On Semicontinuity of Convex-valued Multifunctions and Cesari's Property (Q)

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Abstract

We investigate two types of semicontinuity for set-valued maps, Painlevé–Kuratowski semicontinuity and Cesari's property (Q). It is shown that, in the context of convex-valued maps, the concepts related to Cesari's property (Q) have better properties than the concepts in the sense of Painlevé–Kuratowski. In particular, we give a characterization of Cesari's property (Q) by means of upper semicontinuity of the scalarizations by the support function $\sigma_{f(\,\cdot\,)}(y^*) : X \to \overline{\mathbb{R}}$. We compare both types of semicontinuity and show their coincidence in special cases.

1 Introduction

Working in the framework of convex-valued multifunctions we expect that an appropriate notion of an upper semicontinuous hull produces a convex-valued multifunction being upper semicontinuous. This cannot be ensured by upper semicontinuity in the sense of Painlevé and Kuratowski (in [10] called *outer semicontinuity*), as the following examples show. We denote by $\operatorname{LIM\,SUP}_{x' \to x} f(x')$ the Painlevé–Kuratowski upper limit (outer limit) of f at x and by $(\operatorname{USC} f)(x) = \operatorname{LIM\,SUP}_{x'\to x} f(x')$ the corresponding upper (outer) semicontinuous hull, see Section 2 for the exact definitions.

Example 1.1 Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$, $f(x) := \{x \,/\, |x|\}$ if $x \neq 0$ and $f(0) := \{0\}$. Then the upper semicontinuous hull of f, namely $(\operatorname{USC} f) : \mathbb{R} \rightrightarrows \mathbb{R}$, $(\operatorname{USC} f)(x) = f(x)$ if $x \neq 0$ and $(\operatorname{USC} f)(0) = \{-1, 0, 1\}$, is not convex-valued.

This might suggest to redefine the Painlevé–Kuratowski upper semicontinuous hull in the framework of convex-valued multifunctions as follows:

$$(\widetilde{\operatorname{USC}} f)(x) := \operatorname{cl}\operatorname{conv} \operatorname*{LIM\,SUP}_{x' \to x} f(x').$$

However, $(\widetilde{\operatorname{USC}} f)$ is not necessarily upper semicontinuous as the following example shows.

Example 1.2 Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$,

$$f(x) := \begin{cases} \left\{\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left[2^{-2n}, 2^{-2n+1}\right) \\ \left\{-\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left[2^{-2n+1}, 2^{-2n+2}\right) \\ \emptyset & \text{else.} \end{cases}$$

The modified upper semicontinuous hull $(\widetilde{\mathrm{USC}}\, f)$ of f is obtained as

$$(\widetilde{\mathrm{USC}}\, f)(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left(2^{-2n}, 2^{-2n+1}\right) \\ \left\{-\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left(2^{-2n+1}, 2^{-2n+2}\right) \\ \left[-\frac{1}{x}, \frac{1}{x}\right] & \text{if} \quad \exists n \in \mathbb{N} : x = 2^{-n} \\ \emptyset & \text{else.} \end{cases}$$

It is easily seen the graph of $(\widetilde{\mathrm{USC}}\, f)$ is not closed. Indeed, the sequence $(2^{-n}, 0)_{n\in\mathbb{N}}$ belongs to the graph of $(\widetilde{\mathrm{USC}}\, f)$, but its limit $(0,0)$ does not. Hence $(\widetilde{\mathrm{USC}}\, f)$ is not Painlevé–Kuratowski upper semicontinuous.

Let us illuminate another aspect. An important idea of Convex Analysis is the relationship between a convex set $A \subset \mathbb{R}^p$ and its support function $\sigma_A : \mathbb{R}^p \to \overline{\mathbb{R}}$. In particular, for closed convex sets $A, B \subset \mathbb{R}^p$ and $\alpha \in \mathbb{R}_+$ we have the following relationships (in particular, we set $-\infty + \infty = -\infty$, $0 \cdot \emptyset = \{0\}$):

$$\Big(A \subset B \Leftrightarrow \sigma_A \leq \sigma_B\Big), \qquad \sigma_A + \sigma_B = \sigma_{A+B}, \qquad \alpha\sigma_A = \sigma_{\alpha A}.$$

This implies that a set-valued map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is concave (i.e. graph-convex) if and only if the functions $\sigma_{f(\,\cdot\,)}(y^*) : \mathbb{R}^n \to \overline{\mathbb{R}}$ have the same property for all $y^* \in \mathbb{R}^p$. But, what can we say about a corresponding relationship for continuity properties? The usual Painlevé–Kuratowski upper and lower semicontinuity doesn't yield a positive result, as the following example shows.

Example 1.3 Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$, $f(x) := \left\{\frac{1}{x}\right\}$ if $x \neq 0$ and $f(0) := \{0\}$. Then f is Painlevé–Kuratowski upper semicontinuous (in particular at $x = 0$), but $\sigma_{f(\,\cdot\,)}(y^*)$ is not upper semicontinuous at $x = 0$ whenever $y^* \neq 0$.

Motivated by these examples we look for an alternative semicontinuity concept for multifunctions having better properties in this framework. In this article, we show that "Cesari's property (Q)" (for instance, see Cesari [1, 2], Cesari and Suryanarayana [3], Goodman [5], Denkowski [4], Suryanarayana [11], Papageorgiou [8]), which plays an important role in Optimal Control and is well-known in this field, fits all our requirements. This leads to a characterization of Cerari's property (Q) by support functions. Our investigations are based on some results on $\mathcal{C}$-convergence (in connection to Cesari's property (Q) usually called Q-convergence), which were recently obtained by C. Zălinescu and the author [7], [6][1].

This article is organized as follows. In the next section we shortly recall some facts on the two types of semicontinuity, Painlevé–Kuratowski semicontinuity and Cerari's property (Q), and we propose our main tools. In Section 3 we present our main result, a characterization of Cesari's property (Q) and we draw some conclusions. Section 4 is devoted to a comparison of Cerari's property (Q) and the Painlevé–Kuratowski semicontinuity. We show their coincidence under certain local boundedness assumptions. Finally, in Section 5, we discuss the special case of concave (i.e. graph-convex) maps.

[1]For the very impotant hint that the results of [7], [6] and this paper are connected to Cesari's property (Q) and Q-convergence we are greatly indebted to Jean-Paul Penot.

2 Preliminaries

Throughout the paper let Y be a finite dimensional normed vector space Y with dimension $p \geq 1$. For the standard concepts of Convex Analysis we mainly use the notation of Rockafellar's "Convex Analysis" [9].

We denote by $\mathcal{F} := \mathcal{F}(Y)$ the family of closed subsets of Y and by $\mathcal{C} := \mathcal{C}(Y)$ the family of closed convex subsets of Y. It is well-known that $(\mathcal{F}, \subset)$ and $(\mathcal{C}, \subset)$ provide complete lattices, i.e. every nonempty subset of $\mathcal{F}$ (resp. of $\mathcal{C}$) has a supremum and an infimum, denoted by $\operatorname{SUP} \mathcal{A}$ ($\sup \mathcal{A}$) and $\operatorname{INF} \mathcal{A}$ ($\inf A$). Of course, for nonempty sets $\mathcal{A} \subset \mathcal{F}$ and $\mathcal{B} \subset \mathcal{C}$ we have $\operatorname{SUP} \mathcal{A} = \operatorname{cl} \bigcup \{A \mid A \in \mathcal{A}\}$, $\operatorname{INF} \mathcal{A} = \bigcap \{A \mid A \in \mathcal{A}\}$, $\sup \mathcal{B} = \operatorname{cl} \operatorname{conv} \bigcup \{B \mid B \in \mathcal{B}\}$ and $\inf \mathcal{B} = \bigcap \{B \mid B \in \mathcal{B}\}$. Further, we set $\operatorname{INF} \emptyset = \operatorname{SUP} \mathcal{F}$, $\operatorname{SUP} \emptyset = \operatorname{INF} \mathcal{F}$, $\inf \emptyset = \sup \mathcal{C}$ and $\sup \emptyset = \inf \mathcal{C}$.

