On totally Fenchel unstable functions in finite dimensional spaces

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Abstract

We give an answer to the Problem 11.6 posed by Stephen Simons in his book "From Hahn-Banach to Monotonicity": Do there exist a nonzero finite dimensional Banach space and a pair of extended real-valued, proper and convex functions which is totally Fenchel unstable? The answer is negative.

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Consider E a nontrivial real Banach space and E^* its topological dual space. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in E^*$ at $x \in E$. The *Fenchel-Moreau conjugate* of a function $f : E \to \overline{\mathbb{R}}$ is the function $f^* : E^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x\in E}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in E^*$. We denote by $\mathrm{dom}(f) = \{x \in E : f(x) < +\infty\}$ its *domain*. We call f *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in E$.

Having $f, g : E \to \overline{\mathbb{R}}$ two arbitrary proper and convex functions, we say that f and g satisfy *stable Fenchel duality* if for all $x^* \in E^*$, there exists $z^* \in E^*$ such that

$$(f + g)^*(x^*) = f^*(x^* - z^*) + g^*(z^*).$$

If this property holds just for $x^* = 0$, then we obtain the classical Fenchel duality. In this case we say thay f and g *satisfy Fenchel duality.* The pair f, g is called *totally Fenchel unstable* (see [3]) if f and g satisfy Fenchel duality but

$$y^*, z^* \in E^* \text{ and } (f + g)^*(y^* + z^*) = f^*(y^*) + g^*(z^*) \Longrightarrow y^* + z^* = 0.$$

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Obviously, stable Fenchel duality implies Fenchel duality, but the converse is not true (see the example in [1], pp. 2798-2799 and Example 11.1 in [3]). Nevertheless, each of these examples, both given in a finite dimensional setting, fails when one tries to verify total Fenchel unstability.

In the infinite dimensional setting the following example of a pair of proper and convex functions f, g, which is totally Fenchel unstable, has been proposed in Example 11.3 in [3]. Let C be a nonempty, bounded, closed and convex subset of E such that there exists an extreme point x_0 of C which is not a support point of C. Recall that if C is a convex subset of E, then $x \in C$ is a *support point* of C if there exists $x^* \in E^* \setminus \{0\}$ such that $\langle x^*, x\rangle = \sup\langle x^*, C\rangle$. We denote by $\delta_D : E \to \overline{\mathbb{R}}$ the *indicator function* of a set $D \subseteq E$ defined as

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{otherwise.} \end{cases}$$

Taking $A := x_0 - C$, $B := C - x_0$, $f := \delta_A$ and $g := \delta_B$, Simons proved in [3] that the pair f, g is totally Fenchel unstable. Let us also mention that an example of a set C and a point x_0 with the above mentioned properties was given in the space ℓ_2, following an idea due to Jonathan Borwein (see [3]).

In finite dimensional spaces a similar example cannot be given, as every extreme point of a convex set is a support point. This fact determines Stephen Simons to formulate the following open problem (Problem 11.6 in [3]).

Problem. Do there exist a nonzero finite dimensional Banach space E and $f, g : E \to \overline{\mathbb{R}}$ proper and convex functions such that the pair f, g is totally Fenchel unstable?

We show that the answer to this question is negative. This result can be interpreted as follows:

If two proper and convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ satisfy Fenchel duality, then there exists at least one element $x^* \in \mathbb{R}^n \setminus \{0\}$, such that $f - \langle x^*, \cdot\rangle$ and g (or f and $g - \langle x^*, \cdot\rangle$) satisfy Fenchel duality, too.

We start with some preliminary results. For a function $f : E \to \overline{\mathbb{R}}$ we denote by $\text{epi}(f) = \{(x, r) \in E \times \mathbb{R} : f(x) \leq r\}$ its *epigraph* and by $\bar{f}$ its *lower semicontinuous hull* of f, namely the function of which epigraph is the closure of epi(f) in $E \times \mathbb{R}$, that is $\text{epi}(\bar{f}) = \text{cl}(\text{epi}(f))$. We write $\omega(E^*, E)$ for the weak* topology on E^*. Further, when $D \subseteq \mathbb{R}^n$ is a nonempty and convex set by 0^+D we denote its *recession cone*.

The following result (see [1, Theorem 2.1]) is direct a consequence of the classical Moreau-Rockafellar theorem.

