Fractal Geometry

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Contents

Chapter 1

Box dimension

1.1 Two preliminary examples

Let $W = [0,1]^d$ be the d-dimensional unit cube in $\mathbb{R}^d$. Obviously, any notion of dimension should assign dimension d to W. If $\tilde{N}_\delta(W)$ denotes the minimal number of cubes of sidelength δ that are needed to cover W, then we have

$$\tilde{N}_\delta(W) \sim \delta^{-d} \quad \text{as } \delta \to 0 , \tag{1.1.1}$$

where we write $f \sim g$ as $x \to 0$ if there exists constants $C > 0$ and $\delta_0 > 0$ such that $C^{-1} f(x) \leq g(x) \leq C f(x)$ for all $x \in (0, \delta_0)$. Hence, if we do not know the value of d, then we can recover the dimension of W by

$$d = \lim_{\delta \to 0} \frac{\log \tilde{N}_\delta(W)}{-\log \delta} . \tag{1.1.2}$$

Remark 1.1.1. Given a function $f : (0, \delta_0) \to \mathbb{R}$, we will frequently use limits of the form

$$d = \lim_{\delta \to 0} \frac{\log f(\delta)}{-\log \delta}$$

to determine the *polynomial growth rate* of f as $\delta \to 0$. If the limit exists, then $f(\delta)$ behaves more or less like δ^{-d} as δ goes to zero (in an appropriate sense). In particular, this is true if $f \sim \delta^{-d}$ in the above sense. Similarly, a limit of the form

$$a = \lim_{\delta \to 0} \delta \log f(\delta)$$

will give the *exponential growth rate* of f as $\delta \to 0$, that is, if the limit exists then f behaves asymptotically like $\exp(a/\delta)$.

Exercise 1. (a) Show that $f \sim \delta^{-d}$ as $\delta \to 0$ implies $\lim_{\delta \to 0} -\log f(\delta)/\log \delta = d$.

(b) Give an equivalent characterisation of the fact that $\lim_{\delta \to 0} -\log f(\delta)/\log \delta = d$.

(c) Replace $\lim_{\delta \to 0} \log f(\delta)/-\log \delta = d$ by $\lim_{\delta \to 0} \delta \log f(\delta)$ and δ^{-d} by $\exp(a/\delta)$ in (a) and (b).

(d) Show that $\sim$ defines an equivalence relation (on a suitable set of functions).

Exercise 2. Suppose $N : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $\lim_{\delta \to 0} N(\delta) = \infty$. Further, assume that

$$s = \varliminf_{\delta \to 0} \frac{\log N(\delta)}{-\log \delta} \leq \varlimsup_{\delta \to 0} \frac{\log N(\delta)}{-\log \delta} = t .$$

Show that for every $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$\delta^{-(s-\varepsilon)} \leq N(\delta) \leq \delta^{-(t+\varepsilon)}$$

for all $\delta \in (0, \delta_0)$. Note that this means in particular that $\delta^{t'} N(\delta) \xrightarrow{\delta \to 0} 0$ for all $t' > t$ and $\delta^{s'} N(\delta) \xrightarrow{\delta \to 0} \infty$ for all $s' < s$.

Let us now consider the above procedure for a more complicated set, namely the Middle Third Cantor Set C. The latter is usually obtained as a nested intersection $C = \bigcap_{n\in\mathbb{N}_0} C_n$, where each C_n is a finite collection of intervals and the sequence $(C_n)_{n\in\mathbb{N}_0}$ is defined recursively as follows. We first let $C_0 = [0,1]$. If $C_n = \bigcup_{j=1}^{n_k} I_j^n$ is defined, then C_{n+1} is obtained by removing the middle third of each interval I_j belonging to C_n, and taking the union of the remaining intervals to form C_{n+1}. Note that thus every interval in C_n splits up into two new intervals with a third of the original length. By induction, we therefore obtain that C_n is the union of 2^n intervals of length 3^{-n}. It therefore turns out that exactly 2^n intervals of length 3^{-n} are needed to cover C. At least if we restrict to the δ-values $\delta_n = 3^{-n}$, we obtain the number

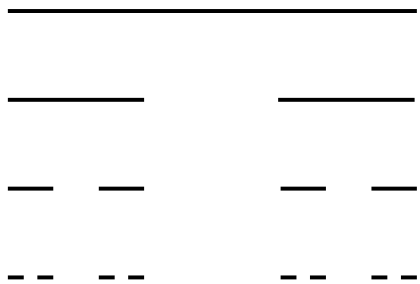

Figure 1.1.1: Construction of the Middle Third Cantor Set.

$$d \;=\; \lim_{n\to\infty} \frac{\log \tilde{N}_{\delta_n}(C)}{-\log \delta_n} \;=\; \lim_{n\to\infty} \frac{\log 2^n}{-\log 3^{-n}} \;=\; \frac{\log 2}{\log 3} \tag{1.1.3}$$

as a possible candidate for the dimension of C.

Remark 1.1.2. An alternative and more formal way of defining C would be the following. Let $\Sigma^+ = \{0,1\}^{\mathbb{N}}$ be the space of one-sided infinite 0-1-sequences. The map $g : \Sigma^+ \to [0,1],\ a = (a_n)_{\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n 2^{-n}$ allows to identify each $a \in \Sigma^+$ with a real number in the unit interval and interpret the sequence as the binary expansion of that number. As a slight modification, we obtain the map

$$h : \Sigma^+ \to [0,1] \quad , \quad a = (a_n)_{\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} 2a_n \cdot 3^{-n} \ ,$$

which maps $[0,1]$ to C.

Exercise 3. Show that $h(\Sigma^+) = C$. (*This requires to formalise the construction of C in some way.*)

Exercise 4. Show that C is a *Cantor set*: it is *compact*, *totally disconnected*[1] and *perfect*[2].

Exercise 5. (a) Show that

$$d(a,b) \;=\; \sum_{n\in\mathbb{N}} 3^{-|a_n - b_n|}$$

defines a metric on Σ, which thus becomes a compact metric space.

(b) Show that with this metric structure Σ^+ is a Cantor set.

(c) Show that the map $h : \Sigma^+ \to C$ is a homeomorphism (in fact, an isometry) between the two spaces.

(d) Show that homeomorphic images of Cantor sets are again Cantor sets. (*Note that this shows again that C is Cantor.*)

[1]This means that all connected components are single points. For subsets of $\mathbb{R}$, this is true if the set contains no non-trivial interval.

[2]All points of the set are accumulation points.

1.2 Definition and equivalent characterisations of box dimension

Suppose (X, d) is a metric space. Given a set $A \subseteq X$ with compact closure and $\delta > 0$, let $N_\delta(A)$ be the smallest number of closed δ-balls $\overline{B_\delta(x)}$ that are needed to cover A. Then the **upper box dimension** of A is defined as

$$\overline{\mathrm{Dim}}_B(A) \;=\; \overline{\lim_{\delta\to 0}} \frac{\log N_\delta(A)}{-\log \delta} \; . \tag{1.2.1}$$

Similarly, the **lower box dimension** of A is defined as

$$\underline{\mathrm{Dim}}_B(A) \;=\; \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(A)}{-\log \delta} \; . \tag{1.2.2}$$

If both quantities coincide, then their common value

$$\mathrm{Dim}_B(A) \;=\; \overline{\mathrm{Dim}}_B(A) \;=\; \underline{\mathrm{Dim}}_B(A) \tag{1.2.3}$$

is called the **box dimension** of A.

Remark 1.2.1. Note that since we assume that A is *relatively compact* (i.e. has compact closure), the numbers $N_\delta(A)$ are all finite. In *σ-compact spaces* (countable unions of compact spaces), one can extend the definition of (lower and upper) box dimension to arbitrary sets by defining it as the supremum of the box dimensions of all relatively compact subsets.

A fact that is quite convenient when it comes to the calculation of box dimension for particular examples is that there exist a number of equivalent definitions, and one may always choose the one that works best in a particular situation. The following variation explains the name *box* dimension.

Lemma 1.2.2. *Suppose $X = \mathbb{R}^d$, $A \subseteq X$ has compact closure and $\tilde{N}_\delta(A)$ denotes the minimal number of boxes (or cubes) of sidelength δ that are needed to cover A. Then*

$$\overline{\mathrm{Dim}}_B(A) \;=\; \overline{\lim_{\delta\to 0}} \frac{\log \tilde{N}_\delta(A)}{-\log \delta} \quad \textit{and} \quad \underline{\mathrm{Dim}}_B(A) \;=\; \underline{\lim}_{\delta\to 0} \frac{\log \tilde{N}_\delta(A)}{-\log \delta} \; . \tag{1.2.4}$$

Proof. Every cube of sidelength δ is contained in a ball of radius $\sqrt{d}\delta$. Therefore, we have $N_{\sqrt{d}\delta}(A) \leq \tilde{N}_\delta(A)$ and hence

$$\overline{\mathrm{Dim}}_B(A) \;=\; \overline{\lim_{\delta\to 0}} \frac{\log N_{\sqrt{d}\delta}(A)}{-\log \sqrt{d}\delta} \;\leq\; \overline{\lim_{\delta\to 0}} \frac{\log \tilde{N}_\delta(A)}{-\log \delta - \log \sqrt{d}} \;=\; \overline{\lim_{\delta\to 0}} \frac{\log \tilde{N}_\delta(A)}{-\log \delta} \; .$$

Conversely, every ball of radius δ is contained in a cube of sidelength 2δ, so that $N_\delta(A) \geq \tilde{N}_{2\delta}(A)$ and thus

$$\overline{\mathrm{Dim}}(A) \;\geq\; \overline{\lim_{\delta\to 0}} \frac{\log \tilde{N}_{2\delta}(A)}{-\log \delta} \;=\; \overline{\lim_{\delta\to 0}} \frac{\log \tilde{N}_\delta(A)}{-\log \delta} \; .$$

The proof for the lower box dimension works in the same way. $\square$

Exercise 6. Let $\mathcal{B}(\delta) = \left\{ \prod_{k=1}^{d} [n_k\delta, (n_k+1)\delta] \mid n_k \in \mathbb{Z} \text{ for } k = 1, \ldots, d \right\}$ be the collection of boxes from the standard grid with sidelengths δ in $\mathbb{R}^d$. Show that the quantity $\tilde{N}_\delta(A)$ in (1.2.4) can also be replaced by the minimal number $\widehat{N}_\delta(A)$ of boxes from $\mathcal{B}(\delta)$ needed to cover A.

Exercise 7. Show that $N_\delta(A)$ in (1.2.1) and (1.2.2) can also be replaced by the minimal number $\bar{N}_\delta(A)$ of sets of diameter at most δ that are needed to cover A.

For the computation of box dimension, the following elementary lemma is often useful.

Lemma 1.2.3. *Let $N : \mathbb{R}^+ \to [1,\infty)$, $\delta \mapsto N_\delta$ be a monotonically decreasing function, and suppose $(\gamma_n)_{n\in\mathbb{N}}$ is a strictly monotonically decreasing sequence of positive real numbers such that $\gamma_1 \le 1$, $\lim_{n\to\infty}\gamma_n = 0$ and γ_n/γ_{n+1} is uniformly bounded. Then*

$$\lim_{n\to\infty} \frac{\log N_{\gamma_n}}{-\log \gamma_n} \;=\; \lim_{\delta\to 0} \frac{\log N_\delta}{-\log\delta}\ ,$$

provided that the limit on the left exist. Otherwise, the same statement holds for the limit inferior and the limit superior.

Proof. We assume that the limit on the left exists and let $L = \lim_{n\to\infty} \frac{\log N_{\gamma_n}}{-\log\gamma_n}$. Let $C > 0$ be such that

$$1 \;\le\; \gamma_n/\gamma_{n+1} \;\le\; C \quad \text{for all } n \in \mathbb{N}\ .$$

Given $\delta \in (0,1)$, let $n(\delta)$ be defined as the unique integer such that $\delta \in [\gamma_{n(\delta)+1}, \gamma_{n(\delta)})$. Then $N_{\gamma_{n(\delta)}} \le N_\delta \le N_{\gamma_{n(\delta)+1}}$ and hence

$$\frac{\log N_{\gamma_{n(\delta)}}}{-\log \gamma_{n(\delta)+1}} \;\le\; \frac{\log N_\delta}{-\log\delta} \;\le\; \frac{\log N_{\gamma_{n(\delta)+1}}}{-\log\gamma_{n(\delta)}}\ .$$

However, we have that

$$\lim_{\delta\to 0} \frac{\log N_{\gamma_{n(\delta)}}}{-\log\gamma_{n(\delta)+1}} \;=\; \lim_{n\to\infty} \frac{\log N_{\gamma_n}}{-\log\gamma_{n+1}} \;\ge\; \lim_{n\to\infty} \frac{\log N_{\gamma_n}}{-\log(\gamma_n/C)} \;=\; L\ .$$

In a similar way, we obtain that $\lim_{\delta\to 0} \frac{\log N_{\gamma_{n(\delta)+1}}}{-\log\gamma_{n(\delta)}} \le L$ and thus $\lim_{\delta\to 0}\frac{\log N_\delta}{-\log\delta} = L$ as required. Finally, if the limit L does not exist, then the same arguments apply to the limit inferior and the limit superior. ☐

Corollary 1.2.4. *Suppose $(\gamma_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Lemma 1.2.3. Then*

$$\overline{\mathrm{Dim}}_B(A) \;=\; \varlimsup_{n\to\infty} \frac{\log N_{\gamma_n}(A)}{-\log\gamma_n} \quad \textit{and} \quad \underline{\mathrm{Dim}}_B(A) \;=\; \varliminf_{n\to\infty}\frac{\log N_{\gamma_n}(A)}{-\log\gamma_n}\ . \tag{1.2.5}$$

Corollary 1.2.5. *The box dimension of the Middle Third Cantor Set equals* $\log 2/\log 3$.

Given a metric space (X,d) and $\delta > 0$, a set $S \subseteq X$ is called **δ-separated** if $d(x,y) \ge \delta$ for all $x \ne y \in S$. For a relatively compact subset $A \subseteq X$, let $M_\delta(A)$ denote the maximal cardinality of a δ-separated set $S \subseteq A$.

Lemma 1.2.6. *We have*

$$\overline{\mathrm{Dim}}_B(A) \;=\; \varlimsup_{\delta\to 0} \frac{\log M_\delta(A)}{-\log\delta} \quad \textit{and} \quad \underline{\mathrm{Dim}}_B(A) \;=\; \varliminf_{\delta\to 0}\frac{\log M_\delta(A)}{-\log\delta}\ .$$

Proof. Suppose $S \subseteq A$ is a δ-separated set of maximal cardinality $M_\delta(A)$. Then every point $x \in A$ must lie in the δ-neighbourhood of some $y \in S$. Otherwise $A \cup \{x\}$ would be δ-separated as well, contradicting the definition of $M_\delta(A)$. Hence, we have that

$$A \;\subseteq\; \bigcup_{x\in S} B_\delta(x)\ ,$$

and there for $N_\delta(A) \le M_\delta(A)$. This implies $\overline{\mathrm{Dim}}_B(A) \le \varlimsup_{\delta\to 0}\frac{\log M_\delta(A)}{-\log\delta}$.

Conversely, if $U_1,\dots,U_{N_\delta(A)}$ is a cover of A by δ-balls $U_i = B_\delta(x_i)$ and S is a 2δ-separated set, then every U_i contains at most one point from S. This means that $M_{2\delta}(A) \le N_\delta(A)$ and thus implies $\overline{\mathrm{Dim}}_B(A) \ge \varlimsup_{\delta\to 0}\frac{\log M_\delta(A)}{-\log\delta}$. Together, we obtain the required equality for the upper box dimension, and the case of the lower box dimension can be treated exactly in the same way. ☐

Example 1.2.7 (The Sierpinski Carpet)**.** In an inductive construction similar to that of the Middle Third Cantor Set, we can obtain another well-known fractal that is called the *Sierpinski Carpet*.

To that end, one divides the unit square $S_0 = [0,1]^2$ into nine equal subsquares and removes the middle one to obtain S_1. The procedure is then repeated recursively by always dividing the 8^n squares of sidelength 3^{-n} in S_n into smaller squares of sidelength $3^{-(n+1)}$ and removing the middle ones in order to obtain S_{n+1}. Finally, we define the Sierpinski Carpet as $\mathcal{S} = \cap_{n\in\mathbb{N}} S_n$. .
By construction, $\mathcal{S}$ is contained in S_n, which is the union of 8^n squares of sidelength 3^{-n}. Hence $\tilde{N}_{3^{-n}}(\mathcal{S}) \leq 8^n$, and Lemma 1.2.2 together with Lemma 1.2.3 yield

$$\overline{\mathrm{Dim}}_B(\mathcal{S}) \;\leq\; \frac{\log 8}{\log 3} \,.$$

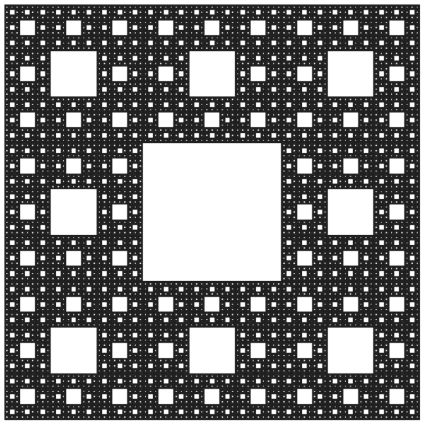

Figure 1.2.1: The Sierpinski Carpet

At the same time, the lower right corners of these squares constitute a 3^{-n}-separated subset of $\mathcal{S}$, so that $M_{3^{-n}}(\mathcal{S}) \geq 8^n$. By Lemma 1.2.6 we obtain

$$\underline{\mathrm{Dim}}_B(\mathcal{S}) \;\geq\; \frac{\log 8}{\log 3} \,.$$

Together, this implies that the box dimension of $\mathcal{S}$ exists and equals $\log 8/\log 3$.

1.3 Basic properties of box dimension

The following lemma collects a few elementary properties of box dimension.

Lemma 1.3.1. *Let (X,d) be a metric space.*

(i) If $A \subseteq B \subseteq X$, then $0 \leq \mathrm{Dim}_B(A) \leq \mathrm{Dim}_B(B)$.

(ii) if $A, B \subseteq X$, then $\mathrm{Dim}_B(A \cup B) = \max\{\mathrm{Dim}_B(A), \mathrm{Dim}_B(B)\}$.

(iii) For any $A \subseteq X$, we have $\mathrm{Dim}_B\left(\overline{A}\right) = \mathrm{Dim}_B(A)$.

(iv) If $A \subseteq \mathbb{R}^d$, then $\mathrm{Dim}_B(A) \leq d$.

All statements apply in an analogous way to the upper and lower box dimensions.

Exercise 8. Prove Lemma 1.3.1.

Example 1.3.2. Note that due to part (iii) of the lemma, we have that $\mathrm{Dim}_B([0,1]\cap\mathbb{Q}) = 1$. In particular, even countable sets can have positive box dimension, and

$$\sup_{n\in\mathbb{N}} \mathrm{Dim}_B(A_n) \;<\; \mathrm{Dim}_B\left(\bigcup_{n\in\mathbb{N}} A_n\right)$$

is possible. One says that box dimension is not *countably stable*. From a theoretical perspective, this is actually a strong disadvantage and one of the main motivations for the development of alternative concepts, including in particular Hausdorff dimension.

Remark 1.3.3. We will next study the box dimension of product sets $A \times B$, where A and B are relatively compact subsets of metric spaces (X, d_X) and (Y, d_Y), respectively. In order to do so, we need to equip the product space $X \times Y$ with a metric. In this context, it needs to be mentioned that there is no unique canonical way in order to do this. The metric we will use is

$$d_{X\times Y}\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right) \;=\; \left(d_X\left(x_1,x_2\right)^2+d_Y\left(y_1,y_2\right)^2\right)^{1/2} \;.$$

This has the advantage that the product of two Euklidean metrics is a Euklidean metric again (on the higher-dimensional product space). Another natural choice would be to use

$$\tilde{d}_{X\times Y}\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right) \;=\; \max\left\{d_X\left(x_1,x_2\right),d_Y\left(y_1,y_2\right)\right\}$$

as a metric on $X\times Y$. However, fortunately these two (and all other) natural choices of product metrics on $X\times Y$ are *equivalent*, in the sense that there exists a constant $C>0$ such that

$$C^{-1}\cdot d_{X\times Y}\left(z,z'\right) \;\leq\; \tilde{d}_{X\times Y}\left(z,z'\right) \;\leq\; C\cdot d_{X\times Y}\left(z,z'\right)$$

for all $z,z'\in X\times Y$, and moreover box dimension is invariant under the change to an equivalent metric.

Exercise 9. (a) Show that the above functions $d_{X\times Y}$ and $\tilde{d}_{X\times Y}$ are indeed metrics on $X\times Y$.

(b) Show that if d and d' are two equivalent metrics on X and $A\subseteq X$ is relatively compact, then the (upper/lower) box dimension of A is the same with respect to d and d'.

(c) Show that the metric $\bar{d}_{X\times Y}$ on $X\times Y$ given by

$$\bar{d}_{X\times Y}\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right) \;=\; d_X\left(x_1,x_2\right)+d_Y\left(y_1,y_2\right)$$

is equivalent to the two product metrics introduced above.

Remark 1.3.4. We also remark, however, that the change to a non-equivalent metric may well have an impact on the value of box dimension. To that end, note that if d is a metric on X, then so is d^α given by

$$d^\alpha(x,y) \;=\; d(x,y)^\alpha$$

for any $\alpha\in(0,1)$.

Exercise 10. Show that d^α is a metric on X and if D is the box dimension of $A\subseteq X$ computed with respect to d and D' is the box dimension of A computed with respect to d^α, then $D'=D/\alpha$.

Proposition 1.3.5 (Product formula)**.** *Suppose (X,d_X) and (Y,d_Y) are metric spaces and $A\subseteq X$ and $B\subseteq Y$ are relatively compact subsets. Then*

$$\overline{\mathrm{Dim}}_B(A\times B) \;\leq\; \overline{\mathrm{Dim}}_B(A)+\overline{\mathrm{Dim}}_B(B) \quad \textit{and} \tag{1.3.1}$$

$$\underline{\mathrm{Dim}}_B(A\times B) \;\geq\; \underline{\mathrm{Dim}}_B(A)+\underline{\mathrm{Dim}}_B(B). \tag{1.3.2}$$

In particular, if both $\mathrm{Dim}_B(A)$ and $\mathrm{Dim}_B(B)$ exist, then $\mathrm{Dim}_B(A\times B)$ exists and equals $\mathrm{Dim}_B(A)+\mathrm{Dim}_B(B)$.

Proof. Let $\delta>0$ and suppose that $U_1,\ldots,U_{N_\delta(A)}$ and $V_1,\ldots,V_{N_\delta(B)}$ are collections of δ-balls in X and Y that cover A and B, respectively. Then each product $U_i\times V_j$ is contained in a ball of radius $\sqrt{2}\delta$ in $X\times Y$, and the union of all such product sets covers $A\times B$. Therefore $N_{\sqrt{2}\delta}(A\times B)\leq N_\delta(A)\cdot N_\delta(B)$, and we obtain

$$\begin{aligned}\overline{\mathrm{Dim}}_B(A\times B) \;&=\; \varlimsup_{\delta\to0}\frac{\log N_{\sqrt{2}\delta}(A\times B)}{-\log(\sqrt{2}\delta)}\\ &\leq\; \varlimsup_{\delta\to0}\frac{\log N_\delta(A)+\log N_\delta(B)}{-\log\sqrt{2}-\log\delta} \;\leq\; \overline{\mathrm{Dim}}_B(A)+\overline{\mathrm{Dim}}_B(B)\;.\end{aligned}$$

For the second inequality, suppose that $S \subseteq A$ and $S' \subseteq B$ are δ-separated sets of cardinality $M_\delta(A)$ and $M_\delta(B)$, respectively. Then $S \times S' \subseteq A \times B$ is δ-separated, and hence $M_\delta(A \times B) \geq M_\delta(A) \cdot M_\delta(B)$. This implies

$$\begin{aligned}\underline{\mathrm{Dim}}_B(A \times B) &= \overline{\lim_{\delta\to 0}} \frac{\log M_\delta(A \times B)}{-\log \delta} \\ &\geq \overline{\lim_{\delta\to 0}} \frac{\log M_\delta(A) + \log M_\delta(B)}{-\log \delta} \;\geq\; \underline{\mathrm{Dim}}_B(A) + \underline{\mathrm{Dim}}_B(B) .\end{aligned}$$

□

Next, we consider how continuous transformation act on the dimenions of sets. Recall that a function $f : X \to Y$ between metric spaces X and Y is called **α-Hölder continuous** (with $\alpha \in (0,1]$) if there exists a constant $C > 0$ such that

$$d(f(x), f(y)) \;\leq\; Cd(x,y)^\alpha$$

for all $x, y \in X$. In this case, C is called the **Hölder constant** of f.

Proposition 1.3.6. *Suppose $f : X \to Y$ is α-Hölder continuous and $A \subseteq X$ is relatively compact. Then*

$$\mathrm{Dim}_B(f(A)) \;\leq\; \frac{\mathrm{Dim}_B(A)}{\alpha} ,$$

provided that both dimensions exist. Otherwise, the analogous estimates apply to the upper and lower box dimensions.

Proof. Suppose $U_1, \ldots, U_{N_\delta(A)}$ is a cover of A by balls of radius δ. Then by assumption each set $f(U_j)$ is contained in a ball of radius $C\delta^\alpha$. Hence, we have $N_{C\delta^\alpha}(f(A)) \leq N_\delta(A)$ and consequently

$$\mathrm{Dim}_B(f(A)) \;=\; \lim_{\delta\to 0} \frac{N_{C\delta^\alpha}(f(A))}{-\log(C\delta^\alpha)} \;\leq\; \lim_{\delta\to 0} \frac{N_\delta(A)}{-\log C - \alpha \log \delta} \;=\; \mathrm{Dim}_B(A)/\alpha .$$

The same argument applies to the upper and lower box dimension. □

A homeomorphism $h : X \to Y$ between metric spaces is called a **bi-Lipschitz map/ transformation** if both f and f^{-1} are Lipschitz continuous.

Corollary 1.3.7. *Bi-Lipschitz transformations preserve the box dimension of sets.*

The construction in Remark 1.3.4 actually shows that the estimate in Proposition 1.3.6 is sharp, in the sense that an α-Hölder continuous function can actually increase the dimension of a set by a factor of $1/\alpha$. In order to see this, it suffices to note that the identity on X as a mapping $\mathrm{Id}_X : (X,d) \to (X, d^\alpha)$ is α-Hölder continuous. The following exercise provides a more explicit example using symbolic spaces.

Exercise 11. Let $\Sigma = \{0,1\}^{\mathbb{N}}$ be the space of 0-1-sequences as in Remark 1.1.2.

- Show that for any $\beta > 1$ the map

$$d_\beta : \Sigma \times \Sigma \to \mathbb{R}^+ \quad , \quad (a,b) \mapsto \beta^{-\min\{n \in \mathbb{N} | a_n \neq b_n\}}$$

  defines a metric on Σ.

- Show that for any $\delta \in (\beta^{-(n+1)}, \beta^{-n}]$ the δ-ball $B_\delta(a)$ in (Σ, d_β) equals the cylinder set

$$[a_1 \ldots a_n] \;=\; \{b \in \Sigma \mid b_j = a_j \forall j = 1, \ldots, n\} .$$

- Compute the box dimension of (Σ, d_β). (*Hint: Show that* $N_{\beta^{-n}}(\Sigma) = 2^b$.)