We frequently use the following notation of [10] (but omitting the index ∞):

$$\mathcal{N} := \{N \subset \mathbb{N} \mid \mathbb{N} \setminus N \text{ finite}\} \quad \text{and} \quad \mathcal{N}^{\#} := \{N \subset \mathbb{N} \mid N \text{ infinite}\}.$$

For a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ the *upper* and *lower $\mathcal{F}$-limits*, respectively, are defined by

$$\operatorname*{LIM\,SUP}_{n\to\infty} A_n = \operatorname*{INF}_{N\in\mathcal{N}} \operatorname*{SUP}_{n\in N} A_n, \qquad \operatorname*{LIM\,INF}_{n\to\infty} A_n = \operatorname*{INF}_{N\in\mathcal{N}^{\#}} \operatorname*{SUP}_{n\in N} A_n.$$

Of course (see [10], in particular Exercise 4.2.(b), for the relationship to alternative definitions), the *upper and lower $\mathcal{F}$-limits* coincide with the upper and lower limits in the sense of Painlevé–Kuratowski (in [10] called outer and inner limits). In the following, all concepts related to upper and lower $\mathcal{F}$-limits are indicated by the prefix $\mathcal{F}$, because $\mathcal{F}$ is the underlying lattice. In formulas we don't use this prefix, instead we consequently use capital letters.

A sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ is *$\mathcal{F}$-convergent* to some $A \in \mathcal{F}$ if $A = \operatorname{LIM\,SUP}_{n\to\infty} A_n = \operatorname{LIM\,INF}_{n\to\infty} A_n$. Then we write $A = \operatorname{LIM}_{n\to\infty} A_n$ or $A_n \xrightarrow{\mathcal{F}} A$.

We proceed analogously in the complete lattice $\mathcal{C}$. The *upper* and *lower $\mathcal{C}$-limits* of a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ are defined, respectively, by

$$\limsup_{n\to\infty} A_n := \inf_{N\in\mathcal{N}} \sup_{n\in N} A_n \quad \text{and} \quad \liminf_{n\to\infty} A_n := \inf_{N\in\mathcal{N}^{\#}} \sup_{n\in N} A_n.$$

Upper and lower $\mathcal{C}$-limits and related concepts were used in the field of Optimal Control, see e.g. [1, 2], [3], [5], [4], [11], [8]. In this area, one speaks about (upper and lower) Q-limits, Q-convergence and so on, because these concepts are related to Cesari's property (Q). In this article, however, we use the prefix $\mathcal{C}$ instead, because $\mathcal{C}$ is the underlying lattice. In formulas we consequently use small letters.

We say a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ is *$\mathcal{C}$-convergent* to some $A \in \mathcal{C}$ if $A = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$ and we write $A = \lim_{n\to\infty} A_n$ or $A_n \xrightarrow{\mathcal{C}} A$.

We next summarize some results related to $\mathcal{C}$-limits. The following initial result is an immediate consequence of [9, Cor. 16.5.1]. By $\sigma_A : Y \to \overline{\mathbb{R}}$, we denote the support function of a set $A \subset Y$.

Proposition 2.1 *Let $\mathcal{A} \subset \mathcal{C}$. Then $\sigma_{\inf \mathcal{A}} \leq \inf_{A\in\mathcal{A}} \sigma_A$ and $\sigma_{\sup \mathcal{A}} = \sup_{A\in\mathcal{A}} \sigma_A$.*

The following characterization of the upper $\mathcal{C}$-limit is useful to show further properties of the upper and lower limits. For simplicity of notation we denote the set $\{m, m+1, \dots, k\} \subset \mathbb{N}$ $(m, k \in \mathbb{N},\ m \le k)$ by $\overline{m,k}$. Further we set $\Delta_p := \left\{\lambda \in [0,1]^p \mid \sum_{i\in\overline{0,p-1}} \lambda_n^i = 1\right\}$.

Proposition 2.2 ([7]) *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ be a sequence. Then,*

$$y \in \limsup_{n\in\mathbb{N}} A_n \iff \left\{ \begin{array}{l} \exists\,(\lambda_n)_{n\in\mathbb{N}} \subset \Delta_{p+1},\ \exists\,(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}^{p+1},\ \exists\,(z_n)_{n\in\mathbb{N}} \subset Y^{p+1}, \\ y = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i,\ \forall n \in \mathbb{N},\ \forall j \in \overline{0,p},\ k_n^j \ge n,\ z_n^j \in A_{k_n^j}. \end{array} \right.$$

The next two theorems give us sufficient conditions for the coincidence of $\mathcal{F}$- and $\mathcal{C}$-convergence. Our condition here is weaker than the one in [8], but the underlying space is a Banach space there. Let $K \subset Y$ be a nonempty closed convex cone. By $\mathcal{C}_K$ we denote the family of all members A of $\mathcal{C} \setminus \{\emptyset\}$ satisfying $0^+A = K$.

Theorem 2.3 ([7]) *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}_K$ be a sequence such that $\sup_{n\in\mathbb{N}} A_n \in \mathcal{C}_K$. Then, $\limsup_{n\to\infty} A_n = \operatorname{cl}\operatorname{conv} \operatorname{LIM\,SUP}_{n\to\infty} A_n$.*

Theorem 2.4 ([7]) *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ be a sequence such that for all $\bar{N} \in \mathcal{N}^\#$ there exists some $\widetilde{N} \in \mathcal{N}^\#$ with $\widetilde{N} \subset \bar{N}$ and some nonempty closed convex cone $K \subset Y$ such that $A_n \in \mathcal{C}_K$ for all $n \in \widetilde{N}$ and $\sup_{n\in\widetilde{N}} A_n \in \mathcal{C}_K$. Then it holds $\liminf_{n\to\infty} A_n = \operatorname{LIM\,INF}_{n\to\infty} A_n$.*

The following lemmas provide the main tools in our investigations.

Lemma 2.5 ([7]) *Let $A, B \subset Y$ be nonempty closed and convex. Then,*

$$A \subset B \quad \Leftrightarrow \quad \forall y^* \in \operatorname{ri}(0^+B)^\circ,\ \sigma_A(y^*) \le \sigma_B(y^*).$$

Lemma 2.6 ([7]) *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $A := \limsup_{n\to\infty} A_n \neq \emptyset$. Then,*

$$\forall y^* \in \operatorname{ri}(0^+A)^\circ, \quad \limsup_{n\to\infty} \sigma_{A_n}(y^*) = \sigma_A(y^*).$$

We next turn to upper and lower $\mathcal{F}$-limits for set-valued maps. Since $\mathcal{F}$-limits are limits in the sense of Painlevé–Kuratowski, this is a collection of well-known results. Analogous concepts and results for the lattice $\mathcal{C}$ are discussed in the next section. Throughout this article let $X = \mathbb{R}^n$, although many assertions are also valid in a more general context.

