Amorphic complexity

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We introduce amorphic complexity as a new topological invariant that measures the complexity of dynamical systems in the regime of zero entropy. Its main purpose is to detect the very onset of disorder in the asymptotic behaviour. For instance, it gives positive value to Denjoy examples on the circle and Sturmian subshifts, while being zero for all isometries and Morse-Smale systems.

After discussing basic properties and examples, we show that amorphic complexity and the underlying asymptotic separation numbers can be used to distinguish almost automorphic minimal systems from equicontinuous ones. For symbolic systems, amorphic complexity equals the box dimension of the associated Besicovitch space. In this context, we concentrate on regular Toeplitz flows and give a detailed description of the relation to the scaling behaviour of the densities of the p-skeletons. Finally, we take a look at strange non-chaotic attractors appearing in so-called pinched skew product systems. Continuous-time systems, more general group actions and the application to cut and project quasicrystals will be treated in subsequent work.

1 Introduction

The paradigm example of a topological complexity invariant for dynamical systems is topological entropy. It measures the exponential growth, in time, of orbits distinguishable at finite precision and can be used to compare the complexity of dynamical systems defined on arbitrary compact metric spaces. Moreover, it is central to the powerful machinery of thermodynamic formalism. There are, however, two situations where entropy does not provide very much information, namely when it is either zero or infinite. In the latter case, mean topological dimension has been identified as a suitable substitute. Its theoretical significance is demonstrated, for example, by the fact that zero mean dimension is one of the few dynamical consequences of unique ergodicity [LW00].

Our focus here lies on the zero entropy regime, and in particular on the very onset of dynamical complexity and the break of equicontinuity. We are looking for a dynamically defined positive real-valued quantity which

(i) is an invariant of topological conjugacy and has other good properties;

(ii) gives value zero to isometries and Morse-Smale systems;

(iii) is able to detect, as test cases, the complexity inherent in the dynamics of Sturmian shifts or Denjoy homeomorphisms on the circle, by taking positive values for such systems.

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The standard approach to measure the complexity of zero entropy systems is to consider subexponential growth rates of distinguishable orbits, instead of exponential ones, leading to the notions of power entropy and modified power entropy [HK02]. In the context of tiling spaces and minimal symbolic subshifts, power entropy is more commonly known as polynomial word complexity and presents a well-established tool to describe the complexity of aperiodic sequences. However, it turns out that power entropy gives positive values to Morse-Smale systems, whereas modified power entropy is too coarse to distinguish Sturmian subshifts or Denjoy examples from irrational rotations. We are thus taking an alternative and complementary direction, which leads us to define the notions of *asymptotic separation numbers* and *amorphic complexity*. Those are based on an asymptotic notion of separation, which is the main qualitative difference to the previous concepts, since the latter rely in their definition on the classical Bowen-Dinaburg/Hamming metrics which consider only finite time-scales. As a consequence, ergodic theorems can be applied in a more or less direct way to compute or estimate amorphic complexity in many situations. In order to fix ideas, we concentrate on the dynamics of continuous maps defined on metric spaces. Continuous-time systems and more general group actions will be treated in future work.

Let (X, d) be a metric space and $f : X \to X$. Given $x, y \in X$, $\delta > 0$, $\nu \in (0, 1]$ and $n \in \mathbb{N}$ we let

$$S_n(f, \delta, x, y) \; := \; \# \left\{ 0 \leq k < n \mid d(f^k(x), f^k(y)) \geq \delta \right\} . \qquad (1)$$

We say that x and y are (f, δ, ν)-*separated* if

$$\varlimsup_{n \to \infty} \frac{S_n(f, \delta, x, y)}{n} \; \geq \; \nu \; .$$

A subset $S \subseteq X$ is said to be (f, δ, ν)-*separated* if all distinct points $x, y \in S$ are (f, δ, ν)-separated. The *(asymptotic) separation number* $\mathrm{Sep}(f, \delta, \nu)$, for distance $\delta > 0$ and frequency $\nu \in (0, 1]$, is then defined as the largest cardinality of an (f, δ, ν)-separated set in X. If these quantities are finite for all $\delta, \nu > 0$, we say f has *finite separation numbers*, otherwise we say it has *infinite separation numbers*. Further, if $\mathrm{Sep}(f, \delta, \nu)$ is uniformly bounded in ν for all $\delta > 0$, we say that f has *bounded separation numbers*, otherwise we say *separation numbers are unbounded*.

These notions provide a first qualitative indication concerning the complexity of a system. Roughly spoken, finite but unbounded separation numbers correspond to dynamics of intermediate complexity, which we are mainly interested in here. Once a system behaves 'chaotically', in the sense of positive entropy or weak mixing, separation numbers become infinite.

Theorem 1.1. *Suppose X is a compact metric space and $f : X \to X$ is continuous. If f has positive topological entropy or is weakly mixing with respect to some invariant probability measure μ with non-trivial support, then it has infinite separation numbers.*

The proof is given in Section 2. Obviously, if f is an isometry or, more generally, equicontinuous, then its separation numbers are bounded. Moving away from equicontinuity, one encounters the class of almost automorphic systems, which are central objects of study in topological dynamics and include many examples of both theoretical and practical importance [Aus88]. At least in the minimal case, separation numbers are suited to describe this transition, as the following result shows. In order to state it, suppose that (X, d) and (Ξ, ρ) are metric spaces and $f : X \to X$ and $g : \Xi \to \Xi$ are continuous. We say that f is an extension of g if there exists a continuous onto map $h : X \to \Xi$ such that $h \circ f = g \circ h$. The map f is called an *almost 1-1 extension* of g if the set $\{\xi \in \Xi \mid \#h^{-1}(\xi) = 1\}$ is dense in Ξ. In the

case that g is minimal, this condition can be replaced by the weaker assumption that there exists one $\xi \in \Xi$ with $\#h^{-1}(\xi) = 1$. We further say that f is an *almost sure 1-1 extension* if the set $\{\xi \in \Xi \mid \#h^{-1}(\xi) > 1\}$ has measure zero with respect to every g-invariant probability measure μ on Ξ.[1] Due to Veech's Structure Theorem [Vee65], almost automorphic minimal systems can be defined as almost 1-1 extensions of equicontinuous minimal systems.

Theorem 1.2. *Suppose X is a compact metric space and $f : X \to X$ is a homeomorphism.*

(a) *If f is minimal and almost automorphic, but not equicontinuous, then f has unbounded separation numbers.*

(b) *If f is an almost sure 1-1 extension of an equicontinuous system, then f has finite separation numbers.*

Again, the proof is given in Section 2. Two examples for case (b) discussed below are regular Toeplitz flows and Delone dynamical systems arising from cut and project quasicrystals. We refer to [LTY14, DG15] for some recent progress on extensions of minimal equicontinuous systems.

In order to obtain quantitative information, we proceed to study the scaling behaviour of separation numbers as the separation frequency ν goes to zero. In principle, one may consider arbitrary growth rates (see Section 3). However, as the examples we discuss all indicate, it is polynomial growth which is the most relevant. Given $\delta > 0$, we let

$$\underline{\mathrm{ac}}(f, \delta) \; := \; \varliminf_{\nu \to 0} \frac{\log \mathrm{Sep}(f, \delta, \nu)}{-\log \nu} \;, \qquad \overline{\mathrm{ac}}(f, \delta) \; := \; \varlimsup_{\nu \to 0} \frac{\log \mathrm{Sep}(f, \delta, \nu)}{-\log \nu} \tag{2}$$

and define the *lower*, respectively *upper amorphic complexity of f* as

$$\underline{\mathrm{ac}}(f) \; := \; \sup_{\delta > 0} \underline{\mathrm{ac}}(f, \delta) \quad \text{and} \quad \overline{\mathrm{ac}}(f) \; := \; \sup_{\delta > 0} \overline{\mathrm{ac}}(f, \delta) \;. \tag{3}$$

If both values coincide, $\mathrm{ac}(f) := \underline{\mathrm{ac}}(f) = \overline{\mathrm{ac}}(f)$ is called the *amorphic complexity of f*. We note once more that the main difference to the notion of (modified) power entropy is the fact that we use an asymptotic concept of separation, and the scaling behaviour that is measured is not the one with respect to time, but that with respect to the separation frequency. Somewhat surprisingly, this makes amorphic complexity quite well-accessible to rigorous computations and estimates. The reason is that separation frequencies often correspond to certain ergodic averages or visiting frequencies, which can be determined by the application of ergodic theorems. We have the following basic properties.

Proposition 1.3. *Suppose X, Y are compact metric spaces and $f : X \to X$, $g : Y \to Y$ continuous maps. Then the following statements hold.*

(a) Factor relation: *If g is a factor of f, then $\overline{\mathrm{ac}}(f) \geq \overline{\mathrm{ac}}(g)$ and $\underline{\mathrm{ac}}(f) \geq \underline{\mathrm{ac}}(g)$. In particular, amorphic complexity is an invariant of topological conjugacy.*

(b) Power invariance: *For all $m \in \mathbb{N}$ we have $\overline{\mathrm{ac}}(f^m) = \overline{\mathrm{ac}}(f)$ and $\underline{\mathrm{ac}}(f^m) = \underline{\mathrm{ac}}(f)$.*

(c) Product formula: *If upper and lower amorphic complexity coincide for both f and g, then the same holds for $f \times g$ and we have $\mathrm{ac}(f \times g) = \mathrm{ac}(f) + \mathrm{ac}(g)$. Otherwise, we have $\overline{\mathrm{ac}}(f \times g) \leq \overline{\mathrm{ac}}(f) + \overline{\mathrm{ac}}(g)$ and $\underline{\mathrm{ac}}(f \times g) \geq \underline{\mathrm{ac}}(f) + \underline{\mathrm{ac}}(g)$.*

(d) Commutation invariance: $\overline{\mathrm{ac}}(f \circ g) = \overline{\mathrm{ac}}(g \circ f)$ *and* $\underline{\mathrm{ac}}(f \circ g) = \underline{\mathrm{ac}}(g \circ f)$.

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[1]Note that if g is equicontinuous and minimal, then it is uniquely ergodic. Hence, there is only one measure to consider in this case.

As the power invariance indicates, amorphic complexity behaves quite different from topological entropy in some aspects. In this context, it should also be noted that no variational principle can be expected for amorphic complexity. This is a direct consequence of requirement (iii) above, which is met by amorphic complexity (see Proposition 1.4). The reason is that since Sturmian subshifts, Denjoy examples and irrational rotations are uniquely ergodic and measure-theoretically isomorphic, they cannot be distinguished on a measure-theoretic level. Hence, no reasonable analogue to the variational principle of topological entropy can be satisfied.

Proposition 1.4. *Amorphic complexity is zero for all isometries and Morse-Smale systems, but equals one for Sturmian subshifts and Denjoy examples on the circle.*

The proof is given in Sections 3.5 and 3.6. By means of some elementary examples in Section 3.7, we will also demonstrate that no direct relations – in terms of inequalities – exist between amorphic complexity and the notions of power entropy and modified power entropy.

The arguments in the proof of Theorem 1.2 can be quantified, at least to some extent, to obtain an upper bound on amorphic complexity for minimal almost sure 1-1 extensions of isometries. In rough terms, the result reads as follows. Details will be given in Section 4. By $\overline{\mathrm{Dim}}_B(A)$ we denote the upper box dimension of a totally bounded subset A of a metric space.

Theorem 1.5. *Suppose X and Ξ are compact metric spaces and $f : X \to X$ is an almost sure 1-1 extension of a minimal isometry $g : \Xi \to \Xi$, with factor map h. Further, assume that the upper box dimension of Ξ is finite and strictly positive. Then*

$$\overline{\mathrm{ac}}(f) \ \leq \ \frac{\gamma(h) \cdot \overline{\mathrm{Dim}}_B(\Xi)}{\overline{\mathrm{Dim}}_B(\Xi) - \sup_{\delta > 0} \overline{\mathrm{Dim}}_B(E_\delta)} \ , \tag{4}$$

where $E_\delta = \{\xi \in \Xi \mid \mathrm{diam}(h^{-1}(\xi)) \geq \delta\}$ and $\gamma(h)$ is a scaling factor depending on the local properties of the factor map h.

The proof is given in Section 4. It should be mentioned, at least according to our current understanding, that this result is of rather abstract nature. The reason is the fact that the scaling factor $\gamma(h)$, defined by (30), seems to be difficult to determine in specific examples. However, it turns out that in many cases direct methods can be used instead to obtain improved explicit estimates.

In this direction, we will first investigate *regular Toeplitz flows* in Section 5. Given a finite alphabet A, a sequence $\omega = (\omega_k)_{k \in \mathbb{I}} \in A^{\mathbb{I}}$ with $\mathbb{I} = \mathbb{N}_0$ or $\mathbb{Z}$ is called *Toeplitz* if for all $k \in \mathbb{I}$ there exists $p \in \mathbb{N}$ such that $\omega_{k+p\ell} = \omega_k$ for all $\ell \in \mathbb{N}$. In other words, every symbol in a Toeplitz sequence occurs periodically. Thus, if we let $\mathrm{Per}(p, \omega) = \{k \in \mathbb{I} \mid \omega_{k+p\ell} = \omega_k$ for all $\ell \in \mathbb{N}\}$, then $\bigcup_{p \in \mathbb{N}} \mathrm{Per}(p, \omega) = \mathbb{I}$. By $D(p) = \#(\mathrm{Per}(p, \omega) \cap [0, p-1])/p$ we denote the density of the p-periodic positions. If $\lim_{p \to \infty} D(p) = 1$, then the Toeplitz sequence is called *regular*. A well-known example of a regular Toeplitz sequence is the paperfolding sequence, also known as the dragon curve sequence [AB92].

We call a sequence $(p_\ell)_{\ell \in \mathbb{N}}$ of integers such that $p_{\ell+1}$ is a multiple of p_ℓ for all $\ell \in \mathbb{N}$ and $\bigcup_{\ell \in \mathbb{N}} \mathrm{Per}(p_\ell, \omega) = \mathbb{I}$ a *weak periodic structure* for ω. More details are given in Section 5. We denote the shift orbit closure of ω by Σ_ω such that (Σ_ω, σ) is the subshift generated by ω.

Theorem 1.6. *Suppose ω is a non-periodic regular Toeplitz sequence with weak periodic structure $(p_\ell)_{\ell \in \mathbb{N}}$. Then*

$$\overline{\mathrm{ac}}\left(\sigma|_{\Sigma_\omega}\right) \ \leq \ \varlimsup_{\ell \to \infty} \frac{\log p_{\ell+1}}{\log(1 - D(p_\ell))} \ .$$

In Section 5, we demonstrate by means of examples that this estimate is sharp and that a dense set of values in $[1, \infty)$ is attained (Theorem 5.6 and Corollary 5.7). A more comprehensive treatment of amorphic complexity for symbolic systems of intermediate complexity (zero topological entropy) will be given in [FG] (see also Section 3.8).

In Section 6, we take a closer look at strange non-chaotic attractors (SNA) appearing in so-called pinched skew products. The latter are known as paradigm examples for the occurrence of SNA [GOPY84, Kel96, GJK06]. In this case technical issues prevent a straightforward computation of amorphic complexity, and we are only able to apply a modified version of the concept. However, since the attempt to distinguish SNA and smooth (non-strange) attractors in skew product systems by means of topological invariants has been the origin of our investigations, it seemed important to include these findings.

Finally, we also want to include a research announcement of a result from the forthcoming paper [FGJ], which fits well into the above discussion. Suppose $\tilde{L}$ is a cocompact discrete subgroup of $\mathbb{R}^m \times \mathbb{R}^D$ such that $\pi_1 : \tilde{L} \to \mathbb{R}^m$ is injective and $\pi_2 : \tilde{L} \to \mathbb{R}^D$ has dense image. Further, assume that $W \subset \mathbb{R}^D$ is compact and satisfies $W = \overline{\text{int}(W)}$. The pair $(\tilde{L}, W)$ is called a *cut and project scheme* and defines a Delone subset $\Lambda(W) := \pi_1\big((\mathbb{R}^m \times W) \cap \tilde{L}\big)$ of $\mathbb{R}^m$. A natural $\mathbb{R}^m$-action on the space of Delone sets in $\mathbb{R}^m$ is given by $(t, \Lambda) \mapsto \Lambda - t$. Taking the orbit closure $\Omega(\Lambda(W)) := \overline{\{\Lambda(W) - t \mid t \in \mathbb{R}^m\}}$ of $\Lambda(W)$, in a suitable topology, we obtain a *Delone dynamical system* $(\Omega(\Lambda(W)), \mathbb{R}^m)$ whose dynamical properties are closely related to the geometry of the Delone set $\Lambda(W)$. We refer to [Sch99, Moo00, LP03, BLM07] and references therein for further details. For the amorphic complexity, adapted to general actions of amenable groups, we obtain

Theorem 1.7 ([FGJ]). *Suppose $(\tilde{L}, W)$ is a cut and project scheme in $\mathbb{R}^m \times \mathbb{R}^D$ and $(\Omega(\Lambda(W)), \mathbb{R}^m)$ is the associated Delone dynamical system. Then*

$$\overline{\text{ac}}(\Omega(\Lambda(W)), \mathbb{R}^m) \ \leq \ \frac{D}{D - \overline{\text{Dim}}_B(\partial W)} \ . \tag{5}$$

As in the case of regular Toeplitz flows, it can be demonstrated by means of examples that this estimate is sharp. At the same time, equality does not always hold.