- Show that $\mathrm{Id}_\Sigma : (X, d_\beta) \to (X, d_\gamma)$ is $\frac{\log \gamma}{\log \beta}$-Hölder continuous.

Hint: One may either show the above statements directly or interpret them as a special case of the construction in Remark 1.3.4 (or ideally do both).

Exercise 12. Suppose $A, B \subseteq \mathbb{R}^d$ are relatively compact sets and let

$$A + B \ = \ \{x + y \mid x \in A,\ y \in B\}\ .$$

Prove that

$$\overline{\mathrm{Dim}}_{\mathrm{B}}(A + B) \ \leq \ \overline{\mathrm{Dim}}_{\mathrm{B}}(A) + \overline{\mathrm{Dim}}_{\mathrm{B}}(B)\ .$$

Hint: Use Proposition 1.3.5 to bound $\overline{\mathrm{Dim}}_{\mathrm{B}}(A \times B)$ and the fact that the mapping

$$\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d\ , \quad (x, y) \mapsto x + y$$

is Lipschitz-continuous.

1.4 Minkowski characterisation of box dimension

By $\mathrm{Leb}_{\mathbb{R}^d}$, we denote the Lebesgue measure on $\mathbb{R}^d$. Moreover, given $A \subseteq \mathbb{R}^d$ and $\delta > 0$, we let $B_\delta(A) = \{x \in \mathbb{R}^d \mid B_\delta(x) \cap A \neq \emptyset\}$. Then we have the following further equivalent characterisation of box dimension.

Theorem 1.4.1. *Given a relatively compact set $A \subseteq \mathbb{R}^d$, we have*

$$\overline{\mathrm{Dim}}_B(A) \ = \ d - \underline{\lim}_{\delta\to 0} \frac{\log \mathrm{Leb}_{\mathbb{R}^d}(B_\delta(A))}{\log \delta}\ . \tag{1.4.1}$$

$$\underline{\mathrm{Dim}}_B(A) \ = \ d - \overline{\lim}_{\delta\to 0} \frac{\log \mathrm{Leb}_{\mathbb{R}^d}(B_\delta(A))}{\log \delta} \tag{1.4.2}$$

Proof. If $B_\delta(x_1), \ldots, B_\delta(x_{N_\delta(A)})$ is a cover of A by δ-balls, we have that

$$B_\delta(A) \subseteq \bigcup_{j=1}^{N_\delta(A)} B_{2\delta}(x_j)\ ,$$

so that

$$\mathrm{Leb}_{\mathbb{R}^d}(B_\delta(A)) \ \leq \ N_{2\delta}(A) \cdot 2^d \cdot C_d \cdot \delta^d\ ,$$

where C_d is the volume of the unit ball $B_1(0)$ in $\mathbb{R}^d$. As a consequence, we obtain that

$$\begin{aligned}
\overline{\mathrm{Dim}}_B(A) \ &= \ \overline{\lim}_{\delta\to 0} \frac{\log N_{2\delta}(A)}{-\log 2\delta} \\
&\geq \ \overline{\lim}_{\delta\to 0} \frac{\log\left(\mathrm{Leb}^d_{\mathbb{R}}(B_\delta(A)) \cdot 2^{-d} \cdot C_d^{-1} \cdot \delta^{-d}\right)}{-\log\delta - \log 2} \\
&= \ \overline{\lim}_{\delta\to 0} \left(\frac{-d\log\delta - \log C_d - d\log 2}{-\log\delta} + \frac{\log \mathrm{Leb}_{\mathbb{R}^d}(B_\delta(A)}{-\log\delta}\right) \\
&= \ d \ - \ \underline{\lim}_{\delta\to 0} \frac{\log \mathrm{Leb}_{\mathbb{R}^d}(B_\delta(A))}{\log\delta}\ .
\end{aligned}$$

This proves (1.4.1). Conversely, suppose that $y_1, \ldots, y_{M_\delta(A)}$ are pairwise δ-separated points in A. Then we have that $\bigcup_{j=1}^{M_\delta(A)} B_\delta(y_j) \subseteq B_\delta(A)$, and hence

$$M_\delta(A) \cdot C_d \cdot \delta^d \ \leq \ \mathrm{Leb}_{\mathbb{R}^d}(B_\delta(A))\ .$$

This allows to prove (1.4.2) in an analogous way. $\square$

1.5 Further examples

Example 1.5.1 (Modified Sierpinski Carpets)**.** Suppose the unit square $\hat{S}_{=}[0,1]^2$ is divided into k^2 smaller squares of equal size, $m \in \{1, \dots, k^2 - m\}$ of these squares are retained (and the other $k^2 - m$ omitted) in order to obtain a union $\hat{S}_1$ of m smaller squares and this construction is then repeated recursively to obtain a sequence $\hat{S}_n$. Then the resulting intersection $\hat{\mathcal{S}} = \bigcap_{n\in\mathbb{N}} \hat{S}_n$ has box dimension

$$\mathrm{Dim}_B\left(\hat{\mathcal{S}}\right) \;=\; \frac{\log m}{\log k} \;.$$

A symbolic coding for such fractals can be obtained as follows. Let $A \in M^{k\times k}(\{0,1\}$ be the matrix defined by $m_{ij} = 1$ if the square in the i-th line and j-th column is retained in the construction and $m_{ij} = 0$ otherwise. Consider the alphabet $\mathcal{A} = \{0, \dots, k-1\}^2$ with corresponding shift space $\Omega = \mathcal{A}^{\mathbb{N}}$ and the subset

$$\Omega_0 \;=\; \{(i_n, j_n)_{n\in\mathbb{N}} \in \Omega \mid m_{i_n, j_n} = 1 \text{ for all } n \in \mathbb{N}\} \;.$$

Then

$$\hat{\mathcal{S}} = \left\{ \left(\sum_{n\in\mathbb{N}} i_n k^{-n}, \sum_{n\in\mathbb{N}} j_n k^{-n} \right)_{n\in\mathbb{N}} \;\middle|\; (i_n, j_n) \in \Omega_0 \right\} \;.$$

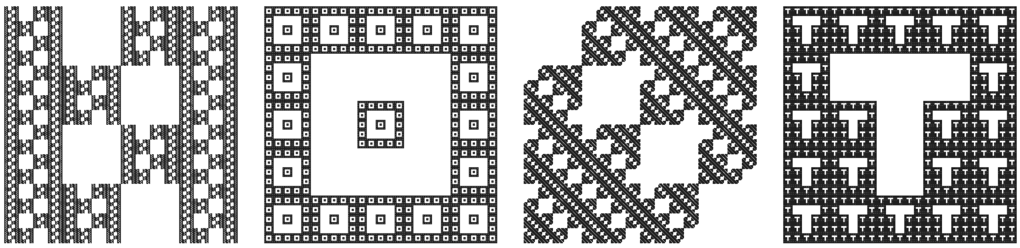

Figure 1.5.1: Some modified Sierpinski carpets.

Exercise 13. Compute the dimensions of the modified Sierpinski carpets shown in Figure 1.5.1.

Example 1.5.2 (Koch Snowflake)**.** The Koch Snowflake is obtained in the following way: First, on starts with $K_0 = [0,1] \times \{0\} \subseteq \mathbb{R}^2$. This interval is then divided into three parts of equal length, and the interior segment is replaced by the two complementing sides of a equal-sided triangle. This yields a piecewise linear curve K_1 consisting of four segments of length $1/3$. The same procedure is then repeated recursively on increasingly smaller scales with all the segments at each level of the recursive construction (see Figure 1.5.2 on the left). This yields a sequence K_n of piecewise linear curves, and the sixth part Koch Snowflake is then defined as the Hausdorff limit $K = \lim_{n\to\infty} K_n$ (we refer to Appendix B for background on the Hausdorff metric). Joining six suitably rotated copies of this set K then yields the complete snowflake, shown on the right in Figure 1.5.2.

In each step of the construction, points move by a distance of at most 3^{-n} when going from K_{n-1} to K_n. In particular $d_{\mathcal{H}}(K_{n-1}, K_n) \le 3^{-n}$, which implies that the K_n form a Cauchy sequence and thus guarantees the existence of the limit K. Moreover, we have that $d_{\mathcal{H}}(K_n, K) \le \frac{1}{2\cdot 3^n}$. As K_n is the union of 4^n segments of length 3^{-n}, this means that K is contained in the union of 4^n balls of radius 3^{-n}, so that $N_{3^{-n}}(K) \le 4^n$. Hence, we obtain $\overline{\mathrm{Dim}}_B(K) \le \log 4/\log 3$. Conversely, the endpoints of all segments in K_n form a 3^{-n}-separated set of cardinality $4^n + 1$ inside K (note here that the endpoints remain in K_j for all $j \ge n$). Thus, we also have $\underline{\mathrm{Dim}}_B(K) \ge \log 4/\log 3$. Altogether, this gives that

$$\mathrm{Dim}_B(K) \;=\; \frac{\log 4}{\log 3} \;.$$

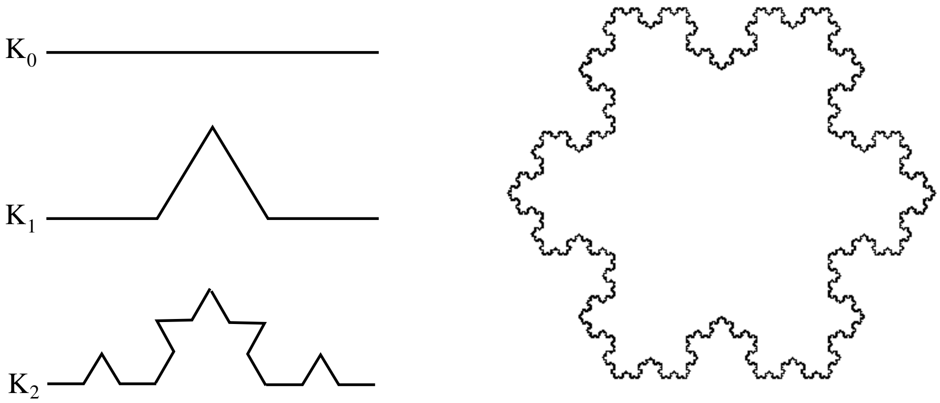

Figure 1.5.2: Construction of (a sixth part of) the Koch Snowflake on the left and the complete snowflake on the right.

Variations of these and similar constructions are abundant. Two further famous examples are the Sierpinski Gasket (or Triangle) or the Sierpinski Cube (or Sponge).

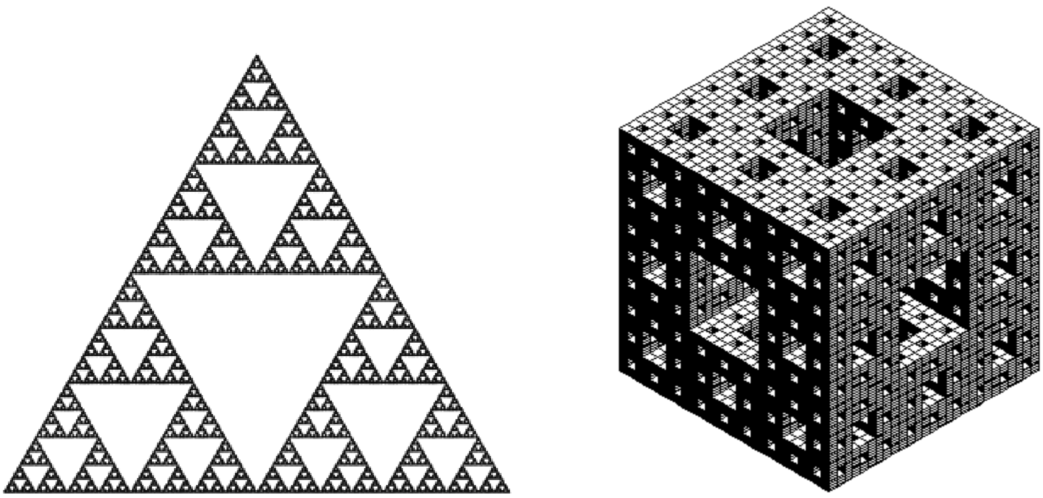

Figure 1.5.3: Sierpinski Gasket and Sierpinski Cube.

Exercise 14. Compute the box dimension of the Sierpinski Gasket and the Sierpinski Cube.

Example 1.5.3 (Countable sets)**.** Let $\alpha \in (0,1)$ and $x_n = C_\alpha \cdot \sum_{k=n}^{\infty} k^{-1/\alpha}$, where $C_\alpha = \left(\sum_{k=1}^{\infty} k^{-1/\alpha}\right)^{-1}$. Then $P_\alpha = \{x_n \mid n \in \mathbb{N}\} \subseteq [0,1]$, and we claim that

$$\mathrm{Dim}_B(P_\alpha) \;=\; \alpha \;.$$

In order to see this, let $\beta = -1/\alpha$. Then $d(x_n, x_{n+1}) = C_\alpha n^\beta$, so that $M_{C_\alpha n^\beta}(P_\alpha) \geq n+1$ (note that $d(x_k, x_{k+1})$ is decreasing in k) and hence

$$\underline{\mathrm{Dim}}_B(P_\alpha) \;\geq\; \lim_{n\to\infty} \frac{\log(n+1)}{-\log C_\alpha n^\beta} \;=\; -1/\beta \;=\; \alpha \;.$$

Conversely, we have

$$x_{n+1} \;=\; C_\alpha \cdot \sum_{k=n+1}^{\infty} k^\beta \;\leq\; C_\alpha \cdot \int_n^\infty \xi^\beta d\xi \;=\; C_\alpha \beta^{-1} n^{\beta+1} \;.$$

Since $\frac{C_\alpha \beta^{-1} n^{\beta+1}}{C_\alpha n^\beta} = \frac{n}{\beta}$, the set P_α can be covered by at most $n+1+\frac{n}{\beta}+1$ intervals of length $C_\alpha n^\beta$. Therefore, we obtain $N_{C_\alpha n^\beta}(P_\alpha) \leq (1+\beta)n+2$ and thus

$$\overline{\mathrm{Dim}}_B(P_\alpha) \ \leq \ \alpha \, .$$

This proves our claim.

Exercise 15. Determine the box dimension of the sets $S_\alpha = \{n^{-\alpha} \mid n \in \mathbb{N}\}$ for $\alpha > 0$.

Exercise 16. Show that for every $\gamma \in [0,d]$ there exists a countable set $A_\gamma \subseteq \mathbb{R}^d$ with $\mathrm{Dim}_B(A_\gamma) = \gamma$.

1.6 Summary

The notion of box dimension allows to quantify the fractal structure of sets. From the mathematical viewpoint, it has the following advantages and disadvantages:

- $\oplus$ Relatively easy and direct definition.
- $\oplus$ Good behaviour with respect to products and unions.
- $\oplus$ Various equivalent definitions are available.
- $\oplus$ Relatively easy to compute for a broad scope of examples.
- $\oplus$ Accesible to numerical computations.
- $\ominus$ No countable stability.
- $\ominus$ Gives positive dimension to certain countable sets.
- $\ominus$ Does not allow to compare the 'size' of two different sets of the same dimension.

Box dimension therefore plays an important role in fractal geometry, but needs to be complemented by alternative notions.

Chapter 2

Hausdorff dimension

2.1 Some basics on measure theory

2.1.1 A non-measurable set

Given a set X, we denote by $\mathcal{P}(X) = \{A \subseteq X\}$ the *power set* of X. The first aim of measure is to define positive real-valued functions, on suitable subsets of $\mathcal{P}(X)$, whose value for a given set A is interpreted as the 'size' of A. As an example, one may define $\mu([a,b]) = b - a$ for all intervals $[a,b] \subseteq \mathbb{R}$ - the 'size' of an interval is simply its length. Another possibility would be to consider an integrable function $f : \mathbb{R} \to \mathbb{R}^+$ and define $\mu([a,b]) = \int_a^b f(x)\, dx$.

The first natural question from the mathematical viewpoint is how far such definitions can be extended to larger families of sets, or even all of $\mathcal{P}(X)$. In order to demonstrate the problems which may arise in this context, we consider the following situation.

We let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z} =$ '$\mathbb{R}$ modulo 1' denote the circle. Alternatively, we may obtain $\mathbb{T}^1$ by identifying the endpoints of the unit interval $[0,1]$ via the equivalence relation given by $x \sim y$ if $x = y$ or $\{x,y\} = \{0,1\}$. Then $\mathbb{T}^1 = [0,1]/\sim$. Denote by $R_\alpha : x \mapsto x + \alpha \bmod 1$ the rotation with angle $\alpha \in \mathbb{T}^1$.

We now want to define a function $\mu : \mathcal{P}(X) \to \mathbb{R}$ such that for any interval $I \subseteq \mathbb{T}^1$ the value of $\mu(I)$ is simply its length.

$$\mu(I) \ = \ \text{length of } I \tag{2.1.1}$$

This includes $I = \mathbb{T}^1$, which has length 1. Since the length of an interval is not changed by a rotation, it is further plausible that the same should be true for any other set. Hence, our function μ should further satisfy

$$\mu(R_\alpha(A)) \ = \ \mu(A) \ \text{ for all } A \subseteq \mathcal{P}(X)\ . \tag{2.1.2}$$

Finally, for a union of two disjoint sets, the sizes should simply add up: $\mu(A \uplus B) = \mu(A) + \mu(B)$. For mathematical reasons, it is important that such a property also extends to countable unions.

$$\mu\left(\biguplus_{n\in\mathbb{N}} A_n\right) \ = \ \sum_{n\in\mathbb{N}} \mu(A_n)\ . \tag{2.1.3}$$

In order to see whether a function with these three properties (2.1.1)–(2.1.3) can be defined on all of $\mathcal{P}(\mathbb{T}^1)$, we introduce another equivalence relation on $\mathbb{T}^1$ by saying $x \sim_{\mathbb{Q}} y$ if and only if $y = R_\alpha(x)$ for some $\alpha \in \mathbb{Q}$. We then choose a set $E \subseteq \mathbb{T}^1$ which contains exactly one representative from each equivalence class of $\sim_{\mathbb{Q}}$.[1] Then the sets $R_q(E)$ with $q \in [0,1) \cap \mathbb{Q}$

[1] Note that this requires the *axiom of choice*.

form a countable family of pairwise disjoint sets whose union is all of $\mathbb{T}^1$. Hence, we should have that

$$1 \;=\; \mu(\mathbb{T}^1) \;=\; \mu\left(\biguplus_{q\in[0,1)\cap Q} \mu(E_q)\right) \stackrel{(2.1.3)}{=} \sum_{q\in[0,1)\cap\mathbb{Q}} \mu(E_q) \stackrel{(2.1.2)}{=} \sum_{q\in[0,1)\cap\mathbb{Q}} \mu(E)\;.$$

This is obviously a contradiction, since the sum on the left can only take the values 0 (if $\mu(E) = 0$) or ∞ (if $\mu(E) > 0$). Hence, we conclude that a function which combines all the three properties (2.1.1)–(2.1.3) cannot be defined on all of $\mathcal{P}(X)$. This is the starting point of measure theory, which first aims to identify suitable set families that are as large as possible and still allow a consistent defintion of functions with analogous properties as the ones above.

2.1.2 σ-algebras and measures

Given a space X, a family of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ is called a **σ-algebra** if it satisfies

($\mathcal{A}$**1**) $X \in \mathcal{A}$

($\mathcal{A}$**2**) $A \in \mathcal{A} \;\Rightarrow\; X \setminus A \in \mathcal{A}$

($\mathcal{A}$**3**) $A_n \in \mathcal{A}\ \forall n \in \mathbb{N} \;\Rightarrow\; \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$

The pair $(X, \mathcal{A})$ is then called a **measurable space**. Further, given $\mathcal{C} \subseteq \mathcal{P}(X)$, the family

$$\sigma(\mathcal{C}) \;=\; \bigcap_{\substack{\mathcal{C}\subseteq\mathcal{A}\subseteq\mathcal{P}(X)\\ \mathcal{A}\text{ is a }\sigma\text{-algebra}}}$$

is called the **σ-algebra generated by** $\mathcal{C}$, and $\mathcal{C}$ is called its **generator** (which is not unique).

Exercise 17. Show that the intersection of an arbitrary family of σ-algebras on a space X is again a σ-algebra. Use this to show that $\sigma(\mathcal{C})$ is the smallest σ-algebra on X that contains $\mathcal{C}$.

If X is a topological space, then the σ-algebra generated by the family of open subsets of X is called the **Borel σ-algebra** on X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a **measure** on $(X, \mathcal{A})$ if it satisfies

$$\mu\left(\biguplus_{n\in\mathbb{N}} A_n\right) \;=\; \sum_{n\in\mathbb{N}} \mu(A_n)$$

for any countable family of pairwise disjoint sets $A_n \in \mathcal{A}$. A family $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a **semiring**, if

($\mathcal{S}$**1**) $\emptyset \in \mathcal{S}$

($\mathcal{S}$**2**) $A, B \in \mathcal{S} \;\Rightarrow A \cap B \in \mathcal{S}$

($\mathcal{S}$**3**) $A, B \in \mathcal{S} \;\Rightarrow \exists A_1, \ldots, A_n \in \mathcal{S} :\; B \setminus A = \biguplus_{k=1}^n A_n$

Examples 2.1.1. (a) The family of half-open ‘intervals’ in $\mathbb{R}^d$,

$$\mathcal{S} \;=\; \left\{ \prod_{i=1}^d (a_i, b_i] \;\middle|\; a_i < b_i \in \mathbb{R} \text{ for } i = 1, \ldots, d \right\}$$

forms a semiring.

(b) The same is true for the family of cylinder sets in Σ^+: Given $a \in \Sigma^+$, set

$$[a]_n \;=\; \{b \in \Sigma^+ \mid b_k = a_k \text{ for all } k = 1, \ldots, n\}\;.$$

Then

$$\mathcal{Z} \;=\; \{[a]_n \mid a \in \Sigma^+,\; n \in \mathbb{N}\}$$

is a semiring.

Given a family of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$, a function $\nu : \mathcal{S} \to \mathbb{R}^+$ is called **(finitely) additive** if for any collection $A_1, \ldots, A_n \in \mathcal{S}$ of pairwise disjoint sets with $A = \biguplus_{i=1}^n A_i \in \mathcal{S}$ we have $\nu(A) = \sum_{i=1}^n \nu(A_i)$. Further, ν is **σ-subadditive** if for any sequence $A_n \in \mathcal{S}$ and any $A \subseteq \bigcup A_n$ with $A \in \mathcal{S}$ we have $\nu(A) \leq \nu\left(\bigcup_{n\in\mathbb{N}} A_n\right)$. The function ν is called **σ-finite** if there exists an increasing sequence $A_n \in \mathcal{S}$ with $\bigcup_{n\in\mathbb{N}} A_n = X$ and $\nu(A_n) < \infty$ for all $n \in \mathbb{N}$.

Theorem 2.1.2 (Generalised Caratheodory Extension Theorem)**.** *Suppose $\mathcal{S} \subseteq \mathcal{P}(X)$ is a semiring and $\nu : \mathcal{S} \to \mathbb{R}^+$ is an additive and σ-subadditive function on $\mathcal{S}$. Then there exists a measure μ on $(X, \sigma(\mathcal{S}))$ such that $\mu_{|\mathcal{S}} = \nu$. Moreover, if ν is σ-finite, then μ is uniquely determined.*

Examples 2.1.3. (a) If $\mathcal{S}$ is the semiring of half-open intervals from Example 2.1.1(a), then the function $\nu : \mathcal{S} \to \mathbb{R}^+$ given by

$$\nu\left(\prod_{i=1}^n (a_i, b_i]\right) \; = \; \prod_{i=1}^n (b_i - a_i)$$

extends uniquely to the so-called *Lebesgue measure* $\mathrm{Leb}_{\mathbb{R}^n}$ on $\mathbb{R}^n$.

(b) Given $p_0, p_1 \in [0,1]$ with $p_0 + p_1 = 1$, we can also use Theorem 2.1.2 in order to define the corresponding Bernoulli measure on Σ^+. To that end, let $\mathcal{Z}$ be the semiring of cylinder sets from Example 2.1.1 and

$$\nu([a]_n) \; = \; \prod_{i=1}^n p_{a_i} \; .$$

Then ν extends in a unique way to the *Bernoulli measure with probabilities p_0 and p_1* on $(\Sigma^+, \sigma(\mathcal{Z}))$.

(c) In a similar way as in (b), one may define Bernoulli measures on shift spaces $\Sigma_k^+ = \{0, \ldots, k-1\}^{\mathbb{N}}$ or $\Sigma_k = \{0, \ldots, k-1\}^{\mathbb{Z}}$ with probability vectors $p = (p_1, \ldots, p_k)$.

Note that in all cases some work is required in order to show that the functions ν satisfy the requirements of Theorem 2.1.2. Further, note that the generated σ-algebras are just the Borel σ-algebras on the respective spaces.

Exercise 18. Show that any measure μ on some measurable space $(X, \mathcal{A})$ is *monotone* ($A \subseteq B$ implies $\mu(A) \leq \mu(B)$), (finitely) additive and σ-subadditive and satisfies $\mu(\emptyset) = 0$.

Exercise 19. Show that the collections $\mathcal{S}$ and $\mathcal{Z}$ from Examples 2.1.1 form semirings and the functions ν defined in Examples 2.1.3 are additive and σ-subadditive as claimed.

Exercise 20. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a map $f : X \to Y$ is called **measurable** if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. Show that in this case, if ν is a measure on $(X, \mathcal{A})$, then

$$\mu(B) \; = \; \nu(f^{-1}(B))$$

defines a measure on $(Y, \mathcal{B})$. This measure μ is also called the **pushforward (measure)** of ν under f and denoted by $\mu = f_*\nu = \nu \circ f^{-1}$.

2.1.3 Outer measures

An **outer measure** on some space X is a σ-subadditive function $\hat{\mu}$ which is defined on all of $\mathcal{P}(X)$ and satisfies $\mu(\emptyset) = 0$.

Lemma 2.1.4. *Suppose $\mathcal{C} \subseteq \mathcal{P}(X)$ is an arbitrary family of subsets of X, $\emptyset \in \mathcal{C}$ and $\nu : \mathcal{C} \to \mathbb{R}^+$ satisfies $\nu(\emptyset) = 0$. Then*

$$\hat{\mu}(A) \; = \; \inf\left\{ \sum_{n\in\mathbb{N}} \nu(C_n) \;\middle|\; C_n \in \mathcal{C} \text{ for all } n \in \mathbb{N} \text{ and } A \subseteq \bigcup_{n\in\mathbb{N}} C_n \right\}$$

defines an outer measure on X.

Proof. Since $\emptyset \in \mathcal{C}$ and $\nu(\emptyset) = 0$, we directly obtain $\hat{\mu}(\emptyset) = 0$ as well. In order to show the σ-subadditivity of $\hat{\mu}$, let $(A_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{P}(X)$ which covers A, that is, $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$. Fix $\varepsilon > 0$ and choose, for each $n \in \mathbb{N}$, a countable cover $(C_j^n)_{j\in\mathbb{N}}$ of A_n with

$$\sum_{j\in\mathbb{N}} \nu(C_j^n) \;\leq\; \hat{\mu}(A_n) + \frac{\varepsilon}{2^n} \; .$$

Then $(C_j^n)_{\substack{j\in\mathbb{N}\\ n\in\mathbb{N}}}$ is a cover of A, and we obtain that

$$\hat{\mu}(A) \;\leq\; \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}} \nu(C_j^n) \;\leq\; \sum_{n\in\mathbb{N}} \left(\hat{\mu}(A_n) + \frac{\varepsilon}{2^n}\right) \;\leq\; \sum_{n\in\mathbb{N}} \hat{\mu}(A_n) + \varepsilon \; .$$

As $\varepsilon > 0$ was arbitrary, this proves the σ-subadditivity of $\hat{\mu}$. $\square$

Exercise 21. Show that if μ is σ-subadditive on $\mathcal{C}$, then the outer measure $\hat{\mu}$ defined in Lemma 2.1.4 satisfies $\hat{\mu}(C) = \mu(C)$ for all $C \in \mathcal{C}$.