Theorem 1. If $f, g : E \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. Then

$$\mathrm{epi}((f+g)^*) = \mathrm{cl}(\mathrm{epi}(f^*) + \mathrm{epi}(g^*)),$$

where the closure is taken in the product topology of $(E^*, \omega(E^*, E)) \times \mathbb{R}$.

Under the hypotheses of Theorem 1 follows that $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ is closed in the product topology of $(E^*, \omega(E^*, E)) \times \mathbb{R}$ if and only if $\mathrm{epi}((f+g)^*) = \mathrm{epi}(f^*) + \mathrm{epi}(g^*)$. By [1, Proposition 2.2]), this is equivalent to saying that f and g satisfy stable Fenchel duality.

Of course, for all $x^*, y^* \in E^*$ it holds

$$(f+g)^*(x^*) \le f^*(x^* - y^*) + g^*(y^*). \tag{1}$$

Therefore, a pair f, g of proper and convex functions is *totally Fenchel unstable* if and only if

$$\exists y^* \in E^* : \ \ (f+g)^*(0) = f^*(-y^*) + g^*(y^*). \tag{2}$$

$$\forall x^* \in E^* \setminus \{0\}, \forall y^* \in E^* : \ \ (f+g)^*(x^*) < f^*(x^* - y^*) + g^*(y^*). \tag{3}$$

Moreover, if the pair f, g is totally Fenchel unstable one must have that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. Indeed, if this is not the case, then $f+g$ is identical $+\infty$ and thus $(f+g)^*$ is identical $-\infty$. By (2) there exists $y^* \in E^*$ such that $f^*(-y^*) + g^*(y^*) = -\infty$. But, f and g being proper we get $f^*(-y^*) > -\infty$ and $g^*(y^*) > -\infty$, a contradiction.

We give now a geometric characterization of the property that the pair f, g is totally Fenchel unstable.

Proposition 2. Let $f, g : E \to \overline{\mathbb{R}}$ be proper functions such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. Then the pair f, g is totally Fenchel unstable if and only if

$$\mathrm{epi}((f+g)^*) \cap (\{0\} \times \mathbb{R}) = (\mathrm{epi}(f^*) + \mathrm{epi}(g^*)) \cap (\{0\} \times \mathbb{R}) \tag{4}$$

and there is no $x^* \in E^* \setminus \{0\}$ such that

$$\mathrm{epi}((f+g)^*) \cap (\{x^*\} \times \mathbb{R}) = (\mathrm{epi}(f^*) + \mathrm{epi}(g^*)) \cap (\{x^*\} \times \mathbb{R}). \tag{5}$$

Proof. We want to notice first that we always have $\mathrm{epi}((f+g)^*) \supseteq \mathrm{epi}(f^*) + \mathrm{epi}(g^*)$. As $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$, $(f+g)^*$ never attains $-\infty$.

"$\Rightarrow$" In case $(f+g)^*(0) = +\infty$, the set $\mathrm{epi}((f+g)^*) \cap (\{0\} \times \mathbb{R})$ is empty and (4) follows automatically. In case $(f+g)^*(0) \in \mathbb{R}$, we consider an arbitrary element $r \in \mathbb{R}$ fulfilling $(f+g)^*(0) \le r$. By (2) there exists $y^* \in E^*$ such that $f^*(-y^*) + g^*(y^*) \le r$ and so

$$(0, r) = (-y^*, f^*(-y^*)) + (y^*, r - f^*(-y^*)) \in (\mathrm{epi}(f^*) + \mathrm{epi}(g^*)) \cap (\{0\} \times \mathbb{R}).$$

Also in this case (4) follows.

Assume now that for $x^* \in E^* \setminus \{0\}$ relation (5) is fulfilled. As (3) implies $(f+g)^*(x^*) < +\infty$, we have $(f+g)^*(x^*) \in \mathbb{R}$. In this case $(x^*, (f+g)^*(x^*)) \in \mathrm{epi}((f+g)^*) \cap (\{x^*\} \times \mathbb{R})$ and so $(x^*, (f+g)^*(x^*)) \in \mathrm{epi}(f^*) + \mathrm{epi}(g^*)$. Thus there exist $(y^*, s) \in \mathrm{epi}(f^*)$ and $(z^*, t) \in \mathrm{epi}(g^*)$ such that $y^* + z^* = x^*$ and $s + t = (f+g)^*(x^*)$. This means that $f^*(y^*) + g^*(z^*) \leq (f+g)^*(y^* + z^*)$ which contradicts (3).

