A Characterization of Maximal Monotone Operators

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Abstract

It is shown that a set-valued map $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ is maximal monotone if and only if the following five conditions are satisfied: (i) M is monotone; (ii) M has a nearly convex domain; (iii) M is convex-valued; (iv) the recession cone of the values $M(x)$ equals the normal cone to the closure of the domain of M at x; (v) M has a closed graph. We also show that the conditions (iii) and (v) can be replaced by Cesari's property (Q).

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1 Introduction

It is well-known (see e.g. [1, 8]) that a maximal monotone mapping $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ has the following properties:

(i) M is monotone;

(ii) M has a nearly convex domain;

(iii) The values $M(x)$ are convex;

(iv) The recession cone of $M(x)$ equals the normal cone to $\operatorname{cl}\operatorname{dom} M$ at every $x \in \operatorname{dom} M$;

(v) The graph of M is closed.

We show that the conditions (i) to (v) are also sufficient for M being maximal monotone. Moreover it is shown that (iii) and (v) can be replaced by

(vi) M is upper $\mathcal{C}$-semicontinuous (everywhere).

Upper $\mathcal{C}$-semicontinuity is also known as Cesari's property (Q). It plays an important role in Optimal Control (see e.g. [2, 3, 4] and the references in [7]). It is known (see e.g. [5]) that a maximal monotone mapping satisfies property (Q).

In [6] we introduced upper and lower limits with respect to a complete lattice (compare also [9]). In the special case of the complete lattice $\mathcal{F}$ of closed subsets of $\mathbb{R}^q$ with respect

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to inclusion, we obtain Painlevé-Kuratowski upper and lower limits (shortly PK-limits or $\mathcal{F}$-limits), but if we consider the complete lattice $\mathcal{C}$ of closed convex subsets of $\mathbb{R}^q$ with respect to inclusion, we obtain the upper and lower $\mathcal{C}$-limits. In [6] it is shown that $\mathcal{C}$-convergence of a sequence of closed convex sets is closely related to scalar convergence (i.e., pointwise convergence of the support functions of these sets). Some related results from [6] are used to prove the result of the present article.

2 Preliminaries

If not stated otherwise, we use the notation of the book "Variational Analysis" by Rockafellar and Wets [8]. Let us recall some concepts which are used in the following. For a convex set $D \subset \mathbb{R}^q$ and some $x \in D$, we denote by

$$N_D(x) := \{x^* \in \mathbb{R}^q |\; \forall w \in D: \; \langle x^*, w - x\rangle \leq 0\}$$

the *normal cone* of D at x. For points $x \notin D$ the normal cone is defined to be the empty set. The *tangent cone* of a convex set D at $x \in D$ is the set

$$T_D(x) := \operatorname{cl} \{w \in \mathbb{R}^q |\; \exists \lambda > 0: \; x + \lambda w \in D\}.$$

It is well-known that $N_D(x)$ is the polar cone of $T_D(x)$. A set $B \subset \mathbb{R}^q$ is said to be *nearly convex* if there exists a convex set C such that $C \subset B \subset \operatorname{cl} C$. The *convex hull* of a set $B \subset \mathbb{R}^q$ is denoted by $\operatorname{co} B$. Furthermore, $\operatorname{bd} B$ is the *boundary* and $\operatorname{lin} B$ the *linear hull* of B. A set-valued mapping $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ is called *monotone* if

$$\forall (x, x^*), (y, y^*) \in \operatorname{gph} M: \; \langle y^* - x^*\,,\, y - x\rangle \geq 0.$$

A monotone mapping $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ is said to be *maximal monotone*, if its graph $\operatorname{gph} M$ is not contained in the graph of any other monotone mapping.