Given an outer measure $\hat{\mu}$ on X, a set $A \subseteq X$ is called **$\hat{\mu}$-measurable** if

$$\hat{\mu}(E \cap A) + \hat{\mu}(E \cap A^c) \;=\; \hat{\mu}(E) \quad \text{ for all } E \in \mathcal{P}(X) \; .$$

Theorem 2.1.5. *Given an outer measure $\hat{\mu}$ on X, the family*

$$\mathcal{M}(\hat{\mu}) \;=\; \{A \in \mathcal{P}(X) \mid A \textit{ is } \hat{\mu}\textit{-measurable}\}$$

forms a σ-algebra, and $\mu = \hat{\mu}_{|\mathcal{M}(\hat{\mu})}$ is a measure on $(X, \mathcal{M}(\hat{\mu}))$.

Lemma 2.1.6. *If $\mathcal{S} \subseteq \mathcal{P}(X)$ is a semiring and $\nu : \mathcal{S} \to \overline{\mathbb{R}^+}$ is (finitely) additive and σ-subadditive and $\hat{\mu}$ is the outer measure defined in Lemma 2.1.4, then $\mathcal{S} \subseteq \mathcal{M}(\hat{\mu})$.*

Note that this implies $\sigma(\mathcal{S}) \subseteq \mathcal{M}(\hat{\mu})$.

Remark 2.1.7. The general version of the Caratheodory Extension Theorem provided by Theorem 2.1.2 follows directly from the two preceeding statements.

First, the function $\nu : \mathcal{S} \to \overline{\mathbb{R}^+}$ from Theorem 2.1.2 defines an outer measure $\hat{\mu} : \mathcal{P}(X) \to \overline{\mathbb{R}^+}$. Due to Lemma 2.1.6, we have that $\sigma(\mathcal{S}) \subseteq \mathcal{M}(\hat{\mu})$, so that by Theorem 2.1.5 we obtain that μ is a measure on $(X, \sigma(\mathcal{S}))$. Finally, it is then relatively easy to show that $\mu = \nu$ on $\mathcal{S}$.

However, in order to introduce Hausdorff measures of fractal dimension, we will need one further concept. An outer measure $\hat{\mu} : \mathcal{P}(X) \to \overline{\mathbb{R}^+}$ on some metric space X is called **metric**, if for any pair of disjoint sets $A, B \subseteq X$ with $d(A,B) > 0$ we have that $\hat{\mu}(A \cup B) = \hat{\mu}(A) + \hat{\mu}(B)$.[2]

Theorem 2.1.8. *If X is a metric space and $\hat{\mu} : \mathcal{P}(X) \to \overline{\mathbb{R}^+}$ is a metric outer measure, then $\mathcal{B}(X) \subseteq \mathcal{M}(\hat{\mu})$. In particular, $\hat{\mu}_{|\mathcal{B}(X)}$ is a measure on $(X, \mathcal{B}(X))$.*

The proof relies on the following

Lemma 2.1.9. *Suppose that X is a metric space and $\hat{\mu}$ is a metric outer measure on X. Given an open set $U \subseteq X$ and $n \in \mathbb{N}$, let $U_n = \{x \in X \mid d(x, U^c) \leq 1/n\}$. Then for any subset $E \subseteq X$ we have*

$$\hat{\mu}(E \cap U_n) \nearrow \hat{\mu}(E \cap U) \quad \textit{as } n \to \infty \; .$$

Proof. The inequality $\lim_{n\to\infty} \hat{\mu}(E \cap U_n) \leq \mu(E \cap U)$ follows by subadditivity of $\hat{\mu}$. Further, the sets $(E \cap U_{n+2}) \setminus U_n$ and $E \cap U_n$, respectively $(E \cap U_n) \setminus U_{n-2}$ have positive distance to each other ($\geq 1/n - 1/(n+2)$) for every $n \in \mathbb{N}$. Therefore, we have that for all $m, k \in \mathbb{N}$

$$\begin{aligned} &\hat{\mu}(E \cap U_m) + \sum_{j=1}^{k} \hat{\mu}((E \cap U_{m+2j}) \setminus U_{m+2j-2}) \\ &= \;\; \hat{\mu}\left((E \cap U_m) \cup \bigcup_{j=1}^{k} (E \cap U_{m+2j}) \setminus U_{m+2j-2}\right) \;\leq\; \mu(E \cap U) \; . \end{aligned}$$

[2]Note that here d is not the Hausdorff distance, which we always denote by $d_{\mathcal{H}}$, but $d(A,B) = \inf\{d(x,y) \mid x \in A,\ y \in B\}$.

Applied to $m = 1$ and $m = 2$, we obtain the convergence of the sum $\sum_{j=1}^{\infty} \hat{\mu}((E \cap U_{j+1}) \setminus U_j)$. This means that in particular

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} \hat{\mu}((E \cap U_{j+1}) \setminus U_j) \;=\; 0 \,.$$

Due to the σ-subadditivity of $\hat{\mu}$ again, we have that for every $n \in \mathbb{N}$

$$\hat{\mu}(E \cap U) \;\leq\; \hat{\mu}(E \cap U_n) + \sum_{j=n}^{\infty} \hat{\mu}((E \cap U_{j+1}) \setminus U_j) \,.$$

Hence, we obtain $\hat{\mu}(E \cap U) \leq \lim_{n\to\infty} \hat{\mu}(E \cap U_n)$. □

Proof of Theorem 2.1.8. The aim is to show that every Borel measurable set is contained in $\mathcal{M}(\hat{\mu})$. As $\mathcal{M}(\hat{\mu})$ is a σ-algebra by Theorem 2.1.5, it suffices to show that all open sets are contained in $\mathcal{M}(\hat{\mu})$, since these form a generator of $\mathcal{B}(X)$. Hence, suppose that $U \subseteq X$ is open and E is an arbitrary subset of E. Then $\hat{\mu}(E) \leq \hat{\mu}(E \cap U) + \hat{\mu}(E \cap U^c)$ follows by subadditivity of $\hat{\mu}$. In order to see the converse inequality, let the sets U_n be defined as in the proof of Lemma 2.1.9. Then for all $n \in \mathbb{N}$ the sets $E \cap U^c$ and $E \cap U_n$ have positive distance ($\geq 1/n$). Hence, we obtain

$$\hat{\mu}(E \cap U^c) + \hat{\mu}(E \cap U_n) \;=\; \hat{\mu}(E \cap (U^c \cup U_n)) \;\leq\; \mu(E)$$

and therefore, using Lemma 2.1.9 above,

$$\hat{\mu}(E \cap U) + \hat{\mu}(E \cap U^c) \;=\; \lim_{n\to\infty} \hat{\mu}(E \cap U_n) + \hat{\mu}(E \cap U^c) \;\leq\; \hat{\mu}(E) \,.$$

□

2.2 Hausdorff measures and dimension

Given a metric space X and $U \subseteq X$, we denote the **diameter** of U by

$$\mathrm{diam}(U) \;=\; \sup_{x,y\in U} d(x,y) \,.$$

If $A \subseteq X$ and $\delta > 0$, then a collection $\mathcal{U} \subseteq \mathcal{P}(X)$ is called a δ-cover of A if $\mathcal{U}$ is countable, $A \subseteq \bigcup_{U\in\mathcal{U}} U$ and $\mathrm{diam}(U) \leq \delta$ for all $U \in \mathcal{U}$. Further, given $s > 0$ and $\delta > 0$, we let

$$\mathcal{H}^s_\delta(A) \;=\; \inf \left\{ \sum_{U\in\mathcal{U}} \mathrm{diam}(U)^s \;\middle|\; \mathcal{U} \text{ is a } \delta\text{-cover of } A \right\}$$

and

$$\mathcal{H}^s(A) \;=\; \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) \;=\; \sup_{\delta>0} \mathcal{H}^s_\delta(A) \,.$$

For the last equality, note that $\mathcal{H}^s_\delta(A)$ is decreasing in δ since the infimum is taken over a larger set when δ is increased.

Exercise 22. (a) Show that if W is the d-dimensional limit cube, then $0 < \mathcal{H}^s(W) < \infty$ if and only if $s = d$.

(b) Show that $H^0(A) = \#A$.

$\mathcal{H}^s$ is called the s-**dimensional Hausdorff measure**. These measures are used to define the notion of *Hausdorff dimension* and to quantify the size of s-dimensional sets. This is accomplished through the following statements.

Theorem 2.2.1. *$\mathcal{H}^s$ is an outer measure, and its restriction $\mathcal{H}^s_{|\mathcal{B}(X)}$ to $\mathcal{B}(X)$ is a measure.*

Proof. Applying Lemma 2.1.4 with $\mathcal{S} = \mathcal{P}(X)$ and $\nu(U) = \mathrm{diam}(U)^s$ allows to see that each of the functions $\mathcal{H}^s_\delta$ is an outer measure. Since therefore $\mathcal{H}^s_\delta(\emptyset) = 0$ for all $\delta > 0$, we also have $\mathcal{H}^s(\emptyset) = 0$. In order to show the σ-subadditivity of $\mathcal{H}^s$, suppose that $\mathcal{U}$ is an arbitrary countable cover of $A \subseteq X$. Then the σ-subadditivity of the outer measures $\mathcal{H}^s_\delta$ implies

$$\mathcal{H}^s_\delta(A) \;\leq\; \sum_{U \in \mathcal{U}} \mathcal{H}^s_\delta(U) \;\leq\; \sum_{U \in \mathcal{U}} \mathcal{H}^s(U) \;,$$

and hence $\mathcal{H}^s(A) \leq \sum_{U \in \mathcal{U}} \mathcal{H}^s(U)$.

In order to prove that the restriction of $\mathcal{H}^s$ to $\mathcal{B}(X)$ is indeed a measure, it suffices to show that $\mathcal{H}^s$ is metric as an outer measure by Theorem 2.1.8. To that end, let $A, B \subseteq X$ with $d(A,B) > 0$. Then $\mathcal{H}^s(A \cup B) \leq \mathcal{H}^s(A) \cup \mathcal{H}^s(B)$ follows by σ-subadditivity, so that it suffices to show the converse inequality. Suppose that $0 < \delta < d(A,B)$ and let $\varepsilon > 0$. Further, note that by definition of $\mathcal{H}^s$ there exists a δ-cover $\mathcal{U}$ of $A \cup B$ with

$$\sum_{\mathcal{U}} \mathrm{diam}(U)^s \;\leq\; \mathcal{H}^s(A \cup B) + \varepsilon \;.$$

Due to the choice of δ, the two collections

$$\mathcal{U}_A \;=\; \{U \in \mathcal{U} \mid U \cup A \neq \emptyset\} \quad \text{and} \quad \mathcal{U}_B \;=\; \{U \in \mathcal{U} \mid U \cup B \neq \emptyset\}$$

are disjoint and provide δ-covers of A and B, respectively. We thus obtain

$$\begin{aligned}
\mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \;&\leq\; \sum_{U \in \mathcal{U}_A} \mathrm{diam}(U)^s + \sum_{U \in \mathcal{U}_B} \mathrm{diam}(U)^s \\
&\leq\; \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s \;\leq \mathcal{H}^s(A \cup B) + \varepsilon
\end{aligned}$$

For $\varepsilon, \delta \to 0$, this implies $\mathcal{H}^s(A) + \mathcal{H}^s(B) \leq \mathcal{H}^s(A \cup B)$ as required. Thus, together with the converse inequality from above, we obtain that $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) \cup \mathcal{H}^s(B)$ whenever $d(A,B) > 0$, so that $\mathcal{H}^s$ is indeed metric. □

Lemma 2.2.2. *Let X be a metric space, $A \subseteq X$ and $s \geq 0$.*

(a) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > s$.

(b) If $\mathcal{H}^s(A) > 0$, then $\mathcal{H}^t(A) = \infty$ for all $t < s$.

Proof. (a) Suppose $\mathcal{H}^s(A) = C < \infty$. Given $\delta > 0$, choose a δ-cover of A with $\sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s < 2C$. Then for $t > s$ we have that

$$\mathcal{H}^t_\delta(A) \;\leq\; \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^t \;\leq\; \delta^{t-s} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s \;\leq\; 2\delta^{t-s} C \;\xrightarrow{\delta \to 0}\; 0 \;,$$

and hence $\mathcal{H}^t(A) = 0$.

(b) Now suppose that $\mathcal{H}^s(A) > 0$ and $t < s$. Then $\mathcal{H}^t(A) < \infty$ would imply $\mathcal{H}^s(A) = 0$ by (a), a contradiction. Hence, we have that $\mathcal{H}^t(A) = \infty$. □

Exercise 23. Show that the measure $\mathcal{H}^d$ on $\mathbb{R}^d$ is equal to a multiple of the Lebesgue measure, $\mathcal{H}^d = C \cdot \mathrm{Leb}_{\mathbb{R}^d}$, where $C = \mathcal{H}^d([0,1]^d)$.

Hint: Use the fact that by construction $\mathcal{H}^d$ is invariant under translations and scales like Lebegue measure under multiplications to show that $\mathcal{H}^d$ and $C \times \mathrm{Leb}_{\mathbb{R}^d}$ coincide on the semiring of half-open rectangles (and hence on the whole Borel σ-algebra by uniqueness of the extension).

Based on the previous lemma, we can now introduce the notion of the **Hausdorff dimension** of a set $A \subseteq X$ as

$$\mathrm{Dim}_H(A) \;=\; \inf\{s > 0 \mid \mathcal{H}^s(A) = 0\} \cup \{\infty\} \;=\; \sup\{s > 0 \mid \mathcal{H}^s(A) = \infty\} \cup \{0\} \;.$$

Remark 2.2.3. In order to compare the definition of Hausdorff dimension with that of box dimension and to better understand the similarities and differences between the two, we make the following considerations.

Given $s, \delta > 0$, define outer measures $\mathcal{G}_\delta^s$ on X by

$$\mathcal{G}_\delta^s(A) \;=\; \inf\left\{ \sum_{U\in\mathcal{U}} \nu_{s,\delta}(U) \;\middle|\; \mathcal{U} \text{ is a } \delta\text{-cover of } A \text{ by } \delta\text{-balls } U = B_\delta(x) \right\} ,$$

where $\nu_{s,\delta}(U) = \delta^s$. Note that this simply means $\mathcal{G}_\delta^s(A) = \delta^s \cdot N_\delta(A)$. Moreover, if $N_\delta(A) = C \cdot \delta^{-s}$, then $G_\delta^s(A) = C$.

The decisive difference to the definition of Hausdorff dimension is that the limit of $\mathcal{G}_\delta^s(A)$ as $\delta \to 0$ does not necessarily exist, since the sequence $G_\delta^s(A)$ is not monotone in δ. Nonwithstanding, one may still define $\mathcal{G}^s(A) = \overline{\lim}_{\delta\to 0} \mathcal{G}_\delta^s(A)$ and show, in a similar way as above, that

$$\overline{\mathrm{Dim}}_B(A) \;=\; \inf\{s > 0 \mid \mathcal{G}^s(A) = 0\} \cup \{\infty\} \;=\; \sup\{s > 0 \mid \mathcal{G}^s(A) < \infty\} \cup \{0\} .$$

However, as the monotonicity of $\mathcal{H}_\delta^s(A)$ in δ was crucial in the proof of Theorem 2.2.1, the argument does not carry over and G^s is usually not a measure. This explains many differences in the structural properties of box and Hausdorff dimension, such as the lack of countable stability of the former.

Exercise 24. Show that

$$\overline{\mathrm{Dim}}_B(A) \;=\; \inf\{s > 0 \mid \mathcal{G}^s(A) = 0\} \;=\; \sup\{s > 0 \mid \mathcal{G}^s(A) < \infty\} \cup \{0\} .$$

2.3 Elementary properties of Hausdorff dimension

Lemma 2.3.1 (Monotonicity). *If X is a metric space and $A \subseteq B \subseteq X$, then $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$ for all $s \geq 0$ and $\mathrm{Dim}_H(A) \leq \mathrm{Dim}_H(B)$.*

Proof. The first statement is a direct consequence the fact that $\mathcal{H}^s$ is an outer measure (and hence monotone), and the second then follows from the definition of Hausdorff dimension. □

Lemma 2.3.2 (Countable stability). *Suppose X is a metric space, $A_n \subseteq X$ for $n \in \mathbb{N}$ and $A = \bigcup_{n\in\mathbb{N}} A_n$. Then*

$$\mathrm{Dim}_H(A) \;=\; \sup_{n\in\mathbb{N}} \mathrm{Dim}_H(A_n) .$$

Proof. On the one hand, we have that $\mathrm{Dim}_H(A_n) \leq \mathrm{Dim}_H(A)$ for all $n \in \mathbb{N}$ by inclusion, so that $\sup_{n\in\mathbb{N}} \mathrm{Dim}_H(A_n) \leq \mathrm{Dim}_H(A)$. Conversely, the σ-subadditivity of the $\mathcal{H}^s$ implies that

$$\mathcal{H}^s(A) \;\leq\; \sum_{n\in\mathbb{N}} \mathcal{H}^s(A_n)$$

for all $s \geq 0$. If $s > \sup_{n\in\mathbb{N}} \mathrm{Dim}_H(A_n)$, then $\mathcal{H}^s(A_n) = 0$ for all $n \in \mathbb{N}$ and therefore also $\mathcal{H}^s(A) = 0$. This implies $s \geq \mathrm{Dim}_H(A)$. Hence, we have that $\mathrm{Dim}_H(A) \leq \sup_{n\in\mathbb{N}} \mathrm{Dim}_H(A_n)$. □

Lemma 2.3.3. *If X and Y are metric spaces and $f : X \to Y$ is α-Hölder continuous with Hölder constant $C > 0$, then*

$$\mathcal{H}^{s/\alpha}(f(A)) \;\leq\; C^{s/\alpha}\mathcal{H}^s(A) . \tag{2.3.1}$$

In particular, $\mathrm{Dim}_H(f(A)) \leq \mathrm{Dim}_H(A)/\alpha$.

Proof. If $\mathcal{U}$ is a δ-cover of A, then $f(\mathcal{U})$ is a $C\delta^\alpha$-cover of $f(A)$. Hence, we have that

$$\mathcal{H}^{s/\alpha}_{C\delta^\alpha}(f(A)) \;\leq\; \sum_{U\in\mathcal{U}} \operatorname{diam}(f(U))^{s/\alpha} \;\leq\; \sum_{U\in\mathcal{U}} (C\operatorname{diam}(U)^\alpha)^{s/\alpha} \;=\; C^{s/\alpha}\sum_{U\in\mathcal{U}} \operatorname{diam}(U)^s \;.$$

Taking the infimum over all δ-covers U of A, we obtain $\mathcal{H}^{s/\alpha}_{C\delta^\alpha}(f(A)) \leq C^{s/\alpha}\mathcal{H}^s_\delta(A)$. Taking the limit $\delta \to 0$ yields (2.3.1), and the statement on the Hausdorff dimensions is a direct consequence. □

Corollary 2.3.4. *If $f : X \to Y$ is Lipschitz continuous, then $\mathrm{Dim}_H(f(A)) \leq \mathrm{Dim}_H(A)$. In particular, the Hausdorff dimension is invariant under bi-Lipschitz transformations.*

A mapping $S : X \to X$ is called a **similarity with scaling factor** $\lambda > 0$ if $d(S(x), S(y)) = \lambda d(x, y)$ for all $x, y \in X$.

Corollary 2.3.5. *If $S : X \to X$ is a similarity with scaling factor $\lambda > 0$, then for every $s \geq 0$ and $A \subseteq X$ we have that $\mathcal{H}^s(S(A)) = \lambda^s\mathcal{H}^s(A)$. In particular, we have that $\mathrm{Dim}_H(S(A)) = \mathrm{Dim}_H(A)$.*

Proof. As any similarity must be injective, we can view S as a bi-Lipschitz homeomorphism between A and $S(A)$, where S has Lipschitz constant λ and S^{-1} has Lipschitz constant λ^{-1}. Applying Lemma 2.3.3 with $f = S$, $\alpha = 1$ and $C = \lambda$ yields $\mathcal{H}^s(S(A)) \leq \lambda^s\mathcal{H}^s(A)$. Another application with $f = S^{-1}$, $\alpha = 1$ and $C = \lambda^{-1}$ yields the converse inequality. □

Exercise 25. Suppose that X is metric, $A \subseteq X$ and $S_1, \ldots, S_n$ are similarities on X with scaling factors $\lambda_1, \ldots, \lambda_n \in (0, 1)$. Further, assume that $S_1(A), \ldots, S_n(A)$ are disjoint and $A = \bigcup_{k=1}^n S_k(A)$. (Sets with this property are often called *self-similar*.)

Show that if A has positive and finite s-dimensional Hausdorff measure $\mathcal{H}^s(A)$, then s is uniquely determined by the equality

$$\sum_{k=1}^n \lambda_k^s \;=\; 1 \;.$$

Lemma 2.3.6. *Suppose X is a metric space and $A \subseteq X$.*

(a) If A is countable, then $\mathrm{Dim}_H(A) = 0$.

(b) If $X = \mathbb{R}^d$ and A has non-empty interior, then $\mathrm{Dim}_H(A) = d$.

(c) If $\mathrm{Dim}_H(A) < 1$, then A is totally disconnected.

Proof. (a) This follows directly from the fact that singletons have zero Hausdorff dimension together with the countable stability of Hausdorff dimension. Alternatively, suppose $A = \{x_n \mid n \in \mathbb{N}\}$ and $s > 0$. Given $\delta > 0$, let $U_n = B_{\delta 2^{-n/s-1}}(x_n)$. Then $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a δ-cover of A and

$$\mathcal{H}^s_\delta(A) \;\leq\; \sum_{n\in\mathbb{N}} (\delta 2^{-n/s})^s \;=\; \delta^s \sum_{n\in\mathbb{N}} 2^{-n} \;=\; \delta^s \;.$$

In the limit $\delta \to 0$, this implies $\mathcal{H}^s(A) = 0$, and hence $\mathrm{Dim}_H(A) \leq s$.

(b) First, consider $W = [0, 1]^d$. For any countable cover $\mathcal{U}$ of W, we have

$$1 \;=\; \mathrm{Leb}_{\mathbb{R}^d}(W) \;\leq\; \sum_{U\in\mathcal{U}} C_d \cdot \operatorname{diam}(U)^d \;\leq\; C_d \sum_{U\in\mathcal{U}} \operatorname{diam}(U)^d \;,$$

where C_d is the volume of the d-dimensional unit ball. Taking the infimum over all δ-covers of A, we obtain $\mathcal{H}^d_\delta(W) \geq 1/C_d$ for all $\delta > 0$. This implies $\mathcal{H}^d(W) \geq 1/C_d$ and thus $\mathrm{Dim}_H(A) \geq d$.

Conversely, W is covered by 2^{nd} d-dimensional cubes of sidelength 2^{-n} and diameter $2^{-n}\sqrt{d}$. For $s > d$, we therefore have

$$\mathcal{H}^s_{2^{-n}\sqrt{d}}(W) \;\leq\; 2^{nd} \cdot 2^{-sn} \cdot d^{s/2} \;=\; d^{s/2} \cdot 2^{-(s-d)n} \;\xrightarrow{n\to\infty}\; 0 \;,$$

so that $\mathcal{H}^s(W) = 0$. This implies $\mathrm{Dim}_H(W) \le d$.

Now, suppose that $A \subseteq \mathbb{R}^d$ is an arbitrary set with non-empty interior. Then A contains the image $S(W)$ under some similarity, and hence $\mathrm{Dim}_H(A) \ge \mathrm{Dim}_H(S(W)) = \mathrm{Dim}_H(W) = d$. Conversely, we have that $\mathbb{R}^d = \bigcup_{n\in\mathbb{N}} S_n(W)$, where S_n denotes the similarity given by $S_n(x) = nx$. Countable stability therefore implies $\mathrm{Dim}_H(A) \le \mathrm{Dim}_H(\mathbb{R}^d) \le \sup_{n\in\mathbb{N}} \mathrm{Dim}_H(S_n(W)) = d$.

(c) Suppose $\mathrm{Dim}_H(A) < 1$ and assume for a contradiction that A has a connected component $C \subseteq A$ with $\#C > 2$. Fix $x \in C$ and define $f : X \to \mathbb{R}$ by $f(y) = d(x, y)$. Then f is obviously Lipschitz-continuous. Moreover, as continuous images of connected sets are connected, $f(C)$ is a non-trivial interval I that contains $[0, d(x, y))$. However, as Lipschitz continuous transformations cannot increase the Hausdorff dimension of a set, we obtain that $\mathrm{Dim}_H(C) \ge \mathrm{Dim}_H(I) = 1$, a contradiction. $\square$

2.4 Computing Hausdorff dimension

First of all, box dimension always provides an upper bound on the Hausdorff dimension.

Theorem 2.4.1. *Suppose that X is a metric space and $A \subseteq X$ is relatively compact. Then*

$$\mathrm{Dim}_H(A) \;\le\; \underline{\mathrm{Dim}}_B(A)\,.$$

Proof. Recall that we have $\underline{\mathrm{Dim}}_B(A) = \lim_{\delta\to 0} \frac{\log N_\delta(A)}{-\log\delta}$. Fix $s < \mathrm{Dim}_H(A)$, so that $\mathcal{H}^s(A) = \infty$. As we have that

$$\mathcal{H}^s_\delta(A) \;\le\; N_{\delta/2}(A)\delta^s$$

by definition, we obtain that

$$\lim_{\delta\to 0} N_{\delta/2}(A)\delta^s \;\ge\; \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) \;=\; \mathcal{H}^s(A) \;=\; \infty\,,$$

and consequently

$$\lim_{\delta\to 0} \log N_{\delta/2}(A) + s\log\delta \;=\; \infty\,.$$

If $\delta > 0$ is sufficiently small, we therefore have $\log N_\delta(A) + s\log\delta \ge 0$, and hence

$$\frac{\log N_{\delta/2}(A)}{-\log\delta} \;\ge\; s\,.$$

In the limit $\delta \to 0$, we obtain $\underline{\mathrm{Dim}}_B(A) \ge s$ for all $s < \mathrm{Dim}_H(A)$, which yields the statement. $\square$

Obtaining a lower bound on the Hausdorff dimension is often a more difficult problem and requires more sophisticated methods. Here, we concentrate on a tool from measure theory.

Theorem 2.4.2 (Mass distribution principle)**.** *Suppose X is a metric space, $A \subseteq X$ and $\hat\mu$ is an outer measure on A. Further, assume that there exist constants $c, s, \varepsilon > 0$ such that*

$$\mathrm{diam}(U) < \varepsilon \;\Rightarrow\; \hat\mu(U) \le c\cdot \mathrm{diam}(U)^s \quad \textit{for all} \;\; U \subseteq A\,.$$

Then $\mathcal{H}^s(A) \ge \hat\mu(A)/c$ and $\mathrm{Dim}_H(A) \ge s$.

