A New Approach to Duality in Vector Optimization

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Abstract

In this article we develop a new approach to duality theory for convex vector optimization problems. We modify a given (set-valued) vector optimization problem such that the image space becomes a complete lattice (a sublattice of the power set of the original image space), where the corresponding infimum and supremum are sets that are related to the set of (minimal and maximal) weakly efficient points. In doing so we can carry over the structures of the duality theory in scalar convex programming. Exemplarily this is demonstrated for the case of Fenchel duality. We also show the relationship to set-valued optimization based on the ordering "set inclusion". Finally some consequences for duality in linear vector optimization are discussed.

1 Introduction

It is an old idea to study additionally to a given optimization problem ($p(x) \to inf$ with infimal value I) a corresponding dual problem ($d(u) \to sup$ with supremal value $S, S \leq I$), remember the dual variational principles of Dirichlet and Thompson or simply the pair of dual programs in linear optimization. The reasons for the introduction of a useful dual problem are the following:

- The dual problem has (under additional conditions) the same optimal value as the given "primal" optimization problem, but solving the dual problem could be done with other methods of analysis or numerical mathematics.
- An approximate solution of the given minimization problem gives an estimation of the infimal value I from above, whereas an approximate solution of the dual problem is an estimation of I from below, so that one gets intervals containing I.
- Recalling Lagrange method, saddle points, equilibrium points of two person games, shadow prices in economics, perturbation methods or dual variational principles, it becomes clear, that optimal dual variables often have a special meaning for the given problem.

Of course, the just listed advantages require a skilfully chosen dual program. Nevertheless, the mentioned points are motivation enough, to look for dual problems in vector optimization with corresponding properties too. There are a lot of papers, which are dedicated to that aim, also a lot of survey papers (see the references in Jahn [10], [11]). In the literature there are several approaches to construct a dual problem for a given vector optimization problem. For instance, Luc [15] distinguishes between Conjugation, Lagrangian and Axiomatic Duality. However, there seems to be no unified approach to dualization in vector optimization. One of the difficulties is in the fact that the efficient solution in multi-objective optimization is not necessarily a single element, but in general becomes a subset of the image space. The definition of infimum (or supremum) of a set with partial order plays a key role in development of duality theory in multi-objective optimization. An interesting discussion of these aspects is given in the book by Pallaschke, Rolewicz [18] and in the paper by Nakayama [16]. There are at least three main ideas which are used for overcoming the difficulties that arise when generalizing well-known duality assertions from the scalar optimization theory to the vector-valued case.

The first one is the usage of scalarization in the formulation of the dual problem (see Schönfeld [23], Breckner [3], Jahn [9, 10]). In this approach, scalarization concepts and corresponding duality assertions from real-valued optimization are often used in order to derive useful dual problems, to prove duality assertions or in order to solve the dual problem. In a lot of papers this procedure is used in the proofs too (see e.g. [2]). However, as shown in [7], this approach has the disadvantage that even in the case of linear vector optimization a duality gap may occur, although the usual assumptions are fulfilled.

A second category of dual problems is based on the observation that a dual vector optimization problem is naturally set-valued (see Tanino, Sawaragi [28], Corley [4], Tanino [26, 27], Luc [15], Nakayama [16], Tammer [25], Dolecki, Malivert [5], Pallaschke, Rolewicz [18], Song [24], Hamel, Heyde, Löhne, Tammer, Winkler [7]). Duality assertions for vector optimization problems are shown without a scalarization "from the beginning". Instead, the dual problem becomes set-valued. For instance, in the paper by Tanino [27] the set-valued structure of the primal and dual vector optimization problem is taken into account: Embedding the primal problem into a family (depending from perturbation parameters) of set-valued optimization problems and applying an extension of Fenchel's inequality, Tanino derives a weak duality assertion and using the relationship between a map and its biconjugate he shows a strong duality statement. Furthermore, in the paper by Dolecki, Malivert [5] a set-valued approach in combination with Lagrangian techniques and perturbations of marginal relations is used in order to show duality assertions for general vector optimization problems where the solution concept is described by a transitive, translation-invariant relation.

A third type of dual problems is based on solution concepts with respect to the supremum and infimum in the sense of a vector lattice. In the book by Pallaschke and Rolewicz [18], special conditions concerning the order in the image space are supposed in order to prove duality assertions. A duality theory for objective functions with values in vector lattices is developed in [18] using corresponding notions of infimum and supremum in the sense of utopia minimum (maximum). However, these infima and suprema may not be solutions in the sense of vector optimization (Pareto) because they may not belong to the set of the objective function's values. In [18] notions of vector-convexity, vector-subgradients and vector duality are introduced. These notions are analogous to the scalar case taking into account the order in the vector lattice.

In the paper by Nieuwenhuis [17] solution concepts on the basis of infimal (supremal) sets are introduced because the assumption that the objective function has its values in a vector lattice is too restrictive for vector optimization. Nieuwenhuis [17] and Taninio [26, 27] derived duality assertions to these solution concepts. These infimal sets are closely related to weakly (Pareto) efficient elements. Dolecki and Malivert [5] extended these concepts to infimal sets being closely related to other kinds of efficiency, too.

In contrast to the these investigations our approach is characterized by an embedding of the the image space of the vector-valued problem into a complete lattice without linear structure, namely a sublattice of the power set of the image space. On the one hand, our primal and dual problems are set-valued and therefore related to the problems of the second type. The infimal (supremal) sets of Nieuwenhuis [17] are involved into the definition of the infimum (supremum) in the lattice which yields a relationship to the papers by Nieuwenhuis [17] and Taninio [26, 27]. On the other hand, a consequent usage of the lattice structure yields that we can carry over the formulations, statements and proof techniques from the scalar optimization theory, even though our assertions are for solution concepts in the sense of vector optimization (weakly (Pareto) efficient elements). In [17, 26, 27] the construction of the dual problem is not completely analogous to the scalar case because the lattice structure is not taken into account.

Furthermore, we point out the relationship between duality in vector optimization and duality for set-valued problems based on the ordering relation "set inclusion", in the sequel called *set inclusion problems* (this approach is also based on the lattice structure of the image space), which were investigated in [12, 13, 14].

Our paper is organized as follows: In Section 2 we introduce some notions. Several properties of infimal sets by Nieuwenhuis [17] and Tanino [26] are recalled. In the third section we introduce the hyperspaces $\mathcal{F}$ of upper closed sets and $\mathcal{I}$ of self-infimal sets and discuss the relationships between them. We point out the lattice structure of both spaces and show that the infimum and supremum in $\mathcal{I}$ can be expressed with the aid of infimal and supremal sets, respectively. In Section 4 we give a reformulation of a given vector optimization problem as an $\mathcal{I}$-valued problem and consider the corresponding dual problem. The relationship to a set inclusion problem, in our setting an $\mathcal{F}$-valued problem, is also discussed. In Section 5 we prove weak and strong duality assertions for our $\mathcal{F}$-valued problem, which is a special type of the set inclusion problems, investigated in [12, 13, 14]. Using the description of the vector optimization problem in complete lattices and the results from Section 5, in Section 6, we derive new duality results for vector optimization problems taking into account the natural set-valued structure of these problems. We show weak and strong duality assertions for weakly efficient elements of upper closed sets. Finally, we study in Section 7 the special case of linear vector optimization problems and apply our duality statements in order to show corresponding results for the linear case. So it is possible to understand better the structure of the dual linear vector optimization problem in comparison with results in the paper by Isermann [8].

In this paper we restrict ourselves to finite dimensional spaces, although there is no basic reason for that. We prefer to draw our attention to the structures rather than generality. An extension to the infinite dimensional case should be possible with some slight modifications.

2 Preliminaries

First we recall the definition of a *partially ordered conlinear space*, which plays an important role in the following. For more details and the links to similar concepts see Hamel [6].

Definition 2.1 ([6]) *A set Y equipped with an addition $+: Y \times Y \to Y$, a multiplication $\cdot : \mathbb{R}_+ \times Y \to Y$, and a partial ordering $\leq$ is said to be a* partially ordered conlinear space *with the neutral element $\theta \in Y$ if for all $y, y_1, y_2 \in Y$ and all $\alpha, \beta \geq 0$ the following axioms are satisfied:*

(C1) $y_1 + (y_2 + y) = (y_1 + y_2) + y,$
(C2) $y + \theta = y,$
(C3) $y_1 + y_2 = y_2 + y_1,$
(C4) $\alpha \cdot (\beta \cdot y) = (\alpha\beta) \cdot y,$
(C5) $1 \cdot y = y,$
(C6) $0 \cdot y = \theta,$
(C7) $\alpha \cdot (y_1 + y_2) = (\alpha \cdot y_1) + (\alpha \cdot y_2),$
(O1) $y_1 \leq y_2 \;\Rightarrow\; y_1 + y \leq y_2 + y,$
(O2) $y_1 \leq y_2 \;\Rightarrow\; \alpha y_1 \leq \alpha y_2.$

The axioms of a partially ordered conlinear space $(Y, +, \cdot, \leq)$ are appropriate to deal with convexity (even though a conlinear space is not a linear space, in general). For our reasons we only need the definition of a convex function with values in Y. Letting X be a linear space, we say that a function $f : X \to Y$ is convex if

$$\forall \lambda \in [0,1], \forall x_1, x_2 \in X : f\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda \cdot f(x_1) + (1-\lambda) \cdot f(x_2).$$

If $(Y, \leq)$ is additionally a complete lattice with largest element $+\infty$ we say that the domain of a function $f : X \to Y$ is the set $\operatorname{dom} f := \{x \in X |\; f(x) \neq +\infty\}$.

Next we collect some assertions on infimal sets, which are due to Nieuwenhuis [17] and were extended by Tanino [26]. We performed some slight modifications concerning the calculus with infinity, because this turned out to be useful for the following considerations.