It is well-known that under the above assumptions the dynamical system $(\Omega(\Lambda(W)), \mathbb{R}^m)$ is an almost 1-1 extension of a minimal and isometric $\mathbb{R}^m$-action on a D-dimensional torus. Moreover, it turns out that with the notions of Theorem 1.5 we have $\overline{\text{Dim}}_B(\partial W) = \overline{\text{Dim}}_B(E_\delta)$ for all $\delta > 0$. Thus, (5) can be interpreted as a special case of (4), with $\gamma(h) = 1$. However, as we have mentioned, the proof is independent and based on more direct arguments.

2 Qualitative behaviour of asymptotic separation numbers

We start by recalling some basic facts from ergodic theory. Let $(X, \mathcal{B}, \mu)$ be a probability space and let μ be invariant with respect to the measurable map $f : X \to X$. We say μ is *ergodic with respect to f* if all sets $B \in \mathcal{B}$ with $f^{-1}(B) = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$. Further, we say μ is *weak-mixing with respect to f* if for all $A, B \in \mathcal{B}$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \mu(f^{-k}(A) \cap B) - \mu(A)\mu(B) \right| = 0 .$$

Theorem 2.1 ([Wal82, Theorem 1.24])**.** *The following statements are equivalent*

(a) *μ is weak-mixing with respect to f,*

(b) *$\mu \times \mu$ is weak-mixing with respect to $f \times f$ and*

(c) *$\mu \times \mu$ is ergodic with respect to $f \times f$.*

Moreover, we have

Theorem 2.2 ([BS02, Theorem 4.10.6])**.** *The measure μ is weak-mixing with respect to f if and only if for all probability measures ν on a measure space $(Y, \mathcal{A}, \nu)$ that are invariant and ergodic with respect to a measurable transformation $g : Y \to Y$, we have that $\mu \times \nu$ is ergodic with respect to $f \times g$.*

In particular, it follows by induction that if μ is weak-mixing, then the measures $\mu^m := \bigtimes_{k=1}^{m} \mu$ are ergodic with respect to $\bigtimes_{k=1}^{m} f$ for all $m \geq 2$.

Theorem 2.3. *Let (X, d) be a metric space. Suppose $f : X \to X$ is Borel measurable and μ is a Borel probability measure invariant under f. Furthermore, assume that μ is weak-mixing with respect to f and its support is not a single point. Then f has infinite separation numbers.*

Note that the assumption that f is only measurable is consistent with the definition of asymptotic separation numbers, since a priori, we do not require any regularity of the map f in their definition.

Proof. For each $\delta > 0$ we define the function $h_\delta : X^2 \to \{0, 1\}$ as $h_\delta(z, w) := \Theta(d(z, w) - \delta)$ where $\Theta : \mathbb{R} \to \{0, 1\}$ is the Heaviside step function. Note that

$$\frac{1}{n} S_n(f, \delta, x, y) = \frac{1}{n} \sum_{k=0}^{n-1} h_\delta\big(f^k(x), f^k(y)\big) .$$

Since μ is not supported on a single point, we can find $\delta_0 > 0$ and $\nu_0 > 0$ such that for all $\delta \leq \delta_0$ we have

$$\int h_\delta d\mu^2 \geq \nu_0 . \tag{6}$$

(Note that $\int h_{\delta'} d\mu^2 \geq \int h_\delta d\mu^2$ for $\delta' \leq \delta$.) Fix $\delta \in (0, \delta_0]$, $\nu \in (0, \nu_0]$ and let, for each $m \geq 2$,

$$\varphi_m : X^m \to \mathbb{R}^{m(m-1)/2} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \mapsto \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \begin{pmatrix} h_\delta(f^k(x_1), f^k(x_2)) \\ h_\delta(f^k(x_1), f^k(x_3)) \\ \vdots \\ h_\delta(f^k(x_{m-1}), f^k(x_m)) \end{pmatrix} . \tag{7}$$

Observe that since h_δ is bounded, the functions $(x_1, \ldots, x_m) \mapsto h_\delta(x_i, x_j)$ with $1 \leq i < j \leq m$ are in $L^1(\mu^m)$. By ergodicity of μ^m, the limits in (7) exist μ^m-almost everywhere. Further, φ_m is μ^m-almost surely constant and all its entries are different from zero, since we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_\delta \left(f^k(x_i), f^k(x_j) \right) = \int_{X^m} h_\delta(x_i, x_j) d\mu^m(x_1, \ldots, x_m)$$

$$= \int_{X^2} h_\delta(x_i, x_j) d\mu(x_i) d\mu(x_j) \geq \nu_0 > 0$$

for $1 \leq i < j \leq m$ by (6). Thus, the above implies that for each $m \in \mathbb{N}$ there exist at least m points that are pairwise (f, δ, ν)-separated, so that

$$\mathrm{Sep}(f, \delta, \nu) \geq m .$$

Since m was arbitrary and the pair (δ, ν) is fixed, we get that $\mathrm{Sep}(f, \delta, \nu)$ is infinite. $\qquad\square$

The analogous statement for maps with positive topological entropy is a direct consequence of a result of Downarowicz in [Dow14]. In order to state it, we say that two points x and y in a metric space (X, d) are *DC2-scrambled with respect to* f if the following two conditions are fulfilled

$$\forall \delta > 0 \quad : \quad \overline{\lim_{n\to\infty}} \frac{\# \left\{ 0 \leq k < n \mid d(f^k(x), f^k(y)) < \delta \right\}}{n} = 1 ,$$

$$\exists \delta_0 > 0 \quad : \quad \underline{\lim_{n\to\infty}} \frac{\# \left\{ 0 \leq k < n \mid d(f^k(x), f^k(y)) < \delta_0 \right\}}{n} < 1 . \tag{8}$$

Furthermore, we say that a subset $S \subseteq X$ is *DC2-scrambled* if any pair $x, y \in S$ with $x \neq y$ is DC2-scrambled. The set S is called *uniformly DC2-scrambled* if the δ_0's and the lower frequencies in (8) are uniform for all pairs $x, y \in S$ with $x \neq y$. Now by [Dow14, Theorem 1.2], if f has positive topological entropy, then there exists an uncountable DC2-scrambled set S, and as stated in [Dow14, Remark 2] this set can be chosen uniformly DC2-scrambled. It is then obvious from (8) that the points in S are pairwise (f, δ, ν)-separated for the respective parameters $\delta, \nu > 0$, i.e. $\mathrm{Sep}(f, \delta, \nu) = \infty$. Thus, we obtain

Theorem 2.4. *Let (X, d) be a compact metric space. Suppose $f : X \to X$ is a continuous map with positive topological entropy. Then f has infinite separation numbers.*

We now turn to the opposite direction and aim to show that almost sure 1-1 extensions of equicontinuous systems have finite separation numbers. In order to do so, we need to introduce some further notions and preliminary statements. Suppose (X, d) and (Ξ, ρ) are compact metric spaces and $f : X \to X$ is an extension of $g : \Xi \to \Xi$ with factor map $h : X \to \Xi$. For $x \in X$, define the *fibre of* x as $F_x := h^{-1}(h(x))$. Denote the collection of fibres by $\mathcal{F} := \{F_x \mid x \in X\}$. Given $\delta > 0$, let

$$\mathcal{F}_\delta := \{x \in X \mid \mathrm{diam}(F_x) \geq \delta\} = \bigcup_{\substack{F \subseteq \mathcal{F} \\ \mathrm{diam}(F) \geq \delta}} F .$$

Further, let $\mathcal{F}_{>0} := \bigcup_{\delta>0} \mathcal{F}_\delta$, $E_\delta := h(\mathcal{F}_\delta)$ and $E := h(\mathcal{F}_{>0})$. Obviously, both $\mathcal{F}_\delta$ and E_δ are decreasing in δ.

The next lemma is well-known and we omit the easy proof.

Lemma 2.5. *The set $\mathcal{F}_\delta$ is closed for all $\delta > 0$.*

Note that as a direct consequence the sets $\mathcal{F}_{>0}$, E_δ and E are Borel measurable. The following basic observation will be crucial in the proof of the next theorem. From now on, we denote by $B_\varepsilon(A)$ for $\varepsilon > 0$ the open ε-neighborhood of a subset A of a metric space.

Lemma 2.6. *For all $\delta > 0$ and $\varepsilon > 0$ there exists $\eta = \eta_\delta(\varepsilon) > 0$ such that if $x, y \in X$ satisfy $d(x, y) \geq \delta$ and $\rho(h(x), h(y)) < \eta$, then $h(x)$ and $h(y)$ are contained in $B_\varepsilon(E_\delta)$.*

Proof. Assume for a contradiction that the statement is false. Then there are $\delta, \varepsilon > 0$ and sequences $(x_k)_{k \in \mathbb{N}}$, $(y_k)_{k \in \mathbb{N}}$ in X such that $h(x_k) \notin B_\varepsilon(E_\delta)$ or $h(y_k) \notin B_\varepsilon(E_\delta)$ and $d(x_k, y_k) \geq \delta$ for all $k \in \mathbb{N}$, but $\rho(h(x_k), h(y_k)) \to 0$ as $k \to \infty$. By going over to subsequences if necessary, we may assume that $(h(x_k))_{k \in \mathbb{N}}$ lies in $X \setminus B_\varepsilon(E_\delta)$ and that $(x_k)_{k \in \mathbb{N}}$ and $(y_k)_{k \in \mathbb{N}}$ converge. Let $x := \lim_{k \to \infty} x_k$ and $y := \lim_{k \to \infty} y_k$. Then $d(x, y) \geq \delta$ and $h(x) = \lim_{k \to \infty} h(x_k) \notin B_\varepsilon(E_\delta)$. However, $h(x) = h(y)$ and thus $\mathrm{diam}(F_x) = \mathrm{diam}(F_y) \geq \delta$, such that $x \in \mathcal{F}_\delta$, which is the required contradiction. $\square$

Theorem 2.7. *Let $f : X \to X$ be a continuous map. Further, assume that f is an almost sure 1-1 extension of an isometry $g : \Xi \to \Xi$. Then f has finite separation numbers.*

Note that any equicontinuous system is an isometry with respect to an equivalent metric. Since finiteness of separation numbers is preserved by topological conjugacy (see also next section), this implies Theorem 1.2(b).

Proof. Denote by $\mathcal{M}(g)$ the set of all g-invariant Borel probability measures on Ξ. Fix $\delta > 0$ and $\nu > 0$. We claim that since $\mu(E_\delta) \leq \mu(E) = 0$ for all $\mu \in \mathcal{M}(g)$, there exists $\varepsilon > 0$ such that

$$\mu\left(\overline{B_\varepsilon(E_\delta)}\right) \;<\; \nu \quad \text{for all } \mu \in \mathcal{M}(g) . \tag{9}$$

Otherwise, it would be possible to find a sequence $\mu_n \in \mathcal{M}(g)$ with $\mu_n(\overline{B_{1/n}(E_\delta)}) \geq \nu$, which can be chosen such that it converges to some $\mu \in \mathcal{M}(g)$ in the weak-$*$-topology. If $\varphi_m(\xi) := \max\{1 - m \cdot d(\xi, B_{1/m}(E_\delta)), 0\}$, then we have $\int_\Xi \varphi_m \, d\mu_n \geq \nu$ for all $n \geq m$ and hence $\int_\Xi \varphi_m \, d\mu \geq \nu$ for all $m \in \mathbb{N}$. However, this implies $\mu(E_\delta) \geq \nu$ by dominated convergence, contradicting our assumptions. Hence, we may choose $\varepsilon > 0$ as in (9).

This, in turn, implies that

$$\varlimsup_{n \to \infty} \frac{\#\left\{0 \leq k < n \mid g^k(\xi) \in \overline{B_\varepsilon(E_\delta)}\right\}}{n} \;<\; \nu \tag{10}$$

for all $\xi \in \Xi$. If this was not the case, it would again be possible to construct a g-invariant measure μ contradicting (9), this time as a limit of finite sums $\mu_\ell := \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \delta_{g^k(\xi)}$ of weighted Dirac measures for some $\xi \in \Xi$ that does not satisfy (10). (Note that in this situation we have $\mu_\ell(\overline{B_\varepsilon(E_\delta)}) \geq \nu$ for all $\ell \in \mathbb{N}$, and this inequality carries over to the limit μ by the Portmanteau Theorem.)

Hence, given any pair $x, y \in X$, the frequency by which both of the iterates of $h(x)$ and $h(y)$ visit $\overline{B_\varepsilon(E_\delta)}$ at the same time is smaller than ν. Together with Lemma 2.6, this implies that if $\rho(h(x), h(y)) < \eta_\delta(\varepsilon)$, then the points x and y cannot be (f, δ, ν)-separated. Thus, if $S \subseteq X$ is an (f, δ, ν)-separated set, then the set $h(S)$ must be $\eta_\delta(\varepsilon)$-separated (compare Section 3.8) with respect to the metric ρ. By compactness, the maximal cardinality N of an $\eta_\delta(\varepsilon)$-separated set in Ξ is bounded. We obtain

$$\mathrm{Sep}(f, \delta, \nu) \;\leq\; N . \tag{11}$$

Since $\delta > 0$ and $\nu > 0$ where arbitrary, this completes the proof. $\square$

As immediate consequences, we obtain

Corollary 2.8. *If for all $\delta > 0$ the set E_δ is finite and contains no periodic point, then f has finite separation numbers.*

Corollary 2.9. *If $\lim_{n \to \infty} \operatorname{diam}\big(F_{f^n(x)}\big) = 0$ for all $x \in X$, then f has finite separation numbers.*

For the second corollary, use Poincaré's Recurrence Theorem to get a contradiction. It remains to prove part (a) of Theorem 1.2, which we restate as

Theorem 2.10. *Let $f : X \to X$ be a continuous map. Further, assume that f is a minimal almost 1-1 extension of an isometry $g : \Xi \to \Xi$ such that the factor map h is not injective. Then f has unbounded separation numbers.*

For the proof, we will again need two preliminary lemmas. Given $x, y \in X$ and $\delta > 0$, we let

$$\nu(f, \delta, x, y) \; := \; \overline{\lim_{n \to \infty}} \, \frac{1}{n} S_n(f, \delta, x, y) \; . \tag{12}$$

Lemma 2.11. *Suppose $V_1, V_2 \subseteq \Xi$ are two open sets which satisfy $d(h^{-1}(V_1), h^{-1}(V_2)) \geq \delta$. Then $\nu(f, \delta, x_1, x_2) > 0$ for all $x_1 \in h^{-1}(V_1)$ and $x_2 \in h^{-1}(V_2)$.*

Proof. Let $\xi_1 := h(x_1)$ and $\xi_2 := h(x_2)$. By assumption, we have that $d(f^k(x_1), f^k(x_2)) \geq \delta$ whenever $g^k(\xi_1) \in V_1$ and $g^k(\xi_2) \in V_2$. Consequently,

$$\nu(f, \delta, x_1, x_2) \; \geq \; \overline{\lim_{n \to \infty}} \, \frac{1}{n} \# \left\{ 0 \leq k < n \mid (g \times g)^k(\xi_1, \xi_2) \in V_1 \times V_2 \right\} \; . \tag{13}$$

However, as g is an isometry, so is $g \times g$. This implies that all points $(\xi_1, \xi_2) \in \Xi \times \Xi$ are almost periodic, and the set of return times to any of their neighbourhoods is syndetic [Aus88]. Hence, the right-hand side of (13) is strictly positive. $\square$

Lemma 2.12. *Suppose f is a minimal almost 1-1 extension of g and $\operatorname{diam}(h^{-1}(\xi)) > \delta$ for some $\xi \in \Xi$. Then for every neighbourhood U of ξ there exist $V_1, V_2 \subseteq U$ such that $d(h^{-1}(V_1), h^{-1}(V_2)) > \delta$.*

Proof. Due to minimality, singleton fibres are dense in X. Hence, it is possible to find $x_1, x_2 \in h^{-1}(U)$ such that $F_{x_i} = \{x_i\}$, $i \in \{1, 2\}$ and $d(x_1, x_2) > \delta$. Then, by continuity, any sufficiently small neighbourhoods V_i of $h(x_i)$ will satisfy $d(h^{-1}(V_1), h^{-1}(V_2)) > \delta$. $\square$

Proof of Theorem 2.10. Since the factor map h is not injective, there exists $\xi \in \Xi$ with $\operatorname{diam}(h^{-1}(\xi)) > \delta$ for some $\delta > 0$. We will construct, by induction on $k \in \mathbb{N}$ with $k \geq 2$, a sequence of finite families of disjoint open sets $V_1^k, \ldots, V_k^k$ with the property that for all $1 \leq i < j \leq k$ there exists $n_{i,j}^k \in \mathbb{N}_0$ such that

$$d \left(h^{-1} \left(g^{n_{i,j}^k} \left(V_i^k \right) \right), h^{-1} \left(g^{n_{i,j}^k} \left(V_j^k \right) \right) \right) \; > \; \delta \; . \tag{14}$$

For any family of points $x_i^k \in h^{-1}(V_i^k)$, $i \in \{1, \ldots, k\}$, and $1 \leq i < j \leq k$ we will then have

$$\nu \left(f, \delta, x_i^k, x_j^k \right) \; = \; \nu \left(f, \delta, f^{n_{i,j}^k} \left(x_i^k \right), f^{n_{i,j}^k} \left(x_j^k \right) \right) \; > \; 0$$

by Lemma 2.11. Thus, if $\nu_k := \min \left\{ \nu \left(f, \delta, x_i^k, x_j^k \right) \mid 1 \leq i < j \leq k \right\}$, then $\{x_1^k, \ldots, x_k^k\}$ is a (f, δ, ν_k)-separated set of cardinality k. This implies that $\sup_{\nu > 0} \operatorname{Sep}(f, \delta, \nu)$ is infinite, as required, since k was arbitrary.