Proof. Given $\delta \in (0, \varepsilon)$, let $\hat{\mathcal{U}}$ be a δ-cover of A and set $\mathcal{U} = \{\hat U \cap A \mid \hat U \in \hat{\mathcal{U}}\}$. Then $\mathcal{U}$ is a δ-cover of A by subsets, and the σ-subadditivity of $\hat\mu$ implies

$$\hat\mu(A) \;\le\; \sum_{U\in\mathcal{U}} \hat\mu(U) \;\le\; \sum_{U\in\mathcal{U}} c\cdot\mathrm{diam}(U)^s \;\le\; c\cdot \sum_{\hat U\in\hat{\mathcal{U}}} \mathrm{diam}(\hat U)^s\,.$$

By taking the infimum over all δ-covers $\hat{\mathcal{U}}$ of A, we obtain $\mathcal{H}^s_\delta(A) \ge \hat\mu(A)/c$. Taking the limit $\delta \to 0$ then yields the statement. $\square$

Remark 2.4.3. Recall that

$$\mathcal{H}^s_\delta(A) \;=\; \inf\left\{ \sum_{U\in\mathcal{U}} \mathrm{diam}(U)^s \;\middle|\; \mathcal{U} \text{ is a } \delta\text{-cover of } A \right\} .$$

Since the diameter of a set and its closure is the same, one may also restrict to δ-covers by closed sets in this definition. This means that we can also restrict to working with such covers by closed sets in the proof of Theorem 2.4.2. In this case, it suffices to have $\hat{\mu}$ defined for all closed subsets of X. This means, in particular, that instead of an outer measure $\hat{\mu}$ we may consider a Borel measure μ on X in the Mass Distribution Principle.

Corollary 2.4.4 (Mass Distribution Principle, version for Borel measures)**.** *Suppose X is a metric space, $A \subseteq X$ and μ is a Borel measure on A. Further, assume that there exist constants $c, s, \varepsilon > 0$ such that*

$$\mathrm{diam}(U) < \varepsilon \;\Rightarrow\; \mu(U) \leq c \cdot \mathrm{diam}(U)^s \quad \textit{for all } \; U \in \mathcal{B}(X) .$$

Then $\mathcal{H}^s(A) \geq \mu(A)/c$ and $\mathrm{Dim}_H(A) \geq s$.

Exercise 26. (a) Show that

$$\mathcal{H}^s_\delta(A) \;=\; \inf\left\{ \sum_{U\in\mathcal{U}} \mathrm{diam}(U)^s \;\middle|\; \mathcal{U} \text{ is a } \delta\text{-cover of } A \text{ by closed sets.} \right\} .$$

(b) Give a direct proof of Corollary 2.4.4.

Example 2.4.5. We consider the Middle Third Cantor Set $C = \left\{\sum_{n\in\mathbb{N}} 2a_n 3^{-n} \mid a \in \{0,1\}^{\mathbb{N}}\right\}$. Since we already know that $\mathrm{Dim}_B(C) = s$ where $s = \log 2/\log 3$, Theorem 2.4.1 implies that $\mathrm{Dim}_H(X) \leq s$.

In order to show that s also presents a lower bound (and hence the precise value) for the Hausdorff dimension of C, we want to apply the Mass Distribution Principle. To that end, we will use the following general fact from Exercise 20: if $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces, $f : X \to Y$ is a measurable map[3] and ν is a measure on X, then a measure $f_*\nu = \nu \circ f^{-1}$ on $(Y, \mathcal{B})$ is defined by

$$f_*\nu(A) \;=\; \nu(f^{-1}(A)) \quad \text{for all } A \in \mathcal{B} .$$

(See Exercise 20). Now, consider the standard Bernoulli measure ν with probabilities $1/2, 1/2$ on $X = \{0,1\}^{\mathbb{N}}$. Let $h : X \to C$, $a \mapsto \sum_{n\in\mathbb{N}} 2a_n 3^{-n}$. Then $\mu = h_*\nu$ defines a Borel measure on C. Moreover, if $U \subseteq C$ has diameter $\mathrm{diam}(U) \in [3^{-n}, 3^{-(n-1)})$, then U is contained in the image of a cylinder set $Z = [a]_{n-1}$ for some $a \in X$. Hence,

$$\mu(U) \;\leq\; \mu(h(Z)) \;=\; \nu(Z) \;=\; 2^{n-1} \;=\; 2 \cdot \left(3^{-n}\right)^s \;\leq\; 2\mathrm{diam}(U)^s .$$

Therefore Theorem 2.4.2 applies and yields that $\mathcal{H}^s(C) \geq 1/2$ and $\mathrm{Dim}_H(C) \geq s$.

Exercise 27. Suppose that C is a modified Sierpinski Carpet that is constructed by dividing the d-dimensional unit cube $[0,1]^d$ into k^d identical smaller cubes and retaining n of them in each step of the construction. Show that $\mathrm{Dim}_H(C) = \log n/\log k$.

Exercise 28. Suppose $\Sigma^+ = \{0,1\}^{\mathbb{N}}$ is equipped with the metric d_β from Exercise 11. Show that the Hausdorff dimension of (Σ^+, d_β) is equal to $\log 2/\log \beta$.

2.5 Summary

The notion of Hausdorff dimension provides an alternative and complementary approach to measure the size of fractals and overcomes some of the shortcomings of box dimension. The prize to pay for this is a more technical machinery that is required.

[3] That is, $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.

- $\oplus$ The definition of Hausdorff dimension is based on a very conceptual approach that roots in measure theory and leads to mathematically very consistent results.
- $\oplus$ In particular, Hausdorff dimension is countably stable.
- $\oplus$ As a byproduct, the s-dimensional Hausdorff measures allow to compare the size of s-dimensional fractals, thus providing additional and refined information.
- $\oplus$ The mass distribution principle provides an efficient tool for the computation of Hausdorff dimension.
- $\ominus$ The definition of Hausdorff dimension is less intuitive and harder to digest than that of box dimension (despite the analogies between the two pointed out in Remark 2.2.3.
- $\ominus$ The computation of Hausdorff dimension is often more complicated than that of box dimension and requires more sophisticated tools.
- $\ominus$ Hausdorff dimension is not very well accessible to numerical computations.

Chapter 3

Iterated Function Systems

3.1 Attractors of iterated function systems

Suppose X is a metric space and $D \subseteq X$ a closed set. An **iterated function system (IFS) on** D is a finite family $\{S_1, \dots, S_m\}$ of contractions[1] on D (with $m \geq 2$). Let $\mathcal{K}(D) = \{K \subseteq D \mid K \text{ is compact}\}$ and equip $\mathcal{K}(D)$ with the Hausdorff metric d_H (see Appendix B), so that $(\mathcal{K}(D), d_H)$ is a complete metric space. Then any IFS $\mathcal{S}$ on D can be interpreted as a map acting on $\mathcal{K}(D)$ by

$$\mathcal{S} : \mathcal{K}(D) \to \mathcal{K}(D) \quad , \quad K \mapsto \bigcup_{k=1}^{m} S_k(K) \; .$$

Theorem 3.1.1 (Existence and uniqueness of an IFS attractor)**.** *Suppose X is a metric space and $\mathcal{S} = \{S_1, \dots, S_m\}$ is an IFS on $D \subseteq X$. Then the mapping $\mathcal{S} : \mathcal{K}(D) \to \mathcal{K}(D)$ is a contraction with contraction constant $c = \sup_{k=1}^{m} c_k$, where the c_k are the contraction constants of the maps S_k. Further, $\mathcal{S}$ has a unique fixed point, that is, there exists a unique set $\mathcal{A}_{\mathcal{S}} \in \mathcal{K}(D)$ such that*

$$\mathcal{A}_{\mathcal{S}} \; = \; \bigcup_{k=1}^{m} S_k(\mathcal{A}_{\mathcal{S}}) \; .$$

Moreover, given any $A \in \mathcal{K}(D)$, we have that $\lim_{n\to\infty} \mathcal{S}^n(A) = \mathcal{A}_{\mathcal{S}}$ and $d_H(\mathcal{S}^n(A), \mathcal{A}_{\mathcal{S}}) \leq c^n \cdot d_H(A, \mathcal{A}_{\mathcal{S}})$. The set $\mathcal{A}_{\mathcal{S}}$ is called the **attractor of the IFS** $\mathcal{S}$.

Proof. Let $A, B \in \mathcal{K}(D)$, $t = d_H(A, B)$ and $\varepsilon > 0$. Then $A \subseteq B_{t+\varepsilon}(B) = \{x \in X \mid \exists y \in B : d(x, y) < t + \varepsilon\}$ and $B \subseteq B_{t+\varepsilon}(A)$. Consequently, we have $S_k(A) \subseteq S_k(B_{t+\varepsilon}(B)) \subseteq B_{c_k(t+\varepsilon)}(S_k(B))$ and $S_k(B) \subseteq B_{c_k(t+\varepsilon)}$. Therefore $d_H(S_k(A), S_k(B)) \leq c_k(t+\varepsilon)$, and in the limit $\varepsilon \to 0$ we obtain $d_H(S_k(A), S_k(B)) \leq c_k \cdot d_H(A, B)$. This further entails

$$d_H(\mathcal{S}(A), \mathcal{S}(B)) \; \leq \; \max_{k=1}^{m} d_H(S_k(A), S_k(B)) \; \leq \; \max_{k=1}^{m} c_k \cdot d_H(A, B) \; .$$

Hence, $\mathcal{S}$ is a contraction with contraction constant c, and the remaining statements follow directly from the Banach Fixed Point Theorem. □

Exercise 29. (a) Show that if A is compact and $\mathcal{S}(A) \subseteq A$ (that is, $S_k(A) \subseteq A$ for all $k = 1, \dots, m$), then $(\mathcal{S}^n(A))_{n\in\mathbb{N}}$ is a decreasing sequence and $\mathcal{A}_{\mathcal{S}} = \bigcap_{n\in\mathbb{N}} \mathcal{S}^n(A)$. (*Hint: It suffices to show that $\mathcal{S}\left(\bigcap_{n\in\mathbb{N}} \mathcal{S}^n(A)\right) = \bigcap_{n\in\mathbb{N}} \mathcal{S}^n(A)$. Why?*)

(b) Show that for every IFS $\mathcal{S}$ on D, there exists $R > 0$ and $x \in D$ such that $A = \overline{B_R(x)} \cap D$ satisfies $\mathcal{S}(A) \subseteq A$.

[1] Recall that a map $S : D \to D$ is called a *contraction on D* if there exists a *contraction constant* $c \in [0, 1)$ such that $d(S(x), S(y)) \leq c d(x, y)$ for all $x, y \in D$.

Example 3.1.2. Suppose that $W = [0,1]^d$ and C is a generalised Sierpinski carpet which is obtained by subdividing W into l^d equal-sized smaller cubes, retaining n of these cubes and repeating this process recursively in the retained cubes. Let x_k denote the smallest vertex of the k-th of the retained cubes (where smallest refers to the lexicographic ordering) and let $S_k(y) = y/l + x_k$. Then $C = \mathcal{A}_{\mathcal{S}}$, where $\mathcal{S}$ is the IFS $\{S_1, \ldots, S_n\}$.

Exercise 30. Provide a precise and short proof that the Middle Third Cantor Set C is the unique attractor of the IFS $\{S_1(x) = x/3, S_2(x) = (x+2)/3\}$ on $\mathbb{R}$.

3.2 IFS and coding sequences

If $\mathcal{S} = \{S_1, \ldots, S_m\}$ is an IFS on D and $a_1, \ldots, a_n \in \{1, \ldots, m\}$, then we let $S_{a_1 \ldots a_n} = S_{a_1} \circ \ldots \circ S_{a_n}$. Note that each $S_{a_1 \ldots a_n}$ is a contraction with contraction constant $c_{a_1} \cdot \ldots c_{a_n}$.

Lemma 3.2.1. *Given $x_0 \in D$ and any sequence $a \in \{1, \ldots, m\}^{\mathbb{N}}$, the limit*

$$\xi_{x_0}(a) \;=\; \lim_{n\to\infty} S_{a_1 \ldots a_n}(x_0)$$

exists and does not depend on x_0. Moreover, we have $\xi_{x_0}(a) \in \mathcal{A}_{\mathcal{S}}$ for all $a \in \{1, \ldots, m\}^{\mathbb{N}}$.

Proof. Let $x_n = S_{a_1 \ldots a_n}(x_0)$. Then for any $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_n) \;=\; d(S_{a_1 \ldots a_n}(S_{a_{n+1}}(x_0)), S_{a_1 \ldots a_n}(x_0)) \;\leq\; c^n \cdot K \;,$$

where $c = \max_{k=1}^m c_k$ is the maximum over the contractions constants c_k of the S_k and $K = \max_{k=1}^m d(x_0, S_k(x_0))$. This shows that the x_n form a Cauchy-sequence and therefore converge to a limit $\xi_{x_0}(a)$.

In order to see that this limit does not depend on x_0, choose a second starting point y_0 and let $y_n = S_{a_1 \ldots a_n}(y_0)$. Then

$$d(x_n, y_n) \;=\; d(S_{a_1 \ldots a_n}(x_0), S_{a_1 \ldots a_n}(y_0)) \;\leq\; c^n \cdot d(x_0, y_0) \;,$$

and hence $\xi_{x_0}(a) = \xi_{y_0}(a)$. The fact that $\xi_{x_0}(a)$ is contained in the attractor $\mathcal{A}_{\mathcal{S}}$ follows by choosing $x_0 \in \mathcal{A}_{\mathcal{S}}$. Then, by invariance of the attractor, we have $S_{a_1 \ldots a_n}(x_0) \in \mathcal{A}_{\mathcal{S}}$ for all $n \in \mathbb{N}$, and as $\mathcal{A}_{\mathcal{S}}$ is compact the same applies to the limit. $\square$

This allows to define a mapping

$$\xi : \Sigma_m^+ = \{1, \ldots, m\}^{\mathbb{N}} \to \mathcal{A}_{\mathcal{S}} \quad , \quad a \mapsto \xi_{x_0}(a) \;, \tag{3.2.1}$$

where the starting point $x_0 \in D$ is arbitrary.

Lemma 3.2.2. *The map $\xi : \Sigma_m^+ \to \mathcal{A}_{\mathcal{S}}$ is continuous and onto.*

Proof. Suppose without loss of generality that $x_0 \in \mathcal{A}_{\mathcal{S}}$ and let $D = \operatorname{diam}(\mathcal{A}_s)$ and $\varepsilon > 0$. Note that due to the invariance of $\mathcal{A}_{\mathcal{S}}$ under $\mathcal{S}$, we have that $S_{a_1 \ldots a_n}(x_0) \in \mathcal{A}_{\mathcal{S}}$ for all $a \in \Sigma_m^+$ and $n \in \mathbb{N}$. Choose $N \in \mathbb{N}$ such that $c^N \cdot D < \varepsilon$ and suppose that $b \in [a]_n$. Then we have that

$$\begin{aligned}
d(\xi(a), \xi(b)) \;&=\; d\left(\lim_{n\to\infty} S_{a_1 \ldots a_n}(x_0), \lim_{n\to\infty} S_{b_1 \ldots b_n}(x_0)\right) \\
&=\; d\left(S_{a_1 \ldots a_N}\left(\lim_{n\to\infty} S_{a_{N+1} \ldots a_N}(x_0)\right), (S_{a_1 \ldots a_N}\left(\lim_{n\to\infty} S_{b_{N+1} \ldots b_N}(x_0)\right)\right) \\
&\leq\; c^N \cdot D \;<\; \varepsilon \;.
\end{aligned}$$

Hence $\xi([a]_n) \subseteq B_\varepsilon(\xi(a))$, and as $[a]_n$ is an open neighbourhood of a, this proves the continuity of ξ.

In order to see that ξ is onto, let $x \in \mathcal{A}_{\mathcal{S}}$. Then $x \in \mathcal{S}^n(\mathcal{A}_{\mathcal{S}})$ for all $n \in \mathbb{N}$, and we can choose a sequence $a \in \Sigma_m^+$ such that $x \in S_{a_1 \ldots a_n}(\mathcal{A}_{\mathcal{S}})$ for all $n \in \mathbb{N}$. (Note that f $a_1, \ldots, a_n$

have been chosen such that $x \in S_{a_1\dots a_n}(\mathcal{A}_\mathcal{S}) = S_{a_1\dots a_n}(\mathcal{S}(\mathcal{A}_\mathcal{S}))$, there always has to be some $a_{n+1} \in \{1,\dots,m\}$ such that $x \in S_{a_1\dots a_n}(S_{a_{n+1}}(\mathcal{A}_\mathcal{S}))$.) Since $\mathrm{diam}(S_{a_1\dots a_n}(\mathcal{A}_\mathcal{S})) \leq c^n \mathrm{diam}(\mathcal{A}_\mathcal{S})$, we obtain that

$$d(x, S_{a_1\dots a_n}(x_0)) \;\leq\; c^n \cdot \mathrm{diam}(\mathcal{A}_\mathcal{S}) \;,$$

where we assume again that $x_0 \in \mathcal{A}_\mathcal{S}$. This shows $x = \lim_{n\to\infty} S_{a_1\dots a_n}(x_0) = \xi(a)$. $\square$

Remark 3.2.3. Suppose that $x_0 \in \mathcal{A}_\mathcal{S}$ and $w = a_1,\dots,a_n \in \{1,\dots,m\}^n$. Then $S_w(x_0) \in S_w(\mathcal{A}_\mathcal{S})$. Moreover, for any sequence $a \in [w]$ and $k \geq n$ we have that $S_{a_1,\dots,a_k}(x_0) = S_w(S_{a_{n+1},\dots,a_k}(x_0)) \in S_w(\mathcal{A}_\mathcal{S})$, as $S_{a_{n+1},\dots,a_k}(x_0) \in \mathcal{A}_\mathcal{S}$ as well. Since $S_w(\mathcal{A}_\mathcal{S})$ is closed, this implies that $\xi(a) = \lim_{k\to\infty} S_{a_1,\dots,a_k}(x_0) \in \mathcal{A}_\mathcal{S}$. Hence, we obtain that $\xi([w]) \subseteq S_w(\mathcal{A}_\mathcal{S})$.

Conversely, suppose that $x \in S_w(\mathcal{A}_\mathcal{S})$. Then $x = S_w(y)$ for some $y \in \mathcal{A}_\mathcal{S}$. If $y = \xi(a)$, then it follows by continuity of S_w that $x = S_w(y) = \xi(wa)$, where wa denotes the sequence given by

$$(wa)_k \;=\; \begin{cases} w_k & 1 \leq k \leq |w| \\ a_{k-|w|}/k < |w| & \end{cases} .$$

Thus, we have $S_w(\mathcal{A}_\mathcal{S}) \subseteq \xi([w])$, and altogether we obtain

$$S_w(\mathcal{A}_\mathcal{S}) \;=\; \xi([w]) \;.$$

Exercise 31. Show that ξ is injective (and hence a homeomorphism) if and only if the images $S_k(\mathcal{A}_\mathcal{S})$ with $k = 1,\dots,m$ are pairwise disjoint. (Here, we assume that all the similarities S_k are invertible, that is, $c_k > 0$ for all $k = 1,\dots,m$.)

Exercise 32. Show that the coding map obtained from the IFS in Exercise 30 coincides with the map h from Remark 1.1.2 when $\Sigma+$ and Σ_2^+ are identified by relabelling the symbols (0 becomes 1 and 1 becomes 2).

3.3 Dimension formulas and the open set condition

In order to determine the dimensions of IFS attractors, we first consider box dimension.

Theorem 3.3.1. *Suppose $\mathcal{S} = \{S_1,\dots,S_m\}$ is an IFS on $D \subseteq X$ with contraction constants $c_1,\dots,c_m \in (0,m)$. Then $\overline{\mathrm{Dim}}_B(\mathcal{A}_\mathcal{S}) \leq s$, where s is uniquely determined by the equation*

$$\sum_{k=1}^{m} c_k^s \;=\; 1 \;. \tag{3.3.1}$$

Remark 3.3.2. (a) Note that since $c_k \in (0,1)$ for all $k = 1,\dots,m$, the right side of (3.3.1) is strictly decreasing in s, so that the equation has a unique solution.

(b) For any $n \in \mathbb{N}$,we have that $\mathcal{A}_\mathcal{S} = \bigcup_{a\in\{1,\dots,m\}^n} S_{a_1\dots a_n}(\mathcal{A}_\mathcal{S})$. If $c = \max_{k=1}^m c_k$, then all these images of $\mathcal{A}_\mathcal{S}$ satisfy $\mathrm{diam}\,(S_{a_1\dots a_n}(\mathcal{A}_\mathcal{S})) \leq c^n \cdot \mathrm{diam}(\mathcal{A}_\mathcal{S})$. This easily allows to conclude that $\overline{\mathrm{Dim}}_B(\mathcal{A}) \leq \log m/\log c$. However, this value is strictly larger than s, unless all the c_k are equal. Hence, the difficulty in the proof will be to improve on this rough estimate by taking into account that some of the images $S_{a_1\dots a_n}(\mathcal{A}_\mathcal{S})$ shrink considerably faster than with rate c.

Proof of Theorem 3.3.1. Let $c_0 = \min_{k=1}^m c_k$. Further, given $\delta > 0$ and $a \in \Sigma_m^+$, let

$$k(a) \;=\; \min\{k \in \mathbb{N} \mid c_{a_1} \cdot \ldots \cdot c_{a_k} \leq \delta\} \;. \tag{3.3.2}$$

Note that we always have $c_{a_1} \cdot \ldots \cdot c_{a_{k(a)}} \geq c_0\delta$. We further define $\mathcal{Q} = \{a_1 \dots a_{k(a)} \mid a \in \Sigma_M^+\}$ and denote the length of a word $w \in \mathcal{Q}$ by $|w|$. Given a finite word w over the alphabet $\{1,\dots,m\}$, denote by $[w]$ the corresponding cylinder set in Σ_m^+ (that is, $[w] = [a]_{|w|}$, where

$a \in \Sigma_m^+$ can be any sequence that starts with w). Then $\bigcup_{w\in\mathcal{Q}}[w] = \Sigma_m^+$, since every $a \in \Sigma_m^+$ is contained in $[a_1 \dots a_{k(a)}]$. Hence, if $S_w = S_{w_1} \circ \dots \circ S_{w_{|a|}}$, we have that

$$\mathcal{A}_\mathcal{S} \;=\; \bigcup_{w\in\mathcal{Q}} S_w(\mathcal{A}_\mathcal{S}) \;.$$

(Note here that $\mathcal{A}_\mathcal{S} = \xi(\Sigma_m^+)$ and $S_w(\mathcal{A}_\mathcal{S}) = \xi([w])$.) Therefore, we obtain a cover $\mathcal{U} = \{S_w(\mathcal{A}_\mathcal{S}) \mid w \in \mathcal{Q}\}$ of $\mathcal{A}_\mathcal{S}$. Moreover, as

$$\operatorname{diam}(S_a(\mathcal{A}_\mathcal{S})) \;\leq\; c_{a_1}\cdot\ldots\cdot c_{a_{|a|}}\cdot\operatorname{diam}(\mathcal{A}_\mathcal{S}) \;\leq\; \delta\cdot\operatorname{diam}(\mathcal{A}_\mathcal{S}),$$

for all $a \in \mathcal{Q}$, we have that $\mathcal{U}$ is a $(\delta\cdot\operatorname{diam}(\mathcal{A}_\mathcal{S}))$-cover of $\mathcal{A}_\mathcal{S}$. In order to estimate the cardinality of $\mathcal{Q}$, we let $\mathcal{Q}^n = \{a_1 \dots a_{\min\{k(a),n\}} \mid a \in \Sigma_m^+\}$. As $|w| \leq \log\delta/\log c_0 + 1$ for all $w \in \mathcal{Q}$, we obtain that $\mathcal{Q}^n = \mathcal{Q}$ if $n \geq \log r/\log c_0 + 1$. We now claim that

$$\sum_{w\in\mathcal{Q}^n} (c_{w_1}\cdot\ldots\cdot c_{w_{|w|}})^s \;=\; 1 \quad \text{ for all } n \in \mathbb{N}\;. \tag{3.3.3}$$

If this holds, then as $(c_{w_1}\cdot\ldots\cdot c_{w_{|w|}})^s \geq (c_0\delta)^s$ for all $w \in \mathcal{Q}_n$, we obtain that $\#Q_n \leq (c_0\delta)^{-s}$ for all $n \in \mathbb{N}$, and thus $\#Q \leq (c_0\delta)^{-s}$ as well. This immediately implies that

$$\overline{\operatorname{Dim}}_B(\mathcal{A}_\mathcal{S}) \;\leq\; \lim_{\delta\to 0}\frac{\log(c_0\delta)^{-s}}{-\log(c_0\delta)} \;=\; s\;.$$

It remains to prove (3.3.3). However, this equality follows immediately from the facts that the cylinders $[w]$ with $w \in \mathcal{Q}^n$ form a partition of Σ_m^+ and $c_{w_1}^s\cdot\ldots\cdot c_{w_{|w|}}^s = \nu([w])$, where ν denotes the Bernoulli measure on Σ_m^+ with probabilities $c_1^s,\dots,c_m^s$.

This proves the claim and thus completes the proof of the theorem. □

In order to obtain a lower bound for the Hausdorff dimension of an IFS attractor, we first consider a simplified case and assume that the images $S_i(\mathcal{A}_\mathcal{S})$ of the attractor under the different mappings of the IFS are pairwise disjoint.

Theorem 3.3.3. *Suppose $\mathcal{S} = \{S_1,\dots,S_m\}$ is an IFS on $D \subseteq X$ and there exist constants $b_1,\dots,b_m \in (0,1)$ such that*

$$d(S_k(x),S_k(y)) \;\geq\; b_k\cdot d(x,y) \quad \textit{for all } x,y \in D\;.$$

Further, let $\mathcal{A}_\mathcal{S}$ be the attractor of the IFS $\mathcal{S}$ and assume that the sets $S_k(\mathcal{A}_\mathcal{S})$ are pairwise disjoint. Then $\operatorname{Dim}_H(\mathcal{A}_\mathcal{S}) \geq s$, where $s \geq 0$ is uniquely determined by the equation

$$\sum_{k=1}^m b_k \;=\; 1\;.$$

Proof. Let ν be the Bernoulli measure with probabilities $b_1^s,\dots,b_m^s$ on Σ_k^+ and $\mu = \xi_*\nu$, where $\xi : \Sigma_k^+ \to \mathcal{A}_\mathcal{S}$ is the coding map associated to the IFS $\mathcal{S}$. Let

$$d = \min_{1\leq i<j\leq m} d(S_i(\mathcal{A}_\mathcal{S}),S_j(\mathcal{A}_\mathcal{S}))\;.$$

Our aim is to show that every Borel measurable $U \subseteq \mathcal{A}_\mathcal{S}$ satisfies $\mu(U) \leq d^{-s}\cdot\operatorname{diam}(U)^s$. The statement then follows from the Mass Distribution Principle. In order to prove this estimate, let $U \subseteq \mathcal{A}_\mathcal{S}$ and $x = \xi(a) \in U$. Choose $k \in \mathbb{N}$ with

$$b_{a_1}\cdot\ldots b_{a_k}\cdot d \;\leq\; \operatorname{diam}(U) \;\leq\; b_{a_1}\cdot\ldots b_{a_{k-1}}\cdot d\;.$$

Given $(\tilde a_1,\dots,\tilde a_k) \neq (a_1,\dots,a_k)$, let $l = \min\{j = 1,\dots,k \mid \tilde a_j \neq a_j\}$. Then

$$d(S_{a_l\dots a_k}(\mathcal{A}_\mathcal{S}),S_{\tilde a_l\dots\tilde a_k}(\mathcal{A}_\mathcal{S})) \;\geq\; d(S_{a_l}(\mathcal{A}_\mathcal{S}),S_{\tilde a_l}(\mathcal{A}_\mathcal{S})) \;\geq\; d$$

and therefore

$$\begin{aligned} & d(d(S_{a_1\dots a_k}(\mathcal{A}_\mathcal{S}), S_{\tilde{a}_1\dots\tilde{a}_k}(\mathcal{A}_\mathcal{S})) \\ \geq\ & d(S_{a_1\dots a_{l-1}}(S_{a_l}(\mathcal{A}_\mathcal{S})), S_{a_1\dots a_{l-1}}(S_{\tilde{a}_l}(\mathcal{A}_\mathcal{S}))) \\ \geq\ & b_1 \cdot \ldots \cdot b_{l-1} \cdot d \ \geq\ \operatorname{diam}(U) \ . \end{aligned}$$

Hence, we obtain that $U \subseteq S_{a_1\dots a_k}(\mathcal{A}_\mathcal{S})$ and $U \cap S_{b_1,\dots,\tilde{a}_k}(\mathcal{A}_\mathcal{S}) = \emptyset$ for all $(\tilde{a}_1,\dots,\tilde{a}_k) \neq (a_1,\dots,a_k)$. The latter implies $\xi^{-1}(U) \subseteq [a_1,\dots,a_k]$ and thus

$$\mu(U) \ \leq\ \mu(\xi[a_1\dots a_k]) \ =\ \nu([a_1\dots a_k]) \ =\ (b_{a_1}\cdot\ldots\cdot b_{a_k})^s \ \leq\ d^{-s}\cdot\operatorname{diam}(U)^s \ ,$$

as required. □

Exercise 33. Show that if the sets $S_k(\mathcal{A}_\mathcal{S})$ are pairwise disjoint for $k = 1,\dots,m$, then the attractor $\mathcal{A}_\mathcal{S}$ of the IFS $\mathcal{S} = \{S_1,\dots,S_m\}$ is totally disconnected.