We now turn to the notion of limits and semicontinuity with respect to the complete lattice $\mathcal{C}$ of all closed convex subsets of $\mathbb{R}^q$ and with respect to set inclusion. We use the following notation of [8] (but omit the index ∞):

$$\mathcal{N} := \{N \subset \mathbb{N} |\; \mathbb{N} \setminus N \text{ finite}\} \quad \text{and} \quad \mathcal{N}^\# := \{N \subset \mathbb{N} |\; N \text{ infinite}\}.$$

Similarly, for an infinite subset M of $\mathbb{N}$ we set

$$\mathcal{N}(M) := \{N \subset M |\; M \setminus N \text{ finite}\} \quad \text{and} \quad \mathcal{N}^\#(M) := \{N \subset M |\; N \text{ infinite}\}.$$

For a sequence (A_n) of subsets of $\mathbb{R}^q$ the *upper* and *lower PK-limits* (in [8] called outer and inner limits) are defined, respectively, by

$$\operatorname*{Lim\,sup}_{n\to\infty} A_n = \bigcap_{N \in \mathcal{N}} \operatorname{cl} \bigcup_{n \in N} A_n, \qquad \operatorname*{Lim\,inf}_{n\to\infty} A_n = \bigcap_{N \in \mathcal{N}^\#} \operatorname{cl} \bigcup_{n \in N} A_n,$$

whereas the *upper* and *lower $\mathcal{C}$-limits* are defined, respectively, by

$$\limsup_{n\to\infty} A_n = \bigcap_{N \in \mathcal{N}} \operatorname{cl} \operatorname{co} \bigcup_{n \in N} A_n, \qquad \liminf_{n\to\infty} A_n = \bigcap_{N \in \mathcal{N}^\#} \operatorname{cl} \operatorname{co} \bigcup_{n \in N} A_n.$$

Note that the sequence (A_n) has the same upper and lower $\mathcal{C}$-limit than the sequence $(\operatorname{cl}\operatorname{co} A_n)$, therefore it is not necessary to restrict ourselves to sequences of closed convex sets. In the following we only consider upper PK-limits and upper $\mathcal{C}$-limits. Let us recall some related results. The following characterization of the upper $\mathcal{C}$-limit was shown in [6, Proposition 3.6].

Proposition 2.1 *Consider a sequence (A_n) in $\mathcal{C}$. Then $x \in \limsup_{n\in\mathbb{N}} A_n$ if and only if the following assertion holds:*

$$\exists\, (\lambda_n)_{n\in\mathbb{N}} \subset [0,1]^{q+1},\ \exists\, (k_n)_{n\in\mathbb{N}} \subset \mathbb{N}^{q+1},\ \exists\, (z_n)_{n\in\mathbb{N}} \subset (\mathbb{R}^q)^{q+1}, \forall n \in \mathbb{N},\ \forall j \in \{0,1,\dots,q\}\ :$$
$$k_n^j \geq n,\ z_n^j \in A_{k_n^j},\quad x = \lim_{n\in\mathbb{N}} \sum_{i=0}^{q} \lambda_n^i z_n^i.$$

As shown in [6, Lemma 4.3], for a sequence (A_n) of closed convex subsets of $\mathbb{R}^q$ and a closed convex set $B \subset \mathbb{R}^m$ it holds

$$\limsup_{n\to\infty} B \times A_n = B \times \limsup A_n. \tag{1}$$

By $\sigma_A : Y \to \overline{\mathbb{R}}$, we denote the support function of a set $A \subset Y$. The *recession cone* (or *horizon cone*) of a convex set A is denoted by A_∞ and the *polar cone* of a cone C is denoted by C°. We write $\operatorname{rint} A$ for the *relative interior* of a set A. The term $\operatorname{rint}(A_\infty)^\circ$ has to be read as $\operatorname{rint}\big((A_\infty)^\circ\big)$. For nonempty closed convex sets $A, B \subset \mathbb{R}^q$ it holds [6, Lemma 5.4]

$$A \subset B \qquad \Longleftrightarrow \qquad \forall y \in \operatorname{rint}(B_\infty)^\circ : \quad \sigma_A(y) \leq \sigma_B(y). \tag{2}$$

The following result [6, Lemma 5.8] plays a key role in the proof of our result.

Lemma 2.2 *For any sequence (A_n) in $\mathcal{C}$ with $A := \limsup_{n\to\infty} A_n \neq \emptyset$ it holds*

$$\forall y \in \operatorname{rint}(A_\infty)^\circ, \quad \limsup_{n\to\infty} \sigma_{A_n}(y) = \sigma_A(y).$$

