ such that $y \leq y^0 + kz$. This implies $Mx - kz \leq y - kz \leq y^0$, i.e., (x, z) is feasible for $(\mathrm{P}_2(y^0))$ and $z < 0$ contradicts the optimality of $(x^0, 0)$.

(i)⇒(ii). If $y^0 \in \operatorname{Min} \mathcal{M}$ then there exists some $x^0 \in \mathcal{X}$ with $Mx^0 \leq y^0$, i.e., $(x^0, 0)$ is feasible for $(\mathrm{P}_2(y^0))$. Assume that there is some $(x, z) \in \mathbb{R}^{n+1}$ with $z < 0$ being feasible for $(\mathrm{P}_2(y^0))$. Let $y := y^0 + zk$ then $y < y^0$ and $Mx \leq y^0 + kz = y$, i.e., $y \in \mathcal{M}$ contradicting the weak minimality of y^0.

(ii)⇔(iii). By duality of $(\mathrm{P}_2(y^0))$ and $(\mathrm{D}_2(y^0))$.

(iii)⇔(iv). We have $y^0 \in \operatorname{Max} \mathcal{H}[\mathcal{U}]$ iff

$$y^0 \in \mathcal{H}[\mathcal{U}] \tag{2}$$

and

$$y^0 \notin \mathcal{H}[\mathcal{U}] - \operatorname{int} \mathbb{R}^q_+. \tag{3}$$

Condition (2) is equivalent to

$$\exists (u^0, c^0) \in \mathcal{U} : {y^0}^T c^0 = b^T u^0, \tag{4}$$

and (3) is equivalent to

$$\forall (u, c) \in \mathcal{U} : {y^0}^T c \geq b^T u. \tag{5}$$

Since (iii) is equivalent to (4) together with (5), the statement follows. □

Now we are able to prove the following theorem which provides sufficient conditions for solutions of (P) and (D).

Theorem 6. *Let $(u^0, c^0) \in \mathcal{U}$ and $x^0 \in \mathcal{X}$ be given. Then x^0 is a solution of* (P) *and (u^0, c^0) is a solution of* (D) *if one of the following equivalent conditions is satisfied.*

(i) $b^T u^0 = {c^0}^T M x^0$,

(ii) *u^0 solves $(\mathrm{D}_1(c^0))$ and x^0 solves $(\mathrm{P}_1(c^0))$,*

(iii) *$(x^0, 0)$ solves $(\mathrm{P}_2(Mx^0))$ and (u^0, c^0) solves $(\mathrm{D}_2(Mx^0))$,*

(iv) *for all $y \in \mathbb{R}^q$ there is some $z^0 \in \mathbb{R}$ such that (x^0, z^0) and (u^0, c^0) are complementary points for $(\mathrm{P}_2(y))$ and $(\mathrm{D}_2(y))$.*

Proof. First we show the equivalence of the four conditions.

(i)⇔(ii). By duality between $(\mathrm{P}_1(c^0))$ and $(\mathrm{D}_1(c^0))$.

(i)⇔(iii). By duality between $(\mathrm{P}_2(Mx^0))$ and $(\mathrm{D}_2(Mx^0))$.

(i)⇔(iv). If $(u^0, c^0) \in \mathcal{U}$, $x^0 \in \mathcal{X}$ and

$$z^0 = b^T u^0 - y^T c^0 \tag{6}$$

then we have

$${u^0}^T (Ax^0 - b) = {c^0}^T (Mx^0 - kz^0 - y) = {c^0}^T Mx^0 - b^T u^0. \tag{7}$$

If (i) holds we define z^0 by (6) and then (i) and (7) imply (iv). If (iv) holds then (6) holds by Lemma 1 and then then (iv) and (7) imply (i).

Now, sufficiency of these equivalent conditions for x^0 and (u^0, c^0) being solutions of (P) and (D) follows from Theorem 4 and Theorem 5. □

In the following we prove some statements showing the relationship between proper faces (in particular facets) of $\mathcal{M}$ and solutions of (D). Let us recall some facts concerning the facial structure of polyhedral sets. Let $\mathcal{A} \subseteq \mathbb{R}^q$ be a convex set. A convex subset $\mathcal{F} \subseteq \mathcal{A}$ is called a *face* of $\mathcal{A}$ iff

$$y^1, y^2 \in \mathcal{A}, \quad \lambda \in (0,1), \quad \lambda y^1 + (1-\lambda) y^2 \in \mathcal{F} \quad \Rightarrow \quad y^1, y^2 \in \mathcal{F}.$$

A face $\mathcal{F}$ of $\mathcal{A}$ is called *proper* iff $\emptyset \neq \mathcal{F} \neq \mathcal{A}$. A set $\mathcal{E} \subseteq \mathcal{A}$ is called an *exposed face* of $\mathcal{A}$ iff there are $c \in \mathbb{R}^q$ and $\gamma \in \mathbb{R}$ such that $\mathcal{A} \subseteq \left\{ y \in \mathbb{R}^q \mid c^T y \geq \gamma \right\}$ and $\mathcal{E} = \left\{ y \in \mathbb{R}^q \mid c^T y = \gamma \right\} \cap \mathcal{A}$. The proper $(r-1)$-dimensional faces of an r-dimensional polyhedral set $\mathcal{A}$ are called *facets* of $\mathcal{A}$. A point $y \in \mathcal{A}$ is called a *vertex* of $\mathcal{A}$ iff $\{y\}$ is a face of $\mathcal{A}$.

Theorem 7 ([21], Theorem 3.2.2)**.** *Let $\mathcal{A}$ be a polyhedral set in $\mathbb{R}^q$. Then $\mathcal{A}$ has a finite number of faces, each of which is exposed and a polyhedral set. Every proper face of $\mathcal{A}$ is the intersection of those facets of $\mathcal{A}$ that contain it, and* $\operatorname{rbd} \mathcal{A}$ *(the relative boundary of $\mathcal{A}$) is the union of all the facets of $\mathcal{A}$. If $\mathcal{A}$ has a nonempty face of dimension s, then $\mathcal{A}$ has faces of all dimensions from s to* $\dim \mathcal{A}$.

Remark. If $\mathcal{M} \neq \emptyset$ then $\mathcal{M}$ is a q-dimensional polyhedral set, hence the facets of $\mathcal{M}$ are the $(q-1)$-dimensional faces of $\mathcal{M}$, i.e., the maximal (w.r.t. inclusion) proper faces. A subset $\mathcal{F} \subseteq \mathcal{M}$ is a proper face iff it is a proper exposed face, i.e., iff there is a supporting hyperplane $\mathcal{H}$ to $\mathcal{M}$ such that $\mathcal{F} = \mathcal{H} \cap \mathcal{M}$. We call a hyperplane $\mathcal{H} := \left\{ y \in \mathbb{R}^q \mid c^T y = \gamma \right\}$ (i.e., $c \neq 0$) supporting to $\mathcal{M}$ iff $c^T y \geq \gamma$ for all $y \in \mathcal{M}$ and there is some $y^0 \in \mathcal{M}$ such that $c^T y^0 = \gamma$.