The *upper* and *lower $\mathcal{F}$-limits* of $f : X \to \mathcal{F}$ at $\bar{x} \in X$ are defined, respectively, by

$$\operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x) := \operatorname*{SUP}_{x_n\to\bar{x}} \operatorname*{LIM\,SUP}_{n\to\infty} f(x_n) \quad \text{and} \quad \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x) := \operatorname*{INF}_{x_n\to\bar{x}} \operatorname*{LIM\,INF}_{n\to\infty} f(x_n).$$

where the index "$x_n \to \bar{x}$" stands for the supremum and infimum over all sequences converging to $\bar{x}$, respectively. The limit of $f : X \to \mathcal{F}$ at $\bar{x}$ exists if the upper and lower $\mathcal{F}$-limits coincide. Then we write

$$\operatorname*{LIM}_{x\to\bar{x}} f(x) = \operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x) = \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x).$$

The upper and lower $\mathcal{F}$-limits can be expressed as

$$\operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x) = \bigcup_{x_n\to\bar{x}} \operatorname*{LIM\,SUP}_{n\to\infty} f(x_n) \quad \text{and} \quad \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x) = \bigcap_{x_n\to\bar{x}} \operatorname*{LIM\,INF}_{n\to\infty} f(x_n).$$

In particular, the union in the first formula is closed, see [10, Proposition 4.4].

The function f is said to be *upper $\mathcal{F}$-semicontinuous* (*$\mathcal{F}$-usc*), *lower $\mathcal{F}$-semicontinuous* (*$\mathcal{F}$-lsc*), *$\mathcal{F}$-continuous* at $\bar{x} \in X$ if $f(\bar{x}) \supset \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x)$, $f(\bar{x}) \subset \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$, $f(\bar{x}) = \operatorname{LIM}_{x\to\bar{x}} f(x)$, respectively. If f is $\mathcal{F}$-usc, $\mathcal{F}$-lsc, $\mathcal{F}$-continuous at every $\bar{x} \in X$ we just say f is $\mathcal{F}$-usc, $\mathcal{F}$-lsc, $\mathcal{F}$-continuous, respectively. The *epigraph* and the *hypograph* of $f : X \to \mathcal{F}$ are defined, respectively, by

$$\operatorname{EPI} f := \{(x, A) \in X \times \mathcal{F} |\ A \supset f(x)\}, \qquad \operatorname{HYP} f := \{(x, A) \in X \times \mathcal{F} |\ A \subset f(x)\}.$$

Note that, for all $x \in X$, we have $(x, \emptyset) \in \operatorname{HYP} f$ and $(x, Y) \in \operatorname{EPI} f$. For a characterization of $\mathcal{F}$-semicontinuity we need to know what is meant by closedness of the epigraph and hypograph. A subset $\mathcal{A} \subset X \times \mathcal{F}$ is said to be $\mathcal{F}$-closed if for every sequence $(x_n, A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $x_n \to \bar{x} \in X$ and $A_n \xrightarrow{\mathcal{F}} \bar{A} \in \mathcal{F}$ it is true that $(\bar{x}, \bar{A}) \in \mathcal{A}$. The *$\mathcal{F}$-closure* of a set $\mathcal{A} \subset X \times \mathcal{F}$, denoted by $\operatorname{CL}\mathcal{A}$, is the set of all such limits $(\bar{x}, \bar{A}) \in X \times \mathcal{F}$ of sequences $(x_n, A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

From [10, Exercise 5.6 (c)] and [10, Theorem 5.7 (a)] we obtain the following characterization of $\mathcal{F}$-upper semicontinuity

$$\operatorname{HYP} f \text{ is } \mathcal{F}\text{-closed} \quad \Leftrightarrow \quad f \text{ is } \mathcal{F}\text{-usc} \quad \Leftrightarrow \quad \operatorname{gr} f \subset X \times Y \text{ is closed.}$$

Likewise, by [10, Exercise 5.6 (d)], lower $\mathcal{F}$-semicontinuity of f is equivalent to the $\mathcal{F}$-closedness of the epigraph. Note that the description by the graph fails in this case, i.e. a function $f : X \to \mathcal{F}$ that is $\mathcal{F}$-lsc has not necessarily a closed graph, see [10, Fig. 5–3. (b)].

Let us collect some basic properties of the *upper $\mathcal{F}$-semicontinuous hull* of f, defined by $(\operatorname{USC} f) : X \to \mathcal{F}$, $(\operatorname{USC} f)(x) := \operatorname{LIM\,SUP}_{x'\to x} f(x')$.

Proposition 2.7 *Let $f : X \to \mathcal{F}$. Then it holds*

(i) $\operatorname{gr}(\operatorname{USC} f) = \operatorname{cl}(\operatorname{gr} f)$,	(iv) $\forall x \in X : \quad (\operatorname{USC} f)(x) \supset f(x)$,
(ii) $\operatorname{HYP}(\operatorname{USC} f) \supset \operatorname{CL}(\operatorname{HYP} f)$,	(v) *f is $\mathcal{F}$-usc at $\bar{x} \in X$* $\Leftrightarrow$ $(\operatorname{USC} f)(\bar{x}) = f(\bar{x})$,
(iii) $(\operatorname{USC} f)$ *is $\mathcal{F}$-usc,*	(vi) $\operatorname{gr} f$ *convex* $\Rightarrow$ $\operatorname{gr}(\operatorname{USC} f)$ *convex.*

Proof. (i) See [10, page 154, 5(2) and 5(3)]. (ii) Let $(\bar{x}, \bar{A}) \in \operatorname{CL}(\operatorname{HYP} f)$. Then, there exist $(x_n)_{n\in\mathbb{N}} \subset X$ and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\bar{x} = \lim_{n\to\infty} x_n$, $\bar{A} = \operatorname{LIM}_{n\to\infty} A_n$ and $A_n \subset f(x_n)$ for all $n \in \mathbb{N}$. Hence, $(\operatorname{USC} f)(\bar{x}) = \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x) \supset \operatorname{LIM\,SUP}_{n\to\infty} f(x_n) \supset \operatorname{LIM\,SUP}_{n\to\infty} A_n = \operatorname{LIM}_{n\to\infty} A_n = \bar{A}$, i.e. $(\bar{x}, \bar{A}) \in \operatorname{HYP}(\operatorname{USC} f)$. (iii) By (i), $\operatorname{gr}(\operatorname{USC} f)$ is closed. Hence, $(\operatorname{USC} f)$ is $\mathcal{F}$-USC. (iv) Choosing the special sequence $x_n \equiv x$, we obtain $(\operatorname{USC} f)(x) = \operatorname{LIM\,SUP}_{x'\to x} f(x') \supset \operatorname{LIM\,SUP}_{n\to\infty} f(x_n) = \operatorname{LIM\,SUP}_{n\to\infty} f(x) = f(x)$. (v) By definition, f is $\mathcal{F}$-usc at $\bar{x}$ if and only if $f(\bar{x}) \supset (\operatorname{USC} f)(\bar{x})$. By (iv), this equivalent to $f(\bar{x}) = (\operatorname{USC} f)(\bar{x})$. (vi) Since $\operatorname{gr} f$ is convex, $\operatorname{cl}(\operatorname{gr} f)$ is convex, too. Hence, the convexity of $\operatorname{gr}(\operatorname{USC} f)$ follows from (i). $\square$

The next example shows that the opposite inclusion in assertion (ii) of the previous proposition does not hold true, in general.

Example 2.8 Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$, $f(x) := \{x / |x|\}$ if $x \neq 0$, $f(0) := \emptyset$. Then, $(0, \{-1, 1\})$ belongs to $\operatorname{HYP}(\operatorname{USC} f)$ but it does not belong to $\operatorname{CL}(\operatorname{HYP} f)$.