"$\Leftarrow$" We prove first that Fenchel duality holds. If $(f+g)^*(0) = +\infty$ this follows automatically from (1). If $(f+g)^*(0) \in \mathbb{R}$, then $(0, (f+g)^*(0)) \in \mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ and so there exist $(-z^*, s) \in \mathrm{epi}(f^*)$ and $(z^*, t) \in \mathrm{epi}(g^*)$ such that $s + t = (f+g)^*(0)$. Thus $f^*(-z^*) + g^*(z^*) \leq (f+g)^*(0)$ and the conclusion follows.

Further assume that there exist $y^*, z^* \in E^*$ such that $y^* + z^* \neq 0$ and $(f+g)^*(y^* + z^*) = f^*(y^*) + g^*(z^*)$. As (5) does not hold with equality, we get $(f+g)^*(y^* + z^*) \in \mathbb{R}$. For all $r \in \mathbb{R}$ such that $(f+g)^*(y^*+z^*) \leq r$ it holds $(y^* + z^*, r) \in (\mathrm{epi}(f^*) + \mathrm{epi}(g^*)) \cap (\{y^* + z^*\} \times \mathbb{R})$. This implies that (5) is satisfied for $x^* = y^* + z^* \neq 0$, a contradiction. □

Proposition 3. Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions such that $\mathrm{int}(\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g})) \neq \emptyset$. Then the pair f, g satisfies stable Fenchel duality.

Proof. Let $x' \in \mathrm{int}(\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g})) \subseteq \mathrm{int}(\mathrm{dom}(\bar{f})) \cap \mathrm{int}(\mathrm{dom}(\bar{g}))$. It holds $\mathrm{int}(\mathrm{dom}(\bar{f})) = \mathrm{ri}(\mathrm{dom}(\bar{f})) = \mathrm{ri}(\mathrm{cl}(\mathrm{dom}(\bar{f}))) = \mathrm{ri}(\mathrm{cl}(\mathrm{dom}(f))) = \mathrm{ri}(\mathrm{dom}(f))$ and the same applies for g. This means that $x' \in \mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g))$. For all $x^* \in \mathbb{R}^n$ we have $\mathrm{dom}(f) = \mathrm{dom}(f - \langle x^*, \cdot \rangle))$. By the Fenchel duality theorem [2, Theorem 31.1], there exists some $y^* \in \mathbb{R}^n$ such that

$$\begin{aligned} -(f+g)^*(x^*) &= \inf_{x \in \mathbb{R}^n} \{f(x) - \langle x^*, x \rangle + g(x)\} \\ &= -(f - \langle x^*, \cdot \rangle)^*(-y^*) - g^*(y^*) \\ &= -f^*(x^* - y^*) - g^*(y^*). \end{aligned}$$ □

It follows the result.

Theorem 4. There are no proper convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that the pair f, g is totally Fenchel unstable.

Proof. We assume the contrary, namely that there exist $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ proper convex functions such that the pair f, g is totally Fenchel unstable. By (3) it follows that $(f+g)^*(x^*) < +\infty$ for all $x^* \in \mathbb{R}^n \setminus \{0\}$. As $(f+g)^*$ is convex, we get $(f+g)^*(0) < +\infty$. As noticed above we have $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$, hence $(f+g)^*(0) > -\infty$.

As noticed above, $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ must be nonempty. Choose some $\bar{x} \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \subseteq \mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g})$ and consider $L = \mathrm{aff}(\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}) - \bar{x}) =$

$\mathrm{lin}(\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}) - \bar{x})$. As $\mathrm{int}(\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g})) = \emptyset$, by Proposition 3, the dimension of L is strictly less than n and this means that the orthogonal space to L, $L^{\perp}$ is nonzero. Of course, we have

$$\mathrm{dom}(f) \cap \mathrm{dom}(g) \subseteq \mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}) \subseteq L + \bar{x} \tag{6}$$

Theorem 1 applies to $\bar{f}$ and $\bar{g}$ and we have $f^* = \bar{f}^*$ and $g^* = \bar{g}^*$. Hence

$$\mathrm{epi}((\bar{f} + \bar{g})^*) = \mathrm{cl}(\mathrm{epi}(f^*) + \mathrm{epi}(g^*)). \tag{7}$$

It follows

$$\mathrm{epi}((f+g)^*) \supseteq \mathrm{epi}((\bar{f} + \bar{g})^*) \supseteq \mathrm{epi}(f^*) + \mathrm{epi}(g^*).$$

