A Minimal Point Theorem in Uniform Spaces

Andreas Hamel Andreas Löhne

Abstract

We present a minimal point theorem in a product space $X \times Y$, X being a separated uniform space, Y a topological vector space under the weakest assumptions up to now. We state Ekeland's variational principle and Kirk-Caristi's fixed point theorem for set-valued maps and show the equivalence of all the three theorems. Using a new characterization of uniform spaces we show that our theorems cover several recent results.

Keywords and phrases. minimal point theorem, set-valued variational principle, fixed point theorem, uniform spaces

Mathematical Subject Classification. 58E30, 46N10

1 Introduction

Ekeland's variational principle [8] is an important tool in nonlinear analysis. In the last decades various theorems had been presented which turned out to be equivalent to Ekeland's principle. One of them, a lemma due to R. R. Phelps (see [29] and especially the version of [30] from 1989) can be considered as the first minimal point theorem. Phelps' lemma yields the existence of minimal points with respect to a partial ordering in a subset of $X \times \mathbb{R}$, where X is a Banach space and $\mathbb{R}$ denotes the reals.

Minimal point theorems in a product space $X \times Y$ were established by Göpfert and Tammer [12], 1995 and generalized by Göpfert, Tammer and Zălinescu in [14], 2000 and in [13], 1999. In the latest version, X is a complete metric space and Y is a separated locally convex space. These theorems are useful tools in vector optimization. In [14], [13] a variational principle for vector-valued functions $f : X \to Y$ was presented to be an easy consequence of the minimal point theorem.

A generalization of Ekeland's variational principle with respect to the space X was given by Fang [9], 1996. He introduced the concept of "F-type topological spaces" generating the topology by families of quasi-metrics.

In this paper we generalize some of the minimal point theorems from [13], [14] with respect to the space X. Instead of complete metric spaces we consider separated uniform spaces. By the way we show that the class of Fang's F-type spaces coincides with the class of separated uniform spaces introduced by Weil [33], 1937.

We use the Brézis-Browder principle combined with a scalarization method for proving our minimal point theorem. This admits a very short proof and shows the power of the Brézis-Browder ordering principle

Sections 5 and 6 are concerned with Ekeland's variational principle and Caristi's fixed point theorem for set-valued maps. We show that our minimal point theorem is equivalent to both of this theorems and obtain a series of known results as corollaries.

A similar variant of a variational principle for set-valued maps was given by Chen, Huang and Hou [7], 2000 but using different assumptions.

Finally, we discuss the relationships of our results to well-known as well as recently published theorems.

2 Uniform Spaces

In this section we present a characterization of uniform spaces via families of quasi-metrics. This result is motivated by Fang [9] who introduced "F-type topological spaces" in this way.

Initially, we shall recall the concept of a uniform space. For further details see Kelly [21] or Köthe [22].

Let X be a nonempty set. We consider a system $\mathfrak{N}$ of subsets N of $X \times X := \{(x,y) : x,y \in X\}$. For $N \subset X \times X$ we denote $N^{-1} := \{(y,x) : (x,y) \in N\}$ and $N \circ N := \{(x,y) \in X \times X : \exists z \in X : (x,z), (z,y) \in N\}$. The set $\Delta := \{(x,x) \in X \times X\}$ is called the *diagonal*. The set X is said to be a *uniform space* iff there exists a filter $\mathfrak{N}$ on $X \times X$ satisfying

(N1) $\forall N \in \mathfrak{N} : \Delta \subset N$;

(N2) $N \in \mathfrak{N} \Rightarrow N^{-1} \in \mathfrak{N}$;

(N3) $\forall N \in \mathfrak{N}\ \exists M \in \mathfrak{N} : M \circ M \subset N$.

The system $\mathfrak{N}$ is called a *uniformity* on X. By the sets $\mathfrak{U}(x) := \{U_N(x) : N \in \mathfrak{N}\}$ where $U_N(x) := \{y \in X : (x,y) \in N\}$ a topology is given, called the *uniform topology* on X. Of course, a uniform space is already well-defined by a base of its uniformity $\mathfrak{N}$, i.e. a filter base $\mathfrak{B}$ of the uniformity $\mathfrak{N}$. The topology of a uniform space is separated iff

(N4) $\displaystyle\bigcap_{N \in \mathfrak{N}} N = \Delta.$

For a proof see [22, p. 32].

We recall a well-established result, the characterization of uniform spaces using *families of pseudo-metrics* (see [21]).

Definition 1 *Let X be a nonempty set. A function $p : X \times X \to [0,\infty)$ is called* pseudo-metric *on X iff for all $x,y,z \in X$ the following conditions are satisfied:*

(P1) $p(x,x) = 0$*;*

(P2) $p(x,y) = p(y,x)$*;*

(P3) $p(x,y) \le p(x,z) + p(z,y)$.

Moreover, let $(\Lambda, \prec)$ be a directed set. A system $\{p_\lambda\}_{\lambda\in\Lambda}$ of pseudo-metrics $p_\lambda : X \times X \to [0,\infty)$ satisfying

(P4) $\lambda \prec \mu \Rightarrow \Big(\forall x,y \in X : p_\lambda(x,y) \le p_\mu(x,y)\Big)$

is called a family of pseudo-metrics. *If additionally the condition*

(P5) $\Big(\forall\lambda\in\Lambda:\ p_\lambda(x,y)=0\Big)\ \Rightarrow\ x=y$

holds, the family of pseudo-metrics is said to be separating.

Proposition 2 *A topological space (X,τ) is a (separated) uniform space iff its topology τ can be generated by a (separating) family of pseudo-metrics.*

Proof. See [21, p. 188, Theorem 15] taking into account that by (P4) we deal with bases instead of subbases. ■

Fang [9] introduced so-called F-type topological spaces using *families of quasi-metrics.* Our definition is slightly more general because (Q5) is an optional condition, not automatically satisfied in our case.