Remark 2.9 As noticed in [10], an analogous definition of the $\mathcal{F}$-lower semicontinuous hull, namely by $(\mathrm{LSC}\, f)(x) := \mathrm{LIM\,INF}_{x' \to x} f(x')$, is not constructive in the sense that $(\mathrm{LSC}\, f)$ is not necessarily $\mathcal{F}$-lsc. In the framework of $\mathcal{C}$-valued functions we will have similar problems. An example is given there.

3 Upper and lower $\mathcal{C}$-Semicontinuity

In this section we deal with upper and lower limits for functions with values in $\mathcal{C}$. The *upper* and *lower* $\mathcal{C}$*-limits* (compare [1, 2], [4], [5], [11]) of a function $f : X \to \mathcal{C}$ at $\bar{x} \in X$ are defined, respectively by

$$\limsup_{x \to \bar{x}} f(x) := \sup_{x_n \to \bar{x}} \limsup_{n \to \infty} f(x_n) \quad \text{and} \quad \liminf_{x \to \bar{x}} f(x) := \inf_{x_n \to \bar{x}} \liminf_{n \to \infty} f(x_n).$$

The $\mathcal{C}$*-limit* of f at $\bar{x}$ exists if the upper and lower limits coincide. Then we write

$$\lim_{x \to \bar{x}} f(x) = \limsup_{x \to \bar{x}} f(x) = \liminf_{x \to \bar{x}} f(x).$$

In case of upper $\mathcal{F}$-limits, the set $\bigcup_{x_n \to \bar{x}} \mathrm{LIM\,SUP}_{n \to \infty} f(x_n)$ is always closed, i.e. the closure operation, which is implicitly contained in the supremum, is superfluous. An analogous result is valid for upper $\mathcal{C}$-limits.

Proposition 3.1 *Let $f : X \to \mathcal{C}$ and $\bar{x} \in X$. Then it holds*

$$\limsup_{x \to \bar{x}} f(x) = \bigcup_{x_n \to \bar{x}} \limsup_{n \to \infty} f(x_n).$$

Proof. We have to show that $A := \bigcup_{x_n \to \bar{x}} \limsup_{n \to \infty} f(x_n)$ is convex and closed.

(i) *Convexity.* Let $y_1, y_2 \in A$ and let $\lambda \in [0,1]$ be given. Hence there exist sequences $(x_n^{(i)})_{n \in \mathbb{N}} \subset X$, $(i = 1, 2)$ with $x_n^{(i)} \to \bar{x}$ such that $y_i \in \limsup_{n \to \infty} f(x_n^{(i)})$. We define a sequence $(x_n^{(3)})_{n \in \mathbb{N}} \subset X$ by $(x_n^{(3)})_{n \in \mathbb{N}} := (x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, x_3^{(1)}, x_3^{(2)}, ...)$. Since $(x_n^{(i)})_{n \in \mathbb{N}}$, $(i = 1, 2)$ are subsequences of $(x_n^{(3)})_{n \in \mathbb{N}}$, we deduce that $\limsup_{n \to \infty} f(x_n^{(i)}) \subset \limsup_{n \to \infty} f(x_n^{(3)})$, $(i = 1, 2)$. Hence we obtain $\lambda y_1 + (1 - \lambda) y_2 \in \limsup_{n \to \infty} f(x_n^{(3)})$. From $x_n^{(3)} \to \bar{x}$ it follows that $\lambda y_1 + (1 - \lambda) y_2 \in A$.

(ii) *Closedness.* Let $(y_m)_{m \in \mathbb{N}} \subset A$ with $y_n \to \bar{y} \in Y$. For all $m \in \mathbb{N}$ there exists a sequence $(x_n^{(m)})_{n \in \mathbb{N}} \subset X$ such that $\bar{x} = \lim_{n \to \infty} x_n^{(m)}$ and $y_m \in \limsup_{n \to \infty} f(x_n^{(m)})$. Thus we can construct a strictly increasing function $n_0 : \mathbb{N} \to \mathbb{N}$ by

$$\forall m \in \mathbb{N},\ \exists n_0(m) \in \mathbb{N},\ \forall n \geq n_0(m),\ \forall k \in \{1, ..., m\},\ \left\| x_n^{(k)} - \bar{x} \right\| < \frac{1}{m}.$$

Consider the (not necessarily strictly) increasing function $m_0 : \mathbb{N} \to \mathbb{N} \cup \{-\infty\}$ being defined by $m_0(n) := \sup \{ m \in \mathbb{N} |\ n \geq n_0(m) \}$. Of course, we have $m_0(n) \to \infty$ for $n \to \infty$. Define a sequence $(\bar{x}_n)_{n \in \mathbb{N}} \subset X$ by

$$(\bar{x}_n)_{n \in \mathbb{N}} := (x_1^{(1)}, x_1^{(2)}, ..., x_1^{(m_0(1))}, x_2^{(1)}, x_2^{(2)}, ..., x_2^{(m_0(2))}, \ldots, x_n^{(1)}, x_n^{(2)}, ..., x_n^{(m_0(n))}, \ldots).$$

where, without loss of generality, it can be assumed that $m_0(n) \neq -\infty$ for all $n \in \mathbb{N}$. Clearly, the sequence $(\bar{x}_n)_{n\in\mathbb{N}}$ converges to $\bar{x}$ and we have $(x_n^{(m)})_{n\geq n_0(m)} \subset (\bar{x}_n)_{n\in\mathbb{N}}$ for all $m \in \mathbb{N}$. It follows that $\limsup_{n\to\infty} f(x_n^{(m)}) \subset \limsup_{n\to\infty} f(\bar{x}_n)$ for all $m \in \mathbb{N}$, whence $(y_m)_{m\in\mathbb{N}} \subset \limsup_{n\to\infty} f(\bar{x}_n)$. Since $\limsup_{n\to\infty} f(\bar{x}_n)$ is closed, we get $\bar{y} \in \limsup_{n\to\infty} f(\bar{x}_n) \subset A$. $\square$

As an easy consequence of the definition we have the following relationship between upper and lower $\mathcal{F}$- and $\mathcal{C}$-limits: $\limsup_{x\to\bar{x}} f(x) \supset \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x)$, $\liminf_{x\to\bar{x}} f(x) \supset \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$.

A function $f : X \to \mathcal{C}$ is said to be *upper* $\mathcal{C}$*-semicontinuous* ($\mathcal{C}$*-usc*), *lower* $\mathcal{C}$*-semicontinuous* ($\mathcal{C}$*-lsc*), $\mathcal{C}$*-continuous* at $\bar{x} \in X$ if $f(\bar{x}) \supset \limsup_{x\to\bar{x}} f(x)$, $f(\bar{x}) \subset \liminf_{x\to\bar{x}} f(x)$, $f(\bar{x}) = \lim_{x\to\bar{x}} f(x)$, respectively. If f is $\mathcal{C}$-lsc, $\mathcal{C}$-usc, $\mathcal{C}$-continuous at every $\bar{x} \in X$ we just say f is $\mathcal{C}$-lsc, $\mathcal{C}$-usc, $\mathcal{C}$-continuous, respectively. It is easy to see that $\mathcal{C}$-usc implies $\mathcal{F}$-usc and $\mathcal{F}$-lsc implies $\mathcal{C}$-lsc.

Remark 3.2 Of course, $\mathcal{C}$-semicontinuity can also be defined for arbitrary set–valued maps, and the following results can be easily rewritten in this case. We prefer to suppose $\mathcal{C}$-valued functions, because this makes our notation easier.

With the aid of Lemma 2.5 and 2.6 we obtain our main result, a characterization of upper $\mathcal{C}$-semicontinuity or in other words a characterization of Cesari's property (Q).