It remains to construct the disjoint open sets V_i^k. For $k = 2$, the sets V_1^2 and V_2^2 can be chosen according to Lemma 2.12 with $n_{1,2}^2 = 0$. Suppose that $V_1^k, \ldots, V_k^k$ have been constructed as above. By minimality, there exists $n \in \mathbb{N}$ such that $g^n(V_k^k)$ is a neighbourhood of ξ. Lemma 2.12 yields the existence of open sets $V, V' \subseteq g^n(V_k^k)$ with $d(h^{-1}(V), h^{-1}(V')) > \delta$. We now set

$$V_i^{k+1} := V_i^k \ \text{ for } \ i \in \{1, \ldots, k-1\} \, , \ V_k^{k+1} := g^{-n}(V) \ \text{ and } \ V_{k+1}^{k+1} := g^{-n}(V') \, ,$$

so that $V_k^{k+1} \cup V_{k+1}^{k+1} \subseteq V_k^k$. Choosing $n_{i,j}^{k+1} := n_{i,j}^k$ if $1 \leq i < j \leq k-1$, $n_{i,j}^{k+1} := n_{i,k}^k$ if $1 \leq i \leq k-1$ and $j \in \{k, k+1\}$ and $n_{k,k+1}^{k+1} := n$, we obtain that (14) is satisfied for all $1 \leq i < j \leq k+1$. $\qquad\square$

3 Properties of amorphic complexity and basic examples

3.1 More general growth rates

As mentioned in the introduction, one may consider more general than just polynomial growth rates in the definition of amorphic complexity. We call $a : \mathbb{R}_+ \times (0,1] \to \mathbb{R}_+$ a *scale function* if $a(s, \nu)$ is increasing in s and ν and $\lim_{\nu \to 0} a(s, \nu) = \infty$ for all $s \in \mathbb{R}_+$. If the separation numbers of f are finite, then we let

$$
\begin{aligned}
\underline{\mathrm{ac}}(f, a, \delta) &:= \sup\left\{ s > 0 \ \Big| \ \varliminf_{\nu \to 0} \frac{\mathrm{Sep}(f, \delta, \nu)}{a(s, \nu)} > 0 \right\} \, , \\
\overline{\mathrm{ac}}(f, a, \delta) &:= \sup\left\{ s > 0 \ \Big| \ \varlimsup_{\nu \to 0} \frac{\mathrm{Sep}(f, \delta, \nu)}{a(s, \nu)} > 0 \right\}
\end{aligned}
\tag{15}
$$

and proceed to define the *lower and upper amorphic complexity of f with respect to the scale function a* as

$$
\begin{aligned}
\underline{\mathrm{ac}}(f, a) &:= \sup_{\delta > 0} \underline{\mathrm{ac}}(f, a, \delta) \, , \\
\overline{\mathrm{ac}}(f, a) &:= \sup_{\delta > 0} \overline{\mathrm{ac}}(f, a, \delta) \, .
\end{aligned}
\tag{16}
$$

As before, if $\underline{\mathrm{ac}}(f, a) = \overline{\mathrm{ac}}(f, a)$, then their common value is denoted by $\mathrm{ac}(f, a)$. If $a(s, \nu) = \nu^{-s}$, then this reduces to the definition given in the introduction.

In order to obtain good properties, however, some regularity has to be imposed on the scale function. We say a scale function a is *O-(weakly) regularly varying (at the origin) with respect to ν* if

$$\varlimsup_{\nu \to 0} \frac{a(s, c\nu)}{a(s, \nu)}$$

is finite for each $s, c > 0$. Under this assumption, a part of the theory can be developed in a completely analogous way, until specific properties of polynomial growth start to play a role. In particular, the statements presented in Section 3.2 remain true in this more general situation. However, we will not make this explicit, for the sake of simplicity of the exposition. For more information on O-regularly varying functions, see for example [AA77, BKS06] and references therein.

3.2 Definition via (f, δ, ν)-spanning sets

As in the case of topological entropy, amorphic complexity can be defined in an equivalent way by using spanning sets instead of separating sets. A subset S of a metric space (X, d)

is said to be (f, δ, ν)-*spanning* if for all $x \in X$ there exists a $y \in S$ such that

$$\varlimsup_{n \to \infty} \frac{S_n(f, \delta, x, y)}{n} \; < \; \nu \; .$$

By $\mathrm{Span}(f, \delta, \nu)$ we denote the smallest cardinality of any (f, δ, ν)-spanning set in X.

Lemma 3.1. *Let $f : X \to X$ be a map, $\delta > 0$ and $\nu \in (0, 1]$. We have that*

$$\mathrm{Sep}(f, \delta, \nu) \; \geq \; \mathrm{Span}(f, \delta, \nu) \quad and \quad \mathrm{Span}(f, \delta, \nu/2) \; \geq \; \mathrm{Sep}(f, 2\delta, \nu) \; . \tag{17}$$

Proof. For the first inequality, assume without loss of generality that $\mathrm{Sep}(f, \delta, \nu) < \infty$. Consider an (f, δ, ν)-separated set $S \subseteq X$ with maximal cardinality $\mathrm{Sep}(f, \delta, \nu)$. Then for any $x \in X$ with $x \notin S$ the set $S \cup \{x\}$ is not (f, δ, ν)-separated. Hence, there exists a $y \in S$ such that

$$\varlimsup_{n \to \infty} \frac{S_n(f, \delta, x, y)}{n} \; < \; \nu \; .$$

In other words, S is also (f, δ, ν)-spanning. Thus, we obtain $\mathrm{Sep}(f, \delta, \nu) \geq \mathrm{Span}(f, \delta, \nu)$.

For the second inequality, assume without loss of generality that $\mathrm{Span}(f, \delta, \nu/2) < \infty$. Let $S \subseteq X$ be an $(f, \delta, \nu/2)$-spanning set of cardinality $\mathrm{Span}(f, \delta, \nu/2)$ and assume for a contradiction that $\tilde{S} \subseteq X$ is an $(f, 2\delta, \nu)$-separated set with $\#\tilde{S} > \#S$. Then for some $y \in S$ there exist $x_1, x_2 \in \tilde{S}$ such that

$$\varlimsup_{n \to \infty} \frac{S_n(f, \delta, x_i, y)}{n} \; < \; \frac{\nu}{2}$$

with $i \in \{1, 2\}$. However, due to the triangle inequality we have that

$$S_n(f, 2\delta, x_1, x_2) \; \leq \; S_n(f, \delta, x_1, y) + S_n(f, \delta, y, x_2)$$

and consequently

$$\varlimsup_{n \to \infty} \frac{S_n(f, 2\delta, x_1, x_2)}{n} \; \leq \; \varlimsup_{n \to \infty} \frac{S_n(f, \delta, x_1, y)}{n} + \varlimsup_{n \to \infty} \frac{S_n(f, \delta, x_2, y)}{n} \; < \; \nu \; .$$

This contradicts the fact that x_1 and x_2 are $(f, 2\delta, \nu)$-separated. $\qquad \square$

Corollary 3.2. *Given a metric space X and $f : X \to X$, we have that*

$$\underline{\mathrm{ac}}(f) \; = \; \sup_{\delta > 0} \varliminf_{\nu \to 0} \frac{\log \mathrm{Span}(f, \delta, \nu)}{-\log \nu} \quad and \quad \overline{\mathrm{ac}}(f) \; = \; \sup_{\delta > 0} \varlimsup_{\nu \to 0} \frac{\log \mathrm{Span}(f, \delta, \nu)}{-\log \nu} \; . \tag{18}$$

Remarks 3.3.

   (a) The above statement remains true if $a(s, \nu) = \nu^{-s}$ is replaced by any O-regularly varying scale function.

   (b) In the definition of (f, δ, ν)-separated sets and (f, δ, ν)-spanning sets one could also use $\liminf$ instead of $\limsup$, and thus define the notions of *strongly (f, δ, ν)-separated sets* and *weakly (f, δ, ν)-spanning sets*, respectively. However, there is no analogue to the second inequality in (17) in this case.

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3.3 Factor relation and topological invariance

We assume that X and Ξ are arbitrary metric spaces, possibly non-compact. The price to pay for this is that we have to assume the uniform continuity of the factor map. All the assertions of this section remain true for arbitrary scale functions.

Proposition 3.4. *Assume $g : \Xi \to \Xi$ is a factor of $f : X \to X$ with a uniformly continuous factor map $h : X \to \Xi$. Then $\underline{\mathrm{ac}}(f) \geq \underline{\mathrm{ac}}(g)$ and $\overline{\mathrm{ac}}(f) \geq \overline{\mathrm{ac}}(g)$.*

Proof. We denote the metric on X and Ξ with d and ρ, respectively. The uniform continuity of h implies that for every $\delta > 0$ there exists $\tilde{\delta} > 0$ such that $\rho(h(z), h(w)) \geq \delta$ implies $d(z, w) \geq \tilde{\delta}$. Suppose $\xi, \xi' \in \Xi$ are (g, δ, ν)-separated. Then there exist $x, x' \in X$ such that $h(x) = \xi$ and $h(x') = \xi'$. Since $\rho(g^k(\xi), g^k(\xi')) \geq \delta$ implies $d(f^k(x), f^k(x')) \geq \tilde{\delta}$, the points x and x' need to be $(f, \tilde{\delta}, \nu)$-separated. Given $\nu \in (0, 1]$, this means that if $S \subseteq \Xi$ is a (g, δ, ν)-separated set, then there exist $\tilde{S} \subseteq X$ with $h(\tilde{S}) = S$ and $\tilde{\delta} > 0$ such that $\tilde{S}$ is a $(f, \tilde{\delta}, \nu)$-separated set. Therefore, for all $\nu \in (0, 1]$ we get

$$\mathrm{Sep}(f, \tilde{\delta}, \nu) \geq \mathrm{Sep}(g, \delta, \nu) \ .$$

The assertions follow easily. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Corollary 3.5. *Suppose X and Ξ are compact and let $f : X \to X$ and $g : \Xi \to \Xi$ be conjugate. Then $\underline{\mathrm{ac}}(f) = \underline{\mathrm{ac}}(g)$ and $\overline{\mathrm{ac}}(f) = \overline{\mathrm{ac}}(g)$.*

For the next corollary, observe that $f \circ g$ is an extension of $g \circ f$ with factor map $h = g$, and conversely $\tilde{h} = f$ is a factor map from $g \circ f$ to $f \circ g$.

Corollary 3.6. *Suppose $f : X \to X$ and $g : X \to X$ are uniformly continuous. Then $\underline{\mathrm{ac}}(f \circ g) = \underline{\mathrm{ac}}(g \circ f)$ and $\overline{\mathrm{ac}}(f \circ g) = \overline{\mathrm{ac}}(g \circ f)$.*

3.4 Power invariance and product rule

We first consider iterates of f. In contrast to topological entropy, taking powers does not affect the amorphic complexity. Throughout this section, we assume that X and Y are metric spaces.

Proposition 3.7. *Assume $f : X \to X$ is uniformly continuous and let $m \in \mathbb{N}$. Then $\underline{\mathrm{ac}}(f^m) = \underline{\mathrm{ac}}(f)$ and $\overline{\mathrm{ac}}(f^m) = \overline{\mathrm{ac}}(f)$.*

Proof. Since all iterates of f are uniformly continuous as well, we have that for every $\delta > 0$ there exists $\tilde{\delta} > 0$ such that $d(f^i(z), f^i(w)) \geq \delta$ implies $d(z, w) \geq \tilde{\delta}$ for all $i \in \{0, \ldots, m-1\}$.

Suppose $x, y \in X$ are (f, δ, ν)-separated. Assume that $d(f^k(x), f^k(y)) \geq \delta$ with $k = m \cdot \tilde{k} + i$, where $\tilde{k} \in \mathbb{N}_0$ and $i \in \{0, \ldots, m-1\}$. Then by the above we have $d(f^{m\tilde{k}}(x), f^{m\tilde{k}}(y)) \geq \tilde{\delta}$. This means that for $\tilde{n} \in \mathbb{N}$ and $n \in \{m \cdot \tilde{n}, \ldots, m(\tilde{n}+1) - 1\}$ we get

$$\frac{1}{n} S_n(f, \delta, x, y) \ \leq \ \frac{1}{n} \left(m \cdot S_{\tilde{n}}(f^m, \tilde{\delta}, x, y) + m \right) \ \leq \ \frac{1}{\tilde{n}} (S_{\tilde{n}}(f^m, \tilde{\delta}, x, y) + 1) \ .$$

By taking the $\limsup$ we get that x and y are $(f^m, \tilde{\delta}, \nu)$-separated. Hence,

$$\mathrm{Sep}(f^m, \tilde{\delta}, \nu) \ \geq \ \mathrm{Sep}(f, \delta, \nu) \ . \tag{19}$$

Conversely, suppose that x and y are (f^m, δ, ν)-separated. Then for $k \geq 1$ it follows from $d(f^{mk}(x), f^{mk}(y)) \geq \delta$ that $d(f^{\tilde{k}}(x), f^{\tilde{k}}(y)) \geq \tilde{\delta}$ for all $\tilde{k} \in \{m(k-1) + 1, \ldots, mk\}$. Each $\tilde{n} \in \mathbb{N}$ belongs to a block $\{m(n-1) + 1, \ldots, m \cdot n\}$ with $n \in \mathbb{N}$ and we have

$$\frac{1}{\tilde{n}} S_{\tilde{n}}(f, \tilde{\delta}, x, y) \ \geq \ \frac{1}{\tilde{n}} \left(m \cdot S_n(f^m, \delta, x, y) - m \right) \ \geq \ \frac{1}{n} \left(S_n(f^m, \delta, x, y) - 1 \right) \ .$$

Again, by taking the lim sup we get that x and y are $(f, \tilde{\delta}, \nu)$-separated. Hence,

$$\text{Sep}(f, \tilde{\delta}, \nu) \geq \text{Sep}(f^m, \delta, \nu) . \tag{20}$$

Using (19) and (20), we get that $\underline{\text{ac}}(f^m) = \underline{\text{ac}}(f)$ and $\overline{\text{ac}}(f^m) = \overline{\text{ac}}(f)$. $\qquad \square$

Remarks 3.8.

(a) The above result remains true for arbitrary scale functions.

(b) If f is not uniformly continuous, then we still have $\text{Sep}(f, \delta, \nu/m) \geq \text{Sep}(f^m, \delta, \nu)$. This yields $\underline{\text{ac}}(f, a) \geq \underline{\text{ac}}(f^m, a)$ and $\overline{\text{ac}}(f, a) \geq \overline{\text{ac}}(f^m, a)$ for a O-regularly varying.

In contrast to the above, the product formula is specific to polynomial growth or, more generally, to scale functions satisfying a product rule of the form $a(s+t, \nu) = a(s, \nu) \cdot a(t, \nu)$.

Proposition 3.9. *Let $f : X \to X$ and $g : Y \to Y$. Then $\underline{\text{ac}}(f \times g) \geq \underline{\text{ac}}(f) + \underline{\text{ac}}(g)$ and $\overline{\text{ac}}(f \times g) \leq \overline{\text{ac}}(f) + \overline{\text{ac}}(g)$. Therefore, if the limits $\text{ac}(f)$ and $\text{ac}(g)$ exist, we get*

$$\text{ac}(f \times g) = \text{ac}(f) + \text{ac}(g) .$$

Proof. We denote the metric on X and Y by d_X and d_Y, respectively. Let d be the maximum metric on the product space $X \times Y$. Using Corollary 3.2, the assertions are direct consequences of the following two inequalities, which we show for all $\delta > 0$ and $\nu \in (0, 1]$

$$\text{Sep}(f \times g, \delta, \nu) \geq \text{Sep}(f, \delta, \nu) \cdot \text{Sep}(g, \delta, \nu) , \tag{21}$$

$$\text{Span}(f \times g, \delta, \nu) \leq \text{Span}(f, \delta, \nu/2) \cdot \text{Span}(g, \delta, \nu/2) . \tag{22}$$

For proving (21) assume that $S_X \subseteq X$ and $S_Y \subseteq Y$ are (f, δ, ν)- and (g, δ, ν)-separated sets, respectively, with cardinalities $\text{Sep}(f, \delta, \nu)$ and $\text{Sep}(g, \delta, \nu)$, respectively. Then $S := S_X \times S_Y \subseteq X \times Y$ is an $(f \times g, \delta, \nu)$-separated set. This implies (21).