Definition 3 *Let X be a nonempty set and let $(\Lambda,\prec)$ be a directed set. A system $\{q_\lambda\}_{\lambda\in\Lambda}$ of functions $q_\lambda:\ X\times X\to[0,\infty)$ satisfying*

(Q1) $\forall\lambda\in\Lambda\ \forall x\in X:\ q_\lambda(x,x)=0;$

(Q2) $\forall\lambda\in\Lambda\ \forall x,y\in X:\ q_\lambda(x,y)=q_\lambda(y,x);$

(Q3) $\forall\lambda\in\Lambda\ \exists\mu\in\Lambda$ *with* $\lambda\prec\mu:\ \forall x,y,z\in X:\ q_\lambda(x,y)\le q_\mu(x,z)+q_\mu(z,y);$

(Q4) $\lambda\prec\mu\ \Rightarrow\ \Big(\forall x,y\in X:\ q_\lambda(x,y)\le q_\mu(x,y)\Big)$

is called a family of quasi-metrics. *If, in addition, the condition*

(Q5) $\Big(\forall\lambda\in\Lambda:\ q_\lambda(x,y)=0\Big)\ \Rightarrow\ x=y$

is satisfied, the family of quasi-metrics is said to be separating.

The stage is set for our first result clarifying the relation between separated uniform spaces and F-type topological spaces.

Theorem 4 *A topological space (X,τ) is a (separated) uniform space iff its topology τ can be generated by a (separating) family of quasi-metrics.*

Proof. Let (X,τ) be a topological space where τ is generated by a family $\{q_\lambda\}_{\lambda\in\Lambda}$ of quasi-metrics, i.e. τ is given by

$$\mathfrak{U}(x):=\{U_x(\lambda,t):\ \lambda\in\Lambda,t>0\}$$

where

$$U_x(\lambda,t):=\{y\in X:\ q_\lambda(x,y)<t\}\,.$$

We claim that a base of a uniformity is given by the system

$$\mathfrak{B}:=\{N(\lambda,t):\ \lambda\in\Lambda,t>0\}$$

where

$$N(\lambda,t):=\{(x,y)\in X\times X:\ q_\lambda(x,y)<t\}\,.$$

To show that $\mathfrak{B}$ is a filter base let $\lambda_1, \lambda_2 \in \Lambda$ and $t_1, t_2 > 0$ be arbitrarily given. Since Λ is a directed set, there exists $\mu \in \Lambda$ with $\lambda_1 \prec \mu$ and $\lambda_2 \prec \mu$. With $t := \min\{t_1, t_2\}$ we show that

$$N(\mu, t) \subset N(\lambda_1, t_1) \cap N(\lambda_2, t_2).$$

Indeed, let $(\bar{x}, \bar{y}) \in N(\mu, t)$, i.e. $q_\mu(\bar{x}, \bar{y}) < t$. It follows that

$$q_{\lambda_i}(\bar{x}, \bar{y}) \overset{(Q4)}{\leq} q_\mu(\bar{x}, \bar{y}) < t \leq t_i \quad (i = 1, 2)\,,$$

hence

$$(\bar{x}, \bar{y}) \in N(\lambda_1, t_1) \cap N(\lambda_2, t_2).$$

Furthermore, $\emptyset \notin \mathfrak{B}$ since each $N(\lambda, t)$ contains the diagonal.

Let $\mathfrak{N}$ be the filter generated by $\mathfrak{B}$. We shall show that for $\mathfrak{N}$ the axioms (N1) to (N3) are satisfied. Obviously, (Q1) and (Q2) imply (N1) and (N2), respectively. To verify (N3) let $N \in \mathfrak{N}$ arbitrarily given. Then there exists $\lambda \in \Lambda$ and $t > 0$ such that $N(\lambda, t) \in \mathfrak{B}$ and $N(\lambda, t) \subset N$. Taking $\mu = \mu(\lambda)$ from (Q3) we set $M := N(\mu, t/2)$. Then we have $M \circ M \subset N$. Indeed, let $(\bar{x}, \bar{y}) \in M \circ M$, i.e.

$$\exists z \in X : \, q_\mu(\bar{x}, z) < \frac{t}{2}, \;\; q_\mu(z, \bar{y}) < \frac{t}{2},$$

hence

$$q_\lambda(\bar{x}, \bar{y}) \overset{(Q3)}{\leq} q_\mu(\bar{x}, z) + q_\mu(z, \bar{y}) < t.$$

It follows that $(\bar{x}, \bar{y}) \in N(\lambda, t) \subset N$. Consequently, $\mathfrak{N}$ is a uniformity generating the topology τ.

If additionally the family $\{q_\lambda\}_{\lambda \in \Lambda}$ of quasi-metrics is separating, (N4) holds, i.e. the uniform space (X, τ) is separated.

The opposite assertion follows by Proposition 2 taking into account that a family of pseudo-metrics is in particular a family of quasi-metrics. ∎

Due to this result we suggest to use the well-established term *uniform space* instead of *F-type topological space*.

An important class of uniform spaces is the class of topological vector spaces. Therefore, Theorem 4 has a counterpart for topological vector spaces where the topology can be generated by families of quasi-norms. This result is due to Hyers [18], 1939 who used the term "*pseudo-norms*" instead of "*quasi-norms*". For more details compare [24].

3 Main Tools

For the convenience of the reader we present the main tools for the proof of our minimal point theorem. The first one is the Brézis-Browder principle.

Theorem 5 *Let $(W, \preceq)$ be a quasi-ordered set (i.e. $\preceq$ is a reflexive and transitive relation) and let $\phi : W \to \mathbb{R}$ be a function satisfying*

(A1) ϕ is bounded below;

(A2) $w_1 \preceq w_2 \Rightarrow \phi(w_1) \le \phi(w_2)$;

(A3) For every $\preceq$-decreasing sequence $\{w_n\}_{n\in\mathbb{N}} \subset W$ there exists some $w \in W$ such that $w \preceq w_n$ for all $n \in \mathbb{N}$.

Then, for every $w_0 \in W$ there exists some $\bar{w} \in W$ such that

(i) $\bar{w} \preceq w_0$;

(ii) $\hat{w} \preceq \bar{w} \Rightarrow \phi(\hat{w}) = \phi(\bar{w})$.

In particular, if we strengthen (A2) to

(A2') $\Big(w_1 \preceq w_2,\ w_1 \neq w_2\Big) \Rightarrow \phi(w_1) < \phi(w_2)$

it holds

(ii') $\hat{w} \preceq \bar{w} \Rightarrow \hat{w} = \bar{w}$, i.e. $\bar{w}$ is $\preceq$-minimal in W.