Lemma 8. *If $\mathcal{H} = \left\{ y \in \mathbb{R}^q \mid c^T y = \gamma \right\}$ is a supporting hyperplane to $\mathcal{M}$ then $c \geq 0$.*

Proof. If $\mathcal{H}$ is a supporting hyperplane to $\mathcal{M}$ then there is some $y^0 \in \mathcal{M}$ with $c^T y^0 = \gamma$ and $c^T y \geq \gamma$ for all $y \in \mathcal{M}$. By definition of $\mathcal{M}$ we have $y^0 + w \in \mathcal{M}$, for all $w \in \mathbb{R}^q_+$, hence $c^T w \geq 0$ for all $w \in \mathbb{R}^q_+$. This implies $c \geq 0$. ☐

Lemma 9. *A set $\mathcal{F} \subseteq \mathcal{M}$ is a proper face of $\mathcal{M}$ if and only if there is a solution $(u,c) \in \mathcal{U}$ of* (D) *such that $\mathcal{F} = \mathcal{F}(u,c)$.*

Proof. "if". If $(u,c) \in \mathcal{U}$ is a solution of (D) then there is some $x^0 \in \mathcal{X}$ such that $Mx^0 \in \mathcal{H}(u,c)$, hence $Mx^0 \in \mathcal{F}(u,c)$. Moreover, if $y \in \mathcal{M}$ then $c^T y \geq b^T u$ by Lemma 2. Consequently, $\mathcal{H}(u,c)$ is a supporting hyperplane to $\mathcal{M}$ and $\mathcal{F}(u,c)$ is a proper face of $\mathcal{M}$.

"only if". If $\mathcal{F}$ is a proper face of $\mathcal{M}$ then there is some $c \in \mathbb{R}^q \setminus \{0\}, \gamma \in \mathbb{R}$ such that $\mathcal{H} := \left\{ y \in \mathbb{R}^q \mid c^T y = \gamma \right\}$ is a supporting hyperplane to $\mathcal{M}$ and $\mathcal{F} = \mathcal{H} \cap \mathcal{M}$. By Lemma 8 we have $c \geq 0$. Since $c \neq 0$ we obtain $k^T c > 0$. Without loss of generality we can assume that $k^T c = 1$. Since $\mathcal{H}$ is a supporting hyperplane, we have $c^T y \geq \gamma$ for all $y \in \mathcal{M}$ and $c^T y^0 = \gamma$ for some $y^0 \in \mathcal{M}$. Hence there is some $x^0 \in \mathcal{X}$ such that $c^T M x^0 = c^T y^0 = \gamma$, i.e.,

$$\gamma = c^T M x^0 = \min \left\{ c^T M x : x \in \mathcal{X} \right\}.$$

By duality between $(\mathrm{P}_1(c))$ and $(\mathrm{D}_1(c))$, problem $(\mathrm{D}_1(c))$ has a solution u with $b^T u = \gamma = c^T M x^0$. Thus $(u,c) \in \mathcal{U}$ is a solution of (D) by Theorem 4, and $\mathcal{H}(u,c) = \mathcal{H}$. Hence $\mathcal{F} = \mathcal{F}(u,c)$. ☐

Corollary 10. *Each proper face of $\mathcal{M}$ is weakly minimal.*

Proof. Let $\mathcal{F}$ be a proper face of $\mathcal{M}$. By the preceding lemma there is a solution $(u,c) \in \mathcal{U}$ of (D) such that $\mathcal{F} = \mathcal{F}(u,c)$. Let $y \in \mathcal{F} = \mathcal{F}(u,c)$, then $y \in \mathcal{M}$ (implying the existence of $x \in \mathcal{X}$ such that $Mx \leq y$, i.e., $(x,0)$ is feasible for $(\mathrm{P}_2(y))$) and $b^T u = c^T y$. Duality between $(\mathrm{P}_2(y))$ and $(\mathrm{D}_2(y))$ implies that (u,c) is optimal in $(\mathrm{D}_2(y))$ and $(x,0)$ is optimal in $(\mathrm{P}_2(y))$ hence y is weakly minimal by Theorem 5. ☐

Corollary 11. $\operatorname{Min} \mathcal{M} \neq \emptyset$ *if and only if* $\emptyset \neq \mathcal{M} \neq \mathrm{I\!R}^q$.

Proof. This is a direct consequence of Corollary 10, Theorem 7 and the fact that a nonempty set in $\mathcal{A} \subseteq \mathrm{I\!R}^q$ has a nonempty boundary iff $\mathcal{A} \neq \mathrm{I\!R}^q$. $\square$

The following lemma shows that facets of $\mathcal{M}$ may be described by extreme solutions of (D) (i.e. solutions of (D) being a vertex of the feasible set $\mathcal{U}$).

Lemma 12. *If* $\mathcal{F}$ *is a facet of* $\mathcal{M}$ *then there is an extreme solution* (u^0, c^0) *of* (D) *such that* $\mathcal{F} = \mathcal{F}(u^0, c^0)$.

Proof. Let

$$\bar{\mathcal{U}} := \{(u,c) \in \mathcal{U} \mid \mathcal{F}(u,c) = \mathcal{F}\}.$$

By Theorem 4, all points of $\bar{\mathcal{U}}$ are solutions of (D) because $\mathcal{F}$ is nonempty as a facet of $\mathcal{M}$. Let $y \in \operatorname{ri} \mathcal{F}$ be arbitrary. Since $\mathcal{F}$ is a $(q-1)$-dimensional face we have $(u,c) \in \bar{\mathcal{U}}$ if and only if $(u,c) \in \mathcal{U}$ and $y \in \mathcal{H}(u,c)$, i.e., $b^T u = y^T c$. Hence $\bar{\mathcal{U}} = \mathcal{U} \cap \mathcal{H}_y$ with

$$\mathcal{H}_y := \left\{(u,c) \in \mathrm{I\!R}^m \times \mathrm{I\!R}^q \mid y^T c - b^T u = 0\right\}.$$