Throughout the paper let $C \subsetneq \mathbb{R}^q$ be a closed convex cone with nonempty interior. By C° we denote its polar cone, i.e., $C^\circ := \{y^* \in \mathbb{R}^q |\; \forall y \in C : \langle y^*, y\rangle \leq 0\}$.

The set of *minimal* or *weakly efficient* points of a subset $A \subseteq \mathbb{R}^q$ (with respect to C) is defined by

$$\operatorname{Min} A := \{y \in A |\; (\{y\} - \operatorname{int} C) \cap A = \emptyset\}.$$

The *upper closure* (with respect to C) of $A \subseteq \mathbb{R}^q$ is defined [5] to be the set

$$\operatorname{Cl}_+ A := \{y \in \mathbb{R}^q |\; \{y\} + \operatorname{int} C \subseteq A + \operatorname{int} C\}.$$

It is an easy task to show that $\operatorname{Cl}_+ A = \operatorname{cl}(A + \operatorname{int} C)$.

If $A \neq \emptyset$ we have [17, Th. I-18]

$$\operatorname{Min} \operatorname{Cl}_+ A = \emptyset \iff A + \operatorname{int} C = \mathbb{R}^q \iff \operatorname{Cl}_+ A = \mathbb{R}^q.$$

Before we recall the definition of infimal sets, we want to extend the upper closure for subsets of the space $\overline{\mathbb{R}}^q := \mathbb{R}^q \cup \{-\infty, \infty\}$. We use the usual calculus rules in $\overline{\mathbb{R}}^q$, in particular, we set $0 \cdot (+\infty) = 0 = 0 \cdot (-\infty)$. Because we mainly consider minimization problems, we will use the *inf-addition* (see [21]), i.e., $+\infty - \infty = -\infty + \infty = \infty$. Transforming a minimization problem into a maximization problem, we have to take into account the replacement of this rule by the *sup-addition*, i.e. $+\infty - \infty = -\infty + \infty = -\infty$. For a subset $A \subseteq \overline{\mathbb{R}}^q$ we set

$$\mathrm{Cl}_+ A := \begin{cases} \mathbb{R}^q & \text{if} \quad -\infty \in A \\ \emptyset & \text{if} \quad A = \{+\infty\} \\ \{y \in \mathbb{R}^q |\ \{y\} + \mathrm{int}\, C \subseteq A + \mathrm{int}\, C\} & \text{else.} \end{cases}$$

Note that the upper closure of a subset of $\overline{\mathbb{R}}^q$ is always a subset in $\mathbb{R}^q$. The *infimal set* of $A \subseteq \overline{\mathbb{R}}^q$ (with respect to C) is defined by

$$\mathrm{Inf}\, A := \begin{cases} \mathrm{Min}\, \mathrm{Cl}_+ A & \text{if} \quad \emptyset \neq \mathrm{Cl}_+ A \neq \mathbb{R}^q \\ \{-\infty\} & \text{if} \quad \mathrm{Cl}_+ A = \mathbb{R}^q \\ \{+\infty\} & \text{if} \quad \mathrm{Cl}_+ A = \emptyset. \end{cases}$$

This means that the *infimal set of A with respect to C* coincides essentially with the set of weakly efficient elements of the set $\mathrm{cl}\,(A + C)$ with respect to C.

By our conventions, $\mathrm{Inf}\, A$ is always a nonempty set. Clearly, if $-\infty$ belongs to A, we have $\mathrm{Inf}\, A = \{-\infty\}$, in particular, $\mathrm{Inf}\,\{-\infty\} = \{-\infty\}$. Moreover, we have $\mathrm{Inf}\, \emptyset = \mathrm{Inf}\,\{+\infty\} = \{+\infty\}$. Furthermore, it holds $\mathrm{Cl}_+ A = \mathrm{Cl}_+(A \cup \{+\infty\})$ and hence $\mathrm{Inf}\, A = \mathrm{Inf}(A \cup \{+\infty\})$ for all $A \subseteq \overline{\mathbb{R}}^q$.

Remark 2.2 The definition of *infimal points* in the paper by Dolecki, Malivert [5] (compare also Postolica [19]) as minimal points of the upper closure of a set is given in a more general way for vector optimization problems with a real linear image space where the solution concept is considered with respect to a transitive relation supposed to be translation invariant.

The following assertions were proved by Nieuwenhuis [17] and, in an extended form, by Tanino [26].

Proposition 2.3 *For $A \subseteq \mathbb{R}^q$ with $\emptyset \neq \mathrm{Cl}_+ A \neq \mathbb{R}^q$ it holds*

(i) $\mathrm{Inf}\, A = \{y \in \mathbb{R}^q |\ y \notin A + \mathrm{int}\, C,\ \{y\} + \mathrm{int}\, C \subseteq A + \mathrm{int}\, C\}$,

(ii) $A + \mathrm{int}\, C = B + \mathrm{int}\, C \iff \mathrm{Inf}\, A = \mathrm{Inf}\, B$,

(iii) $A + \mathrm{int}\, C = \mathrm{Inf}\, A + \mathrm{int}\, C$,

(iv) $\mathrm{Cl}_+ A = \mathrm{Inf}\, A \cup (\mathrm{Inf}\, A + \mathrm{int}\, C)$,

(v) $\mathrm{Inf}\, A$, $(\mathrm{Inf}\, A - \mathrm{int}\, C)$ *and* $(\mathrm{Inf}\, A + \mathrm{int}\, C)$ *are disjoint,*

(vi) $\mathrm{Inf}\, A \cup (\mathrm{Inf}\, A - \mathrm{int}\, C) \cup (\mathrm{Inf}\, A + \mathrm{int}\, C) = \mathbb{R}^q$.

Proposition 2.4 *For $A \subseteq \overline{\mathbb{R}}^q$ it holds*

(i) $\mathrm{Inf}\,\mathrm{Inf}\, A = \mathrm{Inf}\, A$, $\mathrm{Cl}_+\mathrm{Cl}_+ A = \mathrm{Cl}_+ A$, $\mathrm{Inf}\,\mathrm{Cl}_+ A = \mathrm{Inf}\, A$, $\mathrm{Cl}_+\,\mathrm{Inf}\, A = \mathrm{Cl}_+ A$,

(ii) $\operatorname{Inf}(\operatorname{Inf} A + \operatorname{Inf} B) = \operatorname{Inf}(A + B)$,

(iii) $\alpha \operatorname{Inf} A = \operatorname{Inf}(\alpha A)$ *for* $\alpha > 0$.

Analogously, we define the set $\operatorname{Max} A$ of maximal elements of $A \subseteq \mathbb{R}^q$, as well as the lower closure $\operatorname{Cl}_- A$ and the set $\operatorname{Sup} A$ of supremal elements of $A \subseteq \overline{\mathbb{R}}^q$ (underlying the sup-addition) and we have analogous statements.

3 Hyperspaces of upper closed sets and self-infimal sets

Let $\mathcal{F} := \mathcal{F}_C(\mathbb{R}^q)$ be the family of all subsets of $\mathbb{R}^q$ with $\operatorname{Cl}_+ A = A$. In $\mathcal{F}$ we introduce an addition $\oplus_{\mathcal{F}} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a multiplication by nonnegative real numbers $\odot_{\mathcal{F}} : \mathbb{R}_+ \times \mathcal{F} \to \mathcal{F}$ and an order relation $\preccurlyeq_{\mathcal{F}}$ as follows:

$$A \oplus_{\mathcal{F}} B := A \oplus B := \operatorname{cl}(A + B),$$

$$\alpha \odot_{\mathcal{F}} A := \begin{cases} \alpha \cdot A & \text{if} \quad \alpha > 0 \\ C & \text{if} \quad \alpha = 0, \end{cases}$$

$$A \preccurlyeq_{\mathcal{F}} B :\iff A \supseteq B.$$

Proposition 3.1 *The space* $(\mathcal{F}, \oplus, \odot_{\mathcal{F}}, \supseteq)$ *is a partially ordered conlinear space.*

Proof. This is obvious. $\square$

Let $\mathcal{I} := \mathcal{I}_C(\overline{\mathbb{R}}^q)$ be the family of all self-infimal subsets of $\overline{\mathbb{R}}^q$, i.e., all sets $A \subseteq \overline{\mathbb{R}}^q$ satisfying $\operatorname{Inf} A = A$. In $\mathcal{I}$ we introduce an addition $\oplus_{\mathcal{I}} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, a multiplication by nonnegative real numbers $\odot_{\mathcal{I}} : \mathbb{R}_+ \times \mathcal{I} \to \mathcal{I}$ and an order relation $\preccurlyeq_{\mathcal{I}}$ as follows:

$$A \oplus_{\mathcal{I}} B := \operatorname{Inf}(A + B),$$

$$\alpha \odot_{\mathcal{I}} A := \begin{cases} \alpha \cdot A & \text{if} \quad \alpha > 0 \\ \operatorname{bd} C & \text{if} \quad \alpha = 0, \end{cases}$$

$$A \preccurlyeq_{\mathcal{I}} B :\iff A \preccurlyeq B :\iff \operatorname{Cl}_+ A \supseteq \operatorname{Cl}_+ B.$$

Note that the definition of $\oplus_{\mathcal{I}}$ is based on the inf-addition in $\overline{\mathbb{R}}^q$. As a consequence we obtain $\{-\infty\} \oplus_{\mathcal{I}} \{\infty\} = \{\infty\}$. In the space of self-supremal sets we would have to underlie the sup-addition in $\overline{\mathbb{R}}^q$.