Proof. See [2, Corollary 1]. ■

Note that (A2') implies the antisymmetry of the relation $\preceq$.

A second important tool is a scalarization method established by Gerstewitz (Tammer), Iwanow [11] and Gerth (Tammer), Weidner [10] and applied in [14], [13], for instance.

Theorem 6 *Let Y be a topological vector space, $K \subset Y$ a convex cone and $k^0 \in K \setminus -\operatorname{cl} K$. The functional $z : Y \to \mathbb{R} \cup \{\infty\}$, defined as $z(y) := \inf\{t \in \mathbb{R} : y \in tk^0 - \operatorname{cl} K\}$ has the following properties*

(i) z is sublinear;

(ii) $y_1 \le_K y_2 \ \Rightarrow\ z(y_1) \le z(y_2)$;

(iii) $\forall y \in Y\ \forall \alpha \in \mathbb{R} :\ z(y + \alpha k^0) = z(y) + \alpha$;

(iv) If $Y_0 \subset Y$ is $\le_K$-bounded below then z is bounded below on Y_0.

Proof. See [14, Lemma 7] taking into account that Y has not to be separated for the proof. Moreover, in the definition of the functional the closed cone can be replaced by the closure of a not necessarily closed cone (since $y_1 \le_K y_2$ implies $y_1 \le_{\operatorname{cl} K} y_2$). Then, if Y is not separated, we have to choose $k^0 \in K \setminus -\operatorname{cl} K$ to avoid $k^0 \in \operatorname{cl}\{0\}$. If Y is separated it suffices to suppose $k^0 \in K \setminus -K$. Besides, (iv) is an easy consequence of (ii). ■

Let Y be a topological vector space and $K \subset Y$ a convex cone. We use the following assumption to derive strong (in [14] called "authentic") variants of the minimal point theorem.

(C) There exists a proper convex cone $C \subset Y$ satisfying $K \setminus \{0\} \subset \operatorname{int} C$;

Remark 1 If a cone K satisfies (C) it is pointed, i.e. $K \cap -K = \{0\}$. To see this let $y \in K \cap (-K)$. Assuming that $y \neq 0$ we have $y, -y \in \operatorname{int} C$. Since C is convex, $\operatorname{int} C$ is convex, hence $0 \in \operatorname{int} C$. Consequently, there exists an absorbing neighborhood U of $0 \in Y$ such that $U \subset \operatorname{int} C$. Hence, C must be the whole space. This contradicts the assumption that C is proper.

Theorem 7 *Let Y be a topological vector space, $K \subset Y$ a convex cone satisfying assumption (C). Let $k^0 \in K \setminus \{0\}$. The functional $z_C : Y \to \mathbb{R}$, defined by $z_C(y) := \inf\{t \in \mathbb{R} : \ y \in tk^0 - \operatorname{cl} C\}$ has the following properties*

(i) z_C *is sublinear;*

(ii) $\Big(y_1 \leq_K y_2 \ , \ y_1 \neq y_2\Big) \ \Rightarrow \ z_C(y_1) < z_C(y_2)$*;*

(iii) $\forall y \in Y \ \forall \alpha \in \mathbb{R} : \ z_C(y + \alpha k^0) = z_C(y) + \alpha$*;*

(iv) For $Y_0 \subset Y$, $\tilde{y} \in Y$ the condition $Y_0 \cap (\tilde{y} - \operatorname{int} C) = \emptyset$ implies that z_C is bounded below on Y_0.

Proof. See [14, Lemma 7] and [13, proof of Theorem 1]) taking into account that $z_C(y) = \infty$ is not possible under our assumptions. Note that we have $k^0 \in \operatorname{int} C \setminus -\operatorname{cl} C$. Therefore, as above, Y has not to be separated. ■

Remark 2 The assumption (C) introduced above is strong enough to play two roles simultaneously. On the one hand, it allows to weaken the boundedness assumption, compare (iv) in Theorem 6 and 7, respectively. On the other hand it ensures the strong monotonicity of the functional z_C, compare (ii) in Theorem 6 and 7, respectively. It is possible to pursue this two goals by different assumptions. This leads to slightly different variants of the minimal point theorem. Compare the forthcoming paper [17].

4 Minimal Point Theorem

Minimal point theorems in product spaces $X \times Y$ were presented by Göpfert and Tammer [12] and by Göpfert, Tammer and Zălinescu [14], [13] being a useful generalization of Ekeland's variational principle. We wish to generalize some of these theorems with respect to the spaces, i.e. we consider separated uniform spaces X instead of complete metric spaces and also topological vector spaces Y instead of separated locally convex spaces.

According to [13] we avoid using Zorn's Lemma with the advantage that all assumptions involve sequences instead of nets. Furthermore, the Brézis-Browder principle together with the scalarization method due to Chr. Tammer and her collaborators [10], [31] turn out to be powerful enough to prove one of the the most general minimal point theorems up to now.

In the following let $(X, \{q_\lambda\}_{\lambda \in \Lambda})$ be a separated uniform space and let Y be a topological vector space. We write $w = (w_X, w_Y) \in W$ to deal with the two components of an element w of the product space $W := X \times Y$.

It is well-known that a convex cone $K \subset Y$ generates a quasi-ordering on Y (i.e. a reflexive and transitive relation) by

$$y_1 \leq_K y_2 \Leftrightarrow y_2 - y_1 \in K.$$

If K is pointed the relation is also antisymmetric, therefore a partial ordering. Using an element $k^0 \in K \setminus -\operatorname{cl} K$ we introduce a relation $\preceq_{k^0}$ on W:

$$(x_1, y_1) \preceq_{k^0} (x_2, y_2) \;\Leftrightarrow \forall \lambda \in \Lambda : \; y_1 + k^0 q_\lambda (x_1, x_2) \leq_K y_2. \tag{1}$$

Lemma 8 *If $K \subset Y$ is a convex cone, a reflexive and transitive relation on W is defined by (1). If additionally K is pointed, the relation $\preceq_{k^0}$ is a partial ordering on W.*