Since $y \in \operatorname{Min} \mathcal{M}$ by Corollary 10, Theorem 5 implies that $\mathcal{H}_y$ is a supporting hyperplane to $\mathcal{U}$, hence $\bar{\mathcal{U}}$ is a nonempty face of $\mathcal{U}$. Since $\bar{\mathcal{U}} \subseteq \mathcal{U} \subseteq \mathrm{I\!R}^{m+q}_+$ contains no lines there is a vertex (u^0, c^0) of $\bar{\mathcal{U}}$ (see [17, Cor. 18.5.3]). Hence (u^0, c^0) is also a vertex of $\mathcal{U}$, i.e. an extreme solution of (D). $\square$

We define the following sets.

$$\begin{aligned}
\operatorname{pFaces}(\mathcal{M}) &:= \{\mathcal{F} \subseteq \mathcal{M} \mid \mathcal{F} \text{ is a proper face of } \mathcal{M}\}, \\
\operatorname{Facets}(\mathcal{M}) &:= \{\mathcal{F} \subseteq \mathcal{M} \mid \mathcal{F} \text{ is a facet of } \mathcal{M}\}, \\
\operatorname{Sol}(\mathrm{D}) &:= \{(u,c) \in \mathcal{U} \mid (u,c) \text{ is a solution of (D)}\}, \\
\operatorname{ExtrSol}(\mathrm{D}) &:= \{(u,c) \in \operatorname{Sol}(\mathrm{D}) \mid (u,c) \text{ is a vertex of } \mathcal{U}\}.
\end{aligned}$$

Now we can extend the strong duality result in Theorem 5. In the next section we interprete the following result as the attainment of the supremum in the dual problem in extreme solutions.

Theorem 13. *We have the following chain of equalities.*

$$\operatorname{Min} \mathcal{M} = \operatorname{bd} \mathcal{M} = \bigcup_{(u,c) \in \operatorname{ExtrSol}(\mathrm{D})} \mathcal{F}(u,c) = \operatorname{Max} \mathcal{H}[\operatorname{ExtrSol}(\mathrm{D})] = \operatorname{Max} \mathcal{H}[\mathcal{U}].$$

Proof. Theorem 7, Lemma 12, Lemma 9 and Corollary 10 imply the following chain of inclusions

$$\begin{aligned}
\operatorname{bd} \mathcal{M} = \bigcup_{\mathcal{F} \in \operatorname{Facets}(\mathcal{M})} \mathcal{F} \subseteq \bigcup_{(u,c) \in \operatorname{ExtrSol}(\mathrm{D})} \mathcal{F}(u,c) \subseteq \bigcup_{(u,c) \in \operatorname{Sol}(\mathrm{D})} \mathcal{F}(u,c) \\
= \bigcup_{\mathcal{F} \in \operatorname{pFaces}(\mathcal{M})} \mathcal{F} \subseteq \operatorname{Min} \mathcal{M} \subseteq \operatorname{bd} \mathcal{M}.
\end{aligned}$$

Hence the first two equalities hold.

The equality $\operatorname{Min} \mathcal{M} = \operatorname{Max} \mathcal{H}[\mathcal{U}]$ was already shown in Theorem 5. Thus it remains to show that $\bigcup_{(u,c) \in \operatorname{ExtrSol}(\mathrm{D})} \mathcal{F}(u,c) = \operatorname{Max} \mathcal{H}[\operatorname{ExtrSol}(\mathrm{D})]$.

If $y \in \bigcup_{(u,c) \in \operatorname{ExtrSol}(\mathrm{D})} \mathcal{F}(u,c)$ then there exists some $(u,c) \in \operatorname{ExtrSol}(\mathrm{D})$ such that $y \in \mathcal{F}(u,c) = \mathcal{H}(u,c) \cap \mathcal{M}$, i.e. $y \in \mathcal{H}[\operatorname{ExtrSol}(\mathrm{D})]$. Since (u,c) is a solution of (D) we have $(y + \operatorname{int} \mathrm{I\!R}^q_+) \cap \mathcal{H}[\mathcal{U}] = \emptyset$, hence $(y + \operatorname{int} \mathrm{I\!R}^q_+) \cap \mathcal{H}[\operatorname{ExtrSol}(\mathrm{D})] = \emptyset$ implying $y \in \operatorname{Max} \mathcal{H}[\operatorname{ExtrSol}(\mathrm{D})]$.

On the other hand, if $y \in \operatorname{Max} \mathcal{H}[\operatorname{ExtrSol}(\mathrm{D})]$ then $y \in \mathcal{H}[\operatorname{ExtrSol}(\mathrm{D})]$ and $y \notin \mathcal{H}[\operatorname{ExtrSol}(\mathrm{D})] - \operatorname{int} \mathrm{I\!R}^q_+$. This is equivalent to

$$\exists (\bar{u}, \bar{c}) \in \operatorname{ExtrSol}(\mathrm{D}) : y^T \bar{c} = b^T \bar{u} \tag{8}$$

and

$$\forall (u,c) \in \mathrm{ExtrSol}(\mathrm{D}) : y^T c \geq b^T u. \tag{9}$$

By Theorem 4, $\bar{u}$ solves $(\mathrm{D}_1(\bar{c}))$ hence $\mathcal{X} \neq \emptyset$ by duality of $(\mathrm{P}_1(\bar{c}))$ and $(\mathrm{D}_1(\bar{c}))$.Thus the feasible set for $(\mathrm{P}_2(y))$ is nonempty as well. Since $(\bar{u},\bar{c}) \in \mathcal{U}$, i.e. $\mathcal{U} \neq \emptyset$, problem $(\mathrm{D}_2(y))$ has an optimal solution (u^0, c^0) being a vertex of $\mathcal{U}$. Optimality of (u^0, c^0) for $(\mathrm{D}_2(y))$ implies optimality of u^0 for $(\mathrm{D}_1(c^0))$ hence $(u^0, c^0) \in \mathrm{ExtrSol}(\mathrm{D})$ by Theorem 4. Now, (9) implies that $y^T c^0 \geq b^T u^0$. Moreover, optimality of (u^0, c^0) for $(\mathrm{D}_2(y))$ implies $b^T u^0 - y^T c^0 \geq b^T \bar{u} - y^T \bar{c} = 0$, i.e. $y^T c^0 = b^T u^0$. Consequently we have $y \in \mathcal{H}(u^0, c^0)$ and $y \in \mathrm{Min}\,\mathcal{M} \subseteq \mathcal{M}$ by Theorem 5, i.e. $y \in \bigcup_{(u,c)\in \mathrm{ExtrSol}(\mathrm{D})} \mathcal{F}(u,c)$. □

3 Lattice theoretical interpretation

In this section, we discuss the theoretical background of the duality assertions developed in the previous section. On the one hand, this provides a motivation of the solution concepts for the dual problem introduced above, which differs from those in the literature. On the other hand we see that vector optimization and scalar optimization can be considered in a common framework, i.e., duality assertions for vector optimization problems can be expressed in the same way as the corresponding scalar results.