We now use the $\mathcal{C}$-limits to introduce a corresponding semicontinuity notion (compare [2, 3, 4, 7]). Let (X, d) be a metric space. The *upper $\mathcal{C}$-limit* for a set-valued map $f : X \rightrightarrows \mathbb{R}^q$ at $\bar{x} \in X$ is defined as

$$\limsup_{x\to\bar{x}} f(x) = \bigcup_{x_n\to\bar{x}} \bigcap_{N\in\mathcal{N}} \operatorname{cl}\operatorname{co} \bigcup_{n\in N} f(x_n),$$

where $\bigcup_{x_n\to\bar{x}}$ stands for the union over all sequences converging to $\bar{x}$. As shown in [7], the upper $\mathcal{C}$-limit can also be expressed as

$$\limsup_{x\to\bar{x}} f(x) = \bigcap_{\delta>0} \operatorname{cl}\operatorname{co} \bigcup_{d(x,\bar{x})<\delta} f(x). \tag{3}$$

We say $f : X \rightrightarrows \mathbb{R}^q$ is *upper $\mathcal{C}$-semicontinuous* at $\bar{x} \in X$ if $f(\bar{x}) \supset \limsup_{x\to\bar{x}} f(x)$. By (3) it is clear that *upper $\mathcal{C}$-semicontinuity* is the same as Cesari's property (Q) [2, 3, 4]. If f is upper $\mathcal{C}$-semicontinuous at every $\bar{x} \in X$ we just say f is upper $\mathcal{C}$-semicontinuous. By (3), the upper $\mathcal{C}$-limit $\limsup_{x\to\bar{x}} f(x)$ is always a closed convex set. For more details about $\mathcal{C}$-semicontinuity the reader is referred to [7].

3 Results

Throughout this section we denote by M a set-valued mapping $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ and we set $D := \operatorname{cl}(\operatorname{dom} M)$. We start with an auxiliary assertion.

Proposition 3.1 *Let M be monotone, let D be convex and $\bar{x} \in D$. Consider sequences $x_n \to \bar{x}$, $v_n \in M(x_n)$ and $\lambda_n \searrow 0$. If the sequence $\lambda_n v_n$ is bounded, then there is a subsequence of $\lambda_n v_n$ converging to some $v^* \in N_D(\bar{x})$.*

Proof. Take $(y, y^*) \in \operatorname{gph} M$. Then $\langle y - x_n, y^* - x_n^* \rangle \geq 0$, and so $\langle y - x_n, \lambda_n y^* - \lambda_n x_n^* \rangle \geq 0$ for every n. The sequence $(\lambda_n x_n^*)$, being bounded, has a subsequence $(\lambda_n x_n^*)_{n \in P}$ (with $P \in \mathcal{N}^\#$) converging to some $v^* \in \mathbb{R}^q$. Taking the limit for $P \ni n \to \infty$ in the preceding inequality we get $\langle y - \bar{x}, v^* \rangle \leq 0$ for every $y \in \operatorname{dom} M$. The conclusion follows. $\square$

With a slightly more precise notation our conditions (i) to (v) reads as follows.

(i) M is monotone;

(ii) There is a convex set C such that $C \subset \operatorname{dom} M \subset \operatorname{cl} C$;

(iii) $M(x)$ is convex for every x;

(iv) $\forall x \in \operatorname{dom} M : (M(x))_\infty = N_D(x)$;

(v) $\operatorname{gph} M$ is closed.

It is well-known that $\operatorname{gph} M$ is closed if and only if M is upper PK-semicontinuous (everywhere). Moreover, M being upper $\mathcal{C}$-semicontinuous implies that M is upper PK-semicontinuous. In [7] (based on [6]), conditions for the opposite implication are given. Although this result does not apply here, we use a similar proof to obtain the following lemma.

Lemma 3.2 *If M satisfies the conditions* (i) *to* (v)*, then M is upper $\mathcal{C}$-semicontinuous.*

Proof. (A) In this first part of the proof we assume that $\operatorname{int}(\operatorname{dom} M) \neq \emptyset$. Let $\bar{x} \in D$ (the case $\bar{x} \notin D$ is obvious) be arbitrarily chosen and let $\bar{x}^* \in \limsup_{x \to \bar{x}} M(x)$, i.e., there is a sequence $(x_n) \to \bar{x}$ such that $\bar{x}^* \in \limsup_{n \to \infty} M(x_n)$. By Proposition 2.1, there exist sequences $(\lambda_n)_{n \in \mathbb{N}}$ in $[0,1]^{q+1}$, $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}^{q+1}$, $(z_n)_{n \in \mathbb{N}}$ in $(\mathbb{R}^q)^{q+1}$ such that

$$\forall n \in \mathbb{N},\ \forall j \in \{0, 1, \ldots, q\} : \ \sum_{i=0}^{q} \lambda_n^i = 1,\ k_n^j \geq n,\ z_n^j \in M(x_{k_n^j}),\ \ \bar{x}^* = \lim_{n \in \mathbb{N}} \sum_{i=0}^{q} \lambda_n^i z_n^i.$$