Proof. Exemplary, we prove the transitivity. Let $w_i = (x_i, y_i) \in W$ $(i = 1, 2, 3)$ satisfying $w_1 \preceq_{k^0} w_2$ and $w_2 \preceq_{k^0} w_3$. The transitivity of the relation $\leq_K$ yields

$$\forall \alpha \in \Lambda : \; y_1 + k^0 \Big(q_\alpha (x_1, x_2) + q_\alpha (x_2, x_3) \Big) \leq_K y_3. \tag{2}$$

By (Q3) for each $\lambda \in \Lambda$ there exists some $\mu \in \Lambda$ with $\lambda \prec \mu$ such that $q_\lambda(x_1, x_3) \leq q_\mu(x_1, x_2) + q_\mu(x_2, x_3)$. Since (2) holds for all $\alpha \in \Lambda$, we obtain $y_1 + k^0 q_\lambda(x_1, x_3) \leq_K y_3$ for all $\lambda \in \Lambda$, i.e. $w_1 \preceq_{k^0} w_3$. ∎

We continue with our main result, the minimal point theorem in uniform spaces. Just as the Brézis-Browder principle (Theorem 5), the following theorem (as well as its equivalent formulations, Theorems 10, 12/13 below) consists of two parts. The "weak" assertion (ii) yields the existence of an element $\bar{w}$ of a certain set A such that some $\hat{w} \in A$ which is dominated by $\bar{w}$ with respect to a quasi-ordering necessarily has the same X-component. However, the Y-components may be distinct. The "strong" ("authentic") assertion (ii') yields the minimality of some $\bar{w} \in A$ in A with respect to a partial ordering, i.e. $\hat{w} \in A, \hat{w} \preceq_{k^0} \bar{w}$ implies $\hat{w} = \bar{w}$. Note that assumption (C) of Section 3 ensures that we deal in fact with a partial ordering. It plays the key role in establishing the strong assertion and can traced back to the early work of Bishop and Phelps.

Theorem 9 (Minimal Point Theorem) *Let $(X, \{q_\lambda\}_{\lambda \in \Lambda})$ be a separated uniform space, Y a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -\operatorname{cl} K$. Let $A \subset W$ be a nonempty subset of the product space $W = X \times Y$ and let $w_0 \in A$ be given such that for the set $W_0 := \{w \in A : \, w \preceq_{k^0} w_0\}$ the following assumptions are satisfied:*

- *(M1) The set $(W_0)_Y := \{y \in Y : \, w = (x, y) \in W_0\}$ is $\leq_K$-bounded below;*
- *(M2) For any $\preceq_{k^0}$-decreasing sequence $\{w_n\}_{n \in \mathbb{N}} \subset W_0$ there exists some $w \in W_0$, such that $w \preceq_{k^0} w_n$ for all $n \in \mathbb{N}$.*

Then there exists some $\bar{w} \in A$ such that

- *(i) $\bar{w} \preceq_{k^0} w_0$;*
- *(ii) $\Big(\hat{w} \in A, \; \hat{w} \preceq_{k^0} \bar{w}\Big) \Rightarrow \hat{w}_X = \bar{w}_X$.*

Under the additional assumption (C) we can relax assumption (M1) to

(M1') There exists some $\tilde{y} \in Y$ *such that* $(W_0)_Y \cap (\tilde{y} - \operatorname{int} C) = \emptyset$

such that even

(ii') $\bar{w}$ *is* $\preceq_{k^0}$*-minimal in* A

holds.

Proof. By Lemma 8, the relation $\preceq_{k^0}$ is reflexive and transitive on W_0. We apply the Brézis-Browder principle (Theorem 5) on the quasi-ordered set $(W_0, \preceq_{k^0})$ using the functional

$$\phi : W_0 \to \mathbb{R}, \;\; \phi(w) = z(w_Y - (w_0)_Y)$$

where $z : Y \to \mathbb{R} \cup \{\infty\}$ is the scalarization functional of Theorem 6. First, we must have $\phi(w) \neq \infty$. Indeed, for $w \in W_0$ it holds $w_Y \leq_K (w_0)_Y$. Hence $w_Y - (w_0)_Y \in -K \subset -\operatorname{cl} K$. By the definition of z we have $\phi(w) \leq 0$.

By (M1) and property (iv) of z (Theorem 6), ϕ is bounded below on W_0. Let be $w_1 \preceq_{k^0} w_2$, hence $(w_1)_Y \leq_K (w_2)_Y$. Property (ii) of z implies assumption (A2) of Theorem 5. Of course, (M2) implies assumption (A3) of Theorem 5.

Theorem 5 yields the existence of some $\bar{w} \in W_0$ (i.e. $\bar{w} \preceq_{k^0} w_0$) such that

$$\hat{w} \preceq_{k^0} \bar{w} \Rightarrow \phi(\hat{w}) = \phi(\bar{w}). \tag{3}$$

To show (ii) let $\hat{w} \in A$, $\hat{w} \preceq_{k^0} \bar{w}$. The transitivity of $\preceq_{k^0}$ yields $\hat{w} \in W_0$. This implies $\hat{w}_Y - (w_0)_Y + k^0 q_\lambda(\hat{w}_X, \bar{w}_X) \leq_K \bar{w}_Y - (w_0)_Y$ for all $\lambda \in \Lambda$. Applying property (iii) of z we get

$$\forall \lambda \in \Lambda : \; q_\lambda(\hat{w}_X, \bar{w}_X) \leq \phi(\bar{w}) - \phi(\hat{w}) \overset{(3)}{=} 0.$$

Since X is separated, it follows that $\hat{w}_X = \bar{w}_X$.