First we embed the image space $\mathbb{R}^q$ of the given vector-valued objective function in a complete lattice. The appropriate lattice is introduced in the first subsection. Then, we can reformulate our pair of dual problems in terms of this lattice. Finally we obtain duality and dual attainment assertions being analogous to the classical scalar results.

3.1 The space $\mathcal{I}$ of self-infimal sets

Let us recall some facts about self-infimal sets. For a more detailed discussion the reader is referred to [14]. The *infimal set* of a subset A of $\overline{\mathbb{R}}^q := \mathbb{R}^q \cup \{-\infty, +\infty\}$ is defined by

$$\mathrm{Inf}\, A := \begin{cases} \{-\infty\} & \text{if} \quad -\infty \in A \quad \text{or} \quad A + \mathbb{R}^q_+ \supseteq \mathbb{R}^q \\ \{+\infty\} & \text{if} \quad A \subseteq \{+\infty\} \\ \mathrm{Min}\,\mathrm{cl}\left((A \setminus \{+\infty\}) + \mathbb{R}^q_+\right) & \text{otherwise} \end{cases}$$

Note that the closure operation is only necessary for the case that $(A \setminus \{+\infty\})$ is not polyhedral. The supremal set of a set $A \subseteq \overline{\mathbb{R}}^q$ is defined analogously and is denoted by $\mathrm{Sup}\, A$. It holds $\mathrm{Sup}\, A = -\,\mathrm{Inf}(-A)$.

Let $\mathcal{I}$ be the family of all self-infimal subsets of $\overline{\mathbb{R}}^q$, i.e., all sets $A \subseteq \overline{\mathbb{R}}^q$ satisfying $\mathrm{Inf}\, A = A$. In $\mathcal{I}$ we introduce an order relation $\preccurlyeq$ as follows:

$$A \preccurlyeq B :\Longleftrightarrow \begin{cases} \left(A, B \subseteq \mathbb{R}^q \;\text{ and }\; A + \mathrm{int}\,\mathbb{R}^q_+ \supseteq B + \mathrm{int}\,\mathbb{R}^q_+\right) & \text{or} \\ A = \{-\infty\} & \text{or} \\ B = \{+\infty\}. \end{cases}$$

As shown in [14, Proposition 3.4 and Theorem 3.5], $(\mathcal{I}, \preccurlyeq)$ is a complete lattice and for arbitrary sets $\mathcal{A} \subseteq \mathcal{I}$ it holds that

$$\inf \mathcal{A} = \mathrm{Inf} \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \mathrm{Sup} \bigcup_{A \in \mathcal{A}} A.$$

Note that we use $\bigcup_{A \in \emptyset} A = \emptyset$. The preceding result shows that the infimum and supremum in $\mathcal{I}$ are closely related to the usual solution concepts in vector optimization.

3.2 Reformulation of the problems using the space $\mathcal{I}$

In Section 2 we considered the linear vector optimization problem (P). It is easy to see that $\operatorname{Min}(M[\mathcal{X}] + \mathrm{IR}^q_+) = \operatorname{Inf} M[\mathcal{X}]$ iff $\mathcal{X} \neq \emptyset$ and $M[\mathcal{X}] + \mathrm{IR}^q_+ \neq \mathrm{IR}^q$. Our aim is to reformulate problem (P) and its dual problem (D) as optimization problems with $\mathcal{I}$-valued objective function. Consider the function

$$P : \mathrm{IR}^n \to \mathcal{I}, \qquad P(x) := \operatorname{Inf}\{Mx\} = \{Mx\} + \operatorname{bd} \mathrm{IR}^q_+.$$

It holds

$$\operatorname{Inf} M[\mathcal{X}] = \operatorname{Inf} \bigcup_{x\in\mathcal{X}} \{Mx\} = \operatorname{Inf} \bigcup_{x\in\mathcal{X}} \operatorname{Inf}\{Mx\} = \inf_{x\in\mathcal{X}} P(x).$$

Hence, we have

$$\inf_{x\in\mathcal{X}} P(x) = \begin{cases} \{+\infty\} & \text{if} \quad \mathcal{X} = \emptyset \\ \{-\infty\} & \text{if} \quad \mathcal{M} = \mathrm{IR}^q \\ \operatorname{Min}\mathcal{M} & \text{otherwise.} \end{cases}$$

Note that, by Corollary 11, $\operatorname{Min}\mathcal{M} \neq \emptyset$ iff $\emptyset \neq \mathcal{M} \neq \mathrm{IR}^q$. This means, if the set $\operatorname{Min}\mathcal{M}$ is nonempty, it coincides with $\inf_{x\in\mathcal{X}} P(x)$, otherwise if $\operatorname{Min}\mathcal{M}$ is empty, we distinguish between two cases: $\inf_{x\in\mathcal{X}} P(x) = \{+\infty\}$ if $\mathcal{X} = \emptyset$ and $\inf_{x\in\mathcal{X}} P(x) = \{+\infty\}$ otherwise. Thus, (P) is essentially equivalent to

$$(\mathrm{P}') \qquad \inf_{x\in\mathcal{X}} P(x), \qquad \mathcal{X} := \{x \in \mathrm{IR}^n \mid Ax \geq b\}.$$

Moreover, it is easy to see that $x \in \mathcal{X}$ is a (weakly efficient) solution of (P) if and only if

$$\left(x \in \mathcal{X},\ P(x) \preccurlyeq P(x^0)\right) \quad \Rightarrow \quad P(x) = P(x^0) \tag{10}$$

The above considerations show the relationships between the concepts used in the previous section and lattice theoretical solution concepts for the primal problem.

We next want to reformulate the dual problem (D) using the supremum in $\mathcal{I}$. We first consider two auxiliary assertions.

Lemma 14. *The set $\mathcal{H}[\mathcal{U}] - \mathrm{IR}^q_+$ is closed.*

Proof. Let $\{y_i\}_{i\in\mathrm{IN}}$ be a sequence in $\mathcal{H}[\mathcal{U}] - \mathrm{IR}^q_+$ converging to $\bar{y} \in \mathrm{IR}^q$, thus for each i there is some $(u_i, c_i) \in \mathcal{U}$ with $y_i \in \mathcal{H}(u_i, c_i) - \mathrm{IR}^q_+$, i.e., $y_i^T c_i \leq b^T u_i$. We have to show that there is some $(\bar{u}, \bar{c}) \in \mathcal{U}$ with $\bar{y}^T \bar{c} \leq b^T \bar{u}$.