Theorem 3.3 *Let $f : X \to \mathcal{C}$ and $\bar{x} \in \operatorname{dom} f$. Then the following statements are equivalent:*

(i) *f is $\mathcal{C}$-usc at $\bar{x}$,*

(ii) *For all $y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ$ the function $\sigma_{f(\cdot)}(y^*) : X \to \overline{\mathbb{R}}$ is usc at $\bar{x}$.*

Proof. Let be given an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$.

(i) $\Rightarrow$ (ii). Let the sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ be defined by $\tilde{x}_{2n} := x_n$ and $\tilde{x}_{2n+1} := \bar{x}$. From (i) we deduce that $f(\bar{x}) = \limsup_{n\to\infty} f(\tilde{x}_n)$. Lemma 2.6 implies that

$$\forall y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ, \;\; \sigma_{f(\bar{x})}(y^*) = \limsup_{n\to\infty} \sigma_{f(\tilde{x}_n)}(y^*) \leq \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*).$$

(ii) $\Rightarrow$ (i). Without loss of generality we can assume that $A := \limsup_{n\to\infty} f(x_n) \neq \emptyset$. By Lemma 2.6 we obtain

$$\forall y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ, \;\; \sigma_{f(\bar{x})}(y^*) \geq \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*) = \sigma_A(y^*).$$

From Lemma 2.5 we deduce that $f(\bar{x}) \supset A$. $\square$

Remark 3.4 By standard arguments one can show: If $\sigma_{f(\cdot)}(y^*) : X \to \overline{\mathbb{R}}$ upper semicontinuous at $\bar{x}$ for all $y^* \in Y^*$, then f is upper $\mathcal{C}$-semicontinuous at $\bar{x}$. A result of this type in Banach spaces can be found in [8, Proposition 2.1]. The proof of the above result, however, is based on some further arguments, see the proof of Lemma 2.5 and Lemma 2.6, which can be found in [7]. Therefore we reduce ourselves to a finite dimensional setting.

The next assertion about nested upper $\mathcal{C}$-limits is essential for an expedient definition of the *upper $\mathcal{C}$-semicontinuous hull.* An analogous assertion for the lower $\mathcal{C}$-limit is not true, see Example 3.8 below.

Proposition 3.5 *Let $f : X \to \mathcal{C}$ and $\bar{x} \in X$. Then it holds*

$$\limsup_{x\to\bar{x}} f(x) = \limsup_{x\to\bar{x}} \limsup_{w\to x} f(w).$$

Proof. Clearly, we have $f(x) \subset \limsup_{w\to x} f(w)$ for all $x \in X$, which implies the inclusion "$\subset$". It remains to show that $A := \limsup_{n\to\infty} \limsup_{w\to x_n} f(w) \subset \limsup_{n\to\infty} f(x_n) =: B$ for an arbitrarily given sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$. For all $y^* \in \mathrm{ri}\left(0^+B\right)^\circ$ it holds

$$\sigma_A(y^*) \overset{\text{Pr. 2.1}}{\leq} \limsup_{n\to\infty} \limsup_{w\to x_n} \sigma_{f(w)}(y^*) = \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*) \overset{\text{Lem. 2.6}}{=} \sigma_B(y^*).$$

Lemma 2.5 yields that $A \subset B$. $\square$

The *upper $\mathcal{C}$-semicontinuous hull* of a function $f : X \to \mathcal{C}$ is defined by

$$(\mathrm{usc}\, f) : X \to \mathcal{C}, \quad (\mathrm{usc}\, f)(x) := \limsup_{x'\to x} f(x').$$

The *hypograph* of a function $f : X \to \mathcal{C}$ is the set $\mathrm{hyp}\, f := \{(x, A) \in X \times \mathcal{C} |\ A \subset f(x)\}$. A subset $\mathcal{A} \subset X \times \mathcal{C}$ is said to be closed if for every sequence $(x_n, A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $x_n \to \bar{x} \in X$ and $A_n \xrightarrow{\mathcal{C}} \bar{A} \in \mathcal{C}$ it is true that $(\bar{x}, \bar{A}) \in \mathcal{A}$. The *$\mathcal{C}$-closure* of a set $\mathcal{A} \subset X \times \mathcal{C}$, denoted by $\mathrm{cl}\, \mathcal{A}$, is the set of all such limits $(\bar{x}, \bar{A}) \in X \times \mathcal{C}$ of sequences $(x_n, A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

Let us collect some properties of the upper $\mathcal{C}$-semicontinuous hull.

Proposition 3.6 *For $f : X \to \mathcal{C}$ the following statements hold true:*

(i) $\mathrm{gr}\,(\mathrm{usc}\, f) \supset \mathrm{cl}\,(\mathrm{gr}\, f)$,

(ii) $\mathrm{hyp}\,(\mathrm{usc}\, f) \supset \mathrm{cl}\,(\mathrm{hyp}\, f)$,

(iii) $(\mathrm{usc}\, f)$ *is $\mathcal{C}$-usc,*

(iv) $\forall x \in X,\ (\mathrm{usc}\, f)(x) \supset f(x)$,

(v) f *is $\mathcal{C}$-usc at* $\bar{x} \in X \quad \Leftrightarrow \quad (\mathrm{usc}\, f)(\bar{x}) = f(\bar{x})$,

(vi) $\mathrm{gr}\,(\mathrm{usc}\, f)$ *is $\mathcal{C}$-closed,*

(vii) $\mathrm{hyp}\,(\mathrm{usc}\, f)$ *is $\mathcal{C}$-closed.*

Proof. (i) Let $(\bar{x}, \bar{y}) \in \mathrm{cl}\,(\mathrm{gr}\, f)$. Then there exists a sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset \mathrm{gr}\, f$ converging to $(\bar{x}, \bar{y})$. For all $n \in \mathbb{N}$, we have $\{y_n\} \subset f(x_n)$. Hence

$$\{\bar{y}\} = \lim_{n\to\infty} \{y_n\} = \limsup_{n\to\infty} \{y_n\} \subset \limsup_{n\to\infty} f(x_n) \subset \limsup_{x\to\bar{x}} f(x) = (\mathrm{usc}\, f)(\bar{x}),$$

i.e. $(\bar{x}, \bar{y}) \in \mathrm{gr}\,(\mathrm{usc}\, f)$. The proof of (ii) is similar. Statement (iii) follows from Proposition 3.5. The proofs of (iv) and (v) are analogous to those of Proposition 2.7 (iv) and (v). (vi) Let $(x_n, y_n)_{n\in\mathbb{N}} \subset \mathrm{gr}\,(\mathrm{usc}\, f)$ with $(x_n, y_n) \to (\bar{x}, \bar{y}) \in X \times Y$ be given. Proceeding as in (i), but replacing f by $(\mathrm{usc}\, f)$, we obtain $\{\bar{y}\} \subset (\mathrm{usc}\,(\mathrm{usc}\, f))(\bar{x})$. From (iii) we conclude that $(\mathrm{usc}\,(\mathrm{usc}\, f))(\bar{x}) = (\mathrm{usc}\, f)(\bar{x})$. Hence $(\bar{x}, \bar{y}) \in \mathrm{gr}\,(\mathrm{usc}\, f)$. The proof of (vii) is similar to that of (iv). $\square$

The next example shows that neither the $\mathcal{C}$-closedness of $\mathrm{hyp}\, f$ nor the closedness of $\mathrm{gr}\, f$ implies that f is $\mathcal{C}$-usc.

Example 3.7 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be defined by $f(x) := \{1/x\}$ if $x \neq 0$ and $f(0) := \emptyset$. Then, it easily follows that $\operatorname{gr} f \subset \mathbb{R} \times \mathbb{R}$ closed and $\operatorname{hyp} f \subset \mathbb{R} \times \mathcal{C}(\mathbb{R})$ is $\mathcal{C}$-closed, but f is not $\mathcal{C}$-usc.