Now, let assumption (C) be satisfied. We can replace (M1) by (M1') and proceed analogously, but using the functional z_C of Theorem 7 instead of z. In particular, the corresponding functional $\phi_C : W_0 \to \mathbb{R}$, $\phi_C(w) = z_C(w_Y)$ (the functional can be chosen slightly simpler than before, because $z_C(y) \neq \infty$ for all $y \in Y$) is even strict $\preceq_{k^0}$-monotone, i.e. $w_1 \preceq_{k^0} w_2$, $w_1 \neq w_2$ implies $\phi_C(w_1) < \phi_C(w_2)$. Indeed, let $w_1 \preceq_{k^0} w_2$ and $w_1 \neq w_2$. If $(w_1)_X \neq (w_2)_X$ then, since X is separated, there exists some $\mu \in \Lambda$ satisfying $q_\mu((w_1)_X, (w_2)_X) > 0$, hence,

$$(w_2)_Y - (w_1)_Y \in \left\{k^0 q_\mu((w_1)_X, (w_2)_X)\right\} + K \subset (K \setminus -\operatorname{cl} K) + K \subset K \setminus \{0\}.$$

Otherwise, if $(w_1)_X = (w_2)_X$, we have $(w_1)_Y \neq (w_2)_Y$ and it also holds $(w_2)_Y - (w_1)_Y \in K \setminus \{0\}$. Property (ii) of z_C yields $\phi(w_1) < \phi(w_2)$. Therefore, assumption (A2') in Theorem 5 is satisfied, too. The $\preceq_{k^0}$-minimality of $\bar{w}$ in A follows from Theorem 5 (ii') taking into account the transitivity of the relation. ■

Remark 3 Our assumption (M2) is not stronger than (H2) in [14]. Assuming X to be sequentially complete, this is because for every $\preceq_{k^0}$-decreasing sequence $\{w_n\}_{n\in\mathbb{N}} \subset A$ we have $(w_n)_X \to x \in X$ for some $x \in X$. Indeed, let $\varepsilon > 0$ and

$\lambda \in \Lambda$ be arbitrarily given. Since $\{w_n\}_{n\in\mathbb{N}}$ is $\preceq_{k^0}$-decreasing, for any $m, n \in \mathbb{N}$ with $m \geq n$ the inequality

$$(w_m)_Y + k^0 q_\lambda((w_m)_X, (w_n)_X) \leq_K (w_n)_Y$$

is satisfied. This implies

$$(w_m)_Y - (w_0)_Y + k^0 q_\lambda((w_m)_X, (w_n)_X) \leq_K (w_n)_Y - (w_0)_Y .$$

The properties of z (Theorem 6) yield

$$\phi(w_m) + q_\lambda((w_m)_X, (w_n)_X) \overset{(iii)}{=} z\big((w_m)_Y - (w_0)_Y + k^0 q_\lambda((w_m)_X, (w_n)_X)\big) \overset{(ii)}{\leq} \phi(w_n).$$

The sequence $\{\phi(w_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is nonincreasing and bounded, hence convergent, i.e. there exists some $N_0(\varepsilon, \lambda) \in \mathbb{N}$ such that for all $m, n \geq N_0(\varepsilon, \lambda)$ it holds

$$q_\lambda((w_m)_X, (w_n)_X) \leq \phi(w_n) - \phi(w_m) < \varepsilon.$$

This means, the sequence $\{(w_n)_X\}_{n\in\mathbb{N}}$ is Cauchy and by the sequentially completeness of X convergent to some $x \in X$. The same considerations can be done using z_C and Theorem 7 instead of z.

We shall discuss a set of conditions which can be supposed instead of assumption (M2) of Theorem 9 if X is sequentially complete.

(M3) For any sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset A$ where $\{x_n\}_{n\in\mathbb{N}}$ tends to $x \in X$ and $\{y_n\}_{n\in\mathbb{N}}$ is $\leq_K$-decreasing, there exists some $y \in Y$ such that $(x, y) \in A$ and $y \leq_K y_n$ for all $n \in \mathbb{N}$;

(M4) For any $y \in K$ the sets $K \cap (y - \mathbb{R}_+ k^0)$ are sequentially closed;

(M5) For the family $\{q_\lambda\}_{\lambda\in\Lambda}$ of quasi-metrics generating the topology of X and defining the relation $\preceq_{k^0}$, the elements q_λ are sequentially lower semi-continuous with respect to the second variable in the following sense. From $x_n \to x \in X$ for $n \to \infty$ it follows

$$\forall u \in X\ \forall \lambda \in \Lambda :\ q_\lambda(u, x) \leq \liminf_{n\to\infty} q_\lambda(u, x_n).$$

In [13], where X is a complete metric space, supposing (M3) and (M4) is already sufficient for (M2). Generating the topology of a uniform space by a family of quasi-metrics, we need an additional assumption (M5) because the quasi-metrics are not necessarily continuous. Of course, the topology of a uniform space always can be generated by a family of uniformly continuous pseudo-metrics (compare [21, p. 188, Theorem 15]). This means that (M5) does not restrict the choice of the space X. However, it could be that (M5) restricts the choice of the ordering relation.

Remark 4 If X is sequentially complete, then (M2) in Theorem 9 can be replaced by (M3), (M4) and (M5). Indeed, let $\{w_n\}_{n\in\mathbb{N}_0} \subset A$ be a $\preceq_{k^0}$-decreasing sequence in A. By Remark 3 it can be assumed that $(w_n)_X \to x \in X$. We denote $x_n := (w_n)_X$ and $y_n := (w_n)_Y$. Then the sequence $\{y_n\}_{n\in\mathbb{N}}$ is $\leq_K$-decreasing. By assumption

(M3) there exists some $y \in Y$ such that $(x, y) \in A$ and $y \leq_K y_n$ for all $n \in \mathbb{N}$. Hence

$$\forall \lambda \in \Lambda \; \forall n, p \in \mathbb{N} : \; y + k^0 \, q_\lambda(x_n, x_{n+p}) \leq_K y_{n+p} + k^0 \, q_\lambda(x_n, x_{n+p}) \leq_K y_n.$$

Denoting $w := (x, y)$ and letting $p \to \infty$, by (M5) and (M4) it follows that $w \preceq_{k^0} w_n$ for all $n \in \mathbb{N}$, i.e. (M2) is satisfied.

Note that if (M1) holds, (M3) is satisfied if A is sequentially closed and K is a sequential Daniell cone, i.e. every $\leq_K$-decreasing sequence which has a lower bound converges to its infimum. Further sufficient conditions can be found in [14] or in [24].

5 Ekeland's principle for set-valued maps

In this section we present a variant of Ekeland's variational principle for set-valued maps being equivalent to the minimal point theorem as well as some conclusions of it. A similar result was proven by Chen, Huang and Hou [7]. However, our assumptions differ from those in [7]. Moreover, while the proof in [7] is quite complicated our variant is an easy consequence of Theorem 9.

We consider a set-valued mapping $F : X \to 2^Y$. The set $\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\}$ is called *domain* of F and $\operatorname{gr} F := \{(x, y) \in X \times Y : \; y \in F(x)\}$ is the *graph* of F. For $M \subset X$ the set $F(M) := \{y \in Y : \; \exists \, x \in M : \; y \in F(x)\}$ is said to be the *image* of M with respect to F.