Now, in order to prove (22), assume that $\tilde{S}_X \subseteq X$ and $\tilde{S}_Y \subseteq Y$ are $(f, \delta, \nu/2)$- and $(g, \delta, \nu/2)$-spanning sets, respectively, with cardinalities $\text{Span}(f, \delta, \nu/2)$ and $\text{Span}(g, \delta, \nu/2)$, respectively. The set $\tilde{S} := \tilde{S}_X \times \tilde{S}_Y \subseteq X \times Y$ is an $(f \times g, \delta, \nu)$-spanning, since for arbitrary $(x, y) \in X \times Y$ there are $\tilde{x} \in \tilde{S}_X$ and $\tilde{y} \in \tilde{S}_Y$ such that

$$S_n(f \times g, \delta, (x, y), (\tilde{x}, \tilde{y})) = \# \left\{ 0 \leq k < n \mid d\big((f \times g)^k(x, y), (f \times g)^k(\tilde{x}, \tilde{y})\big) \geq \delta \right\}$$

$$\leq \# \left\{ 0 \leq k < n \mid d_X(f^k(x), f^k(\tilde{x})) \geq \delta \right\} + \# \left\{ 0 \leq k < n \mid d_Y(g^k(y), g^k(\tilde{y})) \geq \delta \right\} . \quad \square$$

3.5 Isometries, Morse-Smale systems and transient dynamics

It is obvious that all isometries have bounded separation numbers and zero amorphic complexity, since $\text{Sep}(f, \delta, \nu)$ does not depend on ν in this case. Similarly, amorphic complexity is zero for Morse-Smale systems. Here, we call a continuous map f on a compact metric space X *Morse-Smale* if its non-wandering set $\Omega(f)$ is finite. This implies that $\Omega(f)$ consists of a finite number of fixed or periodic orbits, and for any $x \in X$ there exists $y \in \Omega(f)$ with $\lim_{n \to \infty} f^{np}(x) = y$, where p is the period of y. Since orbits converging to the same periodic orbit cannot be (f, δ, ν)-separated, we obtain $\text{Sep}(f, \delta, \nu) \leq \#\Omega(f)$ for all $\delta, \nu > 0$. Hence, separation numbers are even bounded uniformly in δ and ν.

This shows that amorphic complexity is, in some sense, less sensitive to transient behaviour than power entropy, which gives positive value to Morse-Smale systems (see Section 3.7). However, amorphic complexity is not entirely insensitive to transient dynamics, and the relation $\text{ac}(f) = \text{ac}(f|_{\Omega(f)})$ does not always hold. An example can be given as follows.

Let $f : [0,1] \times \mathbb{T}^1 \to [0,1] \times \mathbb{T}^1$ be of the form $f(x,y) := (g(x), y + \alpha(x) \bmod 1)$, where $\mathbb{T}^1 := \mathbb{R}^1/\mathbb{Z}^1$, $\alpha : [0,1] \to \mathbb{R}$ is continuous and $g : [0,1] \to [0,1]$ is a Morse-Smale homeomorphism with unique attracting fixed point $x_a = 0$ and unique repelling fixed point $x_r = 1$, so that $\lim_{k\to\infty} g^k(x) = 0$ for all $x \in (0,1)$. Let $x_0 \in (0,1)$ and $x_k := g^k(x_0)$ for $k \in \mathbb{N}$ and $x_0' := (x_0 + x_1)/2$. Suppose α is given by

$$
\alpha(x) := \left\{ \begin{array}{ll} 0 & \text{if } x \in \{0\} \cup (x_0, 1]; \\ 1 - 2\frac{|x_0' - x|}{x_0 - x_1} & \text{if } x \in (x_1, x_0]; \\ \frac{1}{k}\alpha\left(g^{-(k-1)}(x)\right) & \text{if } x \in (x_k, x_{k-1}], \ k \geq 2; \end{array} \right. \tag{23}
$$

Then, if $x, x' \in [x_1, x_0']$, we have that

$$
\left| \sum_{k=0}^{n-1} \alpha \circ g^k(x) - \sum_{k=0}^{n-1} \alpha \circ g^k(x') \right| = 2\frac{|x - x'|}{x_0 - x_1} \sum_{k=1}^{n} \frac{1}{k} . \tag{24}
$$

This means that one of the two points $(x,0), (x',0)$ performs infinitely more turns around the annulus $[0,1] \times \mathbb{T}^1$ as $n \to \infty$, and it is not difficult to deduce from (24) that $(x,0), (x',0)$ are (f, δ, ν)-separated for some fixed $\delta, \nu > 0$ independent of x, x'. Hence, $[x_1, x_0'] \times \{0\}$ is an uncountable (f, δ, ν)-separated set, and we obtain $\mathrm{Sep}(f, \delta, \nu) = \infty$.

It should be interesting to describe which types of transient behaviour have an impact on amorphic complexity and which ones do not, and thus to understand whether this quantity may be used to distinguish qualitatively different types of transient dynamics. However, we are not going to pursue this issue further here, but confine ourselves to give a simple criterion for the validity of the equality $\mathrm{ac}(f) = \mathrm{ac}(f|_{\Omega(f)})$.

We say f has the *unique target property* if for every $x \in X \setminus \Omega(f)$ there exists $y \in \Omega(f)$ such that $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$. Then the following statement is easy to prove.

Lemma 3.10. *Assume f has the unique target property, then $\underline{\mathrm{ac}}(f) = \underline{\mathrm{ac}}(f|_{\Omega(f)})$ and $\overline{\mathrm{ac}}(f) = \overline{\mathrm{ac}}(f|_{\Omega(f)})$.*

In fact, the nonwandering set $\Omega(f)$ does not play a special role in the definition of the unique target property nor in the above lemma and can be replaced by any other subset of X (even invariance is not necessary). For later use (see Section 6), we provide a precise formulation. Given $E \subseteq X$, we let

$$
\mathrm{Sep}_E(f, \delta, \nu) := \sup\{\#A \mid A \subseteq E \text{ and } A \text{ is } (f, \delta, \nu)\text{-separated}\} \tag{25}
$$

and define

$$
\begin{aligned}
\underline{\mathrm{ac}}_E(f, \delta) &:= \varliminf_{\nu \to 0} \frac{\log \mathrm{Sep}_E(f, \delta, \nu)}{-\log \nu} \quad , \quad \underline{\mathrm{ac}}_E(f) := \sup_{\delta > 0} \underline{\mathrm{ac}}_E(f, \delta) , \\
\overline{\mathrm{ac}}_E(f, \delta) &:= \varlimsup_{\nu \to 0} \frac{\log \mathrm{Sep}_E(f, \delta, \nu)}{-\log \nu} \quad , \quad \overline{\mathrm{ac}}_E(f) := \sup_{\delta > 0} \overline{\mathrm{ac}}_E(f, \delta) .
\end{aligned} \tag{26}
$$

We say f has the *unique target property with respect to* $E \subseteq X$ if for all $x \in X$ there exists $y \in E$ such that $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

Lemma 3.11. *Suppose $f : X \to X$ has the unique target property with respect to $E \subseteq X$. Then $\underline{\mathrm{ac}}(f) = \underline{\mathrm{ac}}_E(f)$ and $\overline{\mathrm{ac}}(f) = \overline{\mathrm{ac}}_E(f)$.*

Proof. Suppose $S = \{x_1, \ldots, x_m\} \subseteq X$ is an (f, δ, ν)-separated set. Then by assumption there exist $y_1, \ldots, y_m \in E$ such that $\lim_{n\to\infty} d(f^n(x_i), f^n(y_i)) = 0$ for all $i \in \{1, \ldots, m\}$. Hence, the set $\tilde{S} := \{y_1, \ldots, y_m\} \subseteq E$ is (f, δ, ν)-separated as well. This shows that $\mathrm{Sep}_E(f, \delta, \nu) \geq \mathrm{Sep}(f, \delta, \nu)$, and since the reverse inequality is obvious this proves the statement. $\qquad\square$

3.6 Denjoy examples and Sturmian subshifts

We start with some standard notation concerning circle maps and symbolic dynamics. Let $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$ be the circle and denote by d the usual metric on $\mathbb{T}^1$. Further, we denote the open and the closed counter-clockwise interval from a to b in $\mathbb{T}^1$ by (a,b) and $[a,b]$, respectively. The Lebesgue measure on $\mathbb{T}^1$ is denoted by Leb. Moreover, the rigid rotation with angle $\alpha \in \mathbb{R}$ is denoted by $R_\alpha(x) := x + \alpha \mod 1$.

For a finite set A we denote by σ the left shift on $\Sigma_A := A^{\mathbb{I}}$ where $\mathbb{I}$ equals either $\mathbb{N}_0$ or $\mathbb{Z}$. The product topology on Σ_A is induced by the *Cantor metric* $\rho(x,y) := 2^{-j}$ where $x = (x_k)_{k \in \mathbb{I}}$, $y = (y_k)_{k \in \mathbb{I}} \in \Sigma_A$ and $j := \min\{|k| : x_k \neq y_k \text{ with } k \in \mathbb{I}\}$.

We first recall some basics about Sturmian subshifts and Denjoy homeomorphisms of the circle. For Sturmians, we mainly follow [CD05, Section 2.2]. Assume that $\alpha \in (0,1)$ is irrational. Consider the coding map $\varphi_\alpha : \mathbb{T}^1 \to \{0,1\}$ defined via $\varphi_\alpha(x) = 0$ if $x \in I_0 := [0, 1-\alpha)$ and $\varphi_\alpha(x) = 1$ if $x \in I_1 := [1-\alpha, 1)$. Set

$$\Sigma_\alpha := \overline{\{(\varphi_\alpha(R_\alpha^k(x)))_{k \in \mathbb{Z}} \mid x \in \mathbb{T}^1\}} \subset \Sigma_{\{0,1\}} \ .$$

The subshift (Σ_α, σ) is called the *Sturmian subshift generated by α* and its elements are called *Sturmian sequences*. According to [MH40], there exists a map $h : \Sigma_\alpha \to \mathbb{T}^1$ semi-conjugating σ and R_α with the property that $\#h^{-1}(x) = 2$ for $x \in \{k\alpha \mod 1 \mid k \in \mathbb{Z}\}$ and $\#h^{-1}(x) = 1$ otherwise. If $x = k\alpha$, then one of the two alternative sequences in $h^{-1}(x)$ corresponds to the coding with respect to the original partition $\{I_0, I_1\}$, whereas the other one corresponds to the coding with respect to the partition $\{(0, 1-\alpha], (1-\alpha, 1]\}$. Further information is given in [BMN00, Section 1.6].

Poincaré's classification of circle homeomorphisms in [Poi85] states that to each orientation preserving homeomorphism $f : \mathbb{T}^1 \to \mathbb{T}^1$ of the circle we can associate a unique real number $\alpha \in [0,1)$, called the rotation number of f, such that f is semi-conjugate, via an orientation preserving map, to the rigid rotation R_α, provided α is irrational (see also [dMvS93, HK97]). Another classical result by A. Denjoy [Den32] states that if f is a diffeomorphism such that its derivative is of bounded variation, then f is even conjugate to R_α. In this case, the amorphic complexity is zero. However, Denjoy also constructed examples of C^1 circle diffeomorphism with irrational rotation number that are not conjugate to a rotation and later, Herman [Her79] showed that these examples can be made $C^{1+\varepsilon}$ for any $\varepsilon < 1$. Such maps are commonly called *Denjoy examples* or *Denjoy homeomorphisms*. From Poincaré's classification, it is known that in this case there exist *wandering intervals*, that is, open intervals $I \subset \mathbb{T}^1$ such that $f^n(I) \cap I = \emptyset$ for all $n \geq 1$. Any Denjoy example has a unique minimal set C, which is a Cantor set and coincides with the non-wandering set $\Omega(f)$. All connected components of $\mathbb{T}^1 \setminus C$ are wandering intervals, and the length of their n-th iterates goes to zero as $n \to \infty$. Since the endpoints of these intervals belong to the minimal set, this also implies that Denjoy examples have the unique target property.

Not surprisingly, there is an intimate connection between Denjoy examples and Sturmian subshifts. Let f be a Denjoy homeomorphism with rotation number α and suppose it has a unique wandering interval I, in the sense that the minimal set $C = \mathbb{T}^1 \setminus \bigcup_{n \in \mathbb{Z}} f^n(I)$.[2] Given any $x_0 \in I$, let $J_0 := [f(x_0), x_0)$ and $J_1 := [x_0, f(x_0))$. Then for every $x \in \mathbb{T}^1$ the coding $\mathbf{1}_{J_1} \circ f^n(x)$, where $\mathbf{1}_{J_1}$ denotes the indicator function of J_1, is a Sturmian sequence in Σ_α. Moreover, in this situation any point in the minimal set C has a unique coding. This yields the following folklore statement.

Lemma 3.12. *For any Sturmian subshift (Σ_α, σ) there exists a Denjoy homeomorpism f with minimal set C such that $f|_C$ is conjugate to $\sigma|_{\Sigma_\alpha}$.*

---

[2]It is possible to have several connected components of $\mathbb{T}^1 \setminus C$ with pairwise disjoint orbits.

For our purposes, this means that we only have to determine the amorphic complexity of Denjoy examples. Note that the converse to the above lemma is false: if f has multiple wandering intervals with pairwise disjoint orbits, then it is not conjugate to a Sturmian subshift.

Theorem 3.13. *Suppose* $f : \mathbb{T}^1 \to \mathbb{T}^1$ *is a Denjoy homeomorphism. Then* $\mathrm{ac}(f) = 1$.

Since Denjoy examples have the unique target property, Lemma 3.10 yields that $\mathrm{ac}(f|_C) = \mathrm{ac}(f) = 1$. Together with Corollary 3.5 and Lemma 3.12, this implies

Corollary 3.14. *For any Sturmian subshift* (Σ_α, σ) *we have* $\mathrm{ac}(\sigma|_{\Sigma_\alpha}) = 1$.

Theorem 3.13 is a direct consequence of the following two lemmas. However, before we proceed, we want to collect some more facts concerning Denjoy examples, following mainly [Mar70, Section 0] and [HOR12, Section 2]. The Cantor set $C = \Omega(f)$ can be described as

$$ C \;=\; \mathbb{T}^1 \backslash \bigcup_{\ell=1}^{\infty} (a_\ell, b_\ell) \;, $$

where $((a_\ell, b_\ell))_{\ell \in \mathbb{N}}$ is a family of open and pairwise disjoint intervals. The *accessible points* $A \subset \mathbb{T}^1$ *of* C are defined as the union of the endpoints of these intervals and the *inaccessible points of* C are defined as $I := C \backslash A$. A *Cantor function* $p : \mathbb{T}^1 \to \mathbb{T}^1$ *associated to* C is a continuous map satisfying

$$ p(x) = p(y) \iff x = y \text{ or } x, y \in [a_\ell, b_\ell] \text{ for some } \ell \geq 1 \;, $$

that is, p collapses the intervals $[a_\ell, b_\ell]$ to single points and is invertible on I. From this definition it is not difficult to deduce that p is onto and that $p(A)$ is countable and dense in $\mathbb{T}^1$. Furthermore, we can assume without loss of generality that $p \circ f = R_\alpha \circ p$, where $\alpha \in [0,1) \backslash \mathbb{Q}$ is the rotation number of f, see [Mar70, Section 2].

Lemma 3.15. *Let* $f : \mathbb{T}^1 \to \mathbb{T}^1$ *be a Denjoy homeomorphism. Then there exists* $\delta > 0$ *such that* $\mathrm{Sep}(f, \delta, \nu) \geq \lfloor 1/\nu \rfloor$ *for all* $\nu \in (0, 1]$.

Note that by definition this implies that $\underline{\mathrm{ac}}(f) \geq 1$.