In Remark 2.9 (due to [10]) we noticed that the lower $\mathcal{F}$-semicontinuous hull that is defined analogously to the upper $\mathcal{F}$-semicontinuous hull is not necessarily lower $\mathcal{F}$-semicontinuous. There are analogous problems with the lower $\mathcal{C}$-semicontinuous hull. This is due to the fact that there is no analogous assertion to Proposition 3.5 for lower $\mathcal{C}$-limits, as the following example shows.

Example 3.8 For functions $f : X \to \mathcal{C}$, in general, we have

$$\liminf_{x \to \bar{x}} f(x) \neq \liminf_{x \to \bar{x}} \; \liminf_{w \to x} f(w).$$

Indeed, consider the function $f : \mathbb{R}^2 \to \mathcal{C}(\mathbb{R})$, defined by

$$f(x) := \begin{cases} \{\|x\|\} & \text{if} \quad x_1 \geq 0 \\ \{-\|x\|\} & \text{if} \quad x_1 < 0. \end{cases}$$

Then it holds

$$\liminf_{w \to x} f(w) := \begin{cases} \{\|x\|\} & \text{if} \quad x_1 > 0 \text{ or } x_2 = 0 \\ \{-\|x\|\} & \text{if} \quad x_1 < 0 \\ \emptyset & \text{if} \quad x_1 = 0 \text{ and } x_2 \neq 0. \end{cases}$$

Hence we obtain $\{0\} = \liminf_{x \to 0} f(x) \neq \liminf_{x \to 0} \liminf_{w \to x} f(w) = \emptyset$.

4 Locally bounded functions

The concept of local boundedness of a set-valued map plays an important role in Variational Analysis, see [10]. As an easy consequence of the definition ([10, Definition 5.14]), local boundedness of a map $f : X \rightrightarrows Y$ at $\bar{x}$ implies that $f(\bar{x})$ is a bounded subset of Y. This means, local boundedness is (at least locally) adapted to set-valued maps with bounded values. Therefore we introduce a slightly generalized concept, adapted to the framework of $\mathcal{C}$-valued functions. It turns out that this concept provides a sufficient condition for the coincidence of $\mathcal{F}$-limits and $\mathcal{C}$-limits.

A function $f : X \to \mathcal{C}$ is said to be *locally bounded* at $\bar{x} \in \operatorname{dom} f$ if there exists a neighborhood $V \in \mathcal{N}(\bar{x})$ such that the following conditions are satisfied:

(i) $0^+ \sup_{x \in V} f(x) \subset 0^+ f(\bar{x})$,

(ii) $\forall x \in V \cap \operatorname{dom} f,\ 0^+ f(x) \supset 0^+ f(\bar{x})$.

Note that, if $f : X \to \mathcal{C}$ is locally bounded at $\bar{x} \in \operatorname{dom} f$, (i) and (ii) of the previous definition are always satisfied with equality. Moreover, if $f(\bar{x}) \subset Y$ is bounded, our concept coincides with the classical one.

Theorem 4.1 *Let $f : X \to \mathcal{C}$ be locally bounded at $\bar{x} \in \operatorname{dom} f$. Then,*

$$\limsup_{x \to \bar{x}} f(x) = \operatorname{cl} \operatorname{conv} \operatorname*{LIM\,SUP}_{x \to \bar{x}} f(x).$$

Proof. Clearly, we have $\limsup_{x\to\bar{x}} f(x) \supset \operatorname{cl}\operatorname{conv} \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x)$. To show the opposite inclusion let $y \in \limsup_{x\to\bar{x}} f(x)$ be given. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$ such that $y \in \limsup_{n\to\infty} f(x_n)$. Assuming that there exists some $n_0 \in \mathbb{N}$ such that $f(x_n) = \emptyset$ for all $n \geq n_0$, we obtain $\limsup_{n\to\infty} f(x_n) = \emptyset$, which contradicts $y \in \limsup_{x\to\bar{x}} f(x)$. Hence, by $(x_{n_k})_{k\in\mathbb{N}} := (x_n)_{n\in\mathbb{N}} \cap \operatorname{dom} f$, we obtain a subsequence of $(x_n)_{n\in\mathbb{N}}$. Of course, we have $\limsup_{n\to\infty} f(x_n) = \limsup_{k\to\infty} f(x_{n_k})$. By the local boundedness, we find $k_0 \in \mathbb{N}$ such that, $f(x_{n_k}) \in \mathcal{C}_K$ for all $k \geq k_0$ and $\sup_{k\geq k_0} f(x_{n_k}) \in \mathcal{C}_K$, where $K := 0^+ f(\bar{x})$. Theorem 2.3 yields $y \in \operatorname{cl}\operatorname{conv} \operatorname{LIM\,SUP}_{k\to\infty} f(x_{n_k}) \subset \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x)$. □

In the next example we show that the assertion of the preceding theorem can fail if one of the conditions in the definition of the local boundedness concept is not satisfied.

Example 4.2 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be defined by $f(x) := \left\{\frac{1}{x}\right\}$ if $x \neq 0$ and $f(0) := \{0\}$, i.e. (ii) is satisfied, but (i) is not. Then, $\mathbb{R} = \limsup_{x\to 0} f(x) \neq \operatorname{cl}\operatorname{conv} \operatorname{LIM\,SUP}_{x\to 0} f(x) = \{0\}$.

Example 4.3 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R}^2)$ be defined by $f(x) = \left\{y \in \mathbb{R}^2 \mid y_2 = 1, y_1 = 1/x\right\}$ if $x \neq 0$ and $f(0) := \left\{y \in \mathbb{R}^2 \mid y_2 = 0\right\}$, i.e. (i) is satisfied, but (ii) is not. An easy calculation shows that $\left\{y \in \mathbb{R}^2 \mid 0 \leq y_2 \leq 1\right\} = \limsup_{x\to 0} f(x) \neq \operatorname{cl}\operatorname{conv} \operatorname{LIM\,SUP}_{x\to 0} f(x) = \left\{y \in \mathbb{R}^2 \mid y_2 = 0\right\}$.

Local boundedness of a function $f : X \to \mathcal{C}$ at a point $\bar{x} \in \operatorname{dom} f$ also implies that $\liminf_{x\to\bar{x}} f(x) = \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$ (see Corollary 4.6 below). Moreover, as shown in the next theorem, a weaker assumption is already sufficient.

Theorem 4.4 *Let $f : X \to \mathcal{C}$ and $\bar{x} \in \operatorname{dom} f$ such that for all sequences $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to \bar{x}$ there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and a nonempty closed convex cone $K \subset Y$ with $f(x_{n_k}) \in \mathcal{C}_K$ for all $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}} f(x_{n_k}) \in \mathcal{C}_K$. Then it holds $\liminf_{x\to\bar{x}} f(x) = \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$.*

Proof. Of course, $\liminf_{x\to\bar{x}} f(x) \supset \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$. In order to show the opposite inclusion let $y \in Y \setminus \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$ be given (the case $\operatorname{LIM\,INF}_{x\to\bar{x}} f(x) = Y$ is obvious). Hence there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to \bar{x}$ such that $y \notin \operatorname{LIM\,INF}_{n\to\infty} f(x_n)$. Every subsequence of $(x_n)_{n\in\mathbb{N}}$ is again a sequence converging to $\bar{x}$, hence our assumption ensures that Theorem 2.4 is applicable. It follows that $y \notin \liminf_{n\to\infty} f(x_n)$. □

The next example shows that the assertion of the previous theorem can fail if the assumption is not satisfied.