Proposition 3.2 *The space* $(\mathcal{I}, \oplus_{\mathcal{I}}, \odot_{\mathcal{I}}, \preccurlyeq)$ *is a partially ordered conlinear space.*

Proof. Exemplarily we show (C1). Let $A_1, A_2, A_3 \in \mathcal{I}$.

(i) If $\operatorname{Cl}_+ A_i = \emptyset$ for some i we have $(A_1 \oplus_{\mathcal{I}} A_2) \oplus_{\mathcal{I}} A_3 = \{+\infty\} = A_1 \oplus_{\mathcal{I}} (A_2 \oplus_{\mathcal{I}} A_3)$.

(ii) If $\operatorname{Cl}_+ A_i \neq \emptyset$ $(i = 1, 2, 3)$ and $\operatorname{Cl}_+ A_i = \mathbb{R}^q$ for some i we have $(A_1 \oplus_{\mathcal{I}} A_2) \oplus_{\mathcal{I}} A_3 = \{-\infty\} = A_1 \oplus_{\mathcal{I}} (A_2 \oplus_{\mathcal{I}} A_3)$.

(iii) It remains the case where $\emptyset \neq \mathrm{Cl}_+ A_i \neq \mathbb{R}^q$ $(i = 1, 2, 3)$. By Proposition 2.3 (iii), we have $(A_1 \oplus_{\mathcal{I}} A_2) \oplus_{\mathcal{I}} A_3 + \operatorname{int} C = A_1 + A_2 + A_3 + \operatorname{int} C = A_1 \oplus_{\mathcal{I}} (A_2 \oplus_{\mathcal{I}} A_3) + \operatorname{int} C$. By Proposition 2.3 (ii) and 2.4 (i) we obtain $(A_1 \oplus_{\mathcal{I}} A_2) \oplus_{\mathcal{I}} A_3 = A_1 \oplus_{\mathcal{I}} (A_2 \oplus_{\mathcal{I}} A_3)$. $\square$

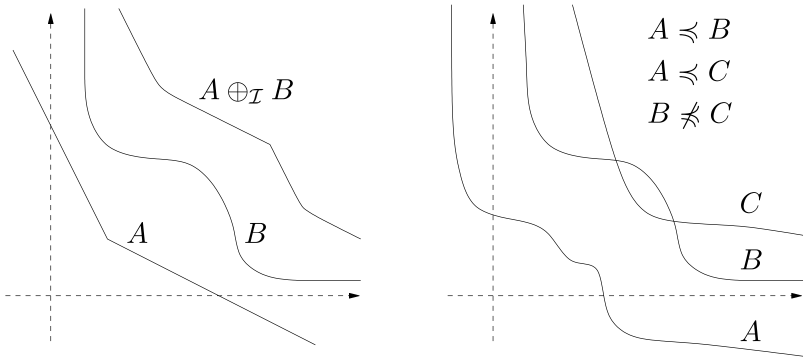

Figure 3.1. The addition and the ordering in $\mathcal{I}$ for $C = \mathbb{R}^2_+$.

Proposition 3.3 *The spaces $(\mathcal{F}, \oplus, \odot_{\mathcal{F}}, \supseteq)$ and $(\mathcal{I}, \oplus_{\mathcal{I}}, \odot_{\mathcal{I}}, \preccurlyeq)$ are isomorphic and isotone. The corresponding bijection is given by*

$$j : \mathcal{F} \to \mathcal{I}, \quad j(\,\cdot\,) = \mathrm{Inf}(\,\cdot\,), \quad j^{-1}(\,\cdot\,) = \mathrm{Cl}_+(\,\cdot\,).$$

Proof. By Proposition 2.4 (i), j is a bijection between $\mathcal{F}$ and $\mathcal{I}$.

For $A_1, A_2 \in \mathcal{F}$, we have $j(A_1) \oplus_{\mathcal{I}} j(A_2) = j(A_1 \oplus A_2)$. In the case where $\emptyset \neq \mathrm{Cl}_+ A_i \neq \mathbb{R}^q$ $(i = 1, 2)$ this follows from Proposition 2.4 (ii), otherwise it can be obtained directly.

Similarly, we can easily verify that for $\alpha \geq 0$ and $A, B \in \mathcal{F}$ we have

$$\alpha \odot_{\mathcal{I}} j(A) = j(\alpha \odot_{\mathcal{F}} A) \quad \text{and} \quad A \supseteq B \iff j(A) \preccurlyeq j(B).$$ $\square$

Proposition 3.4 $(\mathcal{F}, \supseteq)$ *and* $(\mathcal{I}, \preccurlyeq)$ *are complete lattices. For nonempty subsets $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{I}$ the infimum and supremum can be expressed by*

$$\inf \mathcal{A} = \mathrm{cl} \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A,$$

$$\inf \mathcal{B} = \mathrm{Inf} \bigcup_{B \in \mathcal{B}} \mathrm{Cl}_+ B, \qquad \sup \mathcal{B} = \mathrm{Inf} \bigcap_{B \in \mathcal{B}} \mathrm{Cl}_+ B.$$

Proof. For the space $(\mathcal{F}, \supseteq)$ the statements are obvious and for $(\mathcal{I}, \preccurlyeq)$ they follow from Proposition 3.3. $\square$

As usual, if $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{I}$ are empty we define the infimum (supremum) to be the largest (smallest) element in the corresponding complete lattice, i.e., $\inf \mathcal{A} = \emptyset$, $\sup \mathcal{A} = \mathbb{R}^q$, $\inf \mathcal{B} = \{+\infty\}$ and $\sup \mathcal{B} = \{-\infty\}$.

It follows the main result of this section, which shows that the infimum as well as the supremum in $\mathcal{I}$ can be expressed in terms that frequently are used in vector optimization (compare [17], [26], [27], [5]), but up to now not in the framework of complete lattices (see Figure 3.2).

Theorem 3.5 *For nonempty sets $\mathcal{B} \subseteq \mathcal{I}$ it holds*

$$\inf \mathcal{B} = \operatorname{Inf} \bigcup_{B \in \mathcal{B}} B, \qquad \sup \mathcal{B} = \operatorname{Sup} \bigcup_{B \in \mathcal{B}} B.$$

Proof. (i) It holds $\inf \mathcal{B} = \operatorname{Inf} \bigcup_{B \in \mathcal{B}} \operatorname{Cl}_+ B = \operatorname{Inf} \operatorname{Cl}_+ \bigcup_{B \in \mathcal{B}} \operatorname{Cl}_+ B = \operatorname{Inf} \operatorname{Cl}_+ \bigcup_{B \in \mathcal{B}} B = \operatorname{Inf} \bigcup_{B \in \mathcal{B}} B$.

(ii) We have to show that $\operatorname{Sup} \bigcup_{B \in \mathcal{B}} B = \operatorname{Inf} \bigcap_{B \in \mathcal{B}} \operatorname{Cl}_+ B$.

a) If $\{+\infty\} \in \mathcal{B}$ we have $+\infty \in \bigcup_{B \in \mathcal{B}} B$ and hence $\operatorname{Sup} \bigcup_{B \in \mathcal{B}} B = \{+\infty\}$. On the other hand, since $\operatorname{Cl}_+ \{+\infty\} = \emptyset$, we have $\operatorname{Inf} \bigcap_{B \in \mathcal{B}} \operatorname{Cl}_+ B = \operatorname{Inf} \emptyset = \{+\infty\}$.

b) Let $\{+\infty\} \notin \mathcal{B}$ but $\{-\infty\} \in \mathcal{B}$. If $\{-\infty\}$ is the only element in $\mathcal{B}$ the assertion is obvious, otherwise we can omit this element without changing the expressions.

c) Let $\{+\infty\} \notin \mathcal{B}$ and $\{-\infty\} \notin \mathcal{B}$. Then, $B \subseteq \mathbb{R}^q$ and $\emptyset \neq \operatorname{Cl}_+ B \neq \mathbb{R}^q$ for all $B \in \mathcal{B}$, i.e., we can use the statements of Proposition 2.3. Define the sets $V := \bigcup_{B \in \mathcal{B}} (B - \operatorname{int} C) = \left(\bigcup_{B \in \mathcal{B}} B\right) - \operatorname{int} C$ and $W := \bigcap_{B \in \mathcal{B}} \operatorname{Cl}_+ B$.

We show that $V \cap W = \emptyset$ and $V \cup W = \mathbb{R}^q$. Assume there exists some $y \in V \cap W$. Hence there is some $\bar{B} \in \mathcal{B}$ such that $y \in \left(\bar{B} - \operatorname{int} C\right) \cap \operatorname{Cl}_+ \bar{B} = \emptyset$, a contradiction. Let $y \in \mathbb{R}^q \setminus W$ (we have $W \neq \mathbb{R}^q$, because otherwise it holds $\operatorname{Cl}_+ B = \mathbb{R}^q$ all $B \in \mathcal{B}$ and hence $\{-\infty\} \in \mathcal{B}$). Then there exists some $\bar{B} \in \mathcal{B}$ such that $y \notin \operatorname{Cl}_+ \bar{B}$. By Proposition 2.3 (iv), (vi) we obtain $y \in \bar{B} - \operatorname{int} C \subseteq V$.