Proof. Suppose $\nu \in (0, 1/2]$. Since $p(A)$ is dense in $\mathbb{T}^1$, we can choose for each $m \in \{1, 2, 3\}$ a point $\zeta_m \in p(A)$ such that

$$ d(\zeta_m, \zeta_n) \;>\; 1/4 \quad \text{for } m \neq n \;. \tag{27} $$

Note that to each ζ_m we can associate an interval $[a_{\ell_m}, b_{\ell_m}]$ with $p([a_{\ell_m}, b_{\ell_m}]) = \{\zeta_m\}$. Now, choose $\delta > 0$ such that

$$ \delta \;\leq\; \min_{m=1}^{3} d(a_{\ell_m}, b_{\ell_m}) \;. $$

Since $p(I)$ has full Lebesgue measure in $\mathbb{T}^1$, we can choose a set of $\lfloor 1/\nu \rfloor$ points

$$ M \;=\; \{x_1, \ldots, x_{\lfloor 1/\nu \rfloor}\} \;\subset\; I \;, $$

such that $p(M)$ is an equidistributed lattice in $\mathbb{T}^1$ with distance $1/\lfloor 1/\nu \rfloor \geq \nu$ between adjacent vertices. Consider distinct points $x_i, x_j \in M$ and assume without loss of generality that $\mathrm{Leb}([p(x_i), p(x_j)]) \leq 1/2$. Set $P := [p(x_i), p(x_j)]$. If $\zeta_1 \in R_\alpha^k(P)$ for some $k \geq 0$, then

due to (27) we have that $\zeta_2 \in \mathbb{T}^1 \backslash R_\alpha^k(P)$ or $\zeta_3 \in \mathbb{T}^1 \backslash R_\alpha^k(P)$, such that both $[f^k(x_i), f^k(x_j)]$ and $[f^k(x_j), f^k(x_i)]$ contain some interval $[a_{\ell_m}, b_{\ell_m}]$ with $m \in \{1, 2, 3\}$. Hence, we have

$$d(f^k(x_i), f^k(x_j)) \geq \delta .$$

Consequently, we obtain

$$\frac{S_n(f, \delta, x_i, x_j)}{n} \geq \frac{\# \left\{ 0 \leq k < n \mid \zeta_1 \in R_\alpha^k(P) \right\}}{n} .$$

By Weyl's Equidistribution Theorem [EW11, Example 4.18], the right-hand side converges to $p(x_j) - p(x_i) \geq \nu$ as $n \to \infty$. This means that x_i and x_j are (f, δ, ν)-separated, so that M is an (f, δ, ν)-separated set. $\qquad \square$

Lemma 3.16. *Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ be a Denjoy homeomorphism. Then for any $\delta > 0$ there exists a constant $\kappa = \kappa(\delta)$ such that*

$$\mathrm{Span}(f, \delta, \nu) \leq \kappa/\nu \quad \text{for all } \nu \in (0, 1] .$$

Together with Corollary 3.2, this implies that $\overline{\mathrm{ac}}(f) \leq 1$, thus completing the proof of Theorem 3.13.

Proof. We show that if $0 < \tilde{\nu} \leq 1/(2(\lceil 1/\delta \rceil + 1))$, then

$$\mathrm{Span}(f, \delta, 2\tilde{\nu}(\lceil 1/\delta \rceil + 1)) \leq \lceil 1/\tilde{\nu} \rceil .$$

Since $\lceil 1/\tilde{\nu} \rceil \leq 2/\tilde{\nu}$, this implies the statement with $\kappa(\delta) := 4(\lceil 1/\delta \rceil + 1)$.

Let $\mu := \mathrm{Leb} \circ p^{-1}$ and define the function $\varphi_{\tilde{\nu}} : \mathbb{T}^1 \to [0, \infty)$ by

$$\varphi_{\tilde{\nu}}(x) := \mu([x, x + \tilde{\nu}]) .$$

Note that $d(x, y) \leq \mu([p(x), p(y)])$ and that $\varphi_{\tilde{\nu}}(x) = d(p^{-1}(x), p^{-1}(x + \tilde{\nu}))$ almost everywhere. In particular, $\varphi_{\tilde{\nu}}$ is measurable. Now, consider a subset $\tilde{I} \subseteq I$ such that

$$\frac{\# \left\{ 0 \leq k < n \mid \varphi_{\tilde{\nu}}(R_\alpha^k(x)) \geq \delta \right\}}{n} \longrightarrow \mathrm{Leb}(\{ x \in \mathbb{T}^1 \mid \varphi_{\tilde{\nu}}(x) \geq \delta \}) \quad \text{as} \quad n \to \infty \qquad (28)$$

for all $x \in p(\tilde{I})$. Let $\{ \varphi_{\tilde{\nu}} \geq \delta \} := \{ x \in \mathbb{T}^1 \mid \varphi_{\tilde{\nu}}(x) \geq \delta \}$. Using Birkhoff's Ergodic Theorem, we know that $\tilde{I}$ can be chosen such that $p(\tilde{I})$ has full Lebesgue measure. Hence, we can choose a set of $\lceil 1/\tilde{\nu} \rceil$ points

$$M := \left\{ x_1, \ldots, x_{\lceil 1/\tilde{\nu} \rceil} \right\} \subset \tilde{I},$$

such that $p(M)$ is an equidistributed lattice in $\mathbb{T}^1$ with distance $1/\lceil 1/\tilde{\nu} \rceil \leq \tilde{\nu}$ between adjacent vertices. Our aim is to show that M is an $(f, \delta, 2\tilde{\nu}(\lceil 1/\delta \rceil + 1))$-spanning set.

For arbitrary $y \in \mathbb{T}^1$, let $x_i, x_j \in M$ be the two adjacent lattice points with $p(y) \in [p(x_i), p(x_j)]$ (that is, $j = i + 1$ or $i = \lceil 1/\tilde{\nu} \rceil$ and $j = 1$). Then

$$R_\alpha^k[p(x_i), p(y)] \subseteq [R_\alpha^k(p(x_i)), R_\alpha^k(p(x_i)) + \tilde{\nu}]$$

for $k \geq 0$, and this implies

$$\begin{aligned} d\left(f^k(x_i), f^k(y)\right) &\leq \mu\left([p(f^k(x_i)), p(f^k(y))]\right) \\ &= \mu\left(R_\alpha^k[p(x_i), p(y)]\right) \leq \varphi_{\tilde{\nu}}(R_\alpha^k(p(x_i))) . \end{aligned}$$

We get that

$$\frac{S_n(f,\delta,x_i,y)}{n} \leq \frac{\#\left\{0 \leq k < n \mid \varphi_{\tilde{\nu}}(R_\alpha^k(p(x_i))) \geq \delta\right\}}{n}$$

and using (28) we know that the right-hand side convergences to $\mathrm{Leb}(\{\varphi_{\tilde{\nu}} \geq \delta\})$ as $n \to \infty$.

It remains to show that $\mathrm{Leb}(\{\varphi_{\tilde{\nu}} \geq \delta\}) < 2\tilde{\nu}(\lceil 1/\delta \rceil + 1)$. Suppose for a contradiction that this inequality does not hold. Then $\{\varphi_{\tilde{\nu}} \geq \delta\}$ is not contained in a union of less than $\lceil 1/\delta \rceil + 1$ intervals of length $2\tilde{\nu}$. Consequently, there exist at least $\lceil 1/\delta \rceil + 1$ points $\zeta_i \in \mathbb{T}^1$ with $\varphi_{\tilde{\nu}}(\zeta_i) \geq \delta$ and $d(\zeta_i, \zeta_j) \geq \tilde{\nu}$ for $i \neq j$. We thus obtain

$$\mu(\mathbb{T}^1) \geq \sum_{i=1}^{\lceil 1/\delta \rceil + 1} \mu([\zeta_i, \zeta_i + \tilde{\nu}]) = \sum_{i=1}^{\lceil 1/\delta \rceil + 1} \varphi_{\tilde{\nu}}(\zeta_i) \geq 1 + \delta > 1 \,,$$

which is a contradiction.

This means $\overline{\lim}_{n\to\infty} S_n(f,\delta,x_i,y)/n \leq \mathrm{Leb}(\{\varphi_{\tilde{\nu}} \geq \delta\}) < 2\tilde{\nu}(\lceil 1/\delta \rceil + 1)$, and since y was arbitrary this shows that M is an $(f,\delta,2\tilde{\nu}(\lceil 1/\delta \rceil + 1))$-spanning set. This completes the proof. □

3.7 Relations to power entropy

Given a compact metric space (X, d) and a continuous map $f : X \to X$, the *Bowen-Dinaburg metrics* are given by $d_n(x,y) := \max_{i=0}^{n-1} d(f^i(x), f^i(y))$. A set $S \subseteq X$ is called (f,δ,n)-*separated*, for $\delta > 0$ and $n \in \mathbb{N}$, if $d_n(x,y) \geq \delta$ for all $x \neq y \in S$. Let $\widehat{S}(f,\delta,n)$ denote the maximal cardinality of an (f,δ,n)-separated set. Then topological entropy, defined as

$$h_{\mathrm{top}}(f) := \sup_{\delta > 0} \lim_{n\to\infty} \frac{\log \widehat{S}(f,\delta,n)}{n} \,,$$

measures the exponential growth of these numbers, see for example [Wal82] for more information. If topological entropy is zero, then *power entropy* instead simply measures the polynomial growth rate, given by

$$h_{\mathrm{pow}}(f) := \sup_{\delta > 0} \overline{\lim}_{n\to\infty} \frac{\log \widehat{S}(f,\delta,n)}{\log n} \,.$$

We refer to [HK02] and [Mar13] for a more detailed discussion.

Now, suppose $g : [0,1] \to [0,1]$ is a Morse-Smale homeomorphism as in Section 3.5, with two fixed points $x_a = 0$ and $x_r = 1$ and $\lim_{k\to\infty} g^k(x) = 0$ for all $x \in (0,1)$. Fix $x_0 \in (0,1)$, let $x_k := g^k(x_0)$ for all $k \in \mathbb{Z}$ and $\delta := |x_0 - x_{-1}|$. Given $n \in \mathbb{N}$, let $S := \{x_0, x_{-1}, \dots, x_{-(n-1)}\}$. Then for $0 \leq i < j < n$ we have that

$$d_n(x_{-i}, x_{-j}) \geq d_n(x_{-i}, x_{-(i+1)}) \geq \left| g^i(x_{-i}) - g^i(x_{-(i+1)}) \right| = |x_0 - x_{-1}| = \delta \,.$$

Consequently, S is a (g,δ,n)-separated set of cardinality n. We obtain $\widehat{S}(g,\delta,n) \geq n$, which implies $h_{\mathrm{pow}}(g) \geq 1$. Thus, as claimed above, power entropy gives positive value to Morse-Smale systems, where an analogous argument works for the general case. Alternatively, one can use the general fact that already one wandering point is enough to ensure that power entropy is at least bigger than one [Lab13].

The last example shows that we may have $h_{\mathrm{pow}}(f) > \mathrm{ac}(f)$. Conversely, consider the map $f : \mathbb{T}^2 \to \mathbb{T}^2$, $(x,y) \mapsto (x, x+y)$ where $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. Then given $z = (x,y)$ and $z' = (x', y')$, we have that

$$d_n(z, z') \leq n|x - x'| + |y - y'| \,,$$

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which implies that $\widehat{S}(f,\delta,n) \leq \frac{C \cdot n}{\delta^2}$ for some constant $C > 0$. Hence, $h_{\mathrm{pow}}(f) \leq 1$. However, at the same time we have that if $x \neq x'$, then z and z' rotate in the vertical direction with different speeds, and this makes it easy to show that $\mathbb{T}^1 \times \{0\}$ is an (f,δ,ν)-separated set for suitable $\delta, \nu > 0$, so that $\mathrm{Sep}(f,\delta,\nu) = \infty$. Hence, we may also have $\mathrm{ac}(f) > h_{\mathrm{pow}}(f)$, showing that no inequality holds between the two quantities.

Modified power entropy h_{pow}^* is defined in the same way as power entropy, with the only difference that the metrics d_n in the definition are replaced by the *Hamming metrics*

$$ d_n^*(x,y) \; := \; \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), f^i(y)) \; . $$

Since $d_n^* \leq d_n$, modified power entropy is always smaller than power entropy, and it can be shown that for Morse-Smale systems it is always zero. The same is true, however, for Denjoy examples and Sturmian subshifts, so that modified power entropy does not seem suitable to detect topological complexity on the very fine level we are interested in here. The same example $f(x,y) = (x, x+y)$ as above shows that we may have $\mathrm{ac}(f) > h_{\mathrm{pow}}^*(f)$. An example for the opposite inequality is more subtle, but can be made such that it demonstrates at the same time the non-existence of a variational principle for the modified power entropy (a question which was left open in [HK02]). It will be contained in the forthcoming note [GJ].

3.8 Besicovitch space

In this section, we want to state some basic results concerning amorphic complexity in the context of symbolic systems. The corresponding proofs will be included in the forthcoming paper [FG], where amorphic complexity of symbolic systems will be studied more systematically.

Let A be a finite set, $\Sigma_A := A^{\mathbb{N}_0}$ and ρ the Cantor metric on Σ_A (see Section 3.6). For a general continuous map $f : X \to X$ on a compact metric space X and some $\delta > 0$ we can not expect that $\overline{\lim}_{n \to \infty} S_n(f,\delta,\cdot,\cdot)/n$ is a metric (even not a pseudo-metric since the triangle inequality will usually fail). However, this changes in the setting of symbolic dynamics.

Proposition 3.17. *We have that* $\left(\tilde{d}_\delta \right)_{\delta \in (0,1]}$, *defined as*

$$ \tilde{d}_\delta(x,y) \; := \; \varlimsup_{n \to \infty} \frac{S_n(\sigma,\delta,x,y)}{n} \quad \text{for} \quad x,y \in \Sigma_A \; , $$

is a family of bi-Lipschitz equivalent pseudo-metrics.

Note that $\tilde{d}_1$ is usually called the *Besicovitch pseudo-metric* and it turns out to be especially useful for understanding certain dynamical behaviour of cellular automata (see, for example, [BFK97] and [CFMM97]).

Now, following a standard procedure, we introduce the equivalence relation

$$ x \sim y \; :\Leftrightarrow \; \tilde{d}_\delta(x,y) \; = \; 0 \quad \text{for} \quad x,y \in \Sigma_A \; . $$

Due to the previous proposition, this relation is well-defined and independent of the chosen δ. Denote the corresponding projection mapping by $[\cdot]$. We equip $[\Sigma_A]$ with the metric $d_\delta([x],[y]) := \tilde{d}_\delta(x,y)$, $[x],[y] \in [\Sigma_A]$ for some $\delta \in (0,1]$ and call $\left([\Sigma_A], d_\delta \right)$ the *Besicovitch space*. Given a subshift $\Sigma \subseteq \Sigma_A$, we also call $[\Sigma]$ the *Besicovitch space associated to* Σ. We have the following properties.

Theorem 3.18 ([BFK97, CFMM97])**.** *The Besicovitch space* $[\Sigma_A]$ *is perfect, complete, pathwise connected and (topologically) infinite dimensional. However, it is neither locally compact nor separable.*

Note that we can define the shift map on the Besicovitch space as well and that it becomes an isometry. Before we proceed, we need to give the definition of box dimension in general metric spaces (X, d). The *lower* and *upper box dimension* of a totally bounded subset $E \subseteq X$ are defined as

$$\underline{\mathrm{Dim}}_B(E) \; := \; \varliminf_{\varepsilon \to 0} \frac{\log N_\varepsilon(E)}{-\log \varepsilon} \quad \text{and} \quad \overline{\mathrm{Dim}}_B(E) \; := \; \varlimsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(E)}{-\log \varepsilon} \; ,$$

where $N_\varepsilon(E)$ is the smallest number of sets of diameter strictly smaller than ε needed to cover E. If $\underline{\mathrm{Dim}}_B(E) = \overline{\mathrm{Dim}}_B(E)$, then we call their common value $\mathrm{Dim}_B(E)$ the *box dimension of E*. Further, let $M_\varepsilon(E)$ be the maximal cardinality of an ε-separated subset of E, that is, a set $S \subseteq E$ with $d(x, y) \geq \varepsilon$ for all $x \neq y \in S$. Then one can replace $N_\varepsilon(E)$ by $M_\varepsilon(E)$ in the definition of box dimension [Edg98, Proposition 1.4.6].

Now, suppose (Σ, σ) is a subshift of (Σ_A, σ). If $\sigma|_\Sigma$ has finite separation numbers, we observe for each $\delta \in (0, 1]$ that

$$\mathrm{Sep}(\sigma|_\Sigma, \delta, \nu) = M_\nu([\Sigma]) \quad \text{and} \quad \mathrm{Span}(\sigma|_\Sigma, \delta, \nu) = N_\nu([\Sigma]) \quad \text{in} \quad \left([\Sigma_A], d_\delta \right)$$

for all $\nu \in (0, 1]$. This immediately implies

Proposition 3.19. *Let Σ be a subshift of Σ_A. Then*

(a) *$\sigma|_\Sigma$ has finite separation numbers if and only if $[\Sigma]$ is totally bounded in $\left[\Sigma_A \right]$, and*

(b) *in this setting, $\underline{\mathrm{ac}}(\sigma|_\Sigma) = \underline{\mathrm{Dim}}_B([\Sigma])$ and $\overline{\mathrm{ac}}(\sigma|_\Sigma) = \overline{\mathrm{Dim}}_B([\Sigma])$.*

This means for example that all regular Toeplitz subshifts Σ (see Section 5) have a totally bounded associated Besicovitch space, using Theorem 2.7 (in fact one can show by a more direct argument that $[\Sigma]$ is even compact), and that we can find regular Toeplitz subshifts with associated Besicovitch spaces of arbitrarily high box dimension, see Theorem 5.6.

An example of a minimal and uniquely ergodic subshift with zero topological entropy such that its projection is not totally bounded is the subshift generated by the shift orbit closure of the well-known Prouhet-Thue-Morse sequence. (See, for example, [AS99] for the definition of this sequence and further information.) The fact that the projection is not totally bounded follows directly from the strict positivity of the *aperiodicity measure* of the Prouhet-Thue-Morse sequence x, defined as $\inf_{m \in \mathbb{N}} \varliminf_{n \to \infty} S_n(\sigma, 1, x, \sigma^m(x))/n$, see [PU09, MH38].