Example 4.5 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R}^2)$ be defined by $f(x) := \operatorname{conv}\left\{\left(-1, -\frac{1}{x}\right), \left(1, \frac{1}{x}\right)\right\}$ if $x > 0$ and $f(x) := \mathbb{R}^2$ if $x \leq 0$, i.e. the condition in the previous theorem is not satisfied. Then we have $\left\{y \in \mathbb{R}^2 \mid -1 \leq y_2 \leq 1\right\} = \liminf_{x\to 0} f(x) \neq \operatorname{LIM\,INF}_{x\to 0} f(x) = \left\{y \in \mathbb{R}^2 \mid y_2 = 0\right\}$.

Corollary 4.6 *Let $f : X \to \mathcal{C}$ be locally bounded at $\bar{x} \in \operatorname{dom} f$. Then,*

$$\liminf_{x\to\bar{x}} f(x) = \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x).$$

Proof. By the local boundedness of f at $\bar{x}$, for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$ there exists some $n_0 \in \mathbb{N}$ such that $f(x_n) \in \mathcal{C}_K$ for all $n \geq n_0$ and $\sup_{n\geq n_0} f(x_n) \in \mathcal{C}_K$, where $K := 0^+ f(\bar{x})$. Hence, Theorem 4.4 yields the desired assertion. $\square$

Corollary 4.7 *Let $f : X \to \mathcal{C}$ be locally bounded on* $\operatorname{dom} f$. *Then the following statements are equivalent:*

(i) $\operatorname{hyp} f \subset X \times \mathcal{C}$ *is $\mathcal{C}$-closed,*

(ii) f *is $\mathcal{C}$-usc,*

(iii) $\operatorname{gr} f \subset X \times Y$ *is closed.*

Proof. (i) ⇒ (iii). Elementary (see also [7]).
(iii) ⇒ (ii). [10, Theorem 5.7 (a)] yields that f is $\mathcal{F}$-usc. By Theorem 4.1, f is $\mathcal{C}$-usc.
(ii) ⇒ (i). Follows from Proposition 3.6 (v), (vii). $\square$

5 Concave functions

This section is devoted to the special case of concave $\mathcal{C}$-valued functions. We show that $\mathcal{F}$-semicontinuity and $\mathcal{C}$-semicontinuity coincide in this case. A function $f : X \to \mathcal{C}$ is said to be *concave* if

$$\forall \lambda \in [0,1],\, \forall x_1, x_2 \in X,\;\; f(\lambda \cdot x_1 + (1-\lambda) \cdot x_2) \supset \lambda\, f(x_1) + (1-\lambda)\, f(x_2).$$

It is easy to see that a function $f : X \to \mathcal{C}$ is concave if and only if $\operatorname{hyp} f \subset X \times \mathcal{C}$ is convex. Of course, concavity (which is often called convexity) of a set-valued map is equivalent to the convexity of its graph. The following proposition shows that the values of a concave $\mathcal{C}$-valued function essentially have the same recession cone.

Proposition 5.1 *Let $f : X \to \mathcal{C}$ be concave. If $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$, then $0^+ f(x) \subset 0^+ f(\bar{x})$ for all $x \in \operatorname{dom} f$ and $0^+ f(x) = 0^+ f(\bar{x})$ for all $x \in \operatorname{ri}(\operatorname{dom} f)$.*

Proof. Note that $\operatorname{dom} f$ is convex. Let $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$ and $x \in \operatorname{dom} f$. By [9, Theorem 6.4], there exists $\mu > 1$ such that $\hat{x} := \mu\bar{x} + (1-\mu)x \in \operatorname{dom} f$. Set $\lambda := 1/\mu \in (0,1)$. The concavity of f yields $f(\bar{x}) \supset \lambda f(\hat{x}) \oplus (1-\lambda) f(x)$. Since $\hat{x} \in \operatorname{dom} f$ we can choose some $\hat{y} \in f(\hat{x})$, hence $f(\bar{x}) \supset \lambda \{\hat{y}\} + (1-\lambda) f(x) := C_x$. It follows that $0^+ C_x \subset 0^+ f(\bar{x})$. With the aid of [9, Theorem 8.1] we conclude that $0^+ C_x = 0^+ f(x)$, hence $0^+ f(x) \subset 0^+ f(\bar{x})$. Assume there is some $\tilde{x} \in \operatorname{ri}(\operatorname{dom} f)$ with $0^+ f(\tilde{x}) \subsetneq 0^+ f(\bar{x})$, then the first part yields $0^+ f(x) \subset 0^+ f(\tilde{x})$ for all $x \in \operatorname{dom} f$, whence the contradiction $0^+ f(\bar{x}) \subsetneq 0^+ f(\bar{x})$. $\square$

Theorem 5.2 *Let $f : X \to \mathcal{C}$ be concave. Then, for all $\bar{x} \in X$ it holds*

$$\limsup_{x\to\bar{x}} f(x) = \operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x).$$

Proof. Of course, we always have $\limsup_{x\to\bar{x}} f(x) \supset \mathrm{LIM\,SUP}_{x\to\bar{x}} f(x)$. To show the opposite inclusion let $y \in \limsup_{x\to\bar{x}} f(x)$ be given. Hence there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$ such that $y \in \limsup_{n\to\infty} f(x_n)$. By Proposition 2.2 this can be written as

$$\exists\,(\lambda_n)_{n\in\mathbb{N}} \subset \Delta_{p+1},\ \exists\,(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}^{p+1},\ \exists\,(z_n)_{n\in\mathbb{N}} \subset Y^{p+1},$$
$$y = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i,\ \forall n\in\mathbb{N},\ \forall j\in\overline{0,p},\ k_n^j \geq n,\ z_n^j \in A_{k_n^j}.$$

We define two sequences $(y_m)_{m\in\mathbb{N}} \subset Y$ and $(\tilde{x}_m)_{m\in\mathbb{N}} \subset X$ by

$$y_m := \sum_{j\in\overline{0,p}} \lambda_j^{(m)} z_j^{(m)}, \qquad \tilde{x}_m := \sum_{j\in\overline{0,p}} \lambda_j^{(m)} x_{k_j^{(m)}}.$$

Then we have $y_m \to y$, $\tilde{x}_m \to \bar{x}$ and the concavity of f yields that

$$y_m = \sum_{j\in\overline{0,p}} \lambda_j^{(m)} z_j^{(m)} \in \sum_{j\in\overline{0,p}} \lambda_j^{(m)} f\left(x_{k_j^{(m)}}\right) \subset f\left(\sum_{j\in\overline{0,p}} \lambda_j^{(m)} x_{k_j^{(m)}}\right) = f\left(\tilde{x}_m\right)$$

for all $m \in \mathbb{N}$. By [10, 5(1)], this means $y \in \mathrm{LIM\,SUP}_{x\to\bar{x}} f(x)$. $\square$

Corollary 5.3 *Let $f : X \to \mathcal{C}$ be concave. Then the following statements hold true:*

(i) $(\mathrm{usc}\, f) = (\mathrm{USC}\, f)$*,*

(ii) $(\mathrm{usc}\, f)$ *is concave,*

(iii) $(\mathrm{usc}\, f) : X \to \mathcal{C}_K \cup \{\emptyset\}$ *for some nonempty closed convex cone $K \subset Y$.*

Proof. (i) Follows from Theorem 5.2.