If $V = \mathbb{R}^q$ we have $W = \emptyset$, hence $\operatorname{Sup} \bigcup_{B \in \mathcal{B}} B = \operatorname{Sup} V = \{+\infty\} = \operatorname{Inf} \emptyset = \operatorname{Inf} W$. Otherwise, we have $\emptyset \neq V \neq \mathbb{R}^q$ and $\emptyset \neq W \neq \mathbb{R}^q$. By Proposition 2.3, we obtain

$$\begin{aligned} \operatorname{Sup} \bigcup_{B \in \mathcal{B}} B &= \{y \in \mathbb{R}^q \mid y \notin V,\ \{y\} - \operatorname{int} C \subseteq V\} \\ &= \{y \in \mathbb{R}^q \mid y \in W, (\{y\} - \operatorname{int} C) \cap W = \emptyset\} \\ &= \operatorname{Min} W = \operatorname{Min} \operatorname{Cl}_+ W = \operatorname{Inf} W. \end{aligned}$$ $\square$

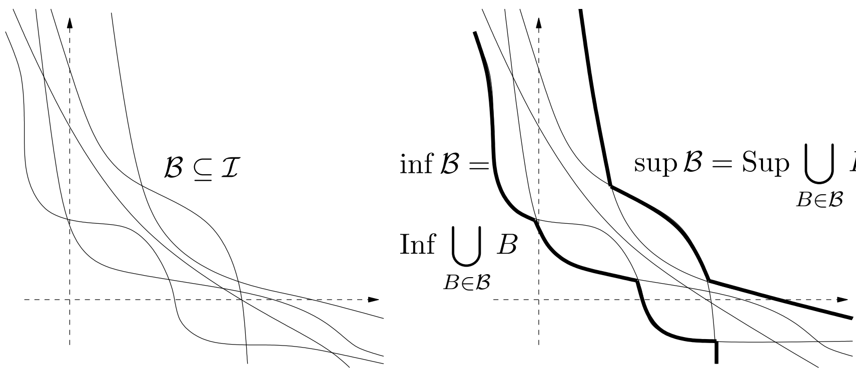

Figure 3.2. The infimum and supremum in $\mathcal{I}$ for $C = \mathbb{R}^2_+$.

In the same manner like $\mathcal{F}$ and $\mathcal{I}$ we define the space $\mathcal{F}^\diamond$ of lower closed subsets of $\mathbb{R}^q$ and the space $\mathcal{S}$ of self-supremal subsets of $\overline{\mathbb{R}}^q$, where we underlie the sup-addition in $\overline{\mathbb{R}}^q$ in the latter case.

4 Duality for $\mathcal{F}$-valued and $\mathcal{I}$-valued functions and application to vector optimization

In this section we discuss the relationship between a vector optimization problem and optimization problems for $\mathcal{F}$-valued and $\mathcal{I}$-valued functions. Since $\mathcal{F}$ and $\mathcal{I}$ are complete lattices we can assign to a given problem a dual problem following the lines of scalar duality theory. In the next sections we show, for the example of Fenchel duality, that under the usual assumptions weak as well as strong duality assertions can be obtained. By the fact $\mathcal{F}$ and $\mathcal{I}$ being isomorphic and isotone it is clear that it is sufficient to prove the duality assertions just for one case, either for the $\mathcal{F}$-valued case or for the $\mathcal{I}$-valued case.

The advantage of the $\mathcal{F}$-valued case is that the operations, the ordering and the infimum and supremum have an easier structure, which is beneficial for proofs. In this case we speak about *set inclusion problems*.

The advantage of the $\mathcal{I}$-valued case is that it is closely related to vector optimization problems, therefore let us speak about *vector optimization problems* in this case. This relationship can be seen as follows. For an arbitrary set X, consider the following vector optimization problem with set-valued objective map $P : X \rightrightarrows \mathbb{R}^q$.

$$\text{(VOP)} \qquad \bar{P} := \operatorname{Inf} \bigcup_{x \in X} P(x).$$

Since $\operatorname{Inf} \bigcup_{x\in X} P(x) = \operatorname{Inf} \bigcup_{x\in X} \operatorname{Inf} P(x)$, (VOP) can be expressed as an $\mathcal{I}$-valued problem; without loss of generality we can assume that the sets $P(x)$ are self-infimal, i.e., $P : X \to \mathcal{I}$. Thus we consider the following problem.

$$\text{(P)} \qquad \bar{P} = \operatorname{Inf} \bigcup_{x \in X} P(x) = \inf_{x \in X} P(x).$$

We assign to (P) a dual problem (D): Let V be a set and $D : V \to \mathcal{I}$,

$$\text{(D)} \qquad \bar{D} := \operatorname{Sup} \bigcup_{v \in V} D(v) = \sup_{v \in V} D(v).$$

As usual, we speak about *weak duality* between (P) and (D) when $\bar{D} \preccurlyeq \bar{P}$ and we speak about *strong duality* when $\bar{D} = \bar{P}$. In contrast to the primal problem the self-infimality of the values of the dual objective function $D(\cdot)$ plays an important role. If we replace a value $D(v)$ by $\hat{D}(v)$ with $\operatorname{Inf} D(v) = \operatorname{Inf} \hat{D}(v)$ but $\hat{D}(v)$ being not self-infimal, $\bar{D}$ might be changed. Therefore we shall understand (D) itself as the dual problem to (VOP).

Remark 4.1 The vector optimization problem (VOP) means that we compute weakly efficient elements of the upper closure (elements belonging to the infimal set) of the image set with respect to the closed convex cone C with nonempty interior. We express the vector optimization problem by the $\mathcal{I}$-valued problem (P) in order to use a complete lattice structure and to derive assertions analogously to the scalar optimization theory.

In the dual problem (D) we study the problem to determine weakly efficient elements of the lower closure (elements belonging to the supremal set) of the dual image set with respect to the cone C.

Also in the case that the primal problem (VOP) is point-valued, the dual problem (D) is always a set-valued problem. Indeed, by many authors (see Corley [4], Tanino [26, 27], Luc [15], Dolecki, Malivert [5], Pallaschke, Rolewicz [18]) it was observed that the dual of a vector optimization problem is "naturally" set-valued. Taking into account the space $\mathcal{I}$ and its lattice structure we give an explanation for this by our approach.

Remark 4.2 In case of $\bar{D}, \bar{P} \subseteq \mathbb{R}^q$, the weak duality inequality $\bar{D} \preccurlyeq \bar{P}$ can be equivalently expressed by $(\bar{D} - \operatorname{int} C) \cap \bar{P} = \emptyset$, what is a well-known relation in vector optimization. Indeed, $\bar{D} \preccurlyeq \bar{P}$ is equivalent to $\bar{P} \subseteq \operatorname{Cl}_+ \bar{D}$ and by Proposition 2.3 (iv), (v), (vi) this is equivalent to $(\bar{D} - \operatorname{int} C) \cap \bar{P} = \emptyset$.

As mentioned above it is useful to consider a dual pair of $\mathcal{F}$-valued problems simultaneously. For $p : X \to \mathcal{F},\ p(x) := \operatorname{Cl}_+ P(x)$ and $d : V \to \mathcal{F},\ d(u) := \operatorname{Cl}_+ D(u)$ consider the problems

(p) $$\bar{p} := \operatorname{cl} \bigcup_{x \in X} p(x) = \inf_{x \in X} p(x),$$

(d) $$\bar{d} := \bigcap_{u \in V} d(u) = \sup_{u \in V} d(u).$$

We define weak and strong duality in the usual way, i.e., by $\bar{d} \supseteq \bar{p}$ (that is $\bar{d} \preccurlyeq_{\mathcal{F}} \bar{p}$) and $\bar{d} = \bar{p}$, respectively.

Of course, by Proposition 3.3 we have $\bar{P} = \operatorname{Inf} \bar{p}$, $\bar{D} = \operatorname{Inf} \bar{d}$, $\operatorname{Cl}_+ \bar{P} = \bar{p}$ and $\operatorname{Cl}_+ \bar{D} = \bar{d}$ and we have weak duality between (P) and (D) if and only if we have weak duality between (p) and (d), and we have strong duality between (P) and (D) if and only if we have strong duality between (p) and (d), i.e., we have

$$\bar{D} \preccurlyeq \bar{P} \qquad \Longleftrightarrow \qquad \bar{d} \supseteq \bar{p},$$

$$\bar{D} = \bar{P} \qquad \Longleftrightarrow \qquad \bar{d} = \bar{p}.$$

In the next section we prove weak and strong duality assertions for the set inclusion problems (p) and (d) in order to derive corresponding duality assertions for the vector optimization problems (P) and (D).

5 Fenchel duality for set inclusion problems

In this section we prove a duality theorem for optimization problems with $\mathcal{F}$-valued objective function and being based on the order relation "set inclusion". A more general result for closed (but not necessarily upper closed) sets was recently obtained in [14], [13].

In the sequel we set $X = X^* = \mathbb{R}^n$ and $U = U^* = \mathbb{R}^m$ (see Remark 5.5 below). Let $f : X \to \mathcal{F}$ and $c \in \mathbb{R}^q$. The function $f_c^* : X^* \to \mathcal{F}^\diamond$ (where $\mathcal{F}^\diamond$ is the space of lower closed subsets of $\mathbb{R}^q$), defined by

$$f_c^*(x^*) := - \inf_{x \in X} \left\{ f(x) - \langle x^*, x \rangle \cdot \{c\} \right\},$$

is said to be the *conjugate of* f with respect to c.

Remark 5.1 In order to avoid calculations in the space $\mathcal{F}^\diamond$ of lower closed subsets of $\mathbb{R}^q$ we will prefer to use the term $-f_c^*(x^*) \in \mathcal{F}$ rather than $f_c^*(x^*) \in \mathcal{F}^\diamond$ in the following. Nevertheless a calculus in $\mathcal{F}^\diamond$ is possible in the same way as in $\mathcal{F}$, if we replace the inf-addition in $\overline{\mathbb{R}}^q$ by the sup-addition in $\overline{\mathbb{R}}^q$, C by $-C$, and $\subseteq$ by $\supseteq$ in the definition of the order relation. By a consequent usage of the notions in $\mathcal{F}^\diamond$ we could express the conjugate as $f_c^*(x^*) = \sup_{x\in X} \{\langle x^*, x\rangle \cdot \{c\} - f(x)\}$, where "sup" now means the supremum in $\mathcal{F}^\diamond$. For more details on this kind of duality, using the concept of oriented sets by Rockafellar [20], see [14], [13].