Theorem 10 (Variational Principle) *Let $(X, \{q_\lambda\}_{\lambda \in \Lambda})$ be a separated uniform space, Y a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -\operatorname{cl} K$. For the set-valued mapping $F : X \to 2^Y$, let $w_0 = (x_0, y_0) \in \operatorname{gr} F$ be given such that for the set $W_0 := \{w \in \operatorname{gr} F : \; w \preceq_{k^0} w_0\}$ the following assumptions are satisfied:*

(E1) The set $\{y \in Y : \; w = (x, y) \in W_0\}$ is $\leq_K$-bounded below;

(E2) For every $\preceq_{k^0}$-decreasing sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subset W_0$ there exists some point $(x, y) \in W_0$ such that $(x, y) \preceq_{k^0} (x_n, y_n)$ for all $n \in \mathbb{N}$.

Then, there exists some point $(\bar{x}, \bar{y}) \in \operatorname{gr} F$ such that

(i) $\forall \lambda \in \Lambda : \; \bar{y} + k^0 q_\lambda(\bar{x}, x_0) \leq_K y_0$;

(ii) $\forall \, (x, y) \in \operatorname{gr} F$ with $x \neq \bar{x}$ $\exists \mu \in \Lambda : \; y + k^0 q_\mu(x, \bar{x}) \not\leq_K \bar{y}$.

If additionally assumption (C) is satisfied, (E1) can be relaxed to

(E1') There exists some $\tilde{y} \in Y$ such that $F(W_0) \cap (\tilde{y} - \operatorname{int} C) = \emptyset$;

and, additionally, $\bar{y}$ is $\leq_K$-minimal in $F(\bar{x})$.

Proof. Setting $A := \operatorname{gr} F$ all assumptions coincide with those of Theorem 9. Therefore, by Theorem 9 for $(x_0, y_0) \in \operatorname{gr} F$ there exists some $(\bar{x}, \bar{y}) \in \operatorname{gr} F$ such that $(\bar{x}, \bar{y}) \preceq_{k^0} (x_0, y_0)$ (i.e. (i) holds), and such that $(\hat{x}, \hat{y}) \in A$, $(\hat{x}, \hat{y}) \preceq_{k^0} (\bar{x}, \bar{y})$ implies $\hat{x} = \bar{x}$. This is equivalent to assertion (ii).

To show that $\bar{y}$ is $\leq_K$-minimal in $F(\bar{x})$, we can use assertion (ii') of Theorem 9. Let $\hat{y} \leq_K \bar{y}$ for $\hat{y}, \bar{y} \in F(\bar{x})$. Hence, $(\bar{x}, \hat{y}) \preceq_{k^0} (\bar{x}, \bar{y})$ and the $\preceq_{k^0}$-minimality of $(\bar{x}, \bar{y})$ yields $\hat{y} = \bar{y}$. ■

Now, we present a variant of Ekeland's variational principle for vector-valued functions. Note that only (ii) of Theorem 10 (the "weak" part) is necessary for proving the following corollary. As proposed in [14], [13], we extend the space Y by an element ∞ such that $y \leq_K \infty$ for all $y \in Y$.

Corollary 11 *Let $(X, \{q_\lambda\}_{\lambda\in\Lambda})$ be a separated sequentially complete uniform space, Y a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -\operatorname{cl} K$. Let $f : X \to Y \cup \{\infty\}$ be a proper function which is $\leq_K$-bounded below and let for every $x \in \operatorname{dom} f$ the set*

$$S(x) := \{u \in X : \forall\lambda \in \Lambda : \ f(u) + k^0 q_\lambda(u, x) \leq_K f(x)\}$$

be sequentially closed. Then, for each $x_0 \in \operatorname{dom} f$ there exists $\bar{x} \in X$ such that

(i) $\forall\lambda \in \Lambda : \ f(\bar{x}) + k^0 q_\lambda(\bar{x}, x_0) \leq_K f(x_0)$;

(ii) $\forall x \in X \setminus \{\bar{x}\} : \exists\mu \in \Lambda : \ f(x) + k^0 q_\mu(x, \bar{x}) \not\leq_K f(\bar{x})$.

Proof. We consider the set-valued mapping

$$F : X \to 2^Y, \ F(x) := \begin{cases} \{f(x)\} & \text{if } \ f(x) \neq \infty \\ \emptyset & \text{if } \ f(x) = \infty \end{cases}$$

and show that Theorem 10 is applicable. It remains to show that (E2) of Theorem 10 is satisfied. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset W_0$ be a $\preceq_{k^0}$-decreasing sequence. By Remark 3 it can be assumed that $x_n \to x \in X$. We have $x_m \in S(x_n)$ for all $m \in \mathbb{N}$, $m \geq n$. Since $S(x_n)$ is sequentially closed, it follows that $x \in S(x_n)$ for all $n \in \mathbb{N}$. Hence, $(x, f(x)) \preceq_{k^0} (x_n, y_n)$ for all $n \in \mathbb{N}$. Theorem 10 implies all assertions. ■

Corollary 11 covers several known results. First, it extends Corollary 2 of [14] in which X is assumed to be a complete metric space, Y a separated locally convex space. Secondly, Theorem 4 of [20] is a special case of Corollary 2 of [14], consequently of our Corollary 11. In Isac's theorem, the cone K is assumed to be normal. Thirdly, Corollary 11 generalizes a recent result of Li et al. [23, Theorem 4]. In the latter, X is assumed to be a complete metric space and Y an ordered separated topological vector space having an ordering cone with nonempty interior.

Note that if (C) is satisfied we can relax the boundedness condition in Corollary 11 in the same way as in Theorem 10. This variant covers Corollary 2 of [13].

6 Kirk-Caristi fixed point theorem for set-valued maps

An important consequence of Ekeland's variational principle is the Kirk-Caristi fixed point theorem. By several authors [5], [25], [15] the equivalence of both theorems was shown. Our aim is to present a variant of the fixed point theorem which corresponds to our variational principle. On the one hand, we consider a mapping

F to formulate the contraction condition and on the other hand a mapping T is involved for which the existence of a fixed point shall be shown. By Tammer [32] a vector-valued variant, i.e. F is a vector-valued function and T is an operator, was proven. Isac [20] presented a vector-valued equilibrium variant considering a set-valued mapping $T : X \to 2^X$. In our variant both F and T are set-valued maps.