4 Quantitative analysis of almost sure 1-1 extensions of isometries

The aim of this section is to give a quantitative version of the argument in the proof of Theorem 2.7 in order to obtain an upper bound for amorphic complexity in this situation. For the whole section let X and Ξ be compact metric spaces and $f : X \to X$ an almost sure 1-1 extension of $g : \Xi \to \Xi$, with factor map h. Further, assume that g is a minimal isometry, with unique invariant probability measure μ.[3] In this case, it is easy to check that the measure of an ε-ball $B_\varepsilon(\xi)$ does not depend on $\xi \in \Xi$. For the scaling of this measure as $\varepsilon \to 0$, we have

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[3] Note that a minimal isometry is necessarily uniquely ergodic.

Lemma 4.1. *In the above situation, we get*

$$\varlimsup_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(\xi))}{\log \varepsilon} \;=\; \overline{\mathrm{Dim}}_B(\Xi)$$

for all $\xi \in \Xi$ and the analogous equality holds for the limit inferior.

Proof. Recall that we can also use $M_\varepsilon(\Xi)$ in the definition of the box dimension of Ξ (see Section 3.8). Let $\hat\mu(\varepsilon) := \mu(B_\varepsilon(\xi))$, where $\xi \in \Xi$ is arbitrary, and suppose $S \subseteq \Xi$ is an ε-separated subset with cardinality $M_\varepsilon(\Xi)$. Observe that the $\varepsilon/2$-balls $B_{\varepsilon/2}(\xi)$ with $\xi \in S$ are pairwise disjoint. We obtain $1 = \mu(\Xi) \geq \sum_{\xi \in S} \hat\mu(\varepsilon/2)$ and thus $M_\varepsilon(\Xi) \leq 1/\hat\mu(\varepsilon/2)$. Hence,

$$\overline{\mathrm{Dim}}_B(\Xi) \;=\; \varlimsup_{\varepsilon \to 0} \frac{\log M_\varepsilon(\Xi)}{-\log \varepsilon} \;\leq\; \varlimsup_{\varepsilon \to 0} \frac{\log \hat\mu(\varepsilon/2)}{\log \varepsilon} \;=\; \varlimsup_{\varepsilon \to 0} \frac{\log \hat\mu(\varepsilon)}{\log \varepsilon} \;.$$

Conversely, the ε-balls $B_\varepsilon(\xi)$ with centres ξ in S cover Ξ, and this easily leads to the reverse inequality. $\qquad\square$

By the Minkowski characterisation of box dimension, we have for $E \subseteq \Xi$

$$\overline{\mathrm{Dim}}_B(E) \;=\; \overline{\mathrm{Dim}}_B(\Xi) - \varliminf_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(E))}{\log \varepsilon} \;. \tag{29}$$

The proof of this fact in the setting above is the same as in Euclidean space, see, for example, [Fal07]. We denote by $\eta_\delta(\varepsilon)$ the constant given by Lemma 2.6 and let

$$\gamma(h) \;:=\; \varlimsup_{\delta \to 0} \varlimsup_{\varepsilon \to 0} \frac{\log \eta_\delta(\varepsilon)}{\log \varepsilon} \;. \tag{30}$$

This is the scaling factor from Theorem 1.5, which we restate here as

Theorem 4.2. *Suppose that the upper box dimension of Ξ is finite and strictly positive and $\gamma(h) > 0$. Then under the above assumptions, we have*

$$\overline{\mathrm{ac}}(f) \;\leq\; \frac{\overline{\mathrm{Dim}}_B(\Xi) \cdot \gamma(h)}{\overline{\mathrm{Dim}}_B(\Xi) - \sup_{\delta > 0} \overline{\mathrm{Dim}}_B(E_\delta)} \;, \tag{31}$$

where $E_\delta = \{\xi \in \Xi \mid \mathrm{diam}(h^{-1}(\xi)) \geq \delta\}$.

Proof. Without loss of generality, we assume that $\gamma(h)$ is finite and fix $\delta > 0$. Going back to the end of the proof of Theorem 2.7, we find that according to its definition the number N in (11) is equal to $M_{\eta_\delta(\varepsilon)}(\Xi)$. Thus, we have already shown that if $\nu > \mu(B_\varepsilon(E_\delta))$ for some $\varepsilon > 0$, then $\mathrm{Sep}(f, \delta, \nu) \leq M_{\eta_\delta(\varepsilon)}(\Xi)$.

Now, note that $\mu(B_\varepsilon(E_\delta))$ is monotonously decreasing to 0 as $\varepsilon \to 0$. For ν small enough choose $k \in \mathbb{N}$ such that $\mu(B_{2^{-k-1}}(E_\delta)) < \nu \leq \mu(B_{2^{-k}}(E_\delta))$. We obtain

$$
\begin{aligned}
\overline{\mathrm{ac}}(f, \delta) \;&\leq\; \varlimsup_{k \to \infty} \frac{\log M_{\eta_\delta(2^{-k-1})}(\Xi)}{-\log \mu(B_{2^{-k}}(E_\delta))} \\
&=\; \varlimsup_{k \to \infty} \frac{M_{\eta_\delta(2^{-k-1})}(\Xi)}{-\log \eta_\delta(2^{-k-1})} \cdot \frac{\log \eta_\delta(2^{-k-1})}{\log 2^{-k-1}} \cdot \frac{\log 2^{-k-1}}{\log \mu(B_{2^{-k}}(E_\delta))} \\
&\leq\; \overline{\mathrm{Dim}}_B(\Xi) \cdot \gamma(h) \cdot \left(\varliminf_{k \to \infty} \frac{\log \mu(B_{2^{-k}}(E_\delta))}{\log 2^{-k}} \right)^{-1} \\
&=\; \frac{\overline{\mathrm{Dim}}_B(\Xi) \cdot \gamma(h)}{\overline{\mathrm{Dim}}_B(\Xi) - \overline{\mathrm{Dim}}_B(E_\delta)} \;,
\end{aligned}
$$

where we use (29) for the last equality. Taking the supremum over all $\delta > 0$ yields (31). $\qquad\square$

5 Regular Toeplitz flows

Inspired by earlier constructions of almost periodic functions by Toeplitz, the notions of Toeplitz sequences and Toeplitz subshifts or flows were introduced by Jacobs and Keane in 1969 [JK69]. In the sequel, these systems have been used by various authors to provide a series of interesting examples of symbolic dynamics with intriguing dynamical properties, see for example [MP79, Wil84] or [Dow05] and references therein. In what follows, we will study the amorphic complexity for so-called regular Toeplitz subshifts.

Let A be a finite alphabet, $\Sigma_A = A^{\mathbb{I}}$ with $\mathbb{I} = \mathbb{N}_0$ or $\mathbb{Z}$ and ρ the Cantor metric on Σ_A (see Section 3.6). Assume that $\omega \in \Sigma_A$ is a non-periodic Toeplitz sequence with associated Toeplitz subshift (Σ_ω, σ), as defined in Section 1. Given $p \in \mathbb{N}$ and $x = (x_k)_{k \in \mathbb{I}} \in \Sigma_A$, let

$$\mathrm{Per}(p, x) := \{ k \in \mathbb{I} \mid x_k = x_{k+p\ell} \text{ for all } \ell \in \mathbb{N} \} \ .$$

We call the p-periodic part of ω the p-skeleton of ω. To be more precise, define the p-skeleton of ω, denoted by $S(p, \omega)$, as the sequence obtained by replacing ω_k with the new symbol '$*$' for all $k \notin \mathrm{Per}(p, \omega)$. Note that the p-skeletons of two arbitrary points in Σ_ω coincide after shifting one of them by at most $p - 1$ positions. We say that p is an *essential period of ω* if $\mathrm{Per}(p, \omega)$ is non-empty and does not coincide with $\mathrm{Per}(\tilde{p}, \omega)$ for any $\tilde{p} < p$. A *weak periodic structure of ω* is a sequence $(p_\ell)_{\ell \in \mathbb{N}}$ such that each p_ℓ divides $p_{\ell+1}$ and

$$\bigcup_{\ell \in \mathbb{N}} \mathrm{Per}(p_\ell, \omega) = \mathbb{I} \ . \tag{32}$$

If, additionally, all the p_l's are essential, we call $(p_\ell)_{\ell \in \mathbb{N}}$ a *periodic structure of ω*. For every (non-periodic) Toeplitz sequence we can find at least one periodic structure [Wil84].

Remark 5.1. Note that from each weak periodic structure we can obtain a periodic structure in the following way. Suppose $(p_\ell)_{\ell \in \mathbb{N}}$ is a weak periodic structure of ω. Without loss of generality, we can assume that $\mathrm{Per}(p_\ell, \omega) \neq \emptyset$ and $\mathrm{Per}(p_\ell, \omega) \subsetneq \mathrm{Per}(p_{\ell+1}, \omega)$ for all $\ell \in \mathbb{N}$ (recall that ω is non-periodic). For each p_ℓ choose the smallest $\tilde{p}_\ell \in \mathbb{N}$ such that $\mathrm{Per}(\tilde{p}_\ell, \omega)$ coincides with $\mathrm{Per}(p_\ell, \omega)$. Then by definition $\tilde{p}_\ell$ is an essential period. Since p_ℓ divides $p_{\ell+1}$ we have $\mathrm{Per}(\tilde{p}_\ell, \omega) \subset \mathrm{Per}(\tilde{p}_{\ell+1}, \omega)$. The next lemma and the minimality of the $\tilde{p}_\ell$'s imply that $\tilde{p}_\ell$ divides $\tilde{p}_{\ell+1}$ for each $\ell \in \mathbb{N}$, so that $(\tilde{p}_\ell)_{\ell \in \mathbb{N}}$ is a periodic structure.

The next lemma is probably well-known to experts and we omit the proof here.

Lemma 5.2. *If* $\mathrm{Per}(p, x) \subseteq \mathrm{Per}(q, x)$, *then* $\mathrm{Per}(\gcd(p, q), x) = \mathrm{Per}(p, x)$ *where* $x \in \Sigma_A$ *and* $p, q \in \mathbb{N}$.

Given $p \in \mathbb{N}$, we define the relative densitiy of the p-skeleton of ω by

$$D(p) \ := \ \frac{\#(\mathrm{Per}(p, \omega) \cap [0, p-1])}{p} \ .$$

Since ω is non-periodic, we have $D(p) \leq 1 - 1/p$. For a (weak) periodic structure $(p_\ell)_{\ell \in \mathbb{N}}$, the densities $D(p_\ell)$ are non-decreasing in ℓ and we say that (Σ_ω, σ) is a *regular Toeplitz subshift* if $\lim_{\ell \to \infty} D(p_\ell) = 1$. Note that regularity of a Toeplitz subshift does not depend on the chosen (weak) periodic structure (use (32) and Lemma 5.2).

It is well-known that a regular Toeplitz subshift is an almost sure 1-1 extension of a minimal isometry (an odometer) [Dow05]. Thus, we obtain from Theorem 2.7 that its asymptotic separation numbers are finite. However, as mentioned in the introduction, a quantitative analysis is possible and yields the following.

Theorem 5.3. *Suppose (Σ_ω, σ) is a regular Toeplitz subshift and let $(p_\ell)_{\ell \in \mathbb{N}}$ be a (weak) periodic structure of ω. For $\delta, s > 0$ we have*

$$\overline{\lim_{\nu \to 0}} \frac{\mathrm{Sep}(\sigma, \delta, \nu)}{\nu^{-s}} \leq C \cdot \overline{\lim_{\ell \to \infty}} \frac{p_{\ell+1}}{(1 - D(p_\ell))^{-s}} \ ,$$

with $C = C(\delta, s) > 0$.

Note that this directly implies Theorem 1.6.

Proof. Recall that since ω is a regular Toeplitz sequence, the densities $D(p_\ell)$ are non-decreasing and converge to 1. Choose $m \in \mathbb{N}$ with $2^{-m} < \delta \leq 2^{-m+1}$ and $\ell \in \mathbb{N}$ such that

$$(2m+1)2(1 - D(p_{\ell+1})) \ < \ \nu \ \leq \ (2m+1)2(1 - D(p_\ell)) \ . \tag{33}$$

Then we have

$$\mathrm{Sep}(\sigma, \delta, \nu) \ \leq \ \mathrm{Sep}(\sigma, 2^{-m}, (2m+1)2(1 - D(p_{\ell+1})))$$

and claim that the second term is bounded from above by p_{l+1}.

Assume for a contradiction that there exists a $(\sigma, 2^{-m}, (2m+1)2(1 - D(p_{\ell+1})))$-separated set $S \subseteq \Sigma_\omega$ with more than $p_{\ell+1}$ elements. Then, there are at least two points $x = (x_k)_{k \in \mathbb{I}}$, $y = (y_k)_{k \in \mathbb{I}} \in S$ with the same $p_{\ell+1}$-skeleton. This means x and y can differ at most at the remaining positions $k \notin \mathrm{Per}(p_{\ell+1}, x) = \mathrm{Per}(p_{\ell+1}, y)$. Using the fact that $\rho(x, y) \geq 2^{-m}$ if and only if $x_k \neq y_k$ for some $k \in \mathbb{I}$ with $|k| \leq m$, we obtain

$$\overline{\lim_{n \to \infty}} \frac{S_n(\sigma, 2^{-m}, x, y)}{n} \ \leq \ (2m+1) \overline{\lim_{n \to \infty}} \frac{\#\{0 \leq k < n \mid x_k \neq y_k\}}{n}$$

$$\leq \ (2m+1) \overline{\lim_{n \to \infty}} \frac{\#([0, n-1] \setminus \mathrm{Per}(p_{\ell+1}, \omega))}{n} \ = \ (2m+1)(1 - D(p_{\ell+1})) \ .$$

However, this contradicts (33). Hence, we obtain

$$\frac{\mathrm{Sep}(\sigma, \delta, \nu)}{\nu^{-s}} \ \leq \ C(\delta, s) \cdot \frac{p_{\ell+1}}{(1 - D(p_\ell))^{-s}} \ ,$$

where $C(\delta, s) := (2m+1)^s$. Note that m only depends on δ. Taking the limit superior yields the desired result. $\square$

For the remainder of this section, our aim is to provide a class of examples demonstrating that the above estimate is sharp and that the amorphic complexity of regular Toeplitz flows takes at least a dense subset of values in $[1, \infty)$. To that end, we first recall an alternative definition of Toeplitz sequences (cf. [JK69]). Consider the extended alphabet $\mathcal{A} := A \cup \{*\}$ where we can think of $*$ as a hole or placeholder like in the definition of the p-skeleton. Then, $\omega \in \Sigma_A$ is a Toeplitz sequence if and only if there exists an *approximating sequence* $(\omega^\ell)_{\ell \in \mathbb{N}}$ of periodic points in $(\Sigma_{\mathcal{A}}, \sigma)$ such that (i) for all $k \in \mathbb{I}$ we have $\omega_k^{\ell+1} = \omega_k^\ell$ as soon as $\omega_k^\ell \in A$ for some $\ell \in \mathbb{N}$ and (ii) $\omega_k = \lim_{\ell \to \infty} \omega_k^\ell$, see [Ebe71]. Such an approximating sequence of a Toeplitz sequence is not unique. For example, every sequence of p_ℓ-skeletons $(S(p_\ell, \omega))_{\ell \in \mathbb{N}}$ with $(p_\ell)_{\ell \in \mathbb{N}}$ a (weak) periodic structure satisfies these properties.

Let us interpret Theorem 5.3 in this context. For a p-periodic point $x \in \Sigma_{\mathcal{A}}$, we can define the relative density of the holes in x by

$$r(x) \ := \ \frac{\#\{0 \leq k < p \mid x_k = *\}}{p} \ .$$

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Note that $D(p) = 1 - r\left(S(p, \omega)\right)$ for every $p \in \mathbb{N}$. Suppose $(\omega^\ell)_{\ell \in \mathbb{N}}$ is an approximating sequence of ω. We say $(p_\ell)_{\ell \in \mathbb{N}}$ is a *sequence of corresponding periods of* $(\omega^\ell)_{\ell \in \mathbb{N}}$ if p_ℓ divides $p_{\ell+1}$ and $\sigma^{p_\ell}(\omega^\ell) = \omega^\ell$ for each $\ell \in \mathbb{N}$. We have that $r(\omega^\ell) \geq 1/p_\ell$. Moreover, $r(\omega^\ell) \geq 1 - D(p_\ell)$, so that Theorem 5.3 implies

Corollary 5.4. *Assume (Σ_ω, σ) is a regular Toeplitz subshift. Let $(\omega^\ell)_{\ell \in \mathbb{N}}$ be an approximating sequence of ω and let $(p_\ell)_{\ell \in \mathbb{N}}$ be a sequence of corresponding periods of $(\omega^\ell)_{\ell \in \mathbb{N}}$. Furthermore, assume $p_{\ell+1} \leq Cp_\ell^t$ and $r(\omega^\ell) \leq K/p_\ell^u$ for ℓ large enough, where $C, t \geq 1$, $u \in (0, 1]$ and $K > 0$. Then*

$$\overline{\mathrm{ac}}(\sigma) \leq \frac{t}{u} \ .$$

For the construction of examples, it will be convenient to use so-called (p, q)-Toeplitz (infinite) words, as introduced in [CK97]. Let $\mathbb{I} = \mathbb{N}_0$. Suppose v is a finite and non-empty word with letters in $\mathcal{A}$ and at least one entry distinct from $*$. Let $|v|$ be its length and $|v|_*$ be the number of holes in v. We use the notation $\overline{v} \in \Sigma_{\mathcal{A}}$ for the one-sided periodic sequence that is created by repeating v infinitely often. Define the sequence $(T_\ell(v))_{\ell \in \mathbb{N}}$ recursively by

$$T_\ell(v) := F_v(T_{\ell-1}(v)) \ ,$$

where $T_0(v) := \overline{*}$ and $F_v : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ assigns to each $x \in \Sigma_{\mathcal{A}}$ the sequence that is obtained from $\overline{v}$ by replacing the subsequence of all occurrences of $*$ in $\overline{v}$ by x. We get that $(T_\ell(v))_{\ell \in \mathbb{N}}$ is an approximating sequence and denote the corresponding Toeplitz sequence by $T(v)$ [CK97]. Setting $p := |v|$, $q := |v|_*$ and $d := \gcd(p, q)$, we say $T(v)$ is a (p, q)-*Toeplitz word*. One particular nice feature of (p, q)-Toeplitz words is that in order to exclude periodicity one only has to check a short prefix of the sequence.