For given functions $f : X \to \mathcal{F}$ and $g : U \to \mathcal{F}$, a linear map $A : X \to U$ and a vector $c \in \mathbb{R}^q$, let

$$p : X \to \mathcal{F} \quad \text{and} \quad d_c : U^* \to \mathcal{F}$$

be defined, respectively, by

$$p(x) = f(x) \oplus g(Ax) \quad \text{and} \quad d_c(u^*) = -\big(f_c^*(A^T u^*) \oplus g_c^*(-u^*)\big).$$

We consider the following optimization problems, the primal problem

$$\text{(p)} \qquad \bar{p} := \inf_{x\in X} p(x) = \operatorname{cl} \bigcup_{x\in X} p(x),$$

and the dual problem associated to (p)

$$(\mathrm{d_c}) \qquad \bar{d}_c := \sup_{u^*\in U^*} d_c(u^*) = \bigcap_{u^*\in U^*} d_c(u^*).$$

Note that the convexity of a function $f : X \to \mathcal{F}$ (defined in Section 2) is equivalent to the convexity of the graph of the corresponding set-valued map. In particular, if f and g are convex in problem (p), the values $\bar{p}$ and $\bar{d}_c$ are convex sets. The following result is a special case of [14, Theorem 4.3].

Theorem 5.2 *The problems* (p) *and* ($\mathrm{d_c}$) *(with arbitrary $c \in \mathbb{R}^q$) satisfy the weak duality inequality, i.e., $\bar{d}_c \supseteq \bar{p}$. Furthermore, let f and g be convex, let $0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f)$ and $c \in \operatorname{int} C$, then we have strong duality, i.e., $\bar{d}_c = \bar{p}$.*

Proof. The weak duality is obvious from the definition. In order to prove the strong duality assertion we use a scalarization method by the support function $\sigma_A : \mathbb{R}^q \to \overline{\mathbb{R}}$ with respect to $A \subseteq \mathbb{R}^q$,

$$\sigma_A(y^*) := \sigma(y^*|A) := \sup_{y\in A} \langle y^*, y\rangle\,,$$

where $\overline{\mathbb{R}}$ is equipped with the sup-addition (i.e., $\infty - \infty = -\infty$). Note that for (not necessarily nonempty) subsets $A, B \subseteq \mathcal{F}$ we have $\sigma_A + \sigma_B = \sigma_{A\oplus B}$ and $\alpha\sigma_A = \sigma_{\alpha A}$ for $\alpha > 0$. Moreover, for all (not necessarily nonempty) sets $\mathcal{A} \subseteq \mathcal{F}$ we have $-\sigma_{\inf \mathcal{A}} = \inf_{A\in\mathcal{A}} -\sigma_A$ (see e.g. [20, Corollary 16.5.1]).

It holds $-\sigma_{\bar{p}} = \inf_{x\in X} -\sigma_{p(x)}$. By the extended real–valued functions $\bar{f}_{y^*}: X \to \overline{\mathbb{R}}$ and $\bar{g}_{y^*}: U \to \overline{\mathbb{R}}$ being defined, respectively, by $\bar{f}_{y^*}(x) := -\sigma\big(y^* |\, f(x)\big)$ and $\bar{g}_{y^*}(u) := -\sigma\big(y^* |\, g(u)\big)$ this can be rewritten as a collection of scalar optimization problems

$$\forall y^* \in \mathbb{R}^p: \; -\sigma(y^* |\, \bar{p}) = \inf_{x\in X} \big\{\bar{f}_{y^*}(x) + \bar{g}_{y^*}(Ax)\big\}. \tag{1}$$

The convexity of f and g implies the convexity of $\bar{f}_{y^*}$ and $\bar{g}_{y^*}$, respectively. Clearly, we have $\operatorname{dom} f = \operatorname{dom} \bar{f}_{y^*}$ and $\operatorname{dom} g = \operatorname{dom} \bar{g}_{y^*}$, whence $0 \in \operatorname{ri}(\operatorname{dom} \bar{g}_{y^*} - A \operatorname{dom} \bar{f}_{y^*})$. A scalar duality theorem, for instance [1, Theorem 3.3.5], yields that

$$\forall y^* \in \mathbb{R}^p: \; -\sigma(y^* |\, \bar{p}) = \sup_{u^* \in U^*} \left\{-\bar{f}^*_{y^*}(A^T u^*) - \bar{g}^*_{y^*}(-u^*)\right\},$$

where the supremum is attained whenever $-\sigma(y^* |\, \bar{p})$ is finite, i.e.,

$$\forall y^* \in \operatorname{dom} \sigma(\,\cdot\,|\, \bar{p}),\, \exists \bar{u}^* \in U^*: \; -\sigma(y^* |\, \bar{p}) = -\bar{f}^*_{y^*}(A^T \bar{u}^*) - \bar{g}^*_{y^*}(-\bar{u}^*). \tag{2}$$

Let $y^* \in \operatorname{dom} \sigma(\,\cdot\,|\, \bar{p})$ be arbitrarily given (hence $\bar{p} \neq \mathbb{R}^q$). Since $\emptyset \neq \bar{p} \neq \mathbb{R}^q$ and $\bar{p} = \operatorname{Cl}_+ \bar{p}$, we have $\operatorname{dom} \sigma(\,\cdot\,|\, \bar{p}) \subseteq C^\circ$. By the choice $c \in \operatorname{int} C$, it follows that $\langle y^*, c\rangle < 0$. Hence, there exists $\alpha_{y^*} > 0$ such that $\langle \alpha_{y^*} y^*, c\rangle = -1$. This can be rewritten as

$$\forall t \in \mathbb{R}: \;\; -\sigma\big(\alpha_{y^*} y^* |\, \{t \cdot c\}\big) = -\left\langle \alpha_{y^*} y^*, t \cdot c\right\rangle = t. \tag{3}$$

For $\alpha := \alpha_{y^*} > 0$ we have

$$\begin{aligned}
\alpha \cdot \big(-\sigma\,(y^* |\, \bar{p})\big) = -\sigma\,(\alpha y^* |\, \bar{p}) \;\; &\overset{(2)}{=} \;\; -\bar{f}^*_{\alpha y^*}(A^T \bar{u}^*) - \bar{g}^*_{\alpha y^*}(-\bar{u}^*) \\
= \;\; & \inf_{x\in X} \big\{-\left\langle A^T \bar{u}^*, x\right\rangle + \bar{f}_{\alpha y^*}(x)\big\} + \inf_{u \in U} \big\{\langle \bar{u}^*, u\rangle + \bar{g}_{\alpha y^*}(u)\big\} \\
\overset{(3)}{=} \;\; & \inf_{x\in X} \left\{-\sigma\big(\alpha y^* \big| - \left\langle A^T \bar{u}^*, x\right\rangle \cdot \{c\}\big) - \sigma\big(\alpha y^* \big| f(x)\big)\right\} \\
+ \;\; & \inf_{u\in U} \left\{-\sigma\big(\alpha y^* \big| \langle \bar{u}^*, u\rangle \cdot \{c\}\big) - \sigma\big(\alpha y^* \big| g(u)\big)\right\} \\
= \;\; & -\sigma\left(\alpha y^* \middle| \inf_{x\in X} \big\{-\left\langle A^T \bar{u}^*, x\right\rangle\{c\} + f(x)\big\} \oplus \inf_{u\in U} \big\{\langle \bar{u}^*, u\rangle\{c\} + g(u)\big\}\right) \\
= \;\; & -\sigma\big(\alpha y^* \big| - f^*_c(A^T \bar{u}^*) \oplus -g^*_c(-\bar{u}^*)\big) = -\sigma\,(\alpha y^* | d_c(\bar{u}^*)) = \alpha \cdot \big(-\sigma\,(y^* | d_c(\bar{u}^*))\big).
\end{aligned}$$

We deduce that

$$\forall y^* \in \operatorname{dom} \sigma(\,\cdot\,|\, \bar{p}\,),\; \exists \bar{u}^* \in U^*: \quad \sigma(y^* | d_c(\bar{u}^*)) = \sigma(y^* |\, \bar{p}\,). \tag{4}$$

It follows that $\sigma(y^* |\, \bar{d}_c\,) \leq \sigma(y^* |\, \bar{p}\,)$ for all $y^* \in \mathbb{R}^q$, hence ($\bar{d}_c$ and $\bar{p}$ being closed and convex) $\bar{d}_c \subseteq \bar{p}$. By the weak duality inequality we obtain $\bar{d}_c = \bar{p}$. $\square$

Remark 5.3 Note that we have not the usual dual attainment assertion, i.e., the dual value $\bar{d}_c$ is not attained by a single element $\bar{u}^* \in U^*$. Instead we have condition (4), which describes the present situation. In the next section we will use this condition in order to obtain a kind of "dual attainment" in the vector optimization setting.

Remark 5.4 In Theorem 5.2 (and its conclusion Theorem 6.2) we suppose the constraint qualification $0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f)$. In the proof we use this condition in order to obtain the corresponding condition for the family of scalarized problems in (1). If all these problems are polyhedral, the constraint qualification can be replaced by $\operatorname{dom} g \cap A \operatorname{dom} f \neq \emptyset$, compare e.g. [1, Corollary 5.1.9].

Remark 5.5 The previous theorem can be extended to more general origin spaces than $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ as long as the corresponding scalar result, which is used in the proof, is valid in these spaces. Then, one usually has to modify the constraint qualification.