Without loss of generality we can assume that $\|\lambda_n^0 z_n^0\| \leq \|\lambda_n^1 z_n^1\| \leq \ldots \leq \|\lambda_n^q z_n^q\|$ for every $n \in \mathbb{N}$. There exists $N \in \mathcal{N}^\#$ such that

$$\forall j \in \{0, \ldots, q\} : \qquad (\lambda_n^j) \xrightarrow{N} \lambda^j \in [0,1].$$

Assume that the sequence $(\lambda_n^q z_n^q)_{n \in N}$ is unbounded. Hence there exists $N' \in \mathcal{N}^\#(N)$ such that $(\|\lambda_n^q z_n^q\|)_{n \in N'} \to \infty$. Consequently, there exists $N'' \in \mathcal{N}^\#(N')$ such that

$$\forall j \in \{0, \ldots, q\} : \qquad (\|\lambda_n^q z_n^q\|^{-1}\, \lambda_n^j z_n^j) \xrightarrow{N''} y^j \in \mathbb{R}^q.$$

We have $(\lambda_n^j / \|\lambda_n^q z_n^q\|)_{n\in N''} \to 0$ for all $j \in \{0,\dots,q\}$. By Proposition 3.1 it follows that $y^j \in N_D(\bar{x})$ for all j. Setting $v_n := \sum_{i=0}^q \lambda_n^i z_n^i$ we have $v_n \to \bar{x}^*$. Passing to the limit (for $n \in N''$) in the relation

$$\|\lambda_n^q z_n^q\|^{-1} v_n = \sum_{j=0}^{q} \|\lambda_n^q z_n^q\|^{-1} \lambda_n^j z_n^j$$

we obtain $0 = \sum_{j=0}^q y^j$. Thus we get $y^q \in N_D(\bar{x}) \cap -N_D(\bar{x})$. Since $\operatorname{int} D \neq \emptyset$, $N_D(\bar{x})$ is pointed. Whence the contradiction $y^q = 0$ (because $\|y^q\| = 1$). It follows that the sequences $\left(\lambda_n^j z_n^j\right)_{n\in N}$ are bounded for all j. Hence there exists $N' \in \mathcal{N}^{\#}(N)$ such that $\left(\lambda_n^j z_n^j\right) \xrightarrow{N'} w^j$ for all j. If $\lambda^j \neq 0$ we have $z_n^j \xrightarrow{N'} z^j := (\lambda^j)^{-1} w^j$. Since $\operatorname{gph} M$ is closed, we obtain $z^j \in M(\bar{x})$. Otherwise, if $\lambda^j = 0$, Proposition 3.1 yields that $w^j \in N_D(\bar{x})$. As $M(\bar{x})$ and $N_D(\bar{x})$ are convex we get

$$\bar{x}^* = \lim_{n\in\mathbb{N}} \sum_{i=0}^{q} \lambda_n^i z_n^i = \sum_{\substack{i\in\{0,\dots,q\}\\ \lambda^i \neq 0}} \lambda^i z^i + \sum_{\substack{i\in\{0,\dots,q\}\\ \lambda^i = 0}} w^i \in M(\bar{x}) + N_D(\bar{x}) \overset{\text{(iv)}}{=} M(\bar{x}).$$

(B) It remains to prove the case where $\operatorname{int}(\operatorname{dom} M)$ is empty. Without loss of generality we can assume that $0 \in \operatorname{dom} M$. Set $X_0 := \operatorname{lin} D$. We have $X_0^\perp \subset N_D(x) = (M(x))_\infty$ and hence $M(x) + X_0^\perp = M(x)$ for all $x \in D$. We define a map $M_0 : X_0 \rightrightarrows X_0$ as follows:

$$M_0(x) := M(x) \cap X_0.$$

Letting $N_D^0(x)$ be the normal cone relative to X_0, we have

$$(M_0(x))_\infty = (M(x) \cap X_0)_\infty = (M(x))_\infty \cap X_0 \quad \text{and} \quad N_D^0(x) = N_D(x) \cap X_0.$$