Let $T : X \to 2^X$ be a set-valued mapping. A point $\bar{x}$ satisfying $\bar{x} \in T(\bar{x})$ is said to be a *fixed point* of T (compare Isac [20]). In Hamel [15] a variant of the fixed point theorem in F-type topological spaces (i.e. in separated uniform spaces) was proven. In contrast to Isac [20], an assertion with respect to *stationary points* was presented in [15]. A point $\bar{x} \in X$ is said to be a stationary point of a set-valued mapping $T : X \to 2^X$ iff $\{\bar{x}\} = T(\bar{x})$. Of course, any stationary point is a fixed point, too. The contrary is only true for functions in general.

We present assertions with respect to both stationary and fixed points. Moreover, we consider two different variants of the fixed point theorem. At first we prove a fixed point theorem for a mapping $T : X \to 2^X$. Under the additional assumption (C) of Section 3 we obtain a fixed point assertion even for a mapping $T : W \to 2^W$, where $W = X \times Y$. As above, the boundedness condition can be weakened in this case. Moreover, we show that a fixed point assertion for $T : X \to 2^X$ may be regarded as a special case of the second variant (see Corollary 14).

Let $F : X \to 2^Y$ be a set-valued mapping. As above we set $W_0 := \{w \in \operatorname{gr} F : w \preceq_{k^0} w_0\}$. We say that $T : X \to 2^X$ satisfies the *weak contraction condition* iff

$$\forall\, (x, y) \in W_0 : \ \exists\, \hat{x} \in T(x)\ \exists\, \hat{y} \in F(\hat{x}) : \ (\hat{x}, \hat{y}) \preceq_{k^0} (x, y)\,. \tag{4}$$

Moreover, we say $T : X \to 2^X$ satisfies the *strong contraction condition* iff

$$\forall\, (x, y) \in W_0 : \begin{cases} T(x) \neq \emptyset \quad \text{and} \\ \forall\, \hat{x} \in T(x)\ \exists\, \hat{y} \in F(\hat{x}) : \ (\hat{x}, \hat{y}) \preceq_{k^0} (x, y)\,. \end{cases} \tag{5}$$

If the strong contraction condition (5) is satisfied, the weak contraction condition (4) is satisfied as well.

Theorem 12 (Fixed Point Theorem in X) *Let $(X, \{q_\lambda\}_{\lambda \in \Lambda})$ be a separated uniform space, Y a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -\operatorname{cl} K$. For the set-valued mapping $F : X \to 2^Y$ let the assumptions (E1) and (E2) of Theorem 10 be satisfied for $(x_0, y_0) \in \operatorname{gr} F$. Moreover, let $T : X \to 2^X$ be a set-valued mapping.*

If T satisfies the weak contraction condition (4), T has a fixed point. Besides, if T satisfies the strong contraction condition (5), T has a stationary point.

Proof. Let $(x_0, y_0) \in \operatorname{gr} F$ be given. By Theorem 10 there exists some $(\bar{x}, \bar{y}) \in \operatorname{gr} F$ such that the assertions (i) and (ii) hold. Since (i), we have $(\bar{x}, \bar{y}) \in W_0$. We claim that $\bar{x}$ is a fixed point of T. Assuming the contrary, by the weak contraction condition (4) we get the existence of some $(x, y) \in \operatorname{gr} F$ such that $x \in T(\bar{x})$, $x \neq \bar{x}$ and $(x, y) \preceq_{k^0} (\bar{x}, \bar{y})$. This contradicts (ii). Hence $\bar{x}$ is a fixed point.

If T satisfies the strong contraction condition (5), $\bar{x}$ is even a stationary point. Indeed, assuming $x \in T(\bar{x})$ with $x \neq \bar{x}$ the same arguments as above lead to a contradiction. ■

We say that $T : W \to 2^W$ satisfies the *weak contraction condition* iff

$$\forall w \in W_0 : \exists \hat{w} \in T(w) : \hat{w} \preceq_{k^0} w. \tag{6}$$

Moreover, we say $T : W \to 2^W$ satisfies the *strong contraction condition* iff

$$\forall w \in W_0 : \Big(T(w) \neq \emptyset \text{ and } \forall \hat{w} \in T(w) : \hat{w} \preceq_{k^0} w \Big). \tag{7}$$

As above, the weak contraction condition (6) is satisfied if the strong contraction condition (7) is satisfied.

Theorem 13 (Fixed Point Theorem in W) *Let $(X, \{q_\lambda\}_{\lambda \in \Lambda})$ be a separated uniform space, Y a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -\operatorname{cl} K$. For the set-valued mapping $F : X \to 2^Y$ let the assumptions (C) as well as (E1') and (E2) of Theorem 10 be satisfied for $w_0 \in \operatorname{gr} F$. Moreover, let $T : W \to 2^W$ be a set-valued mapping.*

If T satisfies the weak contraction condition (6), T has a fixed point. Besides, if T satisfies the strong contraction condition (7), T has a stationary point.

Proof. Let $w_0 \in \operatorname{gr} F$ be given. By Theorem 10 there exists some $\bar{w} \in \operatorname{gr} F$ such that $\bar{w}_Y$ is $\leq_K$-minimal in $F(\bar{w}_X)$ and the assertions (i) and (ii) hold. Since (i), we have $\bar{w} \in W_0$. We claim that $\bar{w}$ is a fixed point of T. Assuming the contrary, by the weak contraction condition (6) we get the existence of some $w \in T(\bar{w})$ with $w \preceq_{k^0} \bar{w}$ and $w \neq \bar{w}$. In case of $w_X = \bar{w}_X$ we have $w_Y \neq \bar{w}_Y$. Since $w_Y \leq_K \bar{w}_Y$, this contradicts the $\leq_K$-minimality of $\bar{w}_Y$. On the other hand $w_X \neq \bar{w}_X$ contradicts (ii). Hence $\bar{w}$ is a fixed point.