6 Fenchel duality for vector optimization problems

Let $F : X \rightrightarrows \mathbb{R}^q$ and $G : U \rightrightarrows \mathbb{R}^q$ be two set–valued maps, where X and U are as in the previous section. We consider the following vector optimization problem

$$\text{(VOP)} \qquad \bar{P} := \operatorname{Inf} \bigcup_{x \in X} \big(F(x) + G(Ax)\big).$$

In the same manner as in Section 4, we do not loose generality if F and G are considered to be functions $F : X \to \mathcal{I}$, $G : U \to \mathcal{I}$ and the vector optimization problem (VOP) can be equivalently expressed as an $\mathcal{I}$-valued problem

$$\text{(P)} \qquad \bar{P} = \operatorname{Inf} \bigcup_{x \in X} \big(F(x) \oplus_{\mathcal{I}} G(Ax)\big).$$

We define the conjugate (compare [27]) of a function $F : X \to \mathcal{I}$ (with respect to $c \in \mathbb{R}^q$) by

$$F^*_c : X^* \to \mathcal{S}, \qquad F^*_c(x) := \operatorname{Sup} \bigcup_{x \in X} \big(\langle x^*, x\rangle \{c\} - F(x)\big),$$

where $\mathcal{S}$ is the space of self-supremal sets with respect to C, which is defined analogously to $\mathcal{I}$. Following the lines of scalar optimization we consider the following dual problem:

$$(\mathrm{D_c}) \qquad \bar{D}_c := \operatorname{Sup} \bigcup_{u^* \in U^*} \big(-F^*_c(A^T u^*) \oplus_{\mathcal{I}} -G^*_c(-u^*)\big).$$

In view of Proposition 3.3 this is just a transformation of the dual pair of problems considered in the previous section into the framework of functions with self-infimal values. Therefore we immediately obtain the same duality assertions.

Remark 6.1 Note that convexity of a function $F : X \to \mathcal{I}$ (see Section 2) is equivalent to the so-called C-convexity (see e.g. Jahn [10, 11] and Luc [15]) of the corresponding set-valued map $\bar{F} : X \rightrightarrows \mathbb{R}^q$, $\bar{F}(x) := F(x)$ if $F(x) \subseteq \mathbb{R}^q$, $\bar{F}(x) := \emptyset$ if $F(x) = \{+\infty\}$, $\bar{F}(x) := \mathbb{R}^q$ if $F(x) = \{-\infty\}$, where C-convexity means that $\operatorname{gr}\big(\operatorname{cl}\big(\bar{F}(\cdot) + C\big)\big)$ is convex.

Theorem 6.2 *For all $c \in \mathbb{R}^q$ it holds weak duality between* (P) *and* ($\mathrm{D_c}$), *i.e., $\bar{D}_c \preccurlyeq \bar{P}$. If F and G are convex, $0 \in \operatorname{ri}(\operatorname{dom} G - A \operatorname{dom} F)$ and $c \in \operatorname{int} C$, then we have strong duality between* (P) *and* ($\mathrm{D_c}$), *i.e., $\bar{D}_c = \bar{P}$; if additionally $\bar{P} \neq \{-\infty\}$, we have*

$$\bar{D}_c = \operatorname{Max} \bigcup_{u^* \in U^*} \big(-F^*_c(A^T u^*) \oplus_{\mathcal{I}} -G^*_c(-u^*)\big).$$

Proof. The first part (weak and strong duality) follows from Theorem 5.2 and the considerations in Section 3, in particular Proposition 3.3. For the second part let $\bar{P} \neq \{-\infty\}$, hence $\emptyset \neq \mathrm{Cl}_+\bar{P} \neq \mathbb{R}^q$. Let $\bar{y} \in \operatorname{Sup} \bigcup_{u^* \in U^*} D_c(u^*) = \bar{D}_c = \bar{P} = \operatorname{Inf} \bar{p}$, where we use the notations of Section 4. Then $\bar{y} \notin \bar{p} + \operatorname{int} C$. By a separation theorem, there exists some $\bar{y}^* \in C^\circ \setminus \{0\}$ such that $\langle \bar{y}^*, \bar{y} \rangle \geq \sigma_{\bar{p}+\operatorname{int} C}(\bar{y}^*) = \sigma_{\bar{p}}(\bar{y}^*) + \sigma_{\operatorname{int} C}(\bar{y}^*) = \sigma_{\bar{p}}(\bar{y}^*)$. By (4) there exists some $\bar{u}^* \in U^*$ such that $\langle \bar{y}^*, \bar{y} \rangle \geq \sigma_{d_c(\bar{u}^*)}(\bar{y}^*)$. Assuming that $\bar{y} \in d_c(\bar{u}^*) + \operatorname{int} C$ we obtain $\langle y^*, \bar{y} \rangle < \sigma_{d_c(\bar{u}^*)}(\bar{y}^*)$ for all $y^* \in C^\circ \setminus \{0\}$, a contradiction. Hence $\bar{y} \notin d_c(\bar{u}^*) + \operatorname{int} C$. On the other hand, $\bar{y} + \operatorname{int} C \in \operatorname{Inf} \bar{p} + \operatorname{int} C = \bar{p} + \operatorname{int} C \subseteq d_c(\bar{u}^*) + \operatorname{int} C$. This yields that $\bar{y} \in \operatorname{Inf} d_c(\bar{u}^*) = D_c(\bar{u}^*) \subseteq \bigcup_{u^* \in U^*} D_c(u^*)$. Together we have $\bar{y} \in \operatorname{Max} \bigcup_{u^* \in U^*} D_c(u^*)$. □

Remark 6.3 Our dual problem ($\mathrm{D_c}$) differs from the dual problem given by Tanino [27] because of the formulation taking into account the conlinear and the complete lattice structure of the space $\mathcal{I}$ of self-infimal sets. So our formulation is completely analogous to that of scalar Fenchel duality theorems, that means our dual objective function is expressed in terms of the conjugates. Note further that the origin space of the dual problem is just the space U^* instead of the space of linear continuous operators $\mathcal{L}(U^*, \mathbb{R}^q)$.

Remark 6.4 In scalar optimization the primal and dual attainment of the solution are of interest in many cases. Even though we cannot expect that the infimum and supremum are attained by a single $x \in X$ and $u^* \in U^*$, respectively, we speak about primal and dual attainment if we can replace the infimal set by the set of minimal (weakly efficient) elements in (P) and the supremal set by the set of maximal (weakly efficient) elements in ($\mathrm{D_c}$), respectively. Concerning the primal attainment let us mention that even in scalar convex programming the infimum is not attained without additional assumptions. Therefore, also in the case of vector optimization one needs additional assumptions. In the paper by Dolecki, Malivert [5] and in the book by Pallaschke, Rolewicz [18] the submission (domination) property is supposed. This property guarantees the existence of minimal points. Using the infimal set in the primal problem, we do not need such a domination property.

As in the scalar theory the dual attainment in Theorem 6.2 follows without any additional assumption.

Remark 6.5 In many papers on duality for vector optimization problems (compare Jahn [10]) a strong direct and a strong inverse duality assertion is shown. In the formulation of the strong direct duality assertion the existence of a weakly efficient solution of the primal vector optimization problem is supposed and the existence of a weakly efficient solution of the dual problem with the same objective function value is shown. Conversely, in the strong inverse duality assertion a closedness condition concerning the primal problem and the existence of a weakly efficient solution of the dual problem are supposed in order to show the existence of a weakly efficient solution of the primal problem with the same objective function value. Such formulations like strong direct and inverse duality assertions are discussed in Section 7 (Corollaries 7.6, 7.7) for the case of linear vector optimization problems, but could be also obtained for the convex setting, as a conclusion of Theorem 6.2.

7 Linear vector optimization

In this section we investigate the special case of linear vector optimization problems. We show that we can maintain a large part of the structures of scalar linear programming. In particular, we have no duality gap in the case that the right-hand side of the inequality constraints is zero, compare the discussion in [7]. On the other hand, our dual objective maps take their values in the space of self-infimal sets, in particular, we cannot expect a point-valued dual objective map. Nevertheless the values of the dual objective map have a simple structure, namely they are boundary points of translated cones, in some cases even hyperplanes. As usual in vector optimization we use the abbreviation $f[S] := \bigcup_{x\in S} f(x)$.

Consider the linear vector optimization problems

$$(\mathrm{LP}^1) \qquad \operatorname{Inf} M[S], \qquad S := \{x \in \mathbb{R}^n |\; Ax \geq b\}\,,$$

$$(\mathrm{LP}^2) \qquad \operatorname{Inf} M[S], \qquad S := \{x \in \mathbb{R}^n |\; x \geq 0,\; Ax \geq b\}\,,$$

$$(\mathrm{LP}^3) \qquad \operatorname{Inf} M[S], \qquad S := \{x \in \mathbb{R}^n |\; x \geq 0,\; Ax = b\}\,,$$

where $M \in \mathbb{R}^{n\times q}, A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^m$. We calculate the corresponding dual problems following the lines of the previous sections. Exemplarily we show the calculations starting with (LP^2). Starting with (LP^1) or (LP^3) we obtain the result similarly and even easier. We set

$$F(x) := \begin{cases} \{Mx\} + \operatorname{bd} C & \text{if} \quad x \geq 0 \\ \{+\infty\} & \text{else} \end{cases} \qquad \text{and} \qquad G(u) := \begin{cases} \operatorname{bd} C & \text{if} \quad u \geq b \\ \{+\infty\} & \text{else.} \end{cases}$$

For the choice $c \in \operatorname{int} C$, an easy calculation shows that

$$-F^*_c(x^*) = \operatorname{Inf}(M - cx^{*T})\big[\mathbb{R}^n_+\big],$$

$$-G^*_c(-u^*) = \begin{cases} \langle u^*, b\rangle \{c\} + \operatorname{bd} C & \text{if} \quad u^* \geq 0 \\ \{-\infty\} & \text{else.} \end{cases}$$

In order to obtain dual side conditions of a simple structure, it is useful to characterize the condition $\operatorname{Inf}(M - c\,x^{*T})\big[\mathbb{R}^n_+\big] = \{-\infty\}$. For this purpose consider the next two assertions.