Assume on the contrary that $\bar{y}^T c - b^T u > 0$ for all $(u, c) \in \mathcal{U}$. Since $\mathcal{U}$ is polyhedral there is some $\gamma > 0$ with $\bar{y}^T c - b^T u \geq \gamma$ for all $(u, c) \in \mathcal{U}$. Take $i_0 \in \mathrm{IN}$ such that $\|y_{i_0} - \bar{y}\|_\infty < \gamma$, then

$$(\bar{y} - y_{i_0})^T c_{i_0} \leq \|y_{i_0} - \bar{y}\|_\infty \|c_{i_0}\|_1 < \gamma$$

hence

$$\bar{y}^T c_{i_0} - b^T u_{i_0} < y_{i_0}^T c_{i_0} + \gamma - b^T u_{i_0} \leq \gamma,$$

a contradiction. □

Lemma 15. *It holds* $\operatorname{Max}\left(\mathcal{H}[\mathcal{U}] - \mathrm{IR}^q_+\right) = \operatorname{Max}\mathcal{H}[\mathcal{U}]$.

Proof. We have

$$y \in \operatorname{Max}\mathcal{H}[\mathcal{U}] \quad \Longleftrightarrow \quad \left(y \in \mathcal{H}[\mathcal{U}] \text{ and } y \notin \mathcal{H}[\mathcal{U}] - \operatorname{int}\mathrm{IR}^q_+\right)$$

and

$$y \in \operatorname{Max}\left(\mathcal{H}[\mathcal{U}] - \mathrm{IR}^q_+\right) \quad \Longleftrightarrow \quad \left(y \in \mathcal{H}[\mathcal{U}] - \mathrm{IR}^q_+ \text{ and } y \notin \mathcal{H}[\mathcal{U}] - \operatorname{int}\mathrm{IR}^q_+\right).$$

Thus it remains to show that

$$\left(y \in \mathcal{H}[\mathcal{U}] - \mathbb{R}^q_+ \text{ and } y \notin \mathcal{H}[\mathcal{U}] - \operatorname{int} \mathbb{R}^q_+\right) \quad \Longrightarrow \quad y \in \mathcal{H}[\mathcal{U}].$$

Indeed, $y \notin \mathcal{H}[\mathcal{U}] - \operatorname{int} \mathbb{R}^q_+$ implies $y^T c \geq b^T u$ for all $(u,c) \in \mathcal{U}$ and $y \in \mathcal{H}[\mathcal{U}] - \mathbb{R}^q_+$ implies the existence of some $(\bar{u}, \bar{c}) \in \mathcal{U}$ with $y^T \bar{c} \leq b^T \bar{u}$. Thus we obtain $y^T \bar{c} = b^T \bar{u}$, i.e., $y \in \mathcal{H}[\mathcal{U}]$. □

Note that the hyperplane $\mathcal{H}(u,c) \subseteq \mathbb{R}^q$ is a self-infimal set, whenever $(u,c) \in \mathcal{U}$. Therefore the term $\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c)$ is well defined. The next lemma clarifies the relationship between this supremum and the solution concept of problem (D).

Lemma 16. *It holds*

$$\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c) = \begin{cases} \{-\infty\} & \text{if} \quad \mathcal{U} = \emptyset \\ \{+\infty\} & \text{if} \quad \mathcal{H}[\mathcal{U}] - \mathbb{R}^q_+ = \mathbb{R}^q \\ \operatorname{Max} \mathcal{H}[\mathcal{U}] & \text{otherwise.} \end{cases}$$

Proof. (i) If $\mathcal{U} = \emptyset$, we have $\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c) = \operatorname{Sup} \mathcal{H}[\mathcal{U}] = \operatorname{Sup} \emptyset = \{-\infty\}$, by definition.

(ii) The case $\mathcal{H}[\mathcal{U}] - \mathbb{R}^q_+ = \mathbb{R}^q$ follows from the definition of the supremal set.

(iii) Since $\mathcal{H}[\mathcal{U}] \subseteq \mathbb{R}^q$, we have

$$\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c) = \operatorname{Sup} \mathcal{H}[\mathcal{U}] = \operatorname{Max} \operatorname{cl} (\mathcal{H}[\mathcal{U}] - \mathbb{R}^q_+),$$

by the definition of the supremal set. Lemma 14 and Lemma 15 yield that $\operatorname{Max} \operatorname{cl} (\mathcal{H}[\mathcal{U}] - \mathbb{R}^q_+) = \operatorname{Max} \mathcal{H}[\mathcal{U}]$. □

Remark. The preceding three lemmas remain valid if the set $\mathcal{U}$ is replaced by any finite or polyhedral subset.

Lemma 16 shows in fact the relationship between problem (D) and the following problem,

$$(\mathrm{D}') \qquad \sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c), \quad \mathcal{U} := \left\{(u,c) \in \mathbb{R}^m \times \mathbb{R}^q \mid (u,c) \geq 0,\ k^T c = 1,\ A^T u = M^T c\right\}.$$

Indeed, if the set $\operatorname{Max} \mathcal{H}[\mathcal{U}]$ is nonempty, it coincides with $\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c)$ in problem (D′). Otherwise, if $\operatorname{Max} \mathcal{H}[\mathcal{U}]$ is empty, we distinguish between the following two cases:

$$\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c) = \begin{cases} \{-\infty\} & \text{when } \mathcal{U} = \emptyset \\ \{+\infty\} & \text{otherwise.} \end{cases}$$

The solution concept for (D) as introduced in Section 2 can be expressed in terms of the ordering relation in the complete lattice $\mathcal{I}$. This characterization is completely analogous to (10). So we obtain yet another motivation for this solution concept.

Lemma 17. *A point* $(u^0, c^0) \in \mathcal{U}$ *is a (weakly efficient) solution of* (D) *if and only if*

$$\Big((u,c) \in \mathcal{U},\ \mathcal{H}(u^0,c^0) \preccurlyeq \mathcal{H}(u,c)\Big) \quad \Rightarrow \quad \mathcal{H}(u^0,c^0) = \mathcal{H}(u,c). \tag{11}$$

Proof. Let $(u^0, c^0) \in \mathcal{U}$ be a solution of (D). Hence u^0 solves $(\mathrm{D}_1(c^0))$ by Theorem 4. Consider $(u,c) \in \mathcal{U}$ with $\mathcal{H}(u^0,c^0) \preccurlyeq \mathcal{H}(u,c)$. Then we have $c^0 = c$ and $b^T u^0 \leq b^T u$. Since $c^0 = c$, u is feasible for $(\mathrm{D}_1(c^0))$ hence $b^T u \leq b^T u^0$ and consequently $b^T u^0 = b^T u$. This means we have $\mathcal{H}(u^0,c^0) = \mathcal{H}(u,c)$.