Now it is easy to see that the conditions (i) to (v) are satisfied for M_0, and $\operatorname{int}(\operatorname{dom} M_0) \neq \emptyset$. Part (A) yields that M_0 is upper $\mathcal{C}$-semicontinuous. Taking into account the relation $M(x) = M_0(x) \times X_0^\perp$ and (1), we conclude that M is upper $\mathcal{C}$-semicontinuous. □

The preceeding lemma shows that, in the presence of (i), (ii) and (iv), a map $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ has property (Q) if and only if it has a closed graph and convex values.

Corollary 3.3 *A monotone map $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ with convex values is upper PK-semicontinuous at some $\bar{x} \in \operatorname{int} \operatorname{dom} M$ if and only if it is upper $\mathcal{C}$-semicontinuous at this point.*

Proof. Restrict M to an open ball $B \subset \operatorname{dom} M$ around $\bar{x}$. The resulting map M_B satisfies the conditions (i) to (iv). As in the proof of Lemma 3.2 we see that upper PK-semicontinuity at $\bar{x}$ implies upper $\mathcal{C}$-semicontinuity at $\bar{x}$ (because only local continuity properties are used in the proof). The opposite implication is obvious. □

It follows our main result, a characterization of maximal monotone mappings.

Theorem 3.4 *A mapping $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ is maximal monotone if and only if the conditions* (i) *to* (v) *are satisfied.*

Proof. The conditions (i) to (v) are well-known properties of maximal monotone mappings, see e.g. [8]. Therefore it remains to show that the conditions (i) to (v) imply that M is maximal monotone.

(A) In the first part of the proof we assume that $\operatorname{int}\operatorname{dom} M \neq \emptyset$. Assume that M is not maximal monotone. Then there exists a maximal monotone mapping $M' : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ such that $\operatorname{gph} M' \supsetneq \operatorname{gph} M$. Set $D' := \operatorname{cl}\operatorname{dom} M'$. Since M' is maximal monotone, D' is convex. Let $(\bar{x}, \bar{x}^*) \in \operatorname{gph} M' \setminus \operatorname{gph} M$. We distinguish three cases:

a) $\bar{x} \in \operatorname{dom} M$. We have $\bar{x}^* \notin M(\bar{x})$. By (2), there exists some

$$\bar{y} \in \operatorname{rint}\left((M(\bar{x}))_\infty\right)^\circ \overset{\text{(iv)}}{=} \operatorname{rint}\left(N_D(\bar{x})\right)^\circ = \operatorname{rint} T_D(\bar{x})$$

such that

$$\langle \bar{y}, \bar{x}^* \rangle > \sigma_{M(\bar{x})}(\bar{y}).$$

Since $\operatorname{int} T_D(\bar{x}) = \{w \in \mathbb{R}^q | \ \exists \lambda > 0 : \ \bar{x} + \lambda w \in \operatorname{int} D\}$ [8, Theorem 6.9], we have

$$\exists \lambda > 0 : \quad \bar{x} + \lambda \bar{y} \in \operatorname{int} D = \operatorname{int}\operatorname{cl}\operatorname{dom} M \subset \operatorname{int}\operatorname{dom} M,$$

where the latter inclusion follows from the fact that $\operatorname{dom} M$ is nearly convex (i.e., there exists a convex set C such that $C \subset \operatorname{dom} M \subset \operatorname{cl} C$). Consider the sequence $(x_n) \to \bar{x}$ where

$$x_n := \begin{cases} \bar{x} + \frac{\lambda}{n}\bar{y} & \text{if } n \text{ is odd} \\ \bar{x} & \text{if } n \text{ is even .} \end{cases} \tag{4}$$

Since M' is monotone, for n being odd and all $x_n^* \in M(x_n)$ we have

$$\langle \bar{y},\ x_n^* - \bar{x}^* \rangle = \frac{n}{\lambda} \langle x_n - \bar{x},\ x_n^* - \bar{x}^* \rangle \geq 0. \tag{5}$$