Remark 3.3.4. If the images of the contractions S_k overlap too much, then it will not be possible to provide a lower bound on the Hausdorff dimension. Note that if $S_1 = \ldots = S_m$, then $\mathcal{A}_\mathcal{S}$ consists of a single point (the unique fixed point of S_1) and its dimension is zero, independent of the contraction constants c_k.

However, at the same time the disjointness of the sets $S_k(\mathcal{A}_\mathcal{S})$ is quite restrictive, since this implies that the attractor $\mathcal{A}_\mathcal{S}$ of the IFS is totally disconnected. Hence, this excludes to describe fractals like the Koch curve and snowflake, which are connected sets. The same is true for some of the modified Sierpinski Carpets discussed in previous sections.

However, in all these cases, intersections between the images of $\mathcal{A}_\mathcal{S}$ under the S_k only occur at the boundaries. The following condition allows to treat this situation in Euklidean spaces.

We say an IFS $\mathcal{S} = \{S_1,\dots,S_m\}$ on $D \subseteq X$ satisfies the **open set condition (OSC)**, if there exists an open set $U \subseteq D$ such that $\operatorname{diam}(U) < \infty$, the images $S_k(U)$ with $k = 1,\dots,m$ are pairwise disjoint and

$$\biguplus_{k=1}^m S_k(U) \ \subseteq\ U \ .$$

Theorem 3.3.5. *Suppose $X = \mathbb{R}^d$, $D \subseteq X$ and $\mathcal{S} = \{S_1,\dots,S_m\}$ is an IFS with similarities $S_1,\dots,S_m$ and corresponding scaling factors $c_1,\dots,c_m$. Further, assume that $\mathcal{S}$ satisfies the open set conditions. Then $\operatorname{Dim}_H(\mathcal{A}_\mathcal{S}) = \operatorname{Dim}_B(\mathcal{A}_\mathcal{S}) = s$, where $s \geq 0$ is uniquely determined by the equation*

$$\sum_{k=1}^m c_k^s \ =\ 1 \ .$$

Before we turn to the proof, we will need the following technical lemma.

Lemma 3.3.6. *Let $(V_i)_{i\in I}$ be a collection of open sets in $\mathbb{R}^d$ such that each V_i contains a ball of radius $\alpha_1 r$ and is contained in a ball of radius $a_2 r$. Then any ball of radius r intersects at most $(1+2\alpha_2)^d\cdot\alpha_1^{-d}$ of the closures $\overline{V_i}$.*

Proof. Suppose the closures of the sets $V_1\dots V_N$ intersect $B_r(x)$. Then each of these sets is contained in $\overline{B_{(1+2\alpha_2)r}(x)}$. Hence, the latter set contains N disjoint balls of radius $\alpha_1 r$. This is only possible if $N\cdot(\alpha_1 r)^d \leq (1+2\alpha_2)^d r^d$. □

Proof of Theorem 3.3.5. The upper bound $\operatorname{Dim}_B(\mathcal{A}_\mathcal{S}) \leq s$ follows immediately from Theorem 3.3.1. In order to provide a lower bound, we proceed in a similar way as in the proof of Theorem 3.3.3.

We denote by ν the Bernoulli measure on Σ_m^+ with probabilities $c_1^s,\dots,c_m^s$ and let $\mu = \xi_*\nu$. In order to apply the Mass Distribution Principle, suppose that $V \subseteq \mathcal{A}_\mathcal{S}$ is a set of diameter $\delta > 0$. As $\bigcup_{k=1}^m S_k(\overline{U}) \subseteq \overline{U}$, the decreasing sequence of image sets $(\mathcal{S}^n(\bar{)})_{n\in\mathbb{N}}$

converges to $\mathcal{A}_\mathcal{S}$. Given a finite word w over the alphabet $\{1,\ldots,m\}$, we write $\overline{U}_w = S_{w_1,\ldots,w_{|w|}}(\overline{U})$. Using the same notation as in the proof of Theorem 3.3.1, we obtain that

$$\mathcal{A}_\mathcal{S} \;\subseteq\; \bigcup_{w\in\mathcal{Q}} \overline{U}_w \;.$$

Choose $\alpha_1 > 0$ such that U contains a ball of diameter $\alpha_1 > 0$ and is contained in a ball of diameter $\alpha_2 > 0$. Due to the definition of $k(a)$ in (3.3.2) and of $\mathcal{Q}$, the scaling constant of the similarity S_w lies between $c_0\delta$ and δ for any $w \in \mathcal{Q}$. Hence, any of the sets U_w with $w \in \mathcal{Q}$ contains a ball of diameter $c_0\alpha_1\delta$ and is contained in a ball of diameter $\alpha_2\delta$. By Lemma 3.3.6, this implies that V intersects at most $(1+2\alpha_2)^d\cdot(c_0\alpha_1)^{-d}$ of the sets U_w with $w\in\mathcal{Q}$. Let $\mathcal{Q}_V = \{w\in\mathcal{Q} \mid \overline{U}_w \cap V \neq \emptyset\}$ and note that

$$\nu([w]) \;=\; c_{w_1}^s \cdot \ldots \cdot c_{w_{|w|}}^s \;\leq\; \delta^s$$

for all $w\in\mathcal{Q}$. Hence, we obtain

$$\mu(V) \;=\; \nu(\xi^{-1}(V)) \;\leq\; \nu\left(\bigcup_{w\in\mathcal{Q}_V}[w]\right) \;\leq\; (1+2\alpha_2)^d\cdot(c_0\alpha_1)^{-d}\cdot\delta^s \;.$$

The Mass Distribution Principle now yields $\mathrm{Dim}_H(\mathcal{A}_\mathcal{S}) \geq s$. □

Example 3.3.7. As an example, we consider the IFS $\mathcal{S} = \{S_1,\ldots,S_4\}$ on $\mathbb{R}^2$ given by

$$S_1(x,y) = (x,y)/3 \quad , \quad S_2(x,y) = R_{\pi/3}\circ S_1(x,y) + (1/3,0)$$
$$S_3(x,y) = R_{-\pi/3}\circ S_1(x,y) + (1/2, 1/(2\sqrt{3})) \quad , \quad S_4(x,y) = S_1(x,y) + (2/3,0) \;.$$

The attractor of this IFS can be seen to be the Koch curve. Moreover, the open set condition holds, with U the open triangle spanned by the points $(0,0)$, $(1,0)$ and $(1/2,\sqrt{3}/2)$. Applying Theorem 3.3.5 yields $\mathrm{Dim}_H(\mathcal{A}_\mathcal{S}) = \log 4/\log 3$.

3.4 Approximation of IFS attractors by random trajectories

The following statements provides a procedure that often allows to efficiently approximate attractors of IFS numerically.

Theorem 3.4.1. *Suppose $\mathcal{S} = \{S_1,\ldots,S_m\}$ is an IFS on $D\subseteq X$ and ν is a Bernoulli measure on Σ_m^+ with probabilities $p_1,\ldots,p_m > 0$. Then for ν-a.e. $a\in\Sigma_m^+$ and any $x_0\in\mathcal{A}_\mathcal{S}$, we have*

$$\mathcal{A}_\mathcal{S} \;=\; \lim\nolimits^{\mathcal{H}}_{n\to\infty}\{S_{a_k\ldots a_1}(x_0) \mid 1\leq k\leq n\} \;.$$

Remark 3.4.2. (a) Note that compared to the statements about the symbolic coding of points in the IFS attractor, the maps S_{a_j} are now composed in reverse order.

(b) In practice, one may choose x_0 to be a fixed point of one of the maps S_k in order to ensure $x_0\in\mathcal{A}_\mathcal{S}$.

Proof of Theorem 3.4.1. If we think of ν as a probability measure, then the probability that a given word $w = w_1,\ldots,w_q$ with $q\in\mathbb{N}$ and $w_j\in\{1,\ldots,m\}$ for $j=1,\ldots,q$ appears as the l-th block of length p (that is, $a_{lp+j} = w_j$ for $j=1,\ldots,q$) equals $p_{w_1}\cdot\ldots\cdot p_{w_q}$. In particular, this probability is strictly positive. Hence, due to the independence of the different blocks, the word w appears infinitely often in a for ν-a.e. sequence $a\in\Sigma_m^+$. Moreover, since the set of all finite words over the alphabet $\{1,\ldots,m\}$ is countable, every word appears infinitely often with probability one.

We may therefore assume that a is a sequence that contains every finite word infinitely often. In this case, fix $x\in\mathcal{A}_\mathcal{S}$ and $\varepsilon > 0$. Choose $n\in\mathbb{N}$ such that $c^n < \varepsilon/\mathrm{diam}(\mathcal{A}_\mathcal{S})$, where $c = \max_{k=1}^m c_k$. Further, suppose that $x = \xi(b)$ for $b\in\Sigma_m^+$. Then $x\in S_{b_1,\ldots,b_n}(\mathcal{A}_\mathcal{S})$.

However, if $w = b_n, \ldots, b_1$ appears in the sequence a after position l, then we have that $S_{a_{l+n}\ldots a_1}(x_0) = S_{b_1,\ldots,b_n}(S_{a_l\ldots a_1}(x_0)) \in S_{b_1,\ldots,b_n}(\mathcal{A}_\mathcal{S})$ as well. Since $\mathrm{diam}(S_{b_1,\ldots,b_n}(\mathcal{A}_\mathcal{S})) \leq \mathrm{diam}(\mathcal{A}_\mathcal{S}) \cdot c^n < \varepsilon$ and $\varepsilon > 0$ was arbitrary, we obtain that

$$x \;\in\; \lim\nolimits^{\mathcal{H}}_{n\to\infty}\{S_{a_k\ldots a_1}(x_0) \mid 1 \leq k \leq n\}$$

As $x \in \mathcal{A}_\mathcal{S}$ was arbitrary as well, this yields

$$\mathcal{A}_\mathcal{S} \;\subseteq\; \lim\nolimits^{\mathcal{H}}_{n\to\infty}\{S_{a_k\ldots a_1}(x_0) \mid 1 \leq k \leq n\}\ .$$

Since the converse inclusion is obvious from $S_{a_k\ldots a_1}(x_0) \in \mathcal{A}_\mathcal{S}$ for all $k \in \mathbb{N}$, this completes the proof. $\square$

3.5 Examples

All examples in this section are given by affine IFS in the plane $\mathbb{R}^2$, that is, the mappings S_k are of the form

$$S_k : \mathbb{R}^2 \to \mathbb{R}^2 \quad , \quad S_k\begin{pmatrix} x \\ y \end{pmatrix} \;=\; R(\alpha_k)\cdot A_k \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_k$$

where

$$R_{\alpha_k} \;=\; \begin{pmatrix} \cos(2\pi\alpha_k) & -\sin(2\pi\alpha_k) \\ \sin(2\pi\alpha_k) & \cos(2\pi\alpha_k) \end{pmatrix}$$

is the rotation matrix with angle α_k,

$$A_k \;=\; \begin{pmatrix} \lambda_{k,1} & 0 \\ 0 & \lambda_{k,2} \end{pmatrix}$$

is a diagonal matrix with eigenfactors $\lambda_{k,1}, \lambda_{k,2} \in [0,1)$ and $v \in \mathbb{R}^2$ is a translation vector.

Example 3.5.1. We first take a look at IFS attractors that can be produced by just two maps. More precisely, we set

$$A_1 \;=\; A_2 \;=\; \begin{pmatrix} 0.95 & 0 \\ 0 & 0.55 \end{pmatrix}$$

and $\alpha_1 = 1/4$, $\alpha_2 = -1/4$. Then the left picture in Figure 3.5.1 shows the corresponding IFS attractor for $v_1 = (0, -0.3)$ and $v_2 = (0.3, 0)$, whereas that on the right is obtained for $v_1 = (-0.3, 0)$ and $v_2 = (0.3, 0)$.

The contraction constant of both maps is 0.95. Hence, Theorem 3.3.1 yields an upper bound of $\log(1/2)/\log(0.95)$ for the box and Hausdorff dimension. However, the maps are no similarities and there is a stronger contraction in the y-direction. Moreover, the images of the attractor under S_1 and S_2 seem to have considerable overlap. For these reasons, we expect that the precise value of the box dimension is smaller than this upper bound, but we lack the tools to improve this estimate. Moreover, due to the apparent overlap of the attractor images, neither disjointness nor the open set condition can be used to obtain a lower bound for the Hausdorff dimension. In general, there exist no results that would allow a precise computation of the attrators of such *non-conformal* IFS with overlaps.

Example 3.5.2 (Barnsley Fern)**.** A famous example is the so-called *Barnsley Fern*. For this IFS attractor, the IFS consists of four mappings $S_1, \ldots, S_4$, where the products $B_k = R_{\alpha_k} \cdot A_k$ are given by

$$B_1 \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 0.16 \end{pmatrix} \quad , \quad B_2 \;=\; \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix}$$
$$B_3 \;=\; \begin{pmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{pmatrix} \quad , \quad B_4 \;=\; \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix}\ .$$

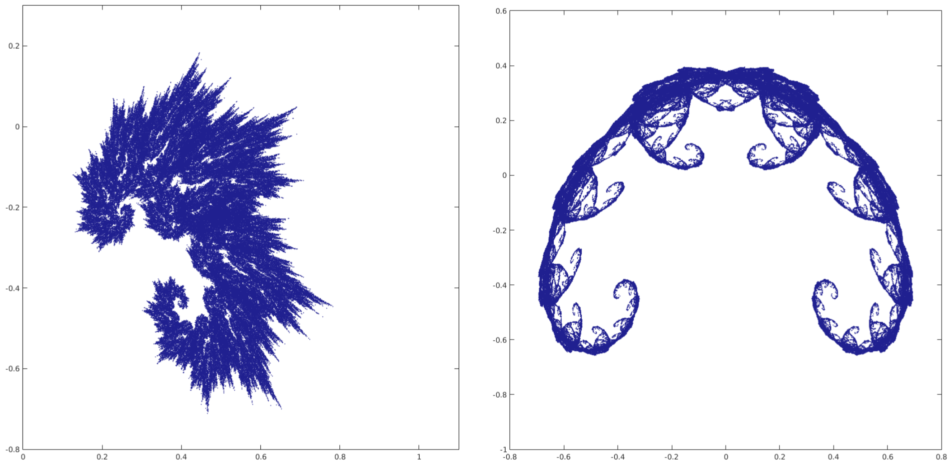

Figure 3.5.1: Attractors of two different iterated function pairs.

The translation vectors are

$$v_1 = \begin{pmatrix} 0 \\ 1.6 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad v_3 = \begin{pmatrix} 0 \\ 1.6 \end{pmatrix} \quad , \quad v_4 = \begin{pmatrix} 0 \\ 0.44 \end{pmatrix} .$$

Roughly spoken, S_1 is responsible for the main part of the fern, S_2 for the stem, S_3 for the

Figure 3.5.2: The Barnsley Fern

first left branch and S_4 for the first branch on the right (which starts at the point $(0, 1.6)$).

Since the contraction rates of the resulting affine maps are quite different (note in particular that S_2 is non-invertible), it is most efficient to plot (an approximation of the) attractor by using random iterations, with a suitably chosen Bernoulli measure. For the picture below, 10^6 random iterates were produced with probabilities $p_1 = 0.84$, $p_2 = 0.02$ and $p_3 = p_4 = 0.07$.

Chapter 4

Further notions of dimension and related aspects

4.1 Modified box dimension

In order to refine the notion of box dimension, one may introduce the concepts of **modified (upper and lower) box dimensions**. Given a subset A of some metrix space X, we denote by $\mathcal{G}(A)$ the family of all countable covers $(G_n)_{n\in\mathbb{N}}$ of A by relatively compact sets G_n. Then, we let

$$\begin{aligned} \overline{\mathrm{Dim}}_{\mathrm{MB}}(A) &= \inf\{\sup_{n\in\mathbb{N}} \overline{\mathrm{Dim}}_{\mathrm{B}}(G_n) \mid (G_n)_{n\in\mathbb{N}} \in \mathcal{G}(A)\} \\ \underline{\mathrm{Dim}}_{\mathrm{MB}}(A) &= \inf\{\sup_{n\in\mathbb{N}} \underline{\mathrm{Dim}}_{\mathrm{B}}(G_n) \mid (G_n)_{n\in\mathbb{N}} \in \mathcal{G}(A)\} \end{aligned}$$

We have that

$$\mathrm{Dim}_H(A) \;\leq\; \underline{\mathrm{Dim}}_{\mathrm{MB}}(A) \;\leq\; \overline{\mathrm{Dim}}_{\mathrm{MB}}(A) \;\leq\; \overline{\mathrm{Dim}}_{\mathrm{B}}(A)\ ,$$

where the first inequality uses the countable stability of Haudorff dimension. Moreover, it is a direct consequence of the definition that countable sets have modified box dimension zero.

In general, modified box dimensions are difficult to compute explicitly, due to the additional step in the definition. However, there is a simple criterion that ensures equality of modifed box dimension and box dimension.

Proposition 4.1.1. *Let X be a complete metric space and suppose that $A \subseteq X$ is a compact set with the property that $\overline{\mathrm{Dim}}_{\mathrm{B}}(A) = \overline{\mathrm{Dim}}_{\mathrm{B}}(A\cap U)$ for all open sets $U \subseteq X$ that intersect A. The same statement holds for the lower (modified) box dimension.*

Proof. Let $\varepsilon > 0$ and suppose $(G_n)_{n\in\mathbb{N}}$ is a cover of A. Without loss of generality, we may assume that all the G_n are closed. Hence, the sets $G_n \cap A$ form a collection of closed subsets of the complete metric space A (equipped with the metric on X) which cover all of A. Baire's Theorem states that there exists some $n \in \mathbb{N}$ such that G_n contains an open subset, which means that $A \cap U \subseteq G_n$ for some open set $U \subseteq X$. By assumption, this implies that

$$\overline{\mathrm{Dim}}_{\mathrm{B}}(G_n) \;\geq\; \overline{\mathrm{Dim}}_{B}(A\cap U) \;\geq =\; \overline{\mathrm{Dim}}_{\mathrm{B}}(A)\ .$$

Hence, we obtain $\overline{\mathrm{Dim}}_{\mathrm{MB}}(A) \geq \overline{\mathrm{Dim}}_{B}(A)$, and the converse equality is obvious. The same argument also works for the case of lower (modified) box dimension. □

Exercise 34. Show that every generalised Sierpinski Carpet has the homogeneity property stated in Proposition 4.1.1.

4.2 Packing measures and packing dimension

Box dimension can be defined either via box coverings or via separated sets. The definition of Hausdorff dimension is then based on a refined version of the coverings. It seems therefore natural to consider a possible refinement of separated sets as well, and this leads to the notions of packing measures and packing dimension.

Thereby, we follow the same pattern as for Hausdorff measures and dimension. Given a metric space X, a subset $A \subseteq X$ and $s \geq 0,\ \delta > 0$, we let

$$\mathcal{P}^s_\delta(A) \ = \ \sup\left\{ \sum_{B\in\mathcal{B}} \mathrm{diam}(U)^s \ \middle|\ \mathcal{B} \text{ is a collection of disjoint } \delta\text{-balls with centres in } A \right\} .$$

Further, as $\mathcal{P}^s_\delta(A)$ is decreasing in δ, we can define the limit

$$\mathcal{P}^s_0(A) \ = \ \lim_{\delta\to 0} \mathcal{P}^s_\delta(A) .$$

However, $\mathcal{P}^s_0$ is not yet a measure. This can be seen, for instance, by considering countable sets.

Exercise 35. Let $A = [0,1]\cap\mathbb{Q}$ and $s \in [0,1]$. Show that $\mathcal{P}^s_0(A) > 0$.

The underlying reason is that the value of $\mathcal{P}^s_0$ of a set A coincides with that for the closure $\overline{A}$.

Exercise 36. Show that $\mathcal{P}^s_0(\overline{A}) = \mathcal{P}^s_0(A)$.

For this reason, an additional step is needed, and we define the **s-dimensional packing measure** $\mathcal{P}^s$ by

$$\mathcal{P}^s(A) \ = \ \inf\left\{ \sum_{U\in\mathcal{U}} \mathcal{P}^s_0(U) \ \middle|\ \mathcal{U} \text{ is a countable cover of } A \right\} .$$

As in the case of Hausdorff measures, one can show that $\mathcal{P}^s$ is a metric outer measure, and therefore its restriction to the Borel σ-algebra is a measure.

Exercise 37. Show that $\mathcal{P}^s$ is a metric outer measure on X.

Exercise 38. Show that for each $A \subseteq \mathbb{R}^d$ there exists a unique $s \geq 0$ such that $\mathcal{P}^t(A) = \infty$ for all $t < s$ and $\mathcal{P}^t(A) = 0$ for all $t > s$.

The definition of **packing dimension** is then again analogous to that of Hausdorff dimension. We let

$$\mathrm{Dim}_{\mathrm{P}}(A) \ = \ \sup\{s \geq 0 \mid \mathcal{P}^s(A) = \infty\} \cup \{0\} \ = \ \inf\{s \geq 0 \mid \mathcal{P}^s(A) = 0\} .$$

It follows immediately from the measure structure that packing dimension is monotone ($\mathrm{Dim}_{\mathrm{P}}(A) \leq \mathrm{Dim}_{\mathrm{P}}(B)$ if $A \subseteq B$) and countably stable. In particular, countable sets have packing dimension zero.

Exercise 39. (a) Show that if $U \subseteq \mathbb{R}^d$ is open and $0 \leq s < d$, then $\mathcal{P}^s_0(U) = \infty$.

(b) Show that $\mathrm{Dim}_{\mathrm{P}}(\mathbb{R}^d) = d$. (*Hint: Use Baire's Theorem to conclude that if $\mathcal{U}$ is a countable cover of $\mathbb{R}^d$ by closed sets, then one of the sets in $\mathcal{U}$ must have non-empty interior.*)

It is important to note that, unlike for the case of box dimension, the two alternative approaches lead to substantially different results. In order to see this, recall that a subset $R \subseteq X$ of some metric space is **residual** if it is a countable intersection $R = \bigcap_{n\in\mathbb{N}} U_n$ of dense open sets U_n. *Baire's Theorem* states that a residual subset of a complete metric space is dense. In some sense, residual sets play the same role in topology as sets of full measure do in measure theory.

Exercise 40. Suppose that X is a complete metric space.

(a) Show that the intersection of two residual subsets is again residual. Show that the same is true for coubtable intersections.

(b) $M \subseteq X$ is called **meager** if $X \setminus M$ is residual. Show that if $(M_n)_{n\in\mathbb{N}}$ is a sequence of meager sets, then $\bigcup_{n\in\mathbb{N}} M_n \neq X$.

(c) Show that if $(A_n)_{n\in\mathbb{N}}$ is a countable family of closed subsets of X such that $\bigcup_{n\in\mathbb{N}} A_n = X$, then there exists $n \in \mathbb{N}$ such that A_n has non-empty interior.

Proposition 4.2.1. *Suppose that $R \subseteq \mathbb{R}^d$ is residual. Then*

$$\mathrm{Dim}_{\mathrm{P}}(R) \;=\; \mathrm{Dim}_{\mathrm{P}}(X)\ .$$

Proof. Suppose for a contradiction that $\mathrm{Dim}_{\mathrm{P}}(R) < \mathrm{Dim}_{\mathrm{P}}(X) = d$. Then there exists some $s < \mathrm{Dim}_{\mathrm{P}}(X)$ such that $\mathcal{P}^s(R) < \infty$. Hence, we can choose a countable cover $\mathcal{U}$ of R such that $\sum_{U\in\mathcal{U}} \mathcal{P}_0^s(U) < \infty$. Since $\mathcal{P}_0^s$ is the same for a set and its closure, we may assume that the set in $\mathcal{U}$ are all closed. However, the countable union of meager sets cannot contain a residual set (which needs to intersect the residual complement of the union). For this reason, one of the sets $U \in \mathcal{U}$ must have non-empty interior. As this implies $\mathcal{P}_0^s = \infty$, we obtain the required contradiction. □

Exercise 41. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence that consists of all rational numbers in $[0,1]$. Let $\varepsilon_n = 2^{-n}$ and $U_n = \bigcup_{k\geq n} B_{\varepsilon_k}(x_k)$. Let $R = \bigcap_{n\in\mathbb{N}} U_n$.

Show that the Hausdorff dimension of R is zero, whereas the packing dimension equals one.

Thus, on the one hand, the previous exercise shows that packing and Hausdorff dimension are different for a large collection of sets. On the other hand, however, it surprisingly turns out that at least in Euklidean space, packing dimension coincides with the modified upper box dimension introduced above. We split the proof into the following two lemmas.

Lemma 4.2.2. *Suppose X is a metric space and $A \subseteq X$. Then* $\mathrm{Dim}_{\mathrm{P}}(A) \leq \overline{\mathrm{Dim}}_{\mathrm{B}}(A)$.

Proof. If $\mathrm{Dim}_{\mathrm{P}}(A) = 0$ the statement is trivial, so we assume $\mathrm{Dim}_{\mathrm{P}}(A) > 0$. Hence, we can choose $0 < t < s < \mathrm{Dim}_{\mathrm{P}}(A)$. Then $\mathcal{P}^s(A) = \infty$, and consequently $\mathcal{P}_0^s(A) = \infty$. This means that for any sufficiently small $\delta > 0$ there exist a collection of disjoint balls $(B_i)_{i\in\mathbb{N}}$ with radius $\leq \delta$ and centres in A such $\sum_{i\in\mathbb{N}} \mathrm{diam}(B_i)^s > 1$.

Suppose that the number of B_i with radius in $(2^{-(k+1)}, 2^{-k}]$ is n_k. Then

$$\sum_{k\in\mathbb{N}} n_k 2^{-ks} \;>\; 1\ , \tag{4.2.1}$$

and therefore there must be some $k \in \mathbb{N}$ such that $n_k \geq 2^{kt} \cdot (1 - 2^{t-s})$. (Note that otherwise the geometric sum in (4.2.1) is smaller than one.)