Let $(u^0,c^0) \in \mathcal{U}$ be no solution of (D). By Theorem 4 there exists some $\bar{u} \geq 0$ with $A^T \bar{u} = M^T c_0$ and $b^T \bar{u} > b^T u^0$. Hence, we have $\mathcal{H}(u^0,c^0) \preccurlyeq \mathcal{H}(\bar{u},c^0)$ but $\mathcal{H}(u^0,c^0) \neq \mathcal{H}(\bar{u},c^0)$, i.e., (11) is not satisfied. □

3.3 Duality and dual attainment

As a consequence of the duality assertion given in Section 2 and the above considerations, we present here duality assertions for vector optimization problems, formulated along the lines of the classical scalar duality theory. The complete lattice $(\mathcal{I}, \preccurlyeq)$ of self-infimal subsets of $\overline{\mathrm{IR}}^q$ plays a key role in these results.

The first result shows that we have weak duality between (P′) and (D′).

Theorem 18 (weak duality)**.** *Let $x \in \mathcal{X}$ and $(u, c) \in \mathcal{U}$. Then it holds*

$$\mathcal{H}(u, c) \preccurlyeq P(x).$$

Proof. For all $y \in P(x) = \{()\, Mx) + \operatorname{bd} \mathrm{IR}^q_+ \subseteq \mathcal{M}$, Lemma 2 yields $y^T c \geq b^T u$, hence $P(x) \subseteq \mathcal{H}(u, c) + \mathrm{IR}^q_+$. This implies $\mathcal{H}(u, c) \preccurlyeq P(x)$. □

The next result shows strong duality between (P′) and (D′). The following distinction between the three cases is well-known from scalar linear programming.

Theorem 19 (strong duality)**.** *Let at least one of the sets $\mathcal{X}$ and $\mathcal{U}$ be nonempty. Then it holds strong duality between* (P′) *and* (D′)*, i.e.,*

$$V := \sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u, c) = \inf_{x\in\mathcal{X}} P(x).$$

Moreover, the following statements are true.

(i) *If $\mathcal{X} \neq \emptyset$ and $\mathcal{U} \neq \emptyset$, then $\{-\infty\} \neq V \neq \{+\infty\}$ and*

$$V = \operatorname{Max} \mathcal{H}[\mathcal{U}] = \operatorname{Min} P[\mathcal{X}] \neq \emptyset.$$

(ii) *If $\mathcal{X} = \emptyset$ and $\mathcal{U} \neq \emptyset$, then $V = \{+\infty\}$.*

(iii) *If $\mathcal{X} \neq \emptyset$ and $\mathcal{U} = \emptyset$, then $V = \{-\infty\}$.*

Proof. By the weak duality we have

$$\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u, c) \preccurlyeq \inf_{x\in\mathcal{X}} P(x).$$

(i) If $\mathcal{X} \neq \emptyset$ and $\mathcal{U} \neq \emptyset$, this implies that neither $\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u, c)$ nor $\inf_{x\in\mathcal{X}} P(x)$ can be $\{-\infty\}$ or $\{+\infty\}$. Hence, Theorem 5 implies

$$\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u, c) = \operatorname{Max} \mathcal{H}[\mathcal{U}] = \operatorname{Min} \mathcal{M} = \inf_{x\in\mathcal{X}} P(x).$$

(ii) If $\mathcal{X} = \emptyset$ and $\mathcal{U} \neq \emptyset$, we have $\inf_{x\in\mathcal{X}} P(x) = \{+\infty\}$. Theorem 5 implies that

$$\operatorname{Max} \mathcal{H}[\mathcal{U}] = \operatorname{Min} \mathcal{M} = \emptyset.$$

Since $\mathcal{U} \neq \emptyset$, we conclude $\mathcal{H}[\mathcal{U}] - \mathrm{IR}^q_+ = \mathrm{IR}^q$ and Lemma 16 yields $\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u, c) = \{+\infty\}$.

(iii) If $\mathcal{X} \neq \emptyset$ and $\mathcal{U} = \emptyset$, we have $\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u, c) = \{-\infty\}$. Theorem 5 implies that

$$\operatorname{Min} \mathcal{M} = \operatorname{Max} \mathcal{H}[\mathcal{U}] = \emptyset.$$

Since $\mathcal{X} \neq \emptyset$, we obtain $\mathcal{M} = \mathrm{IR}^q$, hence $\inf_{x\in\mathcal{X}} P(x) = \{-\infty\}$. □

In scalar linear programming, the attainment of the supremum of the problem in a vertex of the feasible set plays a key role in the simplex algorithm. It is therefore sufficient to search for a solution on a finite subset of the feasible set. The next result shows that we have a corresponding result for our dual problem. Typically, in our case, the supremum in (D′) is not attained in a single vertex, but in a finite number of vertices, namely, in the set of those vertices of $\mathcal{U}$ being solutions of (D), i.e., the set ExtrSol(D) of extreme solutions of (D).

Theorem 20 (dual attainment in vertices). *Let $\mathcal{X} \neq \emptyset$ and $\mathcal{U} \neq \emptyset$. Then the supremum in the dual problem* (D′) *is attained in extreme solutions of* (D)*, i.e.,*

$$\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c) = \sup_{(u,c)\in\mathrm{ExtrSol(D)}} \mathcal{H}(u,c).$$

Proof. Since $\mathcal{U} \neq \emptyset$ and $\mathcal{X} \neq \emptyset$ we have

$$\sup_{(u,c)\in\mathcal{U}} \mathcal{H}(u,c) = \mathrm{Max}\,\mathcal{H}[\mathcal{U}]$$

by Theorem 19. $\mathrm{Max}\,\mathcal{H}[\mathcal{U}] = \mathrm{Max}\,\mathcal{H}[\mathrm{ExtrSol(D)}]$ follows from Theorem 13.

It remains to show that

$$\mathrm{Max}\,\mathcal{H}[\mathrm{ExtrSol(D)}] = \sup_{(u,c)\in\mathrm{ExtrSol(D)}} \mathcal{H}(u,c).$$

If $\mathcal{X} \neq \emptyset$ and $\mathcal{U} \neq \emptyset$ then we conclude from Theorem 19 and Corollary 11 that $\emptyset \neq \mathcal{M} \neq \mathrm{I\!R}^q$. Thus $\mathcal{M}$ has a facet and consequently $\mathrm{ExtrSol(D)} \neq \emptyset$ by Lemma 12. Moreover, $\mathcal{H}[\mathrm{ExtrSol(D)}] - \mathrm{I\!R}^q_+ \subseteq \mathcal{H}[\mathcal{U}] - \mathrm{I\!R}^q_+ \neq \mathrm{I\!R}^q$. Hence the desired statement follows from the remark after Lemma 16. □

4 An example from Mathematical Finance

We consider a Markowitz-type bicriterial portfolio optimization problem, where the expected return of the portfolio should be maximized and the risk of the portfolio, measured by the Conditional Value at Risk, should be minimized. For details about the Conditional Value at Risk (sometimes also called Average Value at Risk) see e.g. [19] or [2, Section 4.4].