Lemma 7.1 *Let $C_1, C_2 \subseteq \mathbb{R}^q$ be nonempty closed convex cones with $\operatorname{int} C_2 \neq \emptyset$. Then*

$$C_1 \cap -\operatorname{int} C_2 = \emptyset \iff C_1^\circ \cap (C_2^\circ \setminus \{0\}) \neq \emptyset.$$

Proof. Note that $y \in -\operatorname{int} C_2$, $z \in (C_2^\circ \setminus \{0\})$ implies that $\langle y, z\rangle > 0$. On the other hand, $y \in C_1$, $z \in C_1^\circ$ implies that $\langle y, z\rangle \leq 0$. This proves the implication "$\Leftarrow$".

If $C_1 \cap -\operatorname{int} C_2 = \emptyset$, by a separation theorem (e.g. [20, Th. 11.3]), we obtain that there exists $y^* \in \mathbb{R}^q \setminus \{0\}$ such that $\inf_{y\in -C_2} \langle y^*, y\rangle = \sup_{y \in C_1} \langle y^*, y\rangle = 0$. Hence we have $y^* \in C_1^\circ \cap (C_2^\circ \setminus \{0\})$. □

Proposition 7.2 *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let $H \in \mathbb{R}^{q\times n}$. Then, the following statements are equivalent:*

(i) $\operatorname{Inf} H[K] \neq \{-\infty\}$;

(ii) $\exists c^* \in C^\circ \setminus \{0\} : H^T c^* \in K^\circ$.

Proof. Set $L := H[K]$. Since $L \subseteq \mathbb{R}^q$ is a nonempty cone, we have

$$\operatorname{Inf} L \neq \{-\infty\} \iff \exists y \in \mathbb{R}^q, \forall \alpha > 0 : \alpha y \notin L + \operatorname{int} C \wedge \alpha y + \operatorname{int} C \subseteq L + \operatorname{int} C \iff 0 \in \operatorname{Inf} L.$$

By Lemma 7.1 we have

$$0 \in \operatorname{Inf} L \iff 0 \notin L + \operatorname{int} C \iff L \cap -\operatorname{int} C = \emptyset \iff L^\circ \cap (C^\circ \setminus \{0\}) \neq \emptyset.$$

Using the bipolar theorem [20, Th. 14.1] we deduce

$$\begin{aligned} L^\circ \cap (C^\circ \setminus \{0\}) \neq \emptyset \quad &\iff \quad \exists c^* \in C^\circ \setminus \{0\} : \forall l \in L : \langle c^*, l \rangle \leq 0 \\ &\iff \quad \exists c^* \in C^\circ \setminus \{0\} : \forall x \in K = K^{\circ\circ} : \langle c^*, Hx \rangle = \left\langle H^T c^*, x \right\rangle \leq 0 \\ &\iff \quad \exists c^* \in C^\circ \setminus \{0\} : H^T c^* \in K^\circ. \end{aligned}$$

Together we obtain the desired assertion. □

Applying the previous result we can write

$$-F_c^*(x^*) = \begin{cases} \operatorname{Inf}(M - c x^{*T}) \left[\mathbb{R}^n_+\right] \subseteq \mathbb{R}^q & \text{if} \quad \exists c^* \in C^\circ \setminus \{0\} : x^* \leq M^T \frac{c^*}{c^T c^*} \\ \{-\infty\} & \text{else.} \end{cases}$$

In the following, the set $B_c := \{c^* \in -C^\circ |\ \langle c, c^* \rangle = 1\}$ is used to express the dual side conditions. In Figure 7.1 we illustrate this set by two examples (with $\|c\| = 1$):

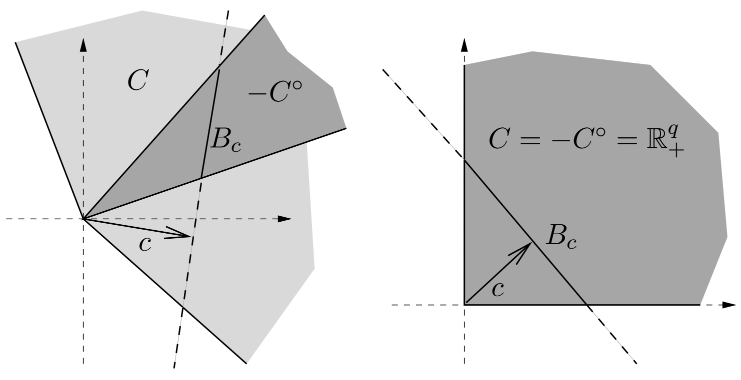

Figure 7.1. Two examples for the set B_c.

Proposition 7.3 *For $c \in \operatorname{int} C$, the set B_c is a compact convex subset of $\mathbb{R}^q$.*

Proof. Obviously, B_c is closed and convex. Thus, it remains to show that B_c is bounded. Assuming the contrary, we obtain a sequence $c_n^* \in B_c$ with $\|c_n^*\| \to \infty$. Without loss of generality we can assume that $c_n^* / \|c_n^*\| \to \bar{c}^* \in -C^\circ \setminus \{0\}$. Since $\langle c_n^* / \|c_n^*\|, c \rangle = 1 / \|c_n^*\| \to 0$, it follows that $\langle \bar{c}^*, c \rangle = 0$. But $\bar{c}^* \in -C^\circ \setminus \{0\}$ and $c \in \operatorname{int} C$ implies that $\langle \bar{c}^*, c \rangle > 0$, a contradiction. $\square$

According to problem ($\mathrm{D_c}$) in the previous section we obtain the dual problem to (LP^2) as

$$(\mathrm{LD}_\mathrm{c}^2) \qquad \begin{cases} \bar{D}_c = \operatorname{Sup} \bigcup_{u^* \in T_c} \left(c\, u^{*T} b \;+\; \operatorname{Inf}(M - c\, u^{*T} A)[\mathbb{R}^n_+] \right) \\ T_c := \left\{ u^* \in \mathbb{R}^m \middle|\, u^* \geq 0,\ \exists c^* \in B_c : A^T u^* \leq M^T c^* \right\}. \end{cases}$$

By a similar calculation we obtain the dual problems to (LP^1) and (LP^3) as

$$(\mathrm{LD}_\mathrm{c}^1) \qquad \begin{cases} \bar{D}_c = \operatorname{Sup} \bigcup_{u^* \in T_c} \left(c\, u^{*T} b \;+\; \operatorname{Inf}(M - c\, u^{*T} A)[\mathbb{R}^n] \right) \\ T_c := \left\{ u^* \in \mathbb{R}^m \middle|\, u^* \geq 0,\ \exists c^* \in B_c : A^T u^* = M^T c^* \right\}, \end{cases}$$

$$(\mathrm{LD}_\mathrm{c}^3) \qquad \begin{cases} \bar{D}_c = \operatorname{Sup} \bigcup_{u^* \in T_c} \left(c\, u^{*T} b \;+\; \operatorname{Inf}(M - c\, u^{*T} A)[\mathbb{R}^n_+] \right) \\ T_c := \left\{ u^* \in \mathbb{R}^m \middle|\, \exists c^* \in B_c : A^T u^* \leq M^T c^* \right\}. \end{cases}$$

Let us collect some properties of the dual feasible set T_c.

Proposition 7.4 *The set T_c in* (LD_c^1)-(LD_c^3) *(where $c \in \operatorname{int} C$) is always a closed convex subset of $\mathbb{R}^m$. Moreover, if C is polyhedral, then T_c is polyhedral, too.*

Proof. Exemplarily we give the proof for T_c in (LD_c^3). The set T_c can be expressed as $T_c = \left\{ u^* \in \mathbb{R}^m \middle|\, A^T u^* \in M[B_c] - \mathbb{R}^n_+ \right\}$. Of course, $M[B_c] - \mathbb{R}^n_+$ is a convex set. By [20, Theorem 3.4], it follows that T_c is convex. Since B_c is compact (Proposition 7.3), the set $M[B_c] - \mathbb{R}^n_+$ is closed. As the map $A^T : \mathbb{R}^m \to \mathbb{R}^n$ is continuous, T_c is closed. If C is polyhedral, we conclude that C°, B_c, $M[B_c]$ and $M[B_c] - \mathbb{R}^n_+$ are polyhedral, too. By [20, Theorem 19.3], it follows that T_c is polyhedral. $\square$

From the Fenchel duality theorem in the previous section and some additional considerations we obtain the following duality result.