Hence, for odd $n \in \mathbb{N}$ we have $\sigma_{M(x_n)}(\bar{y}) \geq \langle \bar{y},\ x_n^* \rangle \geq \langle \bar{y},\ \bar{x}^* \rangle$. It follows that

$$\limsup_{n\to\infty} \sigma_{M(x_n)}(\bar{y}) \geq \limsup_{n\to\infty} \sigma_{M(x_{2n+1})}(\bar{y}) \geq \langle \bar{y},\ \bar{x}^* \rangle > \sigma_{M(\bar{x})}(\bar{y}). \tag{6}$$

From Lemma 3.2 we conclude that $\limsup_{n\to\infty} M(x_n) = M(\bar{x}) \neq \emptyset$, where the equality follows from the fact that (x_n) contains a subsequence all whose members equal $\bar{x}$. But, Lemma 2.2 implies

$$\forall y \in \operatorname{rint}\left((M(\bar{x}))_\infty\right)^\circ : \limsup_{n\to\infty} \sigma_{M(x_n)}(y) = \sigma_{M(\bar{x})}(y),$$

which contradicts (6).

b) $\bar{x} \in D$ and $M(\bar{x}) = \emptyset$. From $\operatorname{int} D \neq \emptyset$ we conclude that $\operatorname{int} T_D(\bar{x})$ is nonempty. Choose an arbitrary point $\bar{y} \in \operatorname{int} T_D(\bar{x})$ and consider the sequence $x_n := \bar{x} + \frac{\lambda}{n}\bar{y}$, where λ is chosen as (4), and a sequence $x_n^* \in M(x_n)$. Since $(\bar{x}, \bar{x}^*) \in \operatorname{gph} M'$, we see as above that (5) holds. Assuming that (x_n^*) is unbounded, we obtain some $N \in \mathcal{N}^\#$ such that $x_n^* / \|x_n^*\| \xrightarrow{N} v^* \neq 0$. By Proposition 3.1 we get $v^* \in N_D(\bar{x})$. It follows that $\langle \bar{y}, v^* \rangle < 0$. But (5) yields the contradiction

$$0 \leq \frac{1}{\|x_n^*\|} \langle \bar{y},\ x_n^* - \bar{x}^* \rangle \xrightarrow{N} \langle \bar{y}, v^* \rangle .$$

On the other hand, if (x_n^*) is bounded, there is some $N' \in \mathcal{N}^\#(N)$ such that $(x_n, x_n^*) \xrightarrow{N'} (\bar{x}, \bar{z}^*)$. As $\operatorname{gph} M$ is closed, we get $\bar{x} \in \operatorname{dom} M$, a contradiction.

c) $\bar{x} \notin D$. Let $x^0 \in \operatorname{int} D$ and let $\hat{x} \in \operatorname{bd} D$ such that $\hat{x} = \lambda x^0 + (1-\lambda)\bar{x} \in \operatorname{bd} D$ where $\lambda \in (0,1)$ is uniquely defined. If $M(\hat{x}) \neq M'(\hat{x})$, we have the situation of either a) or b). Otherwise, $M(\hat{x})$ is nonempty and bounded as $\hat{x} \in \operatorname{int} D'$. But $N_D(\hat{x}) = (M(\hat{x}))_\infty$ is unbounded, a contradiction.

(B) We now prove the case where $\operatorname{int}(\operatorname{dom} M)$ is empty. We consider the map $M_0 : X_0 \rightrightarrows X_0$ as defined in the proof of Lemma 3.2. We have seen there that M_0 satisfies the conditions (i) to (v) and $\operatorname{int}(\operatorname{dom} M_0) \neq \emptyset$. By Part (A) of the proof we conclude that M_0 is maximal monotone. It follows that M is maximal monotone. Indeed, if we assume the contrary, there exists a maximal monotone extension M' of M. As M' satisfies (i) to (v), we get by $M_0' : X_0 \rightrightarrows X_0$, $M_0'(x) := M'(x) \cap X_0$ a maximal monotone extension of M_0 (see Part (B) of the proof of Lemma 3.2). □

We easily conclude the following characterization of maximal monotone mappings by upper $\mathcal{C}$-semicontinuity (Cesari's property (Q)).

Corollary 3.5 *The mapping $M : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ is maximal monotone if and only if the conditions* (i), (ii), (iv) *and*

(vi) *M is upper $\mathcal{C}$-semicontinuous (everywhere);*

are satisfied.

Proof. This follows from Lemma 3.2 and Theorem 3.4 and the fact that condition (vi) implies the conditions (iii) and (v). □

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