Theorem 5.5 ([CK97, Theorem 4]). *Let $T(v)$ be a (p, q)-Toeplitz word. Then $T(v)$ is periodic if and only if its prefix of length p is d-periodic.*

Theorem 5.6. *Suppose $m \in \mathbb{N}$ and let $0^m 1$ be the word starting with m zeros and ending with a single one. Furthermore, let v be a word with letters in $\mathcal{A} = \{0, 1, *\}$ such that $1 \leq |v|_* \leq |v| \leq m$. Then $\omega := T(0^m 1 v)$ is a (p, q)-Toeplitz word and the corresponding regular Toeplitz subshift (Σ_ω, σ) has amorphic complexity*

$$\mathrm{ac}(\sigma) = \frac{\log p/d}{\log p/q} \ .$$

Proof. Define for each $n \in \mathbb{N}$ and $x = (x_k)_{k \in \mathbb{N}_0}$, $y = (y_k)_{k \in \mathbb{N}_0} \in \Sigma_{\mathcal{A}}$

$$S_n(x, y) := \# \left\{ 0 \leq k < n \mid x_k, y_k \neq * \text{ and } x_k \neq y_k \right\} \ .$$

Observe that

$$
\begin{aligned}
&S_p\big(T(0^m 1 v), \sigma^j(T(0^m 1 v))\big) \\
&\geq \ S_p\big(T_1(0^m 1 v), \sigma^j(T_1(0^m 1 v))\big) \ = \ S_p\big(\overline{0^m 1 v}, \sigma^j(\overline{0^m 1 v})\big) \ \geq \ 1
\end{aligned}
$$

for every $0 < j < p$ due to the special form of the prefix $0^m 1$ and the assumption $|v| \leq m$. This directly implies that ω is non-periodic, using Theorem 5.5.

To get an upper bound for $\overline{\mathrm{ac}}(\sigma)$, note that $(p^\ell/d^{\ell-1})_{\ell \in \mathbb{N}}$ is a sequence of corresponding periods of $(T_\ell(0^m 1 v))_{\ell \in \mathbb{N}}$ and $r(T_\ell(0^m 1 v)) = q^\ell/p^\ell$ for each $\ell \in \mathbb{N}$. This is proved easily by induction: The statement is true for $T_1(0^m 1 v) = \overline{0^m 1 v}$. When going from ℓ to $\ell + 1$, by the induction hypothesis each of the $p^\ell/d^{\ell-1}$-periodic blocks of $T_\ell(0^m 1 v)$ has $q^\ell/d^{\ell-1}$ free

positions. In order to accommodate q/d such periodic blocks of $T_\ell(0^{\,m}1v)$ it needs p^ℓ/d^ℓ of the p-periodic blocks of $\overline{0^{\,m}1v}$ with q free positions each. Thus, the resulting periodic block of $T_{\ell+1}(0^{\,m}1v)$ has length $p^{\ell+1}/d^\ell$ and $q^{\ell+1}/d^\ell$ free positions. Now, Corollary 5.4 gives the desired upper bound.

In order to prove the lower bound, we show by a similar induction that

$$S_{p^\ell/d^{\ell-1}}\big(T_\ell(0^{\,m}1v),\sigma^j(T_\ell(0^{\,m}1v))\big)\ \geq\ q^{\ell-1}/d^{\ell-1} \tag{34}$$

for every $0 < j < p^\ell/d^{\ell-1}$ and $\ell \in \mathbb{N}$. If j is not a multiple of p, then by induction assumption each $p^\ell/d^{\ell-1}$-periodic block of $T_\ell(0^{\,m}1v)$ has $p/d \cdot q^{\ell-2}/d^{\ell-2}$ mismatches with $\sigma^j(T_\ell(0^{\,m}1v))$ coming from the mismatches of the p/d contained $p^{\ell-1}/d^{\ell-2}$-periodic blocks of $T_{\ell-1}(0^{\,m}1v)$ with $\sigma^j(T_{\ell-1}(0^{\,m}1v))$. If j is a multiple of p, then the mismatches result in a similar way from the shift in the sequences that are inserted into $\overline{0^{\,m}1v}$, since $\sigma^{ip}(T_\ell(0^{\,m}1v)) = F_v(\sigma^{iq}(T_{\ell-1}(0^{\,m}1v)))$. Note that the fact that $p^\ell/d^{\ell-1}$ is a minimal period comes from the assumption that $d = \gcd(p,q)$.

As a direct consequence from (34), we obtain that for all $\ell \in \mathbb{N}$ and $0 \leq i < j < p^\ell/d^{\ell-1}$

$$S_{p^\ell/d^{\ell-1}}\big(\sigma^i(T_\ell(0^{\,m}1v)),\sigma^j(T_\ell(0^{\,m}1v))\big) \geq q^{\ell-1}/d^{\ell-1}\ .$$

Hence,

$$\{\omega,\sigma(\omega),\ldots,\sigma^{p^\ell/d^{\ell-1}-1}(\omega)\}$$

is a $(\sigma,1,q^{\ell-1}/p^\ell)$-separated set. For ν small enough choose $\ell \in \mathbb{N}$ such that $q^\ell/p^{\ell+1} < \nu \leq q^{\ell-1}/p^\ell$ and observe

$$\frac{\mathrm{Sep}(\sigma,\delta,\nu)}{\nu^{-s}}\ \geq\ \frac{\mathrm{Sep}(\sigma,1,q^{\ell-1}/p^\ell)}{\nu^{-s}} > \frac{p^\ell}{d^{\ell-1}}\cdot\frac{q^{ls}}{p^{(l+1)s}}$$

for $\delta, s > 0$. This yields $\underline{\mathrm{ac}}(\sigma) \geq (\log p/d)/(\log p/q)$. □

As the set $\{\log p/\log(p/q) \mid p,q \in \mathbb{N}, \gcd(p,q)=1\}$ is dense in $[1,\infty)$, we obtain

Corollary 5.7. *In the class of (p,q)-Toeplitz words, amorphic complexity takes (at least) a dense set of values in $[1,\infty)$.*

Remark 5.8. From the results in [CK97, Theorem 5], one can directly conclude that for all (non-periodic) (p,q)-Toeplitz words the power entropy equals $(\log p/d)/(\log p/q)$. Thus, for our examples provided by the last theorem power entropy and amorphic complexity coincide. It would be interesting to know if this is true for all (p,q)-Toeplitz words, or if not, in which cases this equality holds.

6 Strange non-chaotic attractors in pinched skew products

As we have mentioned in previous sections, one of the main reasons for considering amorphic complexity is the fact that it gives value zero to Morse-Smale systems, while power entropy assigns a positive value to these. The latter is unsatisfactory from an abstract viewpoint, since such dynamics should certainly be considered entirely trivial. At the same time, however, this issue may also raise practical problems. In more complicated systems, attractor-repeller dynamics may coexist with other more subtle dynamical mechanisms. In this case, the contribution of the Morse-Smale component to power entropy may overlay other effects, and two systems may not be distinguishable despite a clearly different degree of dynamical complexity.

Of course, the computation of topological complexity invariants in more complex non-linear dynamical systems will generally be difficult and technically involved. Nevertheless, we want to include one classical example in this section which fits the situation described above. In order to keep the exposition brief, we concentrate on a positive qualitative result for amorphic complexity and refrain from going into detail concerning (modified) power entropy.

Recall that $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, d is the usual metric on $\mathbb{T}^1$ and Leb denotes the Lebesgue measure on $\mathbb{T}^1$ (cf. Section 3.6). Suppose $f : \mathbb{T}^1 \times [0,1] \to \mathbb{T}^1 \times [0,1]$ is a continuous map of the form

$$f(\theta, x) \;=\; (\theta + \omega \mod 1, f_\theta(x)) \;, \tag{35}$$

where $\omega \in \mathbb{T}^1$ is irrational. For the sake of simplicity we will suppress 'mod 1' in the following. The maps $f_\theta : [0,1] \to [0,1]$ are called *fibre maps*, f itself is often called a *quasiperiodically forced (qpf) 1D map*. If all the fibre maps in (35) are monotonically increasing, then the topological entropy of f is zero.[4] Notwithstanding, systems of this type may exhibit considerable dynamical complexity. A paradigm example in this context are so-called *pinched skew products*, introduced by Grebogi et al in [GOPY84] and later treated rigorously by Keller [Kel96]. In order to fix ideas, we concentrate on the specific parameter family

$$f(\theta, x) \;=\; (\theta + \omega, \tanh(\alpha x) \cdot \sin(\pi\theta)) \;, \tag{36}$$

which is close to the original example introduced by Grebogi and his coworkers. The crucial features of this system are that

(i) the zero line $\mathbb{T}^1 \times \{0\}$ is f-invariant;

(ii) the fibre maps $f_\theta : x \mapsto \tanh(\alpha x) \cdot \sin(\pi\theta)$ are all concave;

(iii) the fibre map f_0 sends the whole interval $[0,1]$ to 0.

Item (iii) is often refered to as *pinching*. It is the defining property of the general class of pinched skew products, as introduced in [Gle02]. Note that all of the arguments and statements in this section immediately carry over to a whole class of fibre maps and higher-dimensional rotations in the base (see the set $\mathcal{T}^*$ and Example 4.1 in [GJ13]).

A function $\varphi : \mathbb{T}^1 \to [0,1]$ is called an *invariant graph* of (35) if $f_\theta(\varphi(\theta)) = \varphi(\theta + \omega)$ for all $\theta \in \mathbb{T}^1$. In this case, the associated point set $\Phi := \{(\theta, \varphi(\theta)) \mid \theta \in \mathbb{T}^1\}$ is f-invariant.[5] If the fibre maps are all differentiable, the *(vertical) Lyapunov exponent* of φ is defined as

$$\lambda(\varphi) \;:=\; \int_{\mathbb{T}^1} \log |f_\theta'(\varphi(\theta))| \; d\theta \;. \tag{37}$$

If $\lambda(\varphi) \leq 0$, then Φ is an attractor in the sense of Milnor [Mil85] (see, for example, [Jäg03, Proposition 3.3]). If φ is continuous, then it is even a topological attractor and contains an open annular neighbourhood in its basin of attraction. In case φ is not continuous, the attractor Φ combines non-chaotic dynamics (zero entropy, absence of positive Lyapunov exponents) with a complicated topological structure (related to the absence of continuity, see [Sta03, Jäg07] for more information). Due to this combination of properties, it is called a *strange non-chaotic attractor (SNA)*.

As mentioned above, the zero line $\Phi_0 := \mathbb{T}^1 \times \{0\}$ is an invariant graph of (36). An elementary computation yields $\lambda(\varphi_0) = \log \alpha - \log 2$. If $\alpha \leq 2$, so that $\lambda(\varphi_0) \leq 0$, then Φ_0 is the global attractor of the system, meaning that $\Phi_0 = \bigcap_{n \in \mathbb{N}} f^n(\mathbb{T}^1 \times [0,1])$. Accordingly, all orbits converge to the zero line, that is, $\lim_{n \to \infty} f_\theta^n(x) = 0$, where $f_\theta^n = f_{\theta + (n-1)\omega} \circ \ldots \circ f_\theta$.

---

[4]This is a direct consequence of [Bow71, Theorem 17].

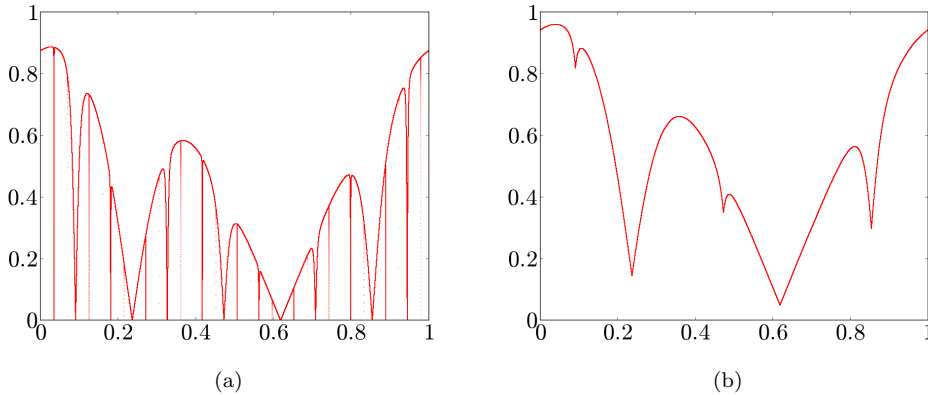[5]Slightly abusing notation, the term *invariant graph* is used both for the function and its graph.

Figure 1: The upper bounding graphs of f (a) and f^ε (b) with $\varepsilon = 0.05$. In both cases, ω is the golden mean and $\alpha = 3$. The horizontal axis is $\mathbb{T}^1$, the vertical axis $[0, 1]$.

If $\alpha > 2$, then this picture changes drastically. Now, a second invariant graph φ^+ with negative Lyapunov exponent appears, which satisfies $\varphi^+(\theta) > 0$ for Leb-a.e. $\theta \in \mathbb{T}^1$ [Kel96]. However, at the same time there exists a dense set of θ's with $\varphi^+(\theta) = 0$. This latter fact is easy to see, since φ^+ is invariant and $f_0(\varphi^+(0)) = 0$ by property (iii) above. Thus, φ^+ is an SNA (see Figure 1(a)). We note that φ^+ can be defined as the upper bounding graph of the global attractor $\mathcal{A} := \bigcap_{n \in \mathbb{N}} f^n(\mathbb{T}^1 \times [0, 1])$, that is,

$$\varphi^+(\theta) := \sup\{x \in [0, 1] \mid (\theta, x) \in \mathcal{A}\} . \tag{38}$$

The rigorous proof of these facts in [Kel96] is greatly simplified by the particular structure of pinched skew products. More natural systems, though, often occur as the time-one-maps of flows generated by scalar differential equations with quasiperiodic right-hand side. In particular, this means that such systems are invertible and the non-invertible pinched skew products have a certain toy-model character. Nowadays, however, established methods of multiscale analysis yield a wealth of results about the existence and structure of SNA in broad classes of invertible systems as well [You97, Bje05, Bje07, Jäg06, Fuh14]. In many cases, it turned out that this machinery allows to transfer results and insights first obtained for pinched systems to a more general setting. One example is the computation of the Hausdorff dimension of SNA [GJ13, FGJ14], another is a question about the structure of their topological closure (*filled-in property*) [Jäg07, Bje07, FGJ14], going back to Herman [Her83]. In this sense, pinched systems have proven to be very adequate models for more general qpf systems.

If (36) is slightly modified by adding a small positive constant $\varepsilon > 0$ to the multiplicative forcing term $\sin(\pi\theta)$, we obtain a new system

$$f^\varepsilon(\theta, x) = (\theta + \omega, \tanh(\alpha x) \cdot (\sin(\pi\theta) + \varepsilon)) . \tag{39}$$

In this case, the Lyapunov exponent $\lambda(\varphi_0)$ still increases strictly with α and there exists a critical value $\alpha_c = \int_{\mathbb{T}^1} \log |\sin(\pi\theta) + \varepsilon| \, d\theta$ at which $\lambda(\varphi_0) = 0$. If $\alpha \leq \alpha_c$, the graph Φ_0 is the global attractor as before, and if $\alpha > \alpha_c$ a second invariant graph φ^+ with negative Lyapunov exponent appears above Φ_0. However, there is one important qualitative difference to the previous situation. Due to the invertibility of the system, it is easy to show that the graph Φ^+ is a continuous curve (see Figure 1(b)).

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The resulting dynamics are much simpler than in the case of an SNA. In particular, the system is conjugate to the direct product of the underlying irrational rotation with a Morse-Smale map g on $[0,1]$, with unique repelling fixed point 0 and unique attracting fixed point $x \in (0,1)$, and all points outside Φ_0 are Lyapunov stable. In contrast to this, the system (36) with $\alpha > 2$ has sensitive dependence on initial conditions [GJK06], and thus has no Lyapunov stable points at all.