If T satisfies the strong contraction condition (7), $\bar{w}$ is even a stationary point. Indeed, assuming $w \in T(\bar{w})$ with $w \neq \bar{w}$ the same argument as above lead to a contradiction. ■

The following corollary is an easy consequence of Theorem 13. As in Theorem 12 we deal with fixed point assertions for a mappings $T : X \to 2^X$. However, we use the assumptions of Theorem 13. The main advantage is that the boundedness condition can be weakened, i.e. we assume (E1') instead of (E1). The price is that we have to suppose the additional assumption (C). However, taking into account Remark 2 in Section 3 we note that in the following corollary assumption (C) could be replaced by a weaker one.

Corollary 14 *Let $(X, \{q_\lambda\}_{\lambda \in \Lambda})$ be a separated uniform space, Y a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -\operatorname{cl} K$. For the set-valued mapping $F : X \to 2^Y$ let the assumptions (C) as well as (E1') and (E2) of Theorem 10 be satisfied. Moreover, let $T : X \to 2^X$ be a set-valued mapping and $(x_0, y_0) \in \operatorname{gr} F$.*

If T satisfies the weak contraction condition (4), T has a fixed point. Besides, if T satisfies the strong contraction condition (5), T has a stationary point.

Proof. We define a mapping $\tilde{T} : W \to 2^W$ by

$$\tilde{T}(w) := \{ \hat{w} \in \operatorname{gr} F : \hat{w}_X \in T(w_X), \hat{w}_Y \in F(\hat{w}_X), \hat{w} \preceq_{k^0} w \}.$$

If T satisfies (4) then $\tilde{T}$ satisfies (6). Theorem 13 yields that $\tilde{T}$ has a fixed point. Hence, T has a fixed point as well.

If T satisfies (5) then $\tilde{T}$ satisfies (7). Theorem 13 yields that $\tilde{T}$ has a stationary point $\bar{w}$. We claim that $\bar{w}_X$ is a stationary point of T. Of course, we have $\bar{w}_X \in T(\bar{w}_X)$. We assume that $x \in T(\bar{w}_X)$ with $x \neq \bar{w}_X$. By (5) there exists some $y \in F(x)$ such that $(x, y) \in \tilde{T}(\bar{w})$. This contradicts the fact that $\bar{w}$ is a stationary point of $\tilde{T}$. ■

Remark 5 We shall show that Theorem 12 implies the weak assertion (ii) of Theorem 9 (minimal point theorem). Indeed, if we define the set-valued mapping $F : X \to 2^Y,\ F(x) := \{y \in Y : (x, y) \in A\}$, then $A = \operatorname{gr} F$ and the assumptions of both theorems coincide. Let $w_0 \in A$ be given. We assume that the assertions (i) and (ii) of Theorem 9 do not hold, i.e. for every $w \in W_0$ there exists some $\hat{w} \preceq_{k^0} w$ such that $\hat{w}_X \neq w_X$. Therefore, the mapping $T : X \to 2^X,\ T(x) := \{\hat{x} \in X \setminus \{x\} : \exists \hat{y} \in F(\hat{x}) : (\hat{x}, \hat{y}) \preceq_{k^0} (x, y)\}$ satisfies the weak contraction condition (4). Obviously, T has no fixed point which contradicts Theorem 12.

Remark 6 In the same way, we can show that Theorem 13 implies the strong assertion (ii') of Theorem 9. Indeed, assuming that (ii') of Theorem 9 does not hold, i.e. for every $w \in W_0$ there exists some $\hat{w} \preceq_{k^0} w$ such that $\hat{w} \neq w$ we obtain that the mapping $T : W \to 2^W,\ T(w) := \{\hat{w} \in W \setminus \{w\} : \hat{w} \preceq_{k^0} w\}$ satisfies the weak contraction condition (6). Obviously, T has no fixed point which contradicts Theorem 13.

Remark 7 The above considerations show that Theorem 9 (minimal point theorem), Theorem 10 (variational principle) and Theorem 12/13 (Kirk-Caristi fixed point theorems) are mutually equivalent.

7 Conclusions and open questions

We proved a minimal point theorem in a product space $X \times Y$ where X is a separated uniform space and Y a topological vector space. The ordering structure in Y is generated by a convex cone $K \subset Y$. For the weak assertion of the minimal point theorem K does not have to be pointed. Moreover, we do not assume that X is complete nor Y is separated. Hence our minimal point theorem covers most of the known results of the field.

On the other hand, there are two main types of results which are not directly comparable to ours. First, results involving cone-valued metrics are not covered by Theorem 9, 10, see [26], [7], [6]. Using cone-valued metrics the class of possible order relations is larger than that one defined by (1).

Secondly, it seems to be not possible to give an equilibrium variant of Ekeland's principle for set-valued maps which is equivalent to Theorem 10. This equivalence has been established for the single-valued case and $Y = \mathbb{R}$ in [19], [15]. It seems to be that a more general principle than the Brézis-Browder principle, for instance the Altman principle [1], might be used to prove an vector-valued equilibrium variant with our assumptions. Moreover, for a set-valued equilibrium variant the "difference of sets" could be the critical point.

8 Historical comments

To the knowledge of the authors, Brønstedt provided an Ekeland-type theorem in uniform spaces first, see Theorem 2 of [4]. It did not involve families of quasi-metrics but a single perturbation function. An additional assumption is necessary to link the uniform structure to the properties of the perturbation function. This approach has been generalized e.g. by Park, see [27] and the references therein. The term "W-distance" is used by Park to denote such kind of perturbation function.

Mizoguchi [25] obtained an Ekeland-type principle on complete uniform spaces using pseudo-metrics. Theorem 2 of [25] is a very special case of our Corollary 11; assume X to be complete uniform, $Y = \mathbb{R}$, f lower semicontinuous, $\{q_\lambda\}_{\lambda\in\Lambda}$ a family of pseudo-metrics. Furthermore, the equivalence of Ekeland's principle, Caristi's fixed point theorem (both in complete uniform spaces, $Y = \mathbb{R}$) and the drop theorem in locally convex spaces was proven in [25].

On the other hand, Fang [9] introduced the concept of F-type topological spaces which we have shown to be an equivalent characterization of uniform spaces. He obtained scalar variants of Ekeland's principle and Caristi's fixed point theorem, so Corollary 11 and Corollary 14 also cover these results.

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