We consider a market with n different financial instruments with returns $r_j, j = 1, ..., n$ being random variables combined in a random vector $r = (r_1, ..., r_n)^T$ with a given probability distribution P. The decision vector $x \in \mathrm{I\!R}^n$ represents a portfolio of these instruments, where the components x_j denote the fraction of the capital invested in instrument j. This yields the constraints

$$x \geq 0, \quad \sum_{j=1}^{n} x_j = 1.$$

The return of a portfolio x equals $r^T x$ so the bicriterial optimization problem consists in minimizing the negative expected return, i.e., $-E(r^T x)$ and the Conditional Value at Risk of the return, i.e., $CVaR_\beta(r^T x)$, for a given risk level $\beta \in [0,1)$. We can approximate this problem by a linear one by sampling the probability distribution of r like it is done in [18]. If $r^1, ..., r^m$ denotes a sample of size m then

$$E(r^T x) \approx \frac{1}{m} \sum_{k=1}^{m} {r^k}^T x$$

and

$$CVaR_\beta(r^T x) \approx \inf \left\{ \alpha + \frac{1}{(1-\beta)m} \sum_{k=1}^{m} z_k \mid \alpha \in \mathrm{I\!R}, \forall k \in \{1, ..., m\} : z_k \in \mathrm{I\!R}_+, {r^k}^T x + \alpha + z_k \geq 0 \right\}.$$

Then the given problem accords essentially with the following linear vector optimization problem:

$$(\mathrm{P_M}) \qquad \mathrm{Min}\left(f[\mathcal{X}] + \mathrm{I\!R}^2_+\right),$$

where

$$\mathcal{X} := \left\{ (x, z, \alpha) \in \mathrm{I\!R}^n_+ \times \mathrm{I\!R}^m_+ \times \mathrm{I\!R} \mid \sum_{j=1}^{n} x_j = 1, \forall k \in \{1, ..., m\} : {r^k}^T x + \alpha + z_k \geq 0 \right\}$$

and

$$f(x,z,\alpha) = \begin{pmatrix} -\dfrac{1}{m}\displaystyle\sum_{k=1}^{m} {r^k}^T x \\ \alpha + \dfrac{1}{(1-\beta)m}\displaystyle\sum_{k=1}^{m} z_k \end{pmatrix}.$$

As already noted in Section 2, finding solutions of $(\mathrm{P_M})$ is equivalent to finding weakly efficient solutions of the problem $\operatorname{Min} f[\mathcal{X}]$.

We set

$$M := \begin{pmatrix} -\frac{1}{m}\mathbf{1}_m^T R^T & 0 & 0 \\ 0 & \frac{1}{(1-\beta)m}\mathbf{1}_m^T & 1 \end{pmatrix}, \qquad A := \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ \mathbf{1}_n^T & 0 & 0 \\ -\mathbf{1}_n^T & 0 & 0 \\ R^T & I_m & \mathbf{1}_m \end{pmatrix}, \qquad b := \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

where

$$R := \begin{pmatrix} r_1^1 & \cdots & r_1^m \\ \vdots & \ddots & \vdots \\ r_n^1 & \cdots & r_n^m \end{pmatrix},$$

I_ℓ is the ℓ-dimensional identity matrix and $\mathbf{1}_\ell$ is the ℓ-dimensional vector with all components being 1. Then the problem $(\mathrm{P_M})$ is equivalent to

$$\operatorname{Min}(M[\mathcal{X}] + \mathbb{R}^q_+), \qquad \mathcal{X} := \left\{\bar{x} \in \mathbb{R}^{n+m+1} \mid A\bar{x} \geq b\right\},$$

a problem of type (P).

As the corresponding dual problem to $(\mathrm{P_M})$ we derive the following problem as a special case of problem (D):

$$(\mathrm{D_M}) \qquad \operatorname{Max} \mathcal{H}[\mathcal{U}], \quad \mathcal{U} = \left\{(\bar{u}, c) \in \mathbb{R}^{n+2m+2}_+ \times \mathbb{R}^2_+ \mid \ c_1 + c_2 = 1,\ A^T u = M^T c\right\}$$

In fact, we have

$$\mathcal{U} = \Bigg\{(w,p,v_1,v_2,u,c) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^m_+ \times \mathbb{R}^2_+ \mid \ c_1 + c_2 = 1,$$
$$w + \mathbf{1}_n(v_1 - v_2) + Ru = R\mathbf{1}_m \frac{-c_1}{m}, \quad p + u = \mathbf{1}_m \frac{c_2}{(1-\beta)m}, \quad \mathbf{1}_m^T u = c_2 \Bigg\}$$

and the set-valued objective map is given

$$\mathcal{H}(w,p,v_1,v_2,u,c) = \left\{y \in \mathbb{R}^2 \mid c_1 y_1 + c_2 y_2 = v_1 - v_2\right\}.$$

Interpreting w and p as slack variables and defining $v := v_1 - v_2$ we arrive at

$$\mathcal{U} = \Bigg\{(v,u,c) \in \mathbb{R} \times \mathbb{R}^m_+ \times \mathbb{R}^2_+ \mid \ c_1 + c_2 = 1, \quad \mathbf{1}_m^T u = c_2,$$
$$\mathbf{1}_n v + Ru \leq R\mathbf{1}_m \frac{-c_1}{m}, \quad u \leq \mathbf{1}_m \frac{c_2}{(1-\beta)m} \Bigg\}$$

and

$$\mathcal{H}(v,u,c) = \left\{ y \in \mathrm{I\!R}^2 \mid c_1 y_1 + c_2 y_2 = v \right\}.$$