We thus obtain that

$$M_{2^{-(k+1)}}(A) \;\geq\; 2^{kt} \cdot (1 - 2^{t-s})\ .$$

Since there must be infinitely many k with this property (note that $\delta > 0$ above can be chosen arbitrarily small), this implies that

$$\overline{\mathrm{Dim}}_{\mathrm{B}}(A) \;\geq\; \lim_{k\to\infty} \frac{\log 2^{kt}}{-\log 2^{-(k+1)}} \;=\; t\ .$$

As the argument works with any $t < \mathrm{Dim}_{\mathrm{P}}(A)$, this proves the statement. □

Lemma 4.2.3. *Suppose that X is a metric space and $A \subseteq X$. Then* $\mathrm{Dim}_{\mathrm{P}}(A) = \overline{\mathrm{Dim}}_{\mathrm{MB}}(A)$.

Proof. If $(G_n)_{n\in\mathbb{N}}$ is a countable cover of A by relatively compact sets, then the countable stability of packing dimension implies that

$$\mathrm{Dim}_{\mathrm{P}}(A) \;\leq\; \sup_{n\in\mathbb{N}} \mathrm{Dim}_{\mathrm{P}}(G_n) \;\leq\; \sup_{n\in\mathbb{N}} \overline{\mathrm{Dim}}_{\mathrm{B}}(G_n)\ .$$

By definition, this further yields $\mathrm{Dim}_\mathrm{P}(A) \leq \overline{\mathrm{Dim}}_\mathrm{MB}(A)$.

Conversely, if $s > \mathrm{Dim}_\mathrm{P}(A)$, then $\mathcal{P}^s(A) = 0$ and hence $A \subseteq \bigcup_{n\in\mathbb{N}} G_n$ for a collection $(G_n)_{n\in\mathbb{N}}$ of sets with $\mathcal{P}_0^s(G_n) < \infty$ for all $n \in \mathbb{N}$. This implies that $M_\delta(G_n) \cdot \delta^s \leq \mathcal{P}_\delta^s(G_n)$ remains bounded as $\delta \to 0$, and hence $\overline{\mathrm{Dim}}_\mathrm{B}(G_n) \leq s$. Hence, we obtain that

$$\overline{\mathrm{Dim}}_\mathrm{MB}(A) \ \leq \ \sup_{n\in\mathbb{N}} \overline{\mathrm{Dim}}_\mathrm{B}(G_n) \ \leq \ s \ .$$

This shows $\overline{\mathrm{Dim}}_\mathrm{MB}(A) \leq \mathrm{Dim}_\mathrm{P}(A)$ and thus completes the proof. □

4.3 Local dimensions and densities of sets and measures

For $\delta > 0$ we have that $\mathrm{Leb}_{\mathbb{R}^d}(B_\delta(x)) = C_d \cdot \delta^d$, where $C_d = \mathrm{Leb}_{\mathbb{R}^d}(B_1(0))$. The dimension d appears in this expression as the exponent of δ. Similarly, given an arbitrary Borel measure μ on some metric space X, we define the **upper and lower local dimension of** μ **in** $x \in X$ as

$$\overline{\mathrm{Dim}}_\mu^{\mathrm{loc}}(x) = \varlimsup_{\delta\to 0} \frac{\log \mu(B_\delta(x))}{\log \delta} \quad \text{and} \quad \underline{\mathrm{Dim}}_\mu^{\mathrm{loc}}(x) = \varliminf_{\delta\to 0} \frac{\log \mu(B_\delta(x))}{\log \delta} \ .$$

If both quantities coincide, we denote the common value by $\mathrm{Dim}_\mu^{\mathrm{loc}}(x)$ and call it the **local dimension of** μ **in** $x \in X$.

Exercise 42. Suppose that $\gamma : [0,1] \to \mathbb{R}^d$ is differentiable and $\mu = \gamma_* \mathrm{Leb}_{[0,1]}$. Show that in this case $\underline{\mathrm{Dim}}_\mu^{\mathrm{loc}}(x) = 1$ for all $x \in \gamma([0,1])$ and $\mathrm{Dim}_\mu(x) = \infty$ otherwise.

Given $s \geq 0$, we further let

$$\overline{D}^s(\mu, x) = \varlimsup_{\delta\to 0} \frac{\mu(B_\delta(x))}{(2\delta)^s} \quad \text{and} \quad \underline{D}^s(\mu, x) = \varliminf_{\delta\to 0} \frac{\mu(B_\delta(x))}{(2\delta)^s} \ .$$

Again, if both values coincide we simply write $D^s(\mu, x)$, which can then be interpreted as density of μ with respect to the s-dimensional Hausdorff measure $\mathcal{H}^s$.

Exercise 43. Show that if $\underline{\mathrm{Dim}}_\mu^{\mathrm{loc}}(x) > s$, then $D^s(\mu, x) = 0$ and if $\overline{\mathrm{Dim}}_\mu^{\mathrm{loc}}(x) < s$, then $D^s(\mu, x) = \infty$.

A classical result from analysis is the

Theorem 4.3.1 (Lebesgue Density Theorem). *If $f \in \mathcal{L}^1(\mathbb{R}^d)$, $f \geq 0$ and $\mu = f \cdot \mathrm{Leb}_{\mathbb{R}^d}$, then $D^d(\mu, x) = f(x)$ μ-almost surely.*

In particular, this implies that $\mathrm{Dim}_\mu^{\mathrm{loc}}(x) = d$ for $\mathrm{Leb}_{\mathbb{R}^d}$-almost all $x \in \mathbb{R}^d$ with $f(x) \neq 0$. For zeros of f, the local dimension of μ depends on the scaling properties of f close to x.

Exercise 44. Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = |x|^q$ with $q > 0$. Show that $\mathrm{Dim}_\mu^{\mathrm{loc}}(0) = q + 1$.

It turns out that the quantities $\overline{D}^s(\mu, x)$ provide a further tool to estimate the Hausdorff measure of a set.

Theorem 4.3.2. *Suppose that μ is a Borel measure on $\mathbb{R}^d$, $A \subseteq \mathcal{B}(\mathbb{R}^d)$ and $0 < c < \infty$.*

(a) If $\overline{D}^s(\mu, x) < c$ for all $x \in A$, then $\mathcal{H}^s(A) \geq \frac{\mu(A)}{c\cdot 2^s}$.

(b) If $\overline{D}^s(\mu, x) > c$ for all $x \in A$, then $\mathcal{H}^s(A) \leq \frac{4^s\cdot\mu(A)}{c}$.

Proof. (a) Let $A_\delta = \{x \in A \mid \mu(B_r(x)) < c\cdot(2r)^s \text{ for all } 0 < r \leq \delta\}$. Then $\bigcup_{\delta>0} A_\delta = A$ and consequently $\lim_{\delta\to 0} \mu(A_\delta) = \mu(A)$. We claim that $\mu(A_\delta) \leq c \cdot \mathcal{H}^s(A)$ for all $\delta > 0$. In order to see this, suppose $\mathcal{U}$ is a δ-cover of A. If $U \in \mathcal{U}$ intersects A_δ and $x \in U \cap A_\delta$, then $\mu(U) \leq \mu(B_{\mathrm{diam}(U)}(x)) \leq c \cdot (2\mathrm{diam}(U))^s$. This implies that

$$\mu(A_\delta) \ \leq \ \sum_{\substack{U\in\mathcal{U} \\ U\cap A_\delta\neq\emptyset}} \ \leq \ \sum_{U\in\mathcal{U}} c \cdot \mathrm{diam}(U)^s \cdot 2^s \ .$$

As $\mathcal{U}$ was an arbitrary δ-cover of A, taking the over all such covers yields

$$\mu(A_\delta) \;\leq\; 2^s c \cdot \mathcal{H}^s_\delta(A) \;\leq\; 2^s c \cdot \mathcal{H}^s(A)\,.$$

This proves part (a).

(b) First, note that since $\mathcal{H}^s(A) = \lim_{R\to\infty} \mathcal{H}^s(A \cap B_R(0))$, it suffices to consider the case that A is bounded. As Borel measures are regular, given $\varepsilon > 0$ we may choose some open set $V \supseteq A$ such that $\mu(V) < \mu(A) + \varepsilon$. Let

$$\mathcal{C} \;=\; \{B_r(x) \mid x \in A,\ 0 < r \leq \delta,\ B_r(x) \subseteq V \text{ and } \mu(B_r(x)) > c \cdot (2r)^s\}\,.$$

Then by assumption we have that $A \subseteq \bigcup_{B\in\mathcal{C}} B$. Due to Lemma 4.3.3 below, we can choose a countable family $(B_n)_{n\in\mathbb{N}}$ of pairwise disjoint balls in $\mathcal{C}$ such that if $\tilde{B}_n$ denotes the ball concentric with B_n, but four times larger in diameter, then $A \subseteq \bigcup_{n\in\mathbb{N}} \tilde{B}_n$. Hence, $(\tilde{B}_n)_{n\in\mathbb{N}}$ is a countable 8δ-cover of A, and we obtain

$$\begin{aligned}\mathcal{H}^s_{8\delta}(A) \;&\leq\; \sum_{n\in\mathbb{N}} \operatorname{diam}(\tilde{B}_n)^s \;\leq\; 4^s \sum_{n\in\mathbb{N}} \operatorname{diam}(B_n)^s \\ &\leq\; 4^s c^{-1} \sum_{n\in\mathbb{N}} \mu(B_n) \;\leq\; 4^s c^{-1}\mu(V) \;\leq 4^s c^{-1}(\mu(A)+\varepsilon)\,.\end{aligned}$$

Taking the limit $\varepsilon \to 0$ yields the statement. □

Lemma 4.3.3. *Suppose that $\mathcal{C}$ is a family of balls in $\mathbb{R}^d$ such that $C = \bigcup_{B\in\mathcal{C}} B$ is bounded. Then there exists a countable family $(B_n)_{n\in\mathbb{N}}$ of pairwise disjoint balls $B_n = B_{\delta_n}(x_n) \in \mathcal{C}\cup\{\emptyset\}$[1] such that we have*

$$C \;\subseteq\; \bigcup_{n\in\mathbb{N}} B_{4\delta_n}(x_n)\,.$$

Proof. We recursively define the sequence $(B_n)_{n\in\mathbb{N}}$ as follows. Given $\kappa \in (0,1)$, we let $r_1 = \kappa \cdot \sup_{B\in\mathcal{C}} \operatorname{diam}(B)$ and choose some $B_1 \in \mathcal{C}$ such that $\operatorname{diam}(B_1) > r_1$. Then, once $B_1,\ldots,B_n$ have been chosen, let $r_{n+1} = \kappa\cdot\sup\{\operatorname{diam}(B) \mid B \in \mathcal{C} \text{ is disjoint from } B_1,\ldots,B_n\}$ and choose $B_{n+1} \in \mathcal{C}$ such that $\operatorname{diam}(B_{n+1}) > r_{n+1}$ and B_{n+1} is disjoint from $B_1,\ldots,B_n$. If there is no such ball in $\mathcal{C}$, we set $B_k = \emptyset$ for all $k > n$.

Now, let $\tilde{B}_n$ the balls concentric to the B_n with $1 + 2/\kappa$ times the same radius. Suppose for a contradiction that $C \not\subseteq \bigcup_{n\in\mathbb{N}} \tilde{B}_n$. Choose some $B \in \mathcal{C}$ and $x \in B$ with $x \notin \bigcup_{n\in\mathbb{N}} \tilde{B}_n$. As C is bounded and the B_n are pairwise disjoint, we must have $\lim_{n\to\infty} \operatorname{diam}(B_n) = 0$. Let $m = \min\{n \in \mathbb{N} \mid \operatorname{diam}(B_n) < \kappa \cdot \operatorname{diam}(B)\}$. Then B needs to intersect one of the balls $B_1,\ldots,B_{m-1}$, since otherwise B_m could not have been selected in the m-th place (as r_m would be at least $\kappa \cdot \operatorname{diam}(B) > \operatorname{diam}(B_n)$ and therefore B would have had to be chosen first in this case). However, if $B \cap B_n \neq \emptyset$ for some $n < m$, then $\operatorname{diam}(B) \leq \operatorname{diam}(B_n)/\kappa$ implies that $B \subseteq \tilde{B}_n$. This yields the desired contradiction and shows that $C \subseteq \bigcup_{n\in\mathbb{N}} \tilde{B}_n$. Choosing $\kappa = 2/3$ completes the proof. □

Given $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}$, let

$$\underline{D}^s(A,x) = \varliminf_{\delta\to 0} \frac{\mathcal{H}^s(A\cap B_\delta(x)}{(2\delta)^s} \quad\text{and}\quad \overline{D}^s(A,x) = \varlimsup_{\delta\to 0} \frac{\mathcal{H}^s(A\cap B_\delta(x)}{(2\delta)^s}\,.$$

Note that thus $\underline{D}^s(A,x) = \underline{D}^s(\mathcal{H}^s_{|A}, x)$ and $\overline{D}^s(A,x) = \overline{D}^s(\mathcal{H}^s_{|A}, x)$. Moreover, we have that $\underline{D}^s(A \uplus B, x) = \underline{D}^s(A,x) + \underline{D}^s(B,x)$, and the same holds for $\overline{D}^s$.

Theorem 4.3.4. *Let $B \in \mathcal{B}(\mathbb{R}^d)$ and $0 < \mathcal{H}^s(B) < \infty$. Then*

(a) $\underline{D}^s(B,x) = \overline{D}^s(B,x) = 0$ for $\mathcal{H}^s$-a.e. $x \in \mathbb{R}^d \setminus B$.

(b) $2^{-s} \leq \overline{D}^s(B,x) \leq 4^s$ for $\mathcal{H}^s$-a.e. $x \in B$.

[1] We use the convention that $B_0(x) = \emptyset$.

Proof. (a) We omit the proof, which is similar to that of the Lebesgue density theorem.

(b) Let $\mu = \mathcal{H}^s_{|B}$, that is, $\mu(S) = \mathcal{H}^s(B \cap S)$. Further, fix $c > 0$ and let

$$B_1 \;=\; \{x \in B \mid \overline{D}^s(B,x) < 2^{-s}c\} \;.$$

Then Theorem 4.3.2(a) with $\mu = \mathcal{H}^s_{|B}$ and $A = B_1$ implies $\mathcal{H}^s(B_1) \geq \mu(B_1)/c = \mathcal{H}^s(B_1)/c$. If $c < 1$, this is only possible when $\mathcal{H}^s(B_1) = 0$. Hence, we obtain $\overline{D}^s(B,x) \geq 2^{-s}$ for $\mathcal{H}^s$-a.e. $x \in B$.

Now, let

$$B_2 \;=\; \{x \in B \mid \overline{D}^s(B,x) > 4^s c\} \;.$$

Then Theorem 4.3.2(b), with $\mu = \mathcal{H}^s_{|B}$ as before and $A = B_2$, implies that $\mathcal{H}^s(B_2) \leq \mu(B_2)/c = \mathcal{H}^s(B_2)/c$. If $c > 1$, this is only possible if $\mathcal{H}^s(B_2) = 0$. Therefore

$$\overline{D}^s(B,x) \leq 4^s$$

for $\mathcal{H}^s$-a.e. $x \in B$. □

4.4 Product formulas

Before we turn to product formulas for dimensions, we first want to consider a counterexample to a straightforward product formula for the Hausdorff dimension. At the same time, this provide an example for a set with lower box dimension strictly smaller than the upper box dimension.

Example 4.4.1. Let $(m_k)_{k\in\mathbb{N}}$ be an increasing sequence of integers such that $\lim_{k\to\infty} \frac{km_k}{m_{k+1}} = 0$. Given $x \in [0,1)$, let $a_n(x)\{0,1\}$ be the n-th symbol of its binary expansion, that is, $x = \sum_{n\in\mathbb{N}} a_n(x)2^{-n}$. Let

$$\begin{aligned} A &= \{x \in [0,1) \mid a_n(x) = 0 \text{ for all } n \in [m_k+1, m_{k+1}] \text{ with } k \text{ even}\} \;, \\ B &= \{x \in [0,1) \mid a_n(x) = 0 \text{ for all } n \in [m_k+1, m_{k+1}] \text{ with } k \text{ odd}\} \;. \end{aligned}$$

Further, when k is even, we let $j_k = (m_2 - m_1) + (m_4 - m_3) + \ldots + (m_k - m_{k-1})$. Then A is covered by 2^{j_k} intervals of length $2^{-m_{k+1}}$, since there are exactly j_k *'free'* positions in the binary expansion of numbers in A up to position m_{k+1}. As $j_k \leq km_k$, we obtain

$$\mathrm{Dim}_{\mathrm{H}}(A) \;\leq\; \underline{\mathrm{Dim}}_{\mathrm{B}}(A) \;\leq\; \lim_{k\to\infty} \frac{\log 2^{j_k}}{-\log 2^{-m_{k+1}}} \;=\; \lim_{k\to\infty} \frac{j_k}{m_{k+1}} \;=\; 0 \;.$$

In an analogue way, it can be shown that $\mathrm{Dim}_{\mathrm{H}}(B) = 0$. At the same time, we obviously have that

$$A + B \;=\; \{x + y \mid x \in A,\; y \in B\} \;=\; [0,1) \;.$$

Hence, if we let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\; (x,y) \mapsto x + y$, then we have $f(A \times B) = [0,1)$, an in particular $\mathrm{Dim}_{\mathrm{H}}(f(A \times B)) = 1$. Moreover, f is obviously Lipschitz continuous, so that it cannot increase the Hausdorff dimension. Therefore, we must have $\mathrm{Dim}_{\mathrm{H}}(A \times B) \geq 1$.

We also note that if k is odd, then $2^{m_{k+1}-m_k}$ intervals of length $2^{-m_{k+1}}$ are needed to cover A, which implies that $\overline{\mathrm{Dim}}_{\mathrm{B}}(A) = 1$ (using $\lim_{k\to\infty} m_k/m_{k+1} = 0$), and similarly $\overline{\mathrm{Dim}}_{\mathrm{B}}(B) = 1$.

In order to estimate the Hausdorff dimension of product sets from below, the following statement is crucial.

Lemma 4.4.2. *Given $A, B \in \mathcal{B}(\mathbb{R}^d)$ with $\mathcal{H}^s(A), \mathcal{H}^t(B) < \infty$, we have that*

$$\mathcal{H}^{s+t}(A \times B) \;\geq\; 8^{-(s+t)}\mathcal{H}^s(A)\mathcal{H}^t(B) \;.$$

Proof. Without loss of generality, suppose that $\mathcal{H}^s(A), \mathcal{H}^t(B) > 0$. Then $\overline{D}^s(A,x) \le 4^s$ for $\mathcal{H}^s$-a.e. $x \in A$, so that there exists a set $A_0 \subseteq A$ with $\mathcal{H}^s(A_0) = \mathcal{H}^s(A)$ and $\overline{D}^s(A,x) \le 4^s$ for all $x \in A_0$. Similarly, there exists a set $B_0 \subseteq B$ such that $\mathcal{H}^t(B_0) = \mathcal{H}^t(B)$ and $\overline{D}^s(B,y) \le 4^s$ for all $y \in B$. If $\mu = \mathcal{H}^s_{|A} \times \mathcal{H}^t_{|B}$, then $B_\delta(x,y) \subseteq B_\delta(x) \times B_\delta(y)$, we have

$$\mu(B_\delta(x,y)) \;\le\; \mu(B_\delta(x) \times B_\delta(y)) \;=\; \mathcal{H}^s(A \cap B_\delta(x)) \cdot \mathcal{H}^t(B \cap B_\delta(y)) \,.$$

Therefore, all $(x,y) \in A_0 \times B_0$ satisfy

$$\begin{aligned} \overline{D}^{s+t}(\mu,(x,y)) \;&=\; \varlimsup_{\delta\to 0} \frac{\mu(B_\delta(x,y))}{(2\delta)^{s+t}} \;\le\; \varlimsup_{\delta\to 0} \frac{\mathcal{H}^s_{|A}(B_\delta(x))}{(2\delta)^s} \cdot \frac{\mathcal{H}^t_{|B}(B_\delta(y))}{(2\delta)^t} \\ &\le\; \overline{D}^s(A,x) \cdot \overline{D}^t(B,y) \;\le\; 4^{s+t} \,. \end{aligned}$$

Theorem 4.3.2(a) now yields

$$\mathcal{H}^{s+t}(A \times B) \;\ge\; \mathcal{H}^{s+t}(A_0 \times B_0) \;\ge\; 2^{-s} \cdot \frac{\mu(A_0 \times B_0)}{4^{s+t}} \;=\; 8^{-(s+t)} \cdot \mathcal{H}^s(A) \cdot \mathcal{H}^s(B) \,.$$

□

Corollary 4.4.3. *Given $A, B \in \mathcal{B}(\mathbb{R}^d)$, we have*

$$\mathrm{Dim}_\mathrm{H}(A \times B) \;\ge\; \mathrm{Dim}_\mathrm{H}(a) + \mathrm{Dim}_\mathrm{H}(b) \,.$$

Proof. If $s < \mathrm{Dim}_\mathrm{H}(A)$ and $t < \mathrm{Dim}_\mathrm{H}(B)$, then $\mathcal{H}^s(A) = \mathcal{H}^t(B) = \infty$. Due to the regularity of Hausdorff measures, we can choose $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $0 < \mathcal{H}^s(A_0), \mathcal{H}^t(B) < \infty$. We obtain that

$$0 \;<\; 8^{-(s+t)} \cdot \mathcal{H}^s(A_0) \cdot \mathcal{H}^t(B_0) \;\le\; \mathcal{H}^{s+t}(A \times B) \,,$$

and consequently $\mathrm{Dim}_\mathrm{H}(A \times B) \ge s + t$. □

Theorem 4.4.4. *Given $A, B \subseteq \mathbb{R}^d$, we have*

$$\mathrm{Dim}_\mathrm{H}(A \times B) \;\le\; \mathrm{Dim}_\mathrm{H}(A) + \overline{\mathrm{Dim}}_\mathrm{B}(B) \,.$$

Proof. Let $s > \mathrm{Dim}_\mathrm{H}(A)$ and $t > \overline{\mathrm{Dim}}_\mathrm{B}(B)$. Choose $\delta_0 > 0$ with $N_\delta(B) < \delta^{-t}$ for all $\delta < \delta_0$ and a δ-cover $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ of A with $\sum_{n\in\mathbb{N}} \mathrm{diam}(U_n)^s < 1$. Further, for every $n \in \mathbb{N}$ we let $\delta_n = \mathrm{diam}(U_n)/2$ and choose a $2\delta_n$-cover $\mathcal{U}^n = (U^n_j)_{j\in\mathbb{N}}$ of B by B by $N_{\delta_n}(B)$ balls of radius δ_n. Then $U_n \times B$ is covered by the $N_{\delta_n}(B)$ product sets $U_n \times U^n_j$, $j = 1, \dots, N_{\delta_n}(B)$, each of which has radius $\le \sqrt{5} \cdot \delta_n$. Hence, we obtain that

$$A \times B \;\subseteq\; \bigcup_{n\in\mathbb{N}} \bigcup_{j=1}^{N_{\delta_n}(B)} U_n \times U^n_j$$

and

$$\begin{aligned} \mathcal{H}^{s+t}_{\sqrt{5}\delta}(A \times B) \;&\le\; \sum_{n\in\mathbb{N}} \sum_{j=1}^{N_{\delta_n}(B)} \mathrm{diam}(U_n \times U^n_j)^{s+t} \;\le\; \sum_{n\in\mathbb{N}} N_{\delta_n}(B) \cdot (\sqrt{5} \cdot \mathrm{diam}(U_n))^{s+t} \\ &\le\; \sum_{n\in\mathbb{N}} \delta_n^{-t} \cdot \delta_n^{s+t} \cdot 5^{(s+t)/2} \;\le\; 5^{(s+t)/2} \,. \end{aligned}$$

As $\delta \to 0$ we obtain that $\mathcal{H}^{s+t}(A \times B) \le 5^{(s+t)/2}$ and therefore $\mathrm{Dim}_\mathrm{H}(A \times B) \le s + t$. □

Corollary 4.4.5. *If $A, B \in \mathcal{B}(\mathbb{R}^d)$ and $\mathrm{Dim}_\mathrm{H}(B) = \mathrm{Dim}_\mathrm{B}(B)$, then*

$$\mathrm{Dim}_\mathrm{H}(A \times B) \;=\; \mathrm{Dim}_\mathrm{H}(A) + \mathrm{Dim}_\mathrm{H}(B) \,.$$

Chapter 5

Fractal geometry and dynamical systems

If X is a metric (or topological space) and $T : X \to X$ a continuous mapping, we call the pair (X,T) a *(topological) dynamical system with discrete time*. Dynamical systems theory focuses on the long-term behaviour of *orbits* $x, T(x), T^2(x), \ldots$ that are generated by the iterations of T. Thereby, different dynamical systems show a broad scope of possible dynamics, ranging from very ordered and predictable behaviour, as in the case of rotations on the circle or higher-dimensional tori, to unpredictable and chaotic behaviour or the appearence of strange attractors.

Another type of dynamical systems is given by flows, which are usually induced by ordinary differential equations. Here a *flow* on a metric space X is a mapping $\varphi : \mathbb{R} \times X \to X,\ (t,x) \mapsto \varphi^t(x)$ that satisfies the flow properties

$$\begin{aligned} \varphi^{s+t}(x) &= \varphi^s(\varphi^t(x)) \\ \varphi^0(x) &= x \end{aligned}$$

for all $x \in X$ and $s,t \in \mathbb{R}$. The pair (X,φ) is called a *continuous-time dynamical system*. For any given $t \geq 0$, one may also consider the discrete-time dynamical system (X,φ^t), and it turns out that in order to understand the long-term behaviour of (X,φ) it often suffices to understand that of (X,φ^t). More generally, it is also possible to replace $\mathbb{R}$ by an arbitrary group G. In this case, the resulting mapping $\varphi : G \times X \to X$, which is assumed to satisfy the analogous equations, is called a *group action* or, more specifically, a G-action.

5.1 Entropy and box dimension

One of the most important notions in dynamical systems theory is that of *entropy*, a concept that is used to quantify the complexity (or chaoticity) of a dynamical system. Given a discrete-time dynamical system (X,T) with compact phase space X, the n*-th Bowen metric* is defined as

$$d_n(x,y) = \max_{i=0}^{n} d_n(T^i x, T^i y) .$$

It is straightforward to show that these are metrics in the originial sense and induce the same topology as $d = d_0$. A set $S \subseteq X$ is called (δ,n)-separated, if $d_n(x,y) \geq \delta$ for all $x,y \in X$. By $S(T,\delta,n)$ we denote the maximal cardinality of a (δ,n)-separated set $S \subseteq X$. As a consequence of compactness, we always have $S(T,n,\delta) < \infty$. We let

$$h_\delta(T) = \limsup_{n\to\infty} \frac{1}{n} \log S(T,\delta,n)$$

and define the *topological entropy of* T as

$$h_{\mathrm{top}}(T) \;=\; \lim_{\delta\searrow 0} h_\delta(T) \;=\; \sup_{\delta>0} h_\delta(T)\ .$$

Note that the quantities $S(T,\delta,n)$ and therefore also the $h_\delta(T)$ are decreasing in δ, which justifies to replace the limit by the supremum in this definition. Depending on the context, we also write $h_{\mathrm{top}}(X,T)$ instead of $h_{\mathrm{top}}(T)$.