Theorem 7.5 *For $c \in \operatorname{int} C$ it holds weak and strong duality between* (LP^i) *and* ($\mathrm{LD}_\mathrm{c}^\mathrm{i}$) *($i = 1, 2, 3$). More precisely we have*

(i) $\bar{D}_c = \bar{P} \subseteq \mathbb{R}^q$ *if $S \neq \emptyset$ and $T_c \neq \emptyset$, where* "Sup" *can be replaced by* "Max" *in this case,*

(ii) $\bar{D}_c = \bar{P} = \{-\infty\}$ *if $S \neq \emptyset$ and $T_c = \emptyset$,*

(iii) $\bar{D}_c = \bar{P} = \{+\infty\}$ *if $S = \emptyset$ and $T_c \neq \emptyset$.*

Proof. In the case where $S \neq \emptyset$ we obtain the strong duality by Theorem 6.2 taking into account Remark 5.4. In case (i), we have $\bar{P} \subseteq \mathbb{R}^q \cup \{-\infty\}$ and $\bar{D}_c \subseteq \mathbb{R}^q \cup \{+\infty\}$, hence $\bar{D}_c = \bar{P} \subseteq \mathbb{R}^q$. In case (ii) we have $\bar{P} = \bar{D}_c = \operatorname{Sup} \emptyset = \{-\infty\}$. In case (iii) we need some additional considerations. Let $\bar{u}^* \in T_c$ and $S = \emptyset$. Exemplarily we show the assertion for (LP^2) and (LD_c^2). By the Farkas Lemma there exists some $\hat{u}^* \in \mathbb{R}^m$ such that $\hat{u}^* \geq 0$, $A^T \hat{u}^* \leq 0$ and $b^T \hat{u}^* > 0$. Hence, for all $\alpha > 0$ we have $\bar{u}^* + \alpha \hat{u}^* \in T_c$. For $\alpha \to \infty$ we have $(\bar{u}^* + \alpha \hat{u}^*)^T b \to \infty$. Hence $\bar{D}_c = \{+\infty\} = \bar{P}$. $\square$

Next we relate our result to classical formulations in vector optimization (see Remark 6.4). Denoting the dual objective functions in $(\mathrm{LD}^1_\mathrm{c})$-$(\mathrm{LD}^3_\mathrm{c})$ by $D_c : \mathbb{R}^m \to \mathcal{I}$, $(i = 1, 2, 3)$, for all three pairs of dual problems we easily obtain the following conclusions from Theorem 7.5.

Corollary 7.6 (direct strong duality) *For every* $c \in \operatorname{int} C$ *it holds* $\operatorname{Min} M[S] \subseteq \operatorname{Max} D_c[T_c]$.

Corollary 7.7 (inverse strong duality) *If* $M[S]$ *is upper closed, then for all* $c \in \operatorname{int} C$ *it holds* $\operatorname{Max} D_c[T_c] \subseteq \operatorname{Min} M[S]$.

Finally, we give an example.

Example 7.8 (see Figure 7.2) Let $q = m = n = 2$, $C = \mathbb{R}^2_+$ and consider the problem (LP^2) with the data

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Let us calculate the dual problem for $c = (1, 1)^T \in \operatorname{int} \mathbb{R}^2_+$. For this choice, we set B_c is given by $B_c = \{c^*_1, c^*_2 \geq 0 \mid c^*_1 + c^*_2 = 1\}$. The dual side condition are as follows

$$u^*_1, u^*_2 \geq 0, \qquad \exists c^*_1 \geq 0 : \; u^*_1 + 2u^*_2 \leq c^*_1, \quad 2u^*_1 + u^*_2 \leq 1 - c^*_1.$$

This can be equivalently expressed by $T_c = \{u^*_1, u^*_2 \geq 0 | \; u^*_1 + u^*_2 \leq 1/3\}$. The vertexes of T_c are the points $v_1 = (0, 0)^T$, $v_2 = (1/3, 0)^T$ and $v_3 = (0, 1/3)^T$. The matrices $H_i := (M - c\, {v_i}^T A)$ can be easily computed as

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad H_3 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

Hence we obtain $\operatorname{Inf} H_1[\mathbb{R}^2_+] = \operatorname{bd} \mathbb{R}^2_+$, $\operatorname{Inf} H_2[\mathbb{R}^2_+] = \big\{y \in \mathbb{R}^2 | \; y_1 + 2y_2 = 0\big\}$, $\operatorname{Inf} H_3[\mathbb{R}^2_+] = \big\{y \in \mathbb{R}^2 | \; 2y_1 + y_2 = 0\big\}$. Consequently, the values of the dual objective function at v_1, v_2, v_3 are $D_c(v_1) = \operatorname{bd} \mathbb{R}^2_+$, $D_c(v_2) = \big\{y \in \mathbb{R}^2 | \; y_1 + 2y_2 = 2\big\}$ and $D_c(v_3) = \big\{y \in \mathbb{R}^2 | \; 2y_1 + y_2 = 2\big\}$. We see that the three dual feasible points $v_1, v_2, v_3 \in T_c$ are already sufficient for strong duality.

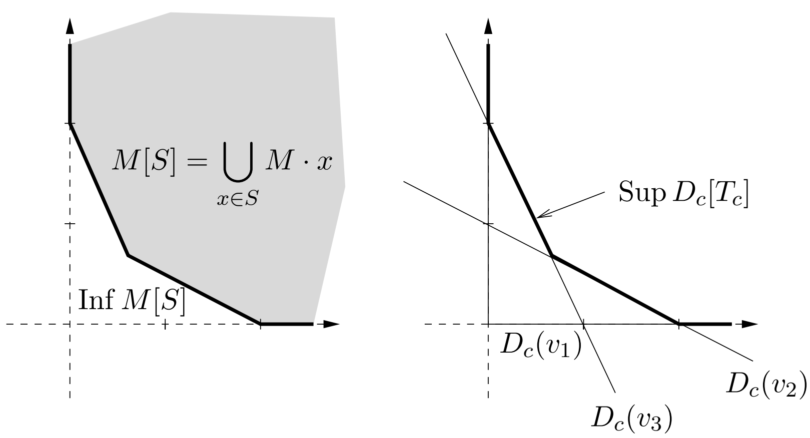

Figure 7.2. The primal and dual values in Example 7.8.

References

[1] Borwein, J.; Lewis, A.: *Convex Analysis and Nonlinear Optimization*, Springer, New York, 2000

[2] Boţ, R. I.; Wanka, G.: An analysis of some dual problems in multiobjective optimization. I, II, *Optimization* **53**, No.3, (2004), 281–300, 301–324

[3] Breckner, W. W.: Dualität bei Optimierungsaufgaben in halbgeordneten topologischen Vektorräumen. I., *Math., Rev. Anal. Numér. Théor. Approximation* **1**, (1972), 5–35

[4] Corley, H. W.: Duality theory for maximizations with respect to cones, *J. Math. Anal. Appl.* **84**, (1981), 560–568

[5] Dolecki, S.; Malivert C.: General duality in vector optimization, *Optimization* **27**, No.1–2, (1993), 97–119

[6] Hamel, A.: *Variational Principles on Metric and Uniform Spaces*, Habilitation thesis, Martin–Luther–Universität Halle–Wittenberg, 2005

[7] Hamel, A.; Heyde, F.; Löhne, A.; Tammer, Chr.; Winkler, K.: Closing the duality gap in linear vector optimization, *Journal of Convex Analysis* **11**, No. 1, (2004), 163–178

[8] Isermann, H.: *On some relations between a dual pair of multiple objective linear programs*, Z. Oper. Res. Ser. A 22, (1978) 33–41

[9] Jahn, J.: Duality in vector optimization, *Math. Program.* **25**, (1983), 343–353

[10] Jahn, J.: *Mathematical Vector Optimization in Partially Ordered Linear Spaces*, Verlag Peter Lang, Frankfurt am Main–Bern–New York, 1986

[11] Jahn, J.: *Vector Optimization. Theory, Applications, and Extensions*, Springer–Verlag, Berlin, 2004

[12] Löhne, A.: On conjugate duality in optimization with set relations, in: Geldermann, J.; Treitz, M. (eds.): *Entscheidungstheorie und -praxis in industrieller Produktion und Umweltforschung*, Shaker, Aachen, 2004

[13] Löhne, A.: Optimization with set relations, PhD thesis, Martin–Luther–Universität Halle–Wittenberg, 2005

[14] Löhne, A.: Optimization with set relations: Conjugate Duality, *Optimization* **54**, No. 3, (2005), 265–282

[15] Luc, D. T.: *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Sciences, 319, Springer–Verlag, Berlin, 1988

[16] Nakayama, H.: Duality in multi-objective optimization, in: Gal, T., Stewart, T.J., Hanne, T. *Multicriteria Decision Making*, Kluwer Academic Publishers, Boston/Dordrecht/London, (1999), 3-1 – 3-29

[17] Nieuwenhuis, J. W.: Supremal points and generalized duality, *Math. Operationsforsch. Stat., Ser. Optimization* **11**, (1980), 41–59

[18] Pallaschke, D., Rolewicz, S.: Foundations of mathematical optimization, Convex analysis without linearity. Kluwer Academic Publishers, Dordrecht, Boston, London 1997

[19] Postolica, V.: *A generalization of Fenchel's duality theorem*, Ann. Sc. math. Quebec 10, 1986, 199–206

[20] Rockafellar, R. T.: *Convex Analysis*, Princeton University Press, Princeton N. J., 1970

[21] Rockafellar, R. T.; Wets, R. J.–B.: *Variational Analysis*,Springer, Berlin, 1998

[22] Sawaragi, Y.; Nakayama, H.; Tanino, T.: *Theory of multiobjective optimization*, Mathematics in Science and Engineering, Vol. 176. Academic Press Inc., Orlando, 1985

[23] Schönfeld, P.: *Some duality theorems for the non-linear vector maximum problem*, Unternehmensforschung 14, 1970, 51-63

[24] Song, W.: A generalization of Fenchel duality in set–valued vector optimization, *Math. Methods Oper. Res.* **48**, (1998), 259–272

[25] Tammer, Chr.: Lagrange–Dualität in der Vektoroptimierung. *Wiss. Z. Tech. Hochsch. Ilmenau* **37**, No.3, (1991), 71–88

[26] Tanino, T.: On the supremum of a set in a multi–dimensional space, *J. Math. Anal. Appl.* **130**, (1988), 386–397

[27] Tanino, T.: Conjugate duality in vector optimization, *J. Math. Anal. Appl.* **167**, No.1, (1992), 84–97

[28] Tanino, T.; Sawaragi, Y.: Duality theory in multiobjective programming, *J. Optimization Theory Appl.* **27**, (1979), 509–529