(ii) f concave $\Leftrightarrow$ $\mathrm{gr}\, f$ convex $\Rightarrow$ $\mathrm{cl}\,(\mathrm{gr}\, f) = \mathrm{gr}\,(\mathrm{USC}\, f)$ convex $\Leftrightarrow$ $\mathrm{USC}\, f = \mathrm{usc}\, f$ convex.

(iii) Since $(\mathrm{usc}\, f)$ is $\mathcal{F}$-usc and concave, its graph is closed and convex. If $\mathrm{dom}\,(\mathrm{usc}\, f) = \emptyset$ there is nothing to prove, otherwise, there exists some $\bar{x} \in \mathrm{ri}\,\mathrm{dom}\,(\mathrm{usc}\, f)$. From Proposition 5.1 we deduce that $0^+(\mathrm{usc}\, f)(x) \subset 0^+(\mathrm{usc}\, f)(\bar{x}) =: K$ for all $x \in \mathrm{dom}\,(\mathrm{usc}\, f)$. It remains to prove the opposite inclusion for all $x \in \mathrm{dom}\,(\mathrm{usc}\, f)$. Indeed, let $\hat{y} \in 0^+(\mathrm{usc}\, f)(\bar{x})$ and $\bar{y} \in (\mathrm{usc}\, f)(\bar{x})$ be arbitrarily chosen. By [9, Theorem 8.3] we have $\bar{y} + \lambda\hat{y} \in (\mathrm{usc}\, f)(\bar{x})$ for all $\lambda \geq 0$ and equivalently $(0, \hat{y}) \in 0^+\mathrm{gr}\,(\mathrm{usc}\, f)$. Given some $x \in \mathrm{dom}\,(\mathrm{usc}\, f)$ we can choose $y \in (\mathrm{usc}\, f)(x)$. Since $(0, \hat{y}) \in 0^+\mathrm{gr}\,(\mathrm{usc}\, f)$, [9, Theorem 8.3] yields that $y + \lambda\hat{y} \in (\mathrm{usc}\, f)(x)$ for all $\lambda \geq 0$ and equivalently $\hat{y} \in 0^+(\mathrm{usc}\, f)(x)$. $\square$

Corollary 5.4 *Let $f : X \to \mathcal{C}_K \cup \{\emptyset\}$. Then the following statements are equivalent:*

(i) *f is concave and $\mathcal{C}$-usc,*

(ii) $\mathrm{gr}\, f \subset X \times Y$ *is convex and closed,*

(iii) $\mathrm{hyp}\, f \subset X \times \mathcal{C}$ *is convex and $\mathcal{C}$-closed,*

(iv) *For all $y^* \in \mathrm{ri}\, K^\circ$ the function $\sigma_{f(\,\cdot\,)}(y^*) : X \to \overline{\mathbb{R}}$ is concave and usc.*

Proof. The equivalence of the convexity/concavity assertions is immediate.

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). The equivalence of the upper $\mathcal{C}$-semicontinuity and closedness assertions follows similarly to the proof of Corollary 4.7 (using Corollary 5.3 (i) instead of Theorem 4.1).

(i) $\Leftrightarrow$ (iv). From Theorem 3.3 taking into account Corollary 5.3 (iii). $\square$

Theorem 5.5 *Let $f : X \to \mathcal{C}$ be concave. Then the following assertions hold true:*

(i) *f is $\mathcal{C}$-usc at every $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$,*

(ii) *f is $\mathcal{C}$-continuous at every $\bar{x} \in \mathrm{int}\,(\mathrm{dom}\, f)$.*

Proof. (i) Let $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$ be given and let $K := 0^+ f(\bar{x})$. By Theorem 3.3, it remains to show that, for all $y^* \in \mathrm{ri}\, K^\circ$, $\sigma_{f(\,\cdot\,)}(y^*)$ is usc at $\bar{x}$. From Proposition 5.1 we deduce that $0^+ f(x) = K$ for all $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$. Hence, for all $y^* \in \mathrm{ri}\, K^\circ$ it is true that $\bar{x} \in \mathrm{ri}\,\big(\mathrm{dom}\, \sigma_{f(\,\cdot\,)}(y^*)\big)$, whence, by [9, Theorem 7.4], $\sigma_{f(\,\cdot\,)}(y^*)$ is usc at $\bar{x}$.

(ii) By [10, Theorem 5.9 (b)], f is $\mathcal{F}$-lsc at $\bar{x} \in \mathrm{int}\,(\mathrm{dom}\, f)$. Hence f is $\mathcal{C}$-lsc at $\bar{x}$. Now the assertion follows from (i). $\square$

We close this article with some assertions concerning the local boundedness of concave functions.

Theorem 5.6 *Let $f : X \to \mathcal{C}$ be concave and $\mathcal{C}$-usc. Then, f is locally bounded on* $\mathrm{dom}\, f$.

Proof. Let $\bar{x} \in \mathrm{dom}\, f$, $V := \{x \in X |\ \|x - \bar{x}\| \le 1\}$ and $K := 0^+ f(\bar{x})$. By Proposition 3.6 (v) and Corollary 5.3 (iii) we have $0^+ f(x) = K$ for all $x \in \mathrm{dom}\, f$. Hence, condition (ii) in the definition of the local boundedness is satisfied. It remains to show $0^+ \sup_{x \in V} f(x) \subset K$.

Since V is convex and f is concave, the set $\bigcup_{x \in V} f(x)$ is convex. Since V is compact and $\mathrm{gr}\, f$ is closed, we deduce that $\bigcup_{x \in V} f(x)$ is closed. Hence $0^+ \sup_{x \in V} f(x) = 0^+ \bigcup_{x \in V} f(x)$. Let $k \in 0^+ \bigcup_{x \in V} f(x)$ be given. By [9, Theorem 8.2], k is the limit of a sequence $(\lambda_n y_n)_{n \in \mathbb{N}}$ where $\lambda_n \downarrow 0$ and $y_n \in \bigcup_{x \in V} f(x)$. Clearly, for all $n \in \mathbb{N}$ there exists $x_n \in V$ such that $y_n \in f(x_n)$. Since V is bounded, we have $(\lambda_n x_n, \lambda_n y_n) \to (0, k)$. Applying [9, Theorem 8.2] to the closed convex set $\mathrm{gr}\, f \subset X \times Y$, we obtain $(0, k) \in 0^+ \mathrm{gr}\, f$. With the aid of [9, Theorem 8.3] we deduce that $\bar{y} + \lambda k \in f(\bar{x} + \lambda \cdot 0) = f(\bar{x})$ for all $\lambda \ge 0$ and arbitrary $\bar{y} \in f(\bar{x})$, which is equivalent to $k \in 0^+ f(\bar{x}) = K$. $\square$

Corollary 5.7 *If $f : X \to \mathcal{C}$ is concave, then f is locally bounded at every $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$.*

Proof. Theorem 5.6 yields that $(\mathrm{usc}\, f)$ is locally bounded at every $x \in \mathrm{dom}\,(\mathrm{usc}\, f)$. By Theorem 5.5 (i), we know that $f(\bar{x}) = (\mathrm{usc}\, f)(\bar{x})$ for all $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$. $\square$

Corollary 5.8 *Let $f : X \to \mathcal{C}$ be concave. Then, for all $\bar{x} \in X$ it holds*

$$\liminf_{x \to \bar{x}} f(x) = \operatorname*{LIM\,INF}_{x \to \bar{x}} f(x).$$

Proof. If $\bar{x} \in \mathrm{int}\,(\mathrm{dom}\, f)$, this follows from Corollary 5.7 and Corollary 4.6. Otherwise, we have $\liminf_{x \to \bar{x}} f(x) = \mathrm{LIM\,INF}_{x \to \bar{x}} f(x) = \emptyset$. $\square$

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