We want to have a look at this concept in the context of *shift dynamics*. Consider the sequence space $\Sigma^+ = \{0,1\}^{\mathbb{N}}$ and the *shift map*

$$\sigma : \Sigma^+ \to \Sigma^+ \quad , \quad \sigma(a)_n \;=\; a_{n+1}\ .$$

Further, we consider the metric

$$d(a,b) \;=\; \exp\left(-\min\{n\in\mathbb{N} \mid a_n \neq b_n\}\right)$$

on Σ^+. Given $\delta \in [\exp -k, \exp -k+1)$, a pair of points $a,b\in\Sigma^+$ is (n,δ)-separated if there exists a mismatch between a and b in the first $n+k$ positions. This implies that we have

$$S(\sigma,n,\delta) \;=\; 2^{n+k}$$

and hence

$$h_\delta(\sigma) \;=\; \log(2)\ .$$

Thus, we obtain $h_{\mathrm{top}}(\sigma) = \log(2)$. At the same time, note that we have $N_{\exp(-n)}(\sigma^+) = 2^n$ and therefore $\mathrm{Dim}_B(\Sigma^+) = \log(2)$ as well.

In order to see whether this is merely a coincidence, we now consider compact subsets $X \subseteq \Sigma^+$ which are *shift-invariant*, that is, which satisfy $\sigma(X) = X$. Then σ acts on X, and the pair (X,σ) is called a *(symbolic) subshift*. Given $k\in\mathbb{N}$, we have that

$$N_{\exp(-k)}(X) \;=\; \sharp\{w\in\{0,1\}^k \mid [w]\cap X \neq \emptyset\}\ .$$

However, at the same time we have that

$$S(X,\sigma,\delta,n) \;=\; \sharp\{w\in\{0,1\}^{n+k} \mid [w]\cap X\neq\emptyset\} \;=\; N_{\exp(-(n+k)}$$

and thus obtain $h_{\mathrm{top}}(X,\sigma) = h_\delta(X,\sigma) = \mathrm{Dim}_B(X)$ for all $\delta\in[0,1/2]$. Hence, we have proved

Theorem 5.1.1. *If Σ^+ is equipped with the above metric and (X,σ) is a symbolic subshift, then*

$$h_{\mathrm{top}}(X,\sigma) \;=\; \mathrm{Dim}_B(X)\ .$$

5.2 Fractal attractors and repellers of dynamical systems

Fractals often appear as attractors and repellers, or more generally as invariant sets, of dynamical systems. Given $T : X \to X$, we call $A \subseteq X$ T-invariant if $T(A) = A$. The following example shows how the Middle Third Cantor Set C can appear as an invariant set of a dynamical system.

Example 5.2.1. Let

$$T:\mathbb{R}\to\mathbb{R} \quad , \quad x \mapsto \begin{cases} T_1(x) = 3x & x\leq 1/2 \\ T_2(x) = 3-3x & x>1/2 \end{cases}\ .$$

If $x < 0$, then $T^n(x) = 3^n x \overset{n\to\infty}{\longrightarrow} -\infty$. Likewise, if $x > 1$, then $T(x) < 0$ and hence $T^n(x) = 3^{n-1}T(x) \overset{n\to\infty}{\longrightarrow} -\infty$. Hence, all points outside of the set

$$A \;=\; \bigcup_{n\in\mathbb{N}} T^{-n}([0,1]) \;=\; \{x \in \mathbb{R} \mid T^n(x) \in [0,1] \;\forall n \in \mathbb{N}\}$$

converge to $-\infty$ under iteration by the map T. Hence, the set A is the maximal invariant set in $[0,1]$, and also the maximal compact invariant set of the system. As all other points move away from A, we say A is a repeller. We claim that A is exactly the Middle Third Cantor Set.

In order to prove $A = C$, recall that C is the unique attractor of the IFS $\mathcal{S} = \{S_1, S_2\}$ on $\mathbb{R}$, with

$$S_1(x) \;=\; x/3 \quad , \quad S_2(x) \;=\; x/3 + 2/3 \;.$$

On the other hand, for any set $B \subseteq [0,1]$ we have that $T^{-1}(B) = T_1^{-1}(B) \cup T_2^{-1}(B)$. As A satisfies $T^{-1}(A) = A$, this means that A is the unique attractor of the IFS $\tilde{\mathcal{S}} = \{\tilde{S}_1, \tilde{S}_2\}$, where

$$\tilde{S}_1(x) \;=\; S_1(x) \quad , \quad \tilde{S}_2(x) = 1 - x/3 \;.$$

Moreover, if we let $H(x) = 1 - x$, then it is easy to check that $S_2 \circ H = \tilde{S}_2$ and, using $H^{-1} = H$, $S_2 = \tilde{S}_2 \circ H$. Similarly, we have

$$H \circ S_1 \circ H \;=\; S_2 \quad , \quad H \circ S_2 \circ H \;=\; S_1 \;.$$

The later implies that

$$\begin{aligned} \mathcal{S}(H(C)) \;&=\; S_1 \circ H(C) \cup S_2 \circ H(C) \\ &=\; H \circ S_2 \circ \underbrace{H \circ H}_{=\mathrm{Id}_{\mathbb{R}}}(C) \cup H \circ S_1 \circ \underbrace{H \circ H}_{=\mathrm{Id}_{\mathbb{R}}}(C) \\ &=\; H \circ S_2(C) \cup H \circ S_1(C) \;=\; H(C) \;. \end{aligned}$$

As C is the unique attractor of $\mathcal{S}$, this implies $H(C) = C$. However, this further yields

$$\tilde{\mathcal{S}}(C) \;=\; \tilde{S}_1(C) \cup \tilde{S}_2(C) \;=\; S_1(C) \cup S_2 \circ H(C) \;=\; S_1(C) \cup S_2(C) \;=\; C \;.$$

As A is the unique attractor of $\tilde{S}$, this implies $A = C$.

Exercise 45. Suppose that $\mathcal{S} = \{S_1, \ldots, S_m\}$ and $\tilde{\mathcal{S}}\{\tilde{S}_1, \ldots, \tilde{S}_m\}$ are two IFS on X and X', respectively. Further, assume that $H : X \to X'$ is a homeomorphism such that $\{H \circ S_j \circ H^{-1}\} = \tilde{\mathcal{S}}$. Show that in this case $\mathcal{A}_{\tilde{\mathcal{S}}} = H(\mathcal{A}_{\mathcal{S}})$.

Example 5.2.2. A similar example is given by the map

$$T : \mathbb{R} \to \mathbb{R} \quad , \quad x \mapsto \begin{cases} 3x & x \le 1/2 \\ 3x - 2 & x > 1/2 \end{cases} .$$

We let $h : \Sigma^+ \to \mathbb{R},\; a \mapsto 2 \cdot \sum_{n\in\mathbb{N}} a_n \cdot 3^{-n}$, so that $h(\Sigma^+) = C$. Given $a \in \Sigma^+$, we have that

$$g \circ h(a) \;=\; \left. \begin{cases} 2 \cdot \sum_{n=2}^{\infty} 3^{-n+1} a_n & h(a) \in [0, 1/3] \\ 3 \cdot \frac{2}{3} + 2 \cdot \sum_{n=2}^{\infty} 3^{-n+1} a_n - 2 & h(a) \in [2/3, 1] \end{cases} \right\} \;=\; h \circ \sigma(a) \;.$$

Hence, we obtain $g \circ h = h \circ \sigma$. In this situation, the maps g and σ are called *conjugate* and the homeomorphism h is called a *conjugation*. As $g = h \circ \sigma \circ h^{-1}$, the map h can be seen as a coordinate change which transforms the one system into the other.

As a consequence, we immediately obtain

$$g(C) \;=\; g(h(\Sigma^+)) \;=\; h(\sigma(\Sigma^+)) \;=\; h(\Sigma^+) \;=\; C \;,$$

so that again C is an invariant set of the system. Note that we could have obtained the same result by a similar argument as in the previous example.

Remark 5.2.3. Similar, but two-dimensional example, is given by so-called *horseshoe maps*. These are diffeomorphism H of the plane which leave invariant a product $C \times C$ of two one-dimensional Cantor sets. Moreover, $H_{C\times C}$ is conjugate to the *two-sided shift* (the shift on the space of two-sided sequences).

5.3 Fractal attractors

Given a topological dynamical system (X, T), a compact set $A \subseteq X$ is called a *topological attractor* if there exists a neighbourhood U of A such that $T(U) \subseteq U$ and $\bigcap_{n\in\mathbb{N}} T^n(U) = A$. If X carries a natural measure (like the Lebesgue measure if $X \subseteq \mathbb{R}^d$ is a set of positive Lebesgue measure), then A is called an *attractor in the sense of Milnor*, or *Milnor attractor*, if

(i) the set $R(A) = \{x \in X \mid \lim_{n\to\infty} d(T^n(x), A) = 0\}$ has positive measure;

(ii) there exists no compact strict subset $A' \subsetneq A$ of A such that $R(A') = R(A)$.

Examples of fractal attractors – in either sense – are more difficult to produce than examples of fractal invariant sets as in the last section. However, the appearence of fractal attractors (often called *strange attractors*) is one of the intriguing phenomena in dynamical systems, and there are various famous examples. One of the best-known is the *Hénon attractor*, which appears as an attractor of the *Hénon map*

$$H_{a,b}(x, y) \;=\; (1 - ax^2 + y, bx)$$

For the classical parameters $a = 1.4$ and $b = 0.3$, numerical estimate for the the box and Hausdorff dimensions range from 1.23 to 1.29. For other parameters (a large, b small), the Hénon map becomes a perturbed version of a standard Horseshoe map with an invariant Cantor set (but no fractal attractor).

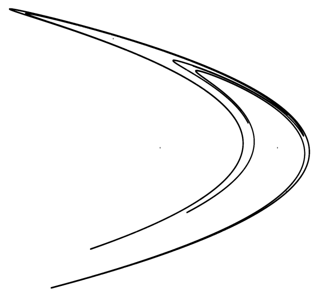

Figure 5.3.1: Attractor of the Hénon map with parameters $a = 1.4$ and $b = 0.3$. (Source: Wikipedia, `https://commons.wikimedia.org/wiki/File:HenonMap.svg`, Creative Commons Attribution-Share Alike 4.0 International license, author Shiyu Ji.)

The Hénon map itself was introduced by Michel Hénon as a simplified model of the Lorenz flow, which is generated by the three-dimensional ODE

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= x(\rho - z) - y \\ z' &= xy - \beta z \end{aligned}$$

with real parameters σ, ρ and β.

Both the Hénon and the Lorenz attractor are strange *chaotic* attractors, where the term 'chaotic' refers to the fact that the system has positive topological entropy. Strange *non-chaotic* attractors have been discovered in skew product systems like

$$f_{\alpha,\beta} : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R} \quad , \quad (x, y) \;=\; (x + \alpha, \tanh(\beta y) \cdot \sin(\pi x)) \; .$$

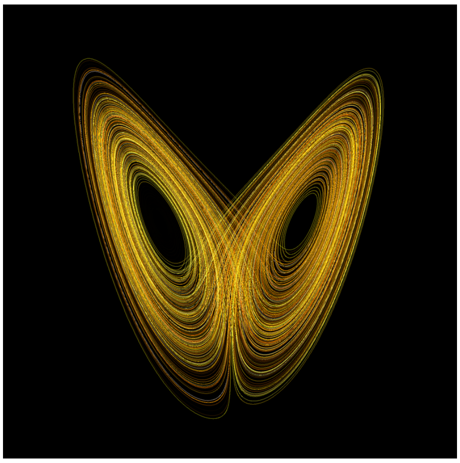

Figure 5.3.2: Attractor of the Lorenz flow with parameters $\sigma = 10, \rho = 28$ and $\beta = 8/3$. (Source: Wikipedia, author Wikimol
https://commons.wikimedia.org/wiki/File:Lorenz_system_r28_s10_b2-6666.png.)

In this case, the term *strange* is not motivated by a non-integer dimension, but by the fact that the Hausdorff dimension equals 1, whereas the box dimension is 2 (Gröger et al, 2013).

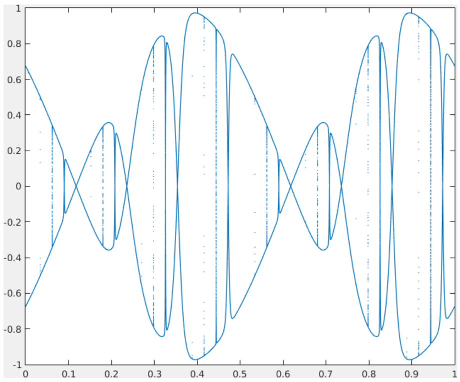

Figure 5.3.3: Attractor of the quasiperiodially forced map $f_{\alpha,\beta}$ with α the golden mean and $\beta = 3$.

5.4 Dimensions of Weierstrass graphs

The *Weierstrass function* is given by the trigonometric series

$$\varphi_{\lambda,b}(\xi) \;=\; \sum_{n=1}^{\infty} \lambda^n \cdot \cos(2\pi b^n \xi)$$

with parameters $\lambda \in (0,1)$ and $b \in \mathbb{N}$, $b > 1/\lambda$. It was introduced by Weierstrass in 1872 in order to provide an example of a continuous function that is nowhere differentiable. Note that $\varphi_{\lambda,b}$ is defined on the real line, but at the same time it is 2π-periodic, so that we will view it as a function $\varphi_{\lambda,b} : [0, 2\pi] \to \mathbb{R}$.

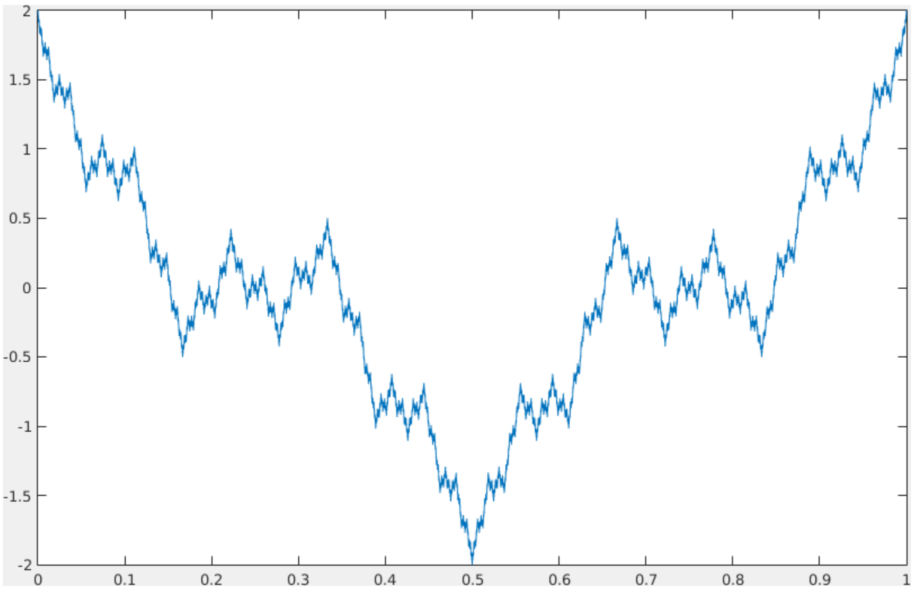

Figure 5.4.1: Graph of the Weierstrass function with parameters $\lambda = 1/2$ and $\beta = 3$.

Proposition 5.4.1. *The Weierstrass function $\varphi_{\lambda,b}$ is α-Hölder continuous with Hölder exponent $\alpha = -\log\lambda/\log b$.*

Theorem 5.4.2. *Suppose that Ξ is a metric space and $\varphi : \Xi \to \mathbb{R}$ is α-Hölder continuous. Then*

$$\begin{aligned} \overline{\mathrm{Dim}}_{\mathrm{B}}(\varphi) &\leq \overline{\mathrm{Dim}}_{\mathrm{B}}(\Xi) + 1 - \alpha && (5.4.1)\\ \underline{\mathrm{Dim}}_{\mathrm{B}}(\varphi) &\leq \underline{\mathrm{Dim}}_{\mathrm{B}}(\Xi) + 1 - \alpha && (5.4.2)\end{aligned}$$

Exercise 46. Show that in the situation of Theorem 5.4.2 we also have

$$\begin{aligned} \overline{\mathrm{Dim}}_{\mathrm{B}}(\varphi) &\leq \overline{\mathrm{Dim}}_{\mathrm{B}}(\Xi)/\alpha \\ \underline{\mathrm{Dim}}_{\mathrm{B}}(\varphi) &\leq \underline{\mathrm{Dim}}_{\mathrm{B}}(\Xi)/\alpha \end{aligned}$$

Why does this estimate only become relevant if $\overline{\mathrm{Dim}}_{\mathrm{B}}(\Xi) < 1$ (respectively $\underline{\mathrm{Dim}}_{\mathrm{B}}(\Xi) < 1$?

Corollary 5.4.3. *We have $\overline{\mathrm{Dim}}_{\mathrm{B}}(\varphi_{\lambda,b} \leq 2 + \log\lambda/\log b$.*

Proposition 5.4.4. *If $\lambda \in (0, 1/4)$ and $b \geq 8$, then we have*

$$\mathrm{Dim}_b(\varphi_{\lambda,b}) \;=\; 2 + \log\lambda/\log b\,.$$

Remark 5.4.5. In fact, both the box dimension and also the Hausdorff dimension of the Weierstrass graph $\varphi_{\lambda,b}$ equal $2+\log\lambda/\log b$, for all admissible parameter pairs λ, b. However, the proofs of these facts have been given by using dynamical methods, for the box dimension by Bedford (Nonlinearity 1989), for the Hausdorff dimension by Barański, Bárány and Romanowska in (Advances in Mathematics, 2014).

The basic observation allowing the application of dynamical methods to this problem is the fact that the Weierstrass graph is an invariant repeller of the skew product system

$$T : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R} \quad , \quad (\xi, x) \mapsto \left(b\xi \bmod 1, \frac{x - \cos(2\pi\xi)}{\lambda} \right) \,.$$

If we let $T_\xi(x) = \frac{x-\cos(2\pi\xi)}{\lambda}$, then an easy computation yields $T_\xi(\varphi_{\lambda,b}(\xi)) = \varphi_{\lambda,b}(b\xi)$ and hence $T(\varphi_{\lambda,b}) = \varphi_{\lambda,b}$ (where $\varphi_{\lambda,b}$ is interpreted as a subset of $X \times \mathbb{R}$).

5.5 Fractal graphs in fibrewise contracting skew product systems

Let Ξ and X be metric spaces and $\tau : \Xi \to \Xi$ a homeomorphism. A **skew product map** T **with base** τ is of the form

$$T : \Xi \times X \quad , \quad (\xi, x) \mapsto (\tau(\xi), T_\xi(y))$$

The map T is said to be **(uniformly) contracting in the fibres** if there exists a constant $\lambda \in [0, 1)$ such that

$$d(T_\xi(x), T_\xi(y)) \;\leq\; \lambda d(x, y)$$

for all $\xi \in \Xi$ and $x, y \in X$. The **graph transform** associated to the skew product T is given by

$$T_* : \mathcal{F}(\xi, X) \to \mathcal{F}(\xi, X) \quad , \quad \varphi \mapsto T_*\varphi(\xi) = T_{\tau^{-1}(\xi)}(\varphi(\tau^{-1}(\xi))) \; .$$

If both φ and $T_*\varphi$ are interpreted as subsets of the product space $\Xi \times X$, then $T(\varphi) = T_*\varphi$.

Theorem 5.5.1. *If $T : \Xi \times X \to \Xi \times X$ is a skew product map that is contracting in the fibres, then the graph transform T_* is a contraction on the Banach space $(\mathcal{F}(\Xi, X), \|.\|_\infty)$ and therefore has a unique fixed point φ_T. Moreover, we have that*

$$d(T_\xi^n(x), \varphi_T(\tau^n(\xi))) \;\leq\; \lambda^n \cdot d(x, \varphi(\xi))$$

for all $(\xi, x) \in \Xi \times X$ and $n \in \mathbb{N}$

Proposition 5.5.2. *Suppose that τ^{-1} is Lipschitz-continuous on Ξ with Lipschitz constant $\gamma > 0$ and T is a fibrewise contracting skew product map on $\Xi \times X$ with uniform contraction constant $\lambda \in (0, 1)$. Then φ_T is α-Hölder continuous for any Hölder exponent $\alpha < \min\{1, \log\lambda / \log\gamma\}$.*

Corollary 5.5.3. *In the situation of 5.5.2, if $\gamma < \lambda$, we have*

$$\overline{\mathrm{Dim}}_{\mathrm{B}}(\varphi_T) \;\leq\; \overline{\mathrm{Dim}}_{\mathrm{B}}(\Xi) + 1 - \log\lambda / \log\gamma \; .$$

Remark 5.5.4. In certain situations where the assumptions of Proposition 5.5.2 hold, but additionally the maps τ and T satisfy a number of further conditions (which are referred to as *partial hyperbolicity* of the map T), it is possible show the equality $\mathrm{Dim}_{\mathrm{B}}(\varphi_T) = \mathrm{Dim}_{\mathrm{B}}(\Xi) + 1 - \log\lambda / \log\gamma$. This relies again on dynamical methods (thermodynamic formalism).

Appendix A

Measure Theory

A.1 Systems of sets

Let X be an arbitrary set and $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ be its power set. Then a family of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a **semiring**, if it satisfied

(S1) $\emptyset \in \mathcal{S}$;

(S2) $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$ ($\cap$-stability);

(S3) if $A, B \in \mathcal{S}$, then there exists a finite number of pairwise disjoint sets $A_1, \ldots, A_k \in \mathcal{S}$ such that $B \setminus A = \biguplus_{i=1}^{k} A_i$

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is called a σ**-algebra**, if it satisfies

(A1) $X \in \mathcal{A}$;

(A2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$;

(A3) if $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

The sets $A \in \mathcal{A}$ are called **measurable (with respect to** $\mathcal{A}$**)** in this case. The pair $(\Omega, \mathcal{A})$ is called a **measurable space**. If (A3) is only satisfied for finite unions, then $\mathcal{A}$ is called an **algebra**.d A family $\mathcal{D} \subseteq \mathcal{P}(X)$ is called **Dynkin system**, if it satisfies

(D1) $X \in \mathcal{D}$;

(D2) $A \in \mathcal{D} \Rightarrow A^c \in \mathcal{D}$I

(D3) if $(A_n)_{n\in\mathbb{N}}$ is a sequence of paarweise disjoint sets in $\mathcal{D}$, then $\biguplus_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

Remark A.1.1. For our purposes, semirings, σ-algebras and Dynkin systems will be the most important types of set systems. However, there exist further closely related notions of sets systems which often play a role in measure theory. For instance, if (S2) is replaced by closedness under (finite) unions and (S3) is replaced by the stronger condition that $A, B \in \mathcal{S}$ implies $A \setminus B \in \mathcal{S}$, then $\mathcal{S}$ is called a ring. A ring is intersection stable, since $A \cap B = A \setminus (A \setminus B)$.

Further, a ring which is closed under countable unions is called a σ-ring. The latter will be needed in the context of Borel measures on topological spaces in Section **??**. The only difference between a σ-ring and a σ-algebra is the fact that a σ-algebra needs to contain X. Hence, a σ-ring which contains X is a σ-algebra.

Lemma A.1.2. *Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a semiring and suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{S}$. Then there exists a sequence $(B_k)_{k\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{S}$ such that*

$$\bigcup_{n\in\mathbb{N}} A_n = \biguplus_{k\in\mathbb{N}} B_k . \tag{A.1.1}$$

Moreover, the sequence $(B_k)_{k\in\mathbb{N}}$ can be chosen such that there exists a sequence of integers $(K_n)_{n\in\mathbb{N}}$ with the property that

$$\bigcup_{i=1}^{n} A_i = \biguplus_{k=1}^{K_n} B_k \; n \in \mathbb{N} \tag{A.1.2}$$

holds for all $n \in \mathbb{N}$.

We will obtain this statement of the consequence of the following, slightly more technical lemma.

Lemma A.1.3. *Suppose $C_1, \dots, C_n$ and $D_1, \dots, D_m$ are two finite sequences of pairwise disjoints sets in $\mathcal{S}$. Then there exist $k \in \mathbb{N}$ and pairwise disjoint sets $R_1, \dots, R_k$ in $\mathcal{S}$ such that*

$$\biguplus_{i=1}^{n} C_i \setminus \biguplus_{j=1}^{m} D_j = \biguplus_{l=1}^{t} R_l .$$

Proof. We proceed by induction on m. Let $m = 1$. According to (S3), for any $i = 1, \dots, n$ we can choose $q_i \in \mathbb{N}$ and pairwise disjoint sets $M_1^i, \dots, M_{q_i}^i \in \mathcal{S}$ such that $C_i \setminus D_1 = \biguplus_{s=1}^{q_i} M_s^i$. Hence, we obtain

$$\biguplus_{i=1}^{n} C_i \setminus D_1 = \biguplus_{i=1}^{n} \biguplus_{s=1}^{l_i} M_s^i = \biguplus_{l=1}^{t} R_l ,$$

where $t = \sum_{i=1}^{n} l_i$ and the sets $R_1, \dots, R_t$ are obtained by relabelling the sets M_s^i. Now, suppose that $m > 1$ and the statement holds for $m - 1$. Then there exist sets $\tilde{R}_1, \dots, \tilde{R}_{\tilde{t}} \in \mathcal{S}$ such that

$$\biguplus_{i=1}^{n} C_i \setminus \biguplus_{j=1}^{m-1} D_j = \biguplus_{l=1}^{\tilde{t}} \tilde{R}_l .$$

We obtain

$$\biguplus_{i=1}^{n} C_i \setminus \biguplus_{j=1}^{m} D_j = \left[\biguplus_{i=1}^{n} C_i \setminus \biguplus_{j=1}^{m-1} D_j \right] \setminus D_m = \biguplus_{l=1}^{\tilde{t}} \tilde{R}_l \setminus D_m .$$

Using the case $m = 1$, for each $l = 1, \dots, \tilde{t}$ we can choose sets $Q_1^l, \dots, Q_{p_l}^l$ such that $\tilde{R}_l \setminus D_m = \biguplus_{s=1}^{p_l} Q_s^l$. Relabelling the Q_s^l then yields the statement. $\square$

Proof of Lemma A.1.2. We prove the statement in its stronger form (A.1.2) induction on n. More precisely, we recursively construct the integers K_n and the sets $B_{K_n+1}, \dots, B_{K_{n+1}}$ for all $n \in \mathbb{N}$. If $n = 1$, we let $K_1 = 1$ and $B_1 = A_1$. If $K_1, \dots, K_{n-1}$ and $B_1, \dots, B_{K_{n-1}}$ are given, then Lemma A.1.3 yields the existence of sets $R_1, \dots, R_t$ such that

$$A_{n+1} \setminus \bigcup_{i=1}^{n} A_i = A_{n+1} \setminus \biguplus_{k=1}^{K_n} B_k = \biguplus_{i=1}^{t} R_i ,$$

We then define $K_{n+1} = K_n + t$ und $B_{K_n+s} = \mathbb{R}_s$ für $s = 1, \dots, l$. $\square$

We say a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is **intersection stable**, if $A, B \in \mathcal{C}$ implies $A \cap B \in \mathcal{C}$.

Lemma A.1.4. *An intersection stable Dynkin system is a σ-algebra.*

Proof. Properties (D1) and (D2) are identical to (A1) and (A2), so that it only remains to prove (A3). Suppose that $A_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. Then we have

$$\bigcup_{n\in\mathbb{N}} A_n = \biguplus_{n\in\mathbb{N}} \left(A_n \setminus \bigcup_{i=1}^{n-1} A_i \right) = \biguplus_{n\in\mathbb{N}} \underbrace{\left(A_n \cap \bigcap_{i=1}^{n-1} A_i^c \right)}_{\in\mathcal{D}},$$

As a countable union of pairwise disjoint sets in $\mathcal{D}$, this is again contained in $\mathcal{D}$. □

The following remark is easy to verify.