The following transformation of the dual variables results in dual variables being interpretable as probabilities. Note that $\mathcal{H}$ does not depend on u and for each $(v,u,c) \in \mathcal{U}$ there is

$$(v,q,c) \in \bar{\mathcal{U}} := \Bigg\{ (v,q,c) \in \mathrm{I\!R} \times \mathrm{I\!R}^m_+ \times \mathrm{I\!R}^2_+ \mid \; c_1 + c_2 = 1, \quad \mathbf{1}_m^T q = 1,$$
$$\mathbf{1}_n v + Rqc_2 \leq R\mathbf{1}_m \frac{-c_1}{m}, \quad q \leq \mathbf{1}_m \frac{1}{(1-\beta)m} \Bigg\},$$

where q is given by $\frac{1}{c_2}u$ if $c_2 \neq 0$ and can be chosen as $q_k = \frac{1}{m}$ for all k if $c_2 = 0$. On the other hand for each $(v,q,c) \in \bar{\mathcal{U}}$ we have $(v, c_2 q, c) \in \mathcal{U}$, hence $\mathcal{U}$ can be replaced by $\bar{\mathcal{U}}$ and problem $(\mathrm{D_M})$ is equivalent to

$$(\bar{\mathrm{D}}_\mathrm{M}) \begin{cases} \operatorname{Max} \mathcal{H}[\bar{\mathcal{U}}] \\ \bar{\mathcal{U}} = \Bigg\{ (v,q,c) \in \mathrm{I\!R} \times \mathrm{I\!R}^m \times \mathrm{I\!R}^2 \mid \; c \geq 0, \quad c_1 + c_2 = 1, \quad \displaystyle\sum_{k=1}^m q_k = 1, \\ \qquad \forall k = 1,...,m: \; 0 \leq q_k \leq \dfrac{1}{(1-\beta)m}, \quad \forall j = 1,...,n: \; v \leq -\displaystyle\sum_{k=1}^m \left(\frac{1}{m} r_j^k c_1 + r_j^k q_k c_2 \right) \Bigg\}. \end{cases}$$

Applying Theorem 4 we can characterize the solutions of $(\bar{\mathrm{D}}_\mathrm{M})$. A triple $(v^*, q^*, c^*) \in \bar{\mathcal{U}}$ is a solution of $(\bar{\mathrm{D}}_\mathrm{M})$ if and only if

$$v^* = \max\left\{ v \mid (v, q^*, c^*) \in \bar{\mathcal{U}} \right\} = \max\left\{ v \mid (v, q, c^*) \in \bar{\mathcal{U}} \right\},$$

i.e., if and only if

$$v^* = \min_{j=1,\dots,n} -\sum_{k=1}^m \left(\frac{1}{m} r_j^k c_1^* + r_j^k q_k^* c_2^* \right) = \max_{q \in \mathcal{Q}} \min_{j=1,\dots,n} -\sum_{k=1}^m \left(\frac{1}{m} r_j^k c_1^* + r_j^k q_k c_2^* \right)$$

with

$$\mathcal{Q} := \left\{ q \in \mathrm{I\!R}^m \mid \sum_{k=1}^m q_k = 1, \quad \forall k = 1,...,m: \; 0 \leq q_k \leq \frac{1}{(1-\beta)m} \right\}$$

Since $q \geq 0$ and $\sum_{k=1}^m q_k = 1$, the numbers q_k may be interpreted as probabilities describing an alternative probability distribution P_q for the samples r^k. Then $\sum_{k=1}^m r_j^k q_k = E^{P_q}(r_j)$, the expectation of r_j under the alternative distribution P_q, and $\sum_{k=1}^m \frac{1}{m} r_j^k = E^P(r_j)$, the expectation of r_j under the given distribution P. The numbers q_k are related to the dual description of the coherent risk measure Conditional Value at Risk. This dual description signifies that the Conditional Value at Risk of some financial position equals the worst case expected loss of this position under a certain set of alternative probability distributions (for deatails see e.g. [2, Theorem 4.47]). Moreover, the scalarization weights c_1 and c_2 describe the model uncertainty, i.e., c_1 can be interpreted as the probability for P being the right probability distribution and c_2 as the probability that P_q provides the appropriate distribution. Then $P_{(c,q)} := c_1 P + c_2 P_q$ describes a probability distribution being a mixture of P and P_q and $E^{P_{(c,q)}}(r_j) = c_1 E^P(r_j) + c_2 E^{P_q}(r_j)$. Hence, a solution for the dual problem consists of some (c^*, q^*) determining an alternative probability distribution $P_{(c^*,q^*)}$ and a number $v^* = \min_{j=1,\dots,n} -E^{P_{(c^*,q^*)}}(r_j)$ where the vector $q^* \in \mathcal{Q}$ must be chosen such that it maximizes $\min_{j=1,\dots,n} -E^{P_{(c^*,q)}}(r_j)$ or minimizes $\max_{j=1,\dots,n} E^{P_{(c^*,q)}}(r_j)$, i.e., the largest expected return of the n given financial instruments, given the value of c^*. That means, (c^*, q^*) provides the worst case for the expected return of the "best" of the given financial instruments under the considered alternative probabilities $P_{(c^*,q)}$.

Using the results of Section 2 we can see that a point $(x^*, z^*, \alpha^*) \in \mathcal{X}$ is a solution of $(\mathrm{P_M})$ if and only if there is a solution (v^*, q^*, c^*) of $(\bar{\mathrm{D}}_\mathrm{M})$ such that

$$-\frac{c_1^*}{m}\sum_{k=1}^{m} r^{k^T} x^* + c_2^*\left(\alpha^* + \frac{1}{(1-\beta)m}\sum_{k=1}^{m} z_k^*\right) = v^* = \min_{j=1,\dots,n} -E^{P_{(c^*,q^*)}}(r_j)$$

or equivalently if

$$-c_1^* E^{appr}(r^T x^*) + c_2^* CVaR_\beta^{appr}(r^T x^*) = v^* = \min_{j=1,\dots,n} -E^{P_{(c^*,q^*)}}(r_j), \tag{12}$$

where E^{appr} and $CVaR_\beta^{appr}$ are the approximations of the expectation and the Conditional Value at Risk with the help of the samples. Thus one can find a solution of the portfolio optimization problem by first determining some "worst case" alternative probability $P_{(c^*,q^*)}$ belonging to a solution (v^*, q^*, c^*) of $(\bar{\mathrm{D}}_\mathrm{M})$ and then searching for a portfolio x^* such that (12) is satisfied.

For vector optimization problems one often does not want to chose a scalarization in advance and prefers computing the whole set of efficient solutions. Concerning the dual problem, it might be also useful to compute all solutions of $(\bar{\mathrm{D}}_\mathrm{M})$ together with the corresponding efficient portfolios and to provide the decision maker (the investor) with this information because from solving the dual problem the investor gets an information about the relationship between the scalarization weights c^* and the "worst case" alternative probability scenario $P_{(c^*,q^*)}$ taken into account under this scalarization.

Acknowledgments. The authors would like to express their gratitude to Matthias Ehrgott for several useful remarks on the manuskript of this article.

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