Since the pair f, g is totally Fenchel unstable, by Proposition 2, one has that

$$\mathrm{epi}(f+g)^* \cap (\{0\} \times \mathbb{R}) = \mathrm{epi}((\bar{f}+\bar{g})^*) \cap (\{0\} \times \mathbb{R}) = (\mathrm{epi}(f^*) + \mathrm{epi}(g^*)) \cap (\{0\} \times \mathbb{R})$$

and so $(f+g)^*(0) = (\bar{f} + \bar{g})^*(0)$. Taking an element $x^* \in L^{\perp} \setminus \{0\}$ we obtain

$$\begin{aligned}
(f+g)^*(x^*) &= \sup_{x \in \mathbb{R}^n} \{\langle x^*, x\rangle - f(x) - g(x)\} \\
&\overset{(6)}{=} \sup_{x \in L + \bar{x}} \{\langle x^*, x\rangle - f(x) - g(x)\} \\
&= \langle x^*, \bar{x}\rangle + (f+g)^*(0) \\
&= \langle x^*, \bar{x}\rangle + (\bar{f}+\bar{g})^*(0) \overset{(6)}{=} (\bar{f}+\bar{g})^*(x^*).
\end{aligned} \tag{8}$$

We distinguish two cases:

(a) If $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ is closed, we obtain from (7) and (8), $(x^*, (f+g)^*(x^*)) \in \mathrm{epi}((\bar{f}+\bar{g})^*) = \mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ and so there exist $(y^*, s) \in \mathrm{epi}(f^*)$ and $(z^*, t) \in \mathrm{epi}(g^*)$ such that $y^* + z^* = x^* \neq 0$ and $s + t = (f+g)^*(x^*)$. This means that $f^*(y^*) + g^*(z^*) \leq (f+g)^*(y^* + z^*)$. As $y^* + z^* = x^* \neq 0$ this contradicts (3).

(b) Otherwise, if $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ is not closed, by [2, Corollary 9.1.2], there exists a direction of recession of $\mathrm{epi}(f^*)$ whose opposite direction is a direction of recession of $\mathrm{epi}(g^*)$. This can be expressed as

$$\exists\, (x^*, r) \neq 0 : \quad (x^*, r) \in 0^+ \mathrm{epi}(f^*) \quad \wedge \quad (-x^*, -r) \in 0^+ \mathrm{epi}(g^*),$$

where r can be chosen nonnegative. It follows $x^* \neq 0$, because otherwise we would have $(0, -r) \in 0^+ \mathrm{epi}(g^*)$ with $r > 0$. But g is proper and so g^* never attains $-\infty$.

Choose some y^* according to (2). Since $(f+g)^*(0), f^*(-y^*), g^*(y^*) \in \mathbb{R}$ and as $\mathrm{epi}(f^*)$ and $\mathrm{epi}(g^*)$ are nonempty convex sets, by [2, Theorem 8.1], it holds

$$\begin{aligned}
\forall \lambda \geq 0 : &\quad (-y^*, f^*(-y^*)) + \lambda \cdot (x^*, r) \in \mathrm{epi}(f^*) \\
\forall \mu \geq 0 : &\quad (y^*, g^*(y^*)) - \mu \cdot (x^*, r) \in \mathrm{epi}(g^*)
\end{aligned}$$

Adding both conditions and taking into account (2) we get

$$\forall \gamma \in \mathbb{R}: \quad (0, (f+g)^*(0)) + \gamma \cdot (x^*, r) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*). \tag{9}$$

Let $\gamma = 1$ in (9). There exist $(u^*, s) \in \operatorname{epi}(f^*)$ and $(v^*, t) \in \operatorname{epi}(g^*)$ such that $u^* + v^* = x^*$ and $s + t = (f+g)^*(0) + r$. It follows

$$(f+g)^*(x^*) \leq f^*(u^*) + g^*(v^*) \leq s + t = (f+g)^*(0) + r \tag{10}$$

Setting $\gamma = -1$ in (9), we obtain analogously

$$(f+g)^*(-x^*) \leq (f+g)^*(0) - r \tag{11}$$

The conditions (10) and (11) must hold with equality. Indeed, adding both inequalities where one of them is strict, we get a contradiction to the fact that $(f+g)^*$ is convex. Hence $(f+g)^*(u^* + v^*) = f^*(u^*) + g^*(v^*)$. This contradicts (3), because of $u^* + v^* = x^* \neq 0$. $\square$

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