Remark A.1.5. The intersection of an arbitrary family of σ-algebras is again a σ-algebra and the intersection of an arbitrary family of Dynkin systems is a Dynkin system.

Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be an arbitrary family of subsets of X. Then

$$\sigma(\mathcal{C}) \;=\; \bigcap_{\substack{\mathcal{A}\ \sigma-\text{algebra} \\ \mathcal{C}\supseteq\mathcal{A}}} \mathcal{A}$$

is called the **σ-algebra induced by** $\mathcal{C}$ and $\mathcal{C}$ is called a **generator** of $\sigma(\mathcal{C})$. Likewise,

$$\mathcal{D}(\mathcal{C}) \;=\; \bigcap_{\substack{\mathcal{D}\ \text{Dynkin system} \\ \mathcal{C}\supseteq\mathcal{D}}} \mathcal{D}$$

is called the Dynkin system **induced by** $\mathcal{C}$. Again, $\mathcal{C}$ is referred to as the **generator** of $\mathcal{D}(\mathcal{C})$. Note that generators not uniqly determined. Moreover, every σ-algebra/Dynkin system is a generator of itself.

Theorem A.1.6. *If $\mathcal{C} \subseteq \mathcal{P}(X)$ is intersection stable, then $\mathcal{D}(\mathcal{C}) = \sigma(\mathcal{C})$.*

Proof. Since a σ-algebra is always a Dynkin system, $\sigma(\mathcal{C})$ is a Dynkin system which contains $\mathcal{C}$, so that $\mathcal{D}(\mathcal{C}) \subseteq \sigma(\mathcal{C})$. In order to obtain the converse inclusion, we show that $\mathcal{D}(\mathcal{C})$ is intersection stable, so that by Lemma A.1.4 it is a σ-algebra that obviously contains $\mathcal{C}$. In order to see this, given $C \in \mathcal{D}(\mathcal{C})$ we let

$$\mathcal{D}_C := \{A \in \mathcal{D}(\mathcal{C}) \mid A \cap C \in \mathcal{D}(\mathcal{C})\} .$$

We claim that $\mathcal{D}_C$ is a Dynkin system:

(D1) $X \cap C = C \in \mathcal{D}(\mathcal{C})$, so that $X \in \mathcal{D}_C$;

(D2) Let $A \in \mathcal{D}_C$. Then $A^c \cap C = (C^c \uplus (A \cap C))^c \in \mathcal{D}(\mathcal{C})$, so $A^c \in \mathcal{D}_C$.

(D3) If $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{D}_C$, then $\left(\biguplus_{n\in\mathbb{N}} A_n\right) \cap C = \biguplus_{n\in\mathbb{N}} (A_n \cap C) \in \mathcal{D}(\mathcal{C})$. Thus, $\biguplus_{n\in\mathbb{N}} A_n \in \mathcal{D}_C$.

By intersection stability of the generator $\mathcal{C}$, this means that for any $C \in \mathcal{C}$ the family $\mathcal{D}_C$ is a Dynkin system that contains $\mathcal{C}$, and hence $\mathcal{D}(\mathcal{C})$. This, in turn, yields that for any $B \in \mathcal{D}(\mathcal{C})$ the generator $\mathcal{C}$ is contained in $\mathcal{D}_B$ and therefore $\mathcal{D}(\mathcal{C}) \subseteq \mathcal{D}_B$. However, this implies that $\mathcal{D}(\mathcal{C})$ is intersection stable and therefore equals $\sigma(\mathcal{C})$. □

A.2 Set functions

Let $\mathcal{C} \subseteq \mathcal{P}(X)$. A mapping $\mu : \mathcal{C} \to [0, +\infty]$ is called a **set function**. The function μ is called **σ-subadditive** on $\mathcal{C}$, if

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{C} \;\Rightarrow\; \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$$

holds for any sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{C}$. If the same holds only for finite unions, then μ is called **(finitely) subadditive**. We say μ is **σ-additive** on $\mathcal{C}$, if

$$\biguplus_{n\in\mathbb{N}} A_n \in \mathcal{C} \;\Rightarrow\; \mu\left(\biguplus_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

holds for any sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise additive sets in $(A_n)_{n\in\mathbb{N}}$. Again, if the same holds only for finite unions, then μ is called **(finitely) additive**. Further, μ is called **monotone** on $\mathcal{C}$ if $A, B \in \mathcal{C}$ and $A \subseteq B$ implies $\mu(A) \leq \mu(B)$.

Remark A.2.1. Note that any subadditive set function on cC is monotone. Further, if $\emptyset \in \mathcal{C}$, then any σ-subadditive set function on $\mathcal{C}$ with $\mu(\emptyset) = 0$ is also subadditive.

A σ-additive set function μ on a σ-algebra $\mathcal{A}$ is called a **measure**. The triple $(\Omega, \mathcal{A}, \mu)$ is then called a **measure space**. When $\mu(X) = 1$, it is called a **probability space**. If $X \in \mathcal{C}$, μ is monoton and $\mu(X) < \infty$, then μ is called **finite**. If there exist an increasing sequence $C_1 \subseteq C_2 \subseteq \ldots$ of sets in $\mathcal{C}$ with $C_n \nearrow X$ (that is, $\bigcup_{n\in\mathbb{N}} C_n = X$) and $\mu(C_n) < \infty$ for all $n \in \mathbb{N}$, then μ is called **σ-finite**.

Lemma A.2.2. *An additive set function μ on a semiring $\mathcal{S}$ is always monotone and subadditive.*

Proof. Let $A, B \in \mathcal{S},\ A \subseteq B$. Then $B \setminus A = \biguplus_{i=1}^n S_i$ with $S_1, \ldots, S_n \in \mathcal{S}$ by (S3), and thus $B = A \uplus \biguplus_{i=1}^n S_i$. The additivity of μ then yields

$$\mu(B) \;=\; \mu(A) + \sum_{i=1}^n \mu(S_i) \;\geq\; \mu(A)$$

This shows that μ is monotone.

In order to see that it is also subadditive, suppose that $A_1, \ldots, A_n \in \mathcal{S}$. By Lemma A.1.2, there exist pairwise disjoint sets $B_1, \ldots, B_k \in \mathcal{S}$ und integers $1 = K_1 < K_2 < \ldots < K_n = K$, such that $B_1 = A_1$ and

$$A_{i+1} \setminus A_i \;=\; \biguplus_{j=K_i+1}^{K_{i+1}} B_j$$

for all $i = 1, \ldots, n-1$, so that

$$A_{i+1} \;=\; \biguplus_{K_i+1}^{K_{i+1}} B_j \uplus \biguplus_{j=1}^{K_i} (B_j \cap A_{i+1})\,.$$

Due to the intersection stability, all these sets are contained in $\mathcal{S}$. Consequently, we obtain

$$\sum_{i=1}^n \mu(A_i) \;=\; \mu(B_1) + \sum_{i=1}^{n-1}\left(\sum_{j=K_i+1}^{K_{i+1}} \mu(B_j) + \sum_{j=1}^{K_i} \mu(B_j \cap A_{i+1})\right) \geq \sum_{j=1}^{k} \mu(B_j) = \mu\left(\bigcup_{i=1}^n A_i\right)$$

$\square$

If$(A_n)_{n\in\mathbb{N}}$ is a sequence of sets, then we write $A_n \nearrow A$ if $(A_n)_{n\in\mathbb{N}}$ is increasing (that is, $A_1 \subseteq A_2 \subseteq \ldots$) and $A = \bigcup_{n\in\mathbb{N}} A_n$. Likewise, we write $A_n \searrow A$ if $(A_n)_{n\in\mathbb{N}}$ is decreasing ($A_1 \supseteq A_2 \supseteq \ldots$) and $A = \bigcap_{n\in\mathbb{N}} A_n$.

Theorem A.2.3. *Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$. Then the following hold.*

(i) If $A_n \nearrow A$, then $\mu(A_n) \nearrow \mu(A)$;

(ii) if $A_n \searrow A$ and $\mu(A_1) < \infty$, then $\mu(A_n) \searrow \mu(A)$;

(iii) if $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$, then $\mu(A) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$; in other words, μ is subadditive.

Beweis.

(i) Let $A_0 = \emptyset$. Due to the monotonicity of μ (Lemma A.2.2, note that σ-algebras are always semirings) we have $\mu(A_1) \leq \mu(A_2) \leq \dots$. Moreover, we have $A_n = \biguplus_{i=1}^n A_i \setminus A_{i-1}$ und $A = \bigcup_{n\in\mathbb{N}} A_n = \biguplus_{n\in\mathbb{N}} A_n \setminus A_{n-1}$. Using σ-additivity, we obtain

$$\mu(A) \;=\; \sum_{n\in\mathbb{N}} \mu(A_n \setminus A_{n-1}) \;=\; \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} \mu(A_n \setminus A_{n-1}) \;=\; \sup_{N\in\mathbb{N}} \mu(A_N)\ .$$

(ii) The fact that $A_n \searrow A$ implies $A_1 \setminus A_n \nearrow A_1 \setminus A$. Using $\mu(A_1) < \infty$, (i) yields

$$\mu(A_1) - \mu(A_n) \;=\; \mu(A_1 \setminus A_n) \;\nearrow\; \mu(A_1 \setminus A) \;=\; \mu(A_1) - \mu(A)$$

and hence $\mu(A_n) \searrow \mu(A)$.

(iii) Let $B_n = A \cap \bigcup_{k=1}^n A_k$. Then $B_n \nearrow A$. Due to (i) and Lemma A.2.2, we obtain

$$\mu(A) = \sup_{n\in\mathbb{N}} \mu(B_n) \;\leq\; \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \mu(A_k) \;=\; \sum_{n\in\mathbb{N}} \mu(A_n)\ .$$

□

A.3 Extension of measures – existence and uniqueness

The aim of this section is to prove the following extension theorem, which allows to define measures on a σ-algebra by determining their values only for sets of a generating semiring.

Theorem A.3.1 (Carathéodory Extension Theorem)**.** *Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a semiring and $\hat{\mu}$ an additive and σ-subadditive set function on $\mathcal{S}$. Then there exists a measure μ on $\sigma(\mathcal{S})$ such that $\mu_{|\mathcal{S}} = \hat{\mu}$. Moreover, if $\hat{\mu}$ is σ-finite, then μ is uniquely determined.*

Example A.3.2. Let $X = \mathbb{R}^d$ and

$$\mathcal{S} \;=\; \left\{ \bigotimes_{i=1}^{d} (a_i, b_i] \;\middle|\; a_i, b_i \in \mathbb{R},\ a_i < b_i \text{ for all } i = 1, \dots, d \right\}\ .$$

Further, define $\hat{\mu}$ on $\mathcal{S}$ by

$$\hat{\mu}\left(\bigotimes_{i=1}^{d} (a_i, b_i] \right) \;=\; \prod_{i=1}^{d} (b_i - a_i)\ .$$

Then it is possible to verify that the assumptions of Theorem A.3.1 are met and therefore $\hat{\mu}$ extents to a unique measure on $\sigma(\mathcal{S})$, which is called the **Lebesgue measure** on $\mathbb{R}^d$ and denoted by $\mathrm{Leb}_{\mathbb{R}^d}$.

We split the proof of Theorem A.3.1 into a number of intermediate statements, starting with one ensuring uniqueness of the extenstion.

Lemma A.3.3. *Suppose $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ is intersection stable and μ, ν are measures on $\sigma(\mathcal{C})$. If μ and ν are σ-finite and coincide on $\mathcal{C}$, then $\mu = \nu$.*

Proof. Given $C \in \mathcal{C}$ with $\mu(C) = \nu(C) < \infty$, we let

$$\mathcal{D}_C \;=\; \{A \in \sigma(\mathcal{C}) \mid \mu(A \cap C) = \nu(A \cap C)\}\;.$$

We claim that $\mathcal{D}_C$ is a Dynkin system. The fact that $\Omega \in \mathcal{D}_C$ is obvious, so (D1) holds. If $A \in \mathcal{D}_C$, then

$$\mu(A^c \cap C) \;=\; \mu(C) - \mu(A \cap C) \;=\; \nu(C) - \nu(A \cap C) \;=\; \nu(A^c \cap C)\;,$$

which shows that $A^c \in \mathcal{D}_C$ as well. Hence, (D2) holds. Finally, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{D}_C$, then

$$\mu\left(\biguplus_{n\in\mathbb{N}} A_n \cap C\right) \;=\; \sum_{n\in\mathbb{N}} \mu(A_n \cap C) \;=\; \sum_{n\in\mathbb{N}} \nu(A_n \cap C) \;=\; \nu\left(\biguplus_{n\in\mathbb{N}} A_n \cap C\right)\;,$$

so that $\biguplus_{n\in\mathbb{N}} A_n \cap C \in \mathcal{D}_C$. Thus, (D3) is satisfied as well and $\mathcal{D}_C$ is a Dynkin system. However, as $\mathcal{C} \subseteq \mathcal{D}_C$ and $\mathcal{C}$ is intersection stable, we have that $\sigma(\mathcal{C}) = \mathcal{D}(\mathcal{C}) = \mathcal{D}_C$.

Now, $A \in \sigma(\mathcal{C})$ and suppose $(C_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ with $C_n \nearrow X$ and $\mu(C_N) = \nu(C_n) < \infty$. Then, by Theorem A.2.3(i), we have that

$$\mu(A) \;=\; \lim_{n\to\infty} \mu(A \cap C_n) \;=\; \lim_{n\to\infty} \nu(A \cap C_n) \;=\; \nu(A)\;.$$

This proves $\mu = \nu$. □

An **outer measure** on a space X is a set function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ which is σ-subadditive and satisfies $\mu^*(\emptyset) = 0$. The following result provides a general construction of outer measures.

Proposition A.3.4. *Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be such that $\emptyset \in \mathcal{C}$ and suppose that $\hat{\mu} : \mathcal{C} \to [0, +\infty]$ is a set function with $\mu(\emptyset) = 0$. Then $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ defined by*

$$\mu^*(A) \;=\; \inf\left\{\sum_{n\in\mathbb{N}} \mu(C_n) \;\middle|\; A \subseteq \bigcup_{n\in\mathbb{N}} C_n,\ C_n \in \mathcal{C} \text{ for all } n \in \mathbb{N}\right\}$$

defines an outer measure on X.

Proof. For the empty set, $\hat{\mu}(\emptyset) = 0$ implies $\mu^*(\emptyset) = 0$. In order to prove the σ-subadditivity, suppose that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$. If $\hat{\mu}(A_n) = +\infty$ for some $n \in \mathbb{N}$, then there is nothing to show. Otherwise, we fix $\varepsilon > 0$ and choose sets C_k^n for $k, n \in \mathbb{N}$ such that

$$\mu^*(A_n) \;\leq\; \left(\sum_{k\in\mathbb{N}} \hat{\mu}(C_k^n)\right) - 2^{-n}\varepsilon\;.$$

Then $A \subseteq \bigcup_{k,n\in\mathbb{N}} C_k^n$, so that by definition of μ^* we obtain the estimate

$$\mu^*(A) \;\leq\; \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \hat{\mu}(C_k^n) \;\;\leq\; \sum_{n\in\mathbb{N}} (\mu^*(A_n) + 2^{-n}\varepsilon) \;=\; \sum_{n\in\mathbb{N}} \mu^*(A_n) + \varepsilon\;.$$

As $\varepsilon > 0$ was arbitrary, this yields $\mu^*(A) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$ and hence the σ-subadditivity of μ^*. □

If μ^* is an outer measure on X, we say $A \subseteq X$ is μ^*-measurable if

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \;=\; \mu(E)$$

holds for all $E \subseteq X$.

Theorem A.3.5. *If μ^* is an outer measure on X, then*

$$\mathcal{M}(\mu^*) \ = \ \{A \subseteq X \mid A \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra and μ^* is a measure on $(X, \mathcal{M}(\mu^*))$.

Proof. Properties (A1) and (A2) are obvious. Hence, we have to prove that $\mathcal{M}(\mu^*)$ is closed under taking countable unions and μ^* is σ-additive on $\mathcal{M}(\mu^*)$.

We first show that $\mathcal{M}(\mu^*)$ is closed under taking finite unions. To that end, let $A, B \in \mathcal{M}(\mu^*)$. Due to the σ-subadditivity of μ^*, we have that

$$\mu^*((A \cup B) \cap E) + \mu^*((A \cup B)^c \cap E) \ \leq \ \mu(E) \ .$$

The reverse inclusion follows from

$$\begin{aligned} &\mu^*((A \cup B) \cap E) + \mu^*((A \cup B)^c \cap E) \\ &\quad \leq \ \mu^*(A \cap B \cap E) + \mu^*(A^c \cap B \cap E) + \mu^*(A \cap B^c \cap E) + \mu^*(A^c \cap B^c \cap E) \\ &\quad = \ \mu^*(B \cap E) + \mu^*(B^c \cap E) \ = \ \mu^*(B) \ , \end{aligned}$$

where we used the μ^*-measurability of A in the step from the second to the third line and that of B in the last step.

Next, we show the σ-additivity of μ^*. Note that if A, B are disjoint, then μ^*-measurability of A implies

$$\mu^*(A \uplus B) \ = \ \mu^*(A \cap (A \uplus B)) + \mu^*(A^c \cap (A \uplus B)) \ = \ \mu^*(A) + \mu^*(B) \ .$$

Hence μ^* is finitely additive. Further, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{M}(\mu^*)$ and $A = \biguplus_{n\in\mathbb{N}} A_n$, then

$$\sum_{n\in\mathbb{N}} \mu^*(A_n) \ = \ \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \mu^*(A_k) \ = \ \sup_{n\in\mathbb{N}} \mu^* \left(\biguplus_{k=1}^{n} A_k \right) \ \leq \ \mu^*(A) \ .$$

Since the reverse inequality follows from σ-subadditivity, this shows the σ-additivity of μ^*. Note that we did not have to assume that $A \in \mathcal{M}(\mu^*)$.

Finally, we need to show the fact that $\mathcal{M}(\mu^*)$ is a σ-algebra. Since it is stable under taking unions and complements, it is also intersection stable. Therefore, by Lemma A.1.4, it suffices to show that $\mathcal{M}(\mu^*)$ is a Dynkin system. (D1) and (D2) are again easy to see. In order to prove (D3), suppose again that $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{M}(\mu^*)$ and let $A = \biguplus_{n\in\mathbb{N}} A_n$. We need to show that A is μ^*-measurable, that is,

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \ = \ \mu(E)$$

holds for all $E \subseteq X$. We start by showing that

$$\mu^* \left(E \cap \biguplus_{k=1}^{n} A_k \right) \ = \ \sum_{k=1}^{n} \mu^*(E \cap A_k) \tag{A.3.1}$$

for all $n \in \mathbb{N}$. If $n = 1$, there is nothing to show. If (A.3.1) holds for some $n \geq 1$, then

$$\begin{aligned} \mu^* \left(E \cap \biguplus_{k=1}^{n+1} A_k \right) \ &= \ \mu^* \left(A_{n+1} \cap E \cap \biguplus_{k=1}^{n+1} A_k \right) + \mu^* \left(A_{n+1}^c \cap E \cap \biguplus_{k=1}^{n+1} A_k \right) \\ &= \ \mu^*(A_{n+1} \cap E) + \mu^* \left(E \cap \biguplus_{k=1}^{n} A_k \right) \ = \ \sum_{k=1}^{n+1} \mu^*(E \cap A_k) \ , \end{aligned}$$

where we used the induction assumption in the last step. This proves that (A.3.1) holds, and by using the monotonicity of μ^* we further obtain

$$\mu^*(E) \;=\; \mu^*\left(E \cap \biguplus_{k=1}^{n} A_k\right) \mu^*\left(E \cap \left(\biguplus_{k=1}^{n} A_k\right)^c\right) \;\geq\; \sum_{k=1}^{n} \mu^*(E \cap A_k) + \mu^*(E \cap A^c)\,.$$

As $n \to \infty$, this yields

$$\mu^*(E) \;\geq\; \sum_{k \in \mathbb{N}} \mu^*(E \cap A_k) + \mu^*(E \cap A^c) \;\geq\; \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

and thus completes the proof. $\square$

The following statement now provides the last step in the proof of the Carathéodory Extension Theorem A.3, since it entails that $\mathcal{M}(\mu^*)$ contains the induced σ-algebra $\sigma(\mathcal{S})$. We can thus obtain the extension measure μ on $\sigma(\mathcal{S})$ as the restriction of the measure μ^* that is defined on the larger σ-algebra $\sigma(\mathcal{S})$.

Lemma A.3.6. *Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a semiring and suppose $\hat{\mu}$ is an additive and σ-subadditive set function on $\mathcal{S}$. Denote by μ^* the outer measure on $\mathcal{P}(X)$ induced by $\hat{\mu}$. Then $\mathcal{S} \subseteq \mathcal{M}(\mu^*)$,*

Proof. Note that additivity of $\hat{\mu}$ implies $\hat{\mu}(\emptyset) = 0$, so that Proposition A.3.4 applies and μ^* is an outer measure. Let $A \in \mathcal{S}$. We have to show that A is μ^*-measurable. To that end, suppose $E \subseteq X$ and assume without loss of generality that $\mu^*(E) \leq \infty$. Given $\varepsilon > 0$, choose sets $E_n \in \mathcal{S}$, $n \in \mathbb{N}$, such that $E \subseteq \bigcup_{n \in \mathbb{N}} E_n$ and $\sum_{n \in \mathbb{N}} \hat{\mu}(E_n) \leq \mu^*(E) + \varepsilon$. Due to Lemma A.1.2, we may assume that the E_n are pairwise disjoint. Moreover, due to the intersection stability of the semiring $\mathcal{S}$ we have $A \cap E_n \in \mathcal{S}$ for all $n \in \mathbb{N}$, so that $(A \cap E_n)_{n \in \mathbb{N}}$ is a covering of $A \cap E$ by sets from $\mathcal{S}$. Further, by (S3), for every $n \in \mathbb{N}$ there exist $C_1^n, \ldots, C_{k_n}^n \in \mathcal{S}$ such that

$$E_n \cap A^c \;=\; E_n \setminus A \;=\; \biguplus_{j=1}^{k_n} C_k^n\,.$$

We thus obtain

$$\begin{aligned} \mu^*(A \cap E) + \mu^*(A^c \cap E) \quad &\leq \quad \sum_{n \in \mathbb{N}} \hat{\mu}(E_n \cap A) + \sum_{n \in \mathbb{N}} \sum_{j=1}^{k_n} \hat{\mu}(C_k^n) \\ &\overset{\text{additivity of } \mu}{=} \quad \sum_{n \in \mathbb{N}} \hat{\mu}(E_n) \;\leq\; \mu^*(E) + \varepsilon\,. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we obtain $\mu^*(A \cap E) + \mu^*(A^c \cap E) \leq \mu^*(E)$. The opposite inequality follows from the subadditivity of μ^*. Together, this shows that A is μ^*-measurable. $\square$

Appendix B

The Hausdorff metric

Suppose (X, d) is a metric space and $A, B \subseteq X$. Then the **Hausdorff distance** between A and B is defined as

$$d_{\mathcal{H}}(A, B) \;=\; \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon(B) \text{ and } B \subseteq B_\varepsilon(A)\}\ ,$$

where $B_\varepsilon(A) = \bigcup_{x\in A} B_\varepsilon(x)$. Let $\mathcal{K}(X) = \{K \subseteq X \mid K \text{ is compact}\}$.

Theorem B.1. *Suppose that X is a compact metric space. Then the Hausdorff distance $d_{\mathcal{H}}$ defines a metric on $\mathcal{K}(X)$ and the resulting* Hausdorff space *$(\mathcal{K}(X), d_{\mathcal{H}})$ is complete. Given a Cauchy-sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(X)$, the Hausdorff limit of $(A_n)_{n\in\mathbb{N}}$ is given by*

$$A \;=\; \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m\geq n} A_m}\ . \tag{B.0.1}$$

For an arbitrary sequence $(A_n)_{n\in\mathbb{N}}$, the set defined in (B.0.1) is called the **upper Hausdorff limit**. We will denote ε-balls in $(\mathcal{K}(X), d_{\mathcal{H}})$ by $B^{\mathcal{H}}_\varepsilon(\cdot)$ and Hausdorff limits by $\lim^{\mathcal{H}}_{n\to\infty}$.

Proof. The fact that $d_{\mathcal{H}}$ defines a metric is easy to see: positive definiteness and symmetry are immediate (using that $d(x, A) > 0$ for all compact sets A and $x \notin A$ for positive definiteness), and the triangle inequality follows from the fact that $B_\varepsilon(B_\delta(A)) \subseteq B_{\varepsilon+\delta}(A)$.

In order to see that $(\mathcal{K}(X), d_{\mathcal{H}})$ is complete, suppose $(A_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\mathcal{K}(X)$. Fix $\varepsilon > 0$ and let $B_n = \overline{\bigcup_{m\geq n} A_m}$, so that $A = \bigcap_{n\in\mathbb{N}} B_n$. Then $\{B_\varepsilon(A)\} \cup \{B_n^c \mid n \in \mathbb{N}$ is an open cover of X and therefore contains a finite subcover. However, as the B_n are decreasing, this means that $B_\varepsilon(A) \cup B_n^c = X$ for some $n \in \mathbb{N}$, and hence $B_m \subseteq B_\varepsilon(A)$ for all $m \geq n$. Conversely, fix $M \in \mathbb{N}$ such that $d_{\mathcal{H}}(A_n, A_m) \leq \varepsilon/2$ for all $n, m \geq M$. Then $A \subseteq B_n \subseteq B_\varepsilon(A_n)$ for all $n \geq M$.

Together, this yields $d_{\mathcal{H}}(A, A_n) < \varepsilon$ for all $n \geq \max\{N, M\}$. As $\varepsilon > 0$ was arbitrary, this proves $A = \lim^{\mathcal{H}}_{n\to\infty} A_n$. □

Corollary B.2. *The metric space $(\mathcal{K}(\mathbb{R}^d), d_{\mathcal{H}})$ is complete.*

Proof. The fact that $d_{\mathcal{H}}$ defines a metric is independent of compactness. Given a Cauchy-sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$, we may assume due to the Cauchy-property that all the A_n are contained in a compact ball $\overline{B_R(0)}$ for some sufficiently large $R > 0$. Then $(A_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{K}(B_R(0)), d_{\mathcal{H}})$ by Theorem B.1, and hence also in $(\mathcal{K}(\mathbb{R}^d), d_{\mathcal{H}})$. □

Theorem B.3. *Suppose X is a compact metric space. Then $(\mathcal{K}(X), d_{\mathcal{H}})$ is compact as well.*

Proof. A metric space is compact if and only if it is complete and precompact (Heine-Borel). As completeness was already proven in Theorem B.1 above, it remains to prove precompactness.

In order to do so, fix $\delta > 0$. Then by compactness there exists a finite δ-dense subset $S \subseteq X$ (where we call as set S δ-dense in X if $\bigcup_{x \in S} B_\delta(x) = X$). Given $K \in \mathcal{K}(X)$, let

$$A_K \;=\; \{x \in S \mid d(x, K) < \delta\} \;.$$

Then by construction we have that $d_{\mathcal{H}}(A_K, K) < \delta$. Therefore, we have that

$$\mathcal{K}(X) \;\subseteq\; \bigcup_{A \subseteq S} B_\delta^{\mathcal{H}}(A) \;.$$

This proves precompactness. $\square$