It is thus reasonable to expect that both cases can be distinguished by means of a suitable topological complexity invariant. However, in this case the rigorous analysis is more difficult than in the previous chapters. The reason is that while the qualitative analysis of pinched skew products is comparatively easy due to their particular structure, a more detailed quantitative study is still rather involved on a technical level. In particular, it typically requires to exclude an exceptional set of measure zero from the considerations, on which the dynamics are hard to control. Since topological complexity invariants in the zero entropy regime typically do not satisfy a variational principle (see Introduction), the lack of control even on a set of measure zero impedes their computation. For this reason, we do not attempt to determine the power entropy or modified power entropy of (36) in a rigorous way. However, based on the intuition gained from previous work on pinched systems in [Jäg07, GJ13] and heuristic arguments, we expect that the power entropy of (36) with $\alpha > 2$ equals 1, whereas the modified power entropy is zero. It is easy to show that the same values are attained by (39) with $\alpha > \alpha_c$, and hence both quantities should not be suitable to distinguish between the two substantially different types of behaviour. As we have mentioned before, this was one of the original motivations for the introduction of amorphic complexity.

In principle, though, the same restrictions as for the computation of (modified) power entropy hold for amorphic complexity, and the existence of the exceptional uncontrolled set does not allow a straightforward application of the concept. What we concentrate on here is to show that amorphic complexity distinguishes between SNAs and continuous attractors. This is the main result of this section. Recall that $\omega \in \mathbb{T}^1$ is called *Diophantine* if there exist constants $c, d > 0$ such that

$$d(n\omega, 0) \geq cn^{-d} \tag{40}$$

for all $n \in \mathbb{N}$.

Theorem 6.1. *Suppose ω is Diophantine and α in (36) is sufficiently large. Then there exists an invariant (under the rotation by angle ω) set $\Omega \subseteq \mathbb{T}^1$ of full Lebesgue measure such that*

$$0 < \underline{\mathrm{ac}}\big(f|_{\Omega \times [0,1]} \big) \leq \overline{\mathrm{ac}}\big(f|_{\Omega \times [0,1]} \big) < \infty \,.$$

In contrast to this, we have $\mathrm{ac}(f^\varepsilon) = 0$ if f^ε is given by (39) with $\varepsilon > 0$ and any $\alpha \geq 0$.

Note that $\mathrm{ac}(f^\varepsilon) = 0$ follows immediatly from the conjugacy between f^ε and the product of the underlying rotation with the Morse-Smale map g from above, using Corollary 3.5.

Remark 6.2. The approach taken by restricting to a subset of full measure in the above statement can be formalized in a more systematic way. Although we do not pursue this issue much further here, we believe that this may make the concept of amorphic complexity applicable to an even broader range of systems. Let (X, d) be a metric space and consider a map $f : X \to X$ and let $E \subseteq X$. For the definition of $\mathrm{Sep}_E(f, \delta, \nu)$ see (25). Further, we say $A \subseteq E$ is (f, δ, ν)-*spanning in* E if for each $x \in E$ there is $y \in A$ such that $\overline{\lim}_{n \to \infty} \frac{1}{n} S_n(f, \delta, x, y) < \nu$. Let $\mathrm{Span}_E(f, \delta, \nu)$ be the smallest cardinality of any (f, δ, ν)-spanning set in E. For the definition of $\underline{\mathrm{ac}}_E(f)$ and $\overline{\mathrm{ac}}_E(f)$ see (26) and note that similarly as before we can use $\mathrm{Span}_E(f, \delta, \nu)$ instead of $\mathrm{Sep}_E(f, \delta, \nu)$ there (see Section 3.2).

Now, suppose we are given a Borel probability measure μ on X. Then we define

$$\underline{\mathrm{ac}}_\mu(f) := \inf\left\{\underline{\mathrm{ac}}_E(f) \mid E \subseteq X \text{ and } \mu(E) = 1\right\},$$
$$\overline{\mathrm{ac}}_\mu(f) := \inf\left\{\overline{\mathrm{ac}}_E(f) \mid E \subseteq X \text{ and } \mu(E) = 1\right\}.$$

With these notions, what we actually show is that under the assumptions of Theorem 6.1 we have

$$0 < \underline{\mathrm{ac}}_\mu(f) \le \overline{\mathrm{ac}}_\mu(f) < \infty,$$

where μ can be either the Lebesgue measure on $\mathbb{T}^1 \times [0,1]$, or the measure μ_{φ^+}, which is the Lebesgue measure on $\mathbb{T}^1$ lifted to the graph Φ^+, i.e. $\mu_{\varphi^+}(A) := \mathrm{Leb}(\pi_\theta(A \cap \Phi^+))$ where $A \subseteq \mathbb{T}^1 \times [0,1]$ is Borel measurable and π_θ is the projection onto the first coordinate. Note that these statements are slightly stronger than the ones given in Theorem 6.1.

We first consider the lower bound. Thereby, we will focus on the SNA Φ^+ and show that the restriction of f to this set already has positive lower amorphic complexity.

Proposition 6.3. *Suppose ω is Diophantine and α in (36) is sufficiently large. Then there is a positive uniform lower bound for $\underline{\mathrm{ac}}_{\Phi^+ \cap (\Omega \times [0,1])}(f)$ for all $\Omega \subseteq \mathbb{T}^1$ with $\mathrm{Leb}(\Omega) = 1$.*

For the proof, we need a number of preliminary statements taken from previous studies of pinched skew products in [Jäg07, GJ13]. First, [GJ13, Lemma 4.2] states that if α in (36) is sufficiently large, then there exist constants $\gamma, L_0, \beta, a, b > 0$ and $m \in \mathbb{N}$ such that the following conditions are satisfied.

$$
\begin{align}
m &\ge 22(1 + 1/\gamma) \tag{41} \\
a &\ge (m+1)^d \tag{42} \\
b &\le c \tag{43} \\
b &< d(n\omega, 0) \text{ for all } n \in \{1, \ldots, m-1\} \tag{44} \\
|f_\theta(x) - f_\theta(y)| &\le \alpha^{-\gamma} |x - y| \text{ for all } \theta \in \mathbb{T}^1,\ x, y \in [L_0, 1] \tag{45} \\
f_\theta(x) &\ge \min\{L_0, ax\} \cdot \min\{1, 2d(\theta, 0)/b\} \text{ for all } (\theta, x) \in \mathbb{T}^1 \times [0,1] \tag{46}
\end{align}
$$

It is worth mentioning that we can choose a proportional to α. Moreover, we note that

$$
\begin{align}
|f_\theta(x) - f_\theta(y)| &\le \alpha |x - y| \text{ for all } \theta \in \mathbb{T}^1,\ x, y \in [0,1], \tag{47} \\
|f_\theta(x) - f_{\theta'}(x)| &\le \pi d(\theta, \theta') \text{ for all } \theta, \theta' \in \mathbb{T}^1,\ x \in [0,1]. \tag{48}
\end{align}
$$

Given any $n \in \mathbb{N}$, let $r_n := \frac{b}{2} a^{-\frac{n-1}{m}}$ and $\tau_n := n\omega$. We will need the following elementary estimate.

Lemma 6.4 ([Jäg07, GJ13]). *Let $n \in \mathbb{N}$ and suppose $d(\tau_n, 0) \le \ell b \cdot a^{-i}$ for some $i > 0$ and $\ell > 0$. Then $n \ge \frac{a^{i/d}}{\ell^{1/d}}$.*

Proof. (40) implies $c \cdot n^{-d} \le \ell b \cdot a^{-i}$, and using (43) we get $n^{-d} \le \ell a^{-i}$. $\qquad\square$

In order to analyse the dynamics of f on Φ^+, it turns out to be crucial that φ^+ is approximated by the so-called *iterated boundary lines* $(\varphi_n)_{n \in \mathbb{N}}$ of (36). These are given by

$$\varphi_n : \mathbb{T}^1 \to [0,1]; \quad \theta \mapsto f^n_{\theta - n\omega}(1),$$

with $f^n_\theta(x) := \pi_x \circ f^n(\theta, x) = f_{\theta + (n-1)\omega} \circ \ldots \circ f_\theta(x)$ where π_x is the projection onto the second coordinate. Note that by the monotonicity of the maps f_θ, the sequence $(\varphi_n)_{n \in \mathbb{N}}$ is decreasing. Further, as a consequence of the definition of φ^+ in (38), it can be shown easily that $\varphi_n \to \varphi^+$ pointwise as $n \to \infty$ [GJ13]. The following proposition tells us to which degree the n-th iterated boundary line approximates the graph φ^+.
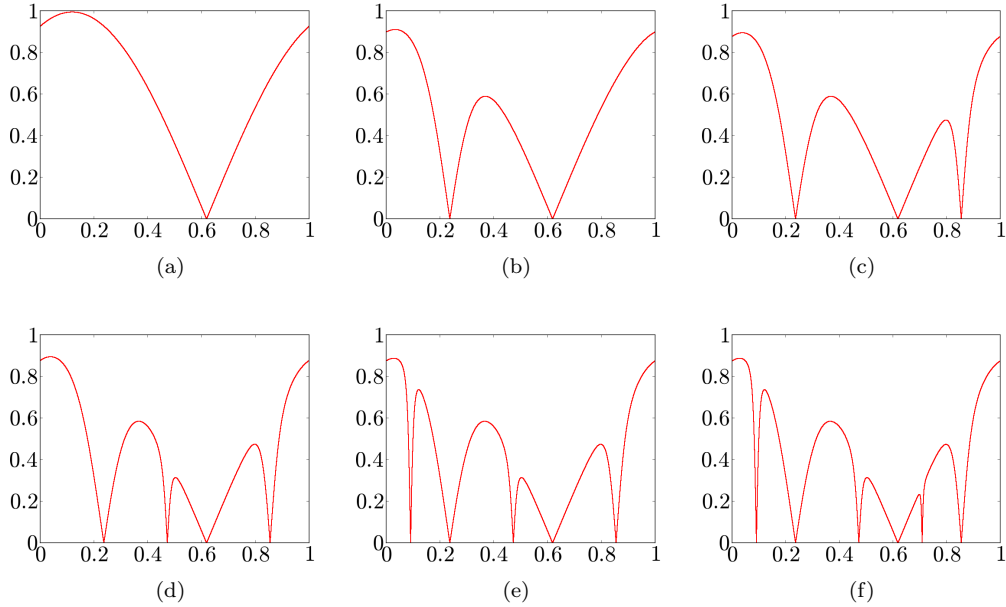
Figure 2: The iterated boundary lines φ_n for $n = 1, \ldots, 6$.

Proposition 6.5 ([Jäg07],[GJ13]). *Given $q \in \mathbb{N}$, the following holds.*

(i) $|\varphi_n(\theta) - \varphi_n(\theta')| \leq \pi \alpha^n d(\theta, \theta')$ *for all $n \in \mathbb{N}$ and $\theta, \theta' \in \mathbb{T}^1$.*

(ii) *There exists $\lambda > 0$ such that if $n \geq mq + 1$ and $\theta \notin \bigcup_{j=q}^n B_{r_j}(\tau_j)$, then $|\varphi_n(\theta) - \varphi_{n-1}(\theta)| \leq \alpha^{-\lambda(n-1)}$.*

Figure 2 shows the development of the iterated upper boundary lines for $n = 1, \ldots, 6$. As can be seen, φ_n has exactly n zeros (at $\tau_1, \ldots, \tau_n$). In order to describe the qualitative behaviour, we refer to $\psi|_{B_{r_j}(\tau_j)}$ as the *j-th peak* of ψ or the *peak of ψ around τ_j*, where $\psi \in \{\varphi^+, \varphi_j, \varphi_{j+1}, \ldots\}$. We say the j-th peak is a *fresh peak* if the $2r_j$-neighbourhood of τ_j does not intersect any previous peak, that is, $B_{2r_j}(\tau_j) \cap B_{r_l}(\tau_l) = \emptyset$ for each $1 \leq l < j$. In the following, we label the fresh peaks by $n_1 < n_2 < \ldots$, that is, there is $j \in \mathbb{N}$ with $n_j = l$ if and only if the l-th peak is fresh.

Lemma 6.6. *If α is large enough, there are infinitely many fresh peaks and they appear with positive density, that is,*

$$\varliminf_{j \to \infty} j/n_j > 0 .$$

Proof. Let

$$N(k) := \left\{ j \in \{2, \ldots, k\} \mid \text{there is } 1 \leq l < j \text{ with } B_{2r_j}(\tau_j) \subseteq B_{2r_l}(\tau_l) \right\} .$$

Thus, $N(n_j)$ contains the complement of $\{n_l \mid 1 \le l \le j\}$. Further,

$$
\begin{aligned}
\#N(k) \;&\le\; \sum_{l=1}^{k-1} \#\left\{ j \in \{l+1,\dots,k\} \mid B_{2r_j}(\tau_j) \subseteq B_{2r_l}(\tau_l) \right\} \\
&\overset{\text{Lemma 6.4}}{\le}\; \#\left\{ j \in \{2,\dots,k\} \mid B_{2r_j}(\tau_j) \subseteq B_{2r_1}(\tau_1) \right\} + \sum_{l=2}^{k-1} \frac{(k-l)2^{1/d}}{a^{(l-1)/md}} \\
&\le\; \#\left\{ j \in \{2,\dots,k\} \mid B_{2r_j}(\tau_j) \subseteq B_{2r_1}(\tau_1) \right\} + 2^{1/d} k \sum_{l=2}^{k-1} a^{-(l-1)/md} \\
&<\; \#\left\{ j \in \{2,\dots,k\} \mid B_{2r_j}(\tau_j) \subseteq B_{2r_1}(\tau_1) \right\} + \frac{2^{1/d} k a^{-1/md}}{1-a^{-1/md}} \;.
\end{aligned}
$$

Thus, for big enough α (and hence big enough a) and due to (44), there are infinitely many fresh peaks and

$$
\varliminf_{j\to\infty} j/n_j \;\ge\; \varliminf_{j\to\infty} \frac{n_j - \#N(n_j)}{n_j} \;\ge\; 1 - \mathrm{Leb}(B_{2r_1}(\tau_1)) - \frac{2^{1/d} a^{-1/md}}{1-a^{-1/md}} \;>\; 0 \;. \qquad \square
$$

Since $(\varphi_n)_{n\in\mathbb{N}}$ is monotonously decreasing and each iterated boundary line is continuous, we know that φ^+ is close to zero in a neighbourhood of each τ_n. However, the next statement tells us that for most θ in a neighbourhood of a fresh peak, φ^+ is bigger than some threshold $\delta_0 > 0$. This dichotomy is the basis for the mechanism by which we prove Proposition 6.3.

Lemma 6.7. *Suppose α is large enough. There exist $\delta_0 > 0$ and a super-exponentially fast decaying sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that*

$$
\mathrm{Leb}\left(\left\{ \theta \in B_{2r_{n_j}}(\tau_{n_j}) \setminus B_{r_{n_j}}(\tau_{n_j}) \mid \varphi^+(\theta) < \delta_0 \right\} \right) \;<\; \varepsilon_{n_j} \;.
$$

Proof. Let $\ell := m+1$. Since φ_ℓ is continuous and $\varphi_\ell(\theta) \ne 0$ for $\theta \notin \{\tau_1,\dots,\tau_\ell\}$, there exists $\delta_0 > 0$ such that

$$
\varphi_\ell(\theta) \;\ge\; 2\delta_0 \tag{49}
$$

for $\theta \notin \bigcup_{j=1}^{\ell} B_{r_j}(\tau_j)$. Due to (44), we have that if α (and hence a) is large enough, then $\mathbb{T}^1 \setminus \bigcup_{j=1}^{\infty} B_{r_j}(\tau_j)$ is non-empty. Let $\theta \notin \bigcup_{j=1}^{\infty} B_{r_j}(\tau_j)$. For $k \in \mathbb{N}$ with $k > \ell$, Proposition 6.5 (ii) yields

$$
|\varphi_k(\theta) - \varphi_\ell(\theta)| \;\le\; \sum_{j=\ell+1}^{k} |\varphi_j(\theta) - \varphi_{j-1}(\theta)| \;\le\; \sum_{j=\ell}^{k-1} \alpha^{-\lambda j} \;\le\; \frac{\alpha^{-\lambda\ell}}{1-\alpha^{-\lambda}} \;.
$$

Together with equation (49), this gives

$$
\varphi_k(\theta) \;\ge\; 2\delta_0 - \frac{\alpha^{-\lambda\ell}}{1-\alpha^{-\lambda}} \;>\; \delta_0 \tag{50}
$$

for sufficiently large α.

Let $j \ge 2$. Since the n_j-th peak is fresh, we have $B_{2r_{n_j}}(\tau_{n_j}) \cap \bigcup_{l=1}^{n_j-1} B_{r_l}(\tau_l) = \emptyset$. Further, Lemma 6.4 yields that the first time $l > n_j$ a peak intersects $B_{2r_{n_j}}(\tau_{n_j})$ is bounded from

below by $n_j + a^{(n_j-1)/md}$. Hence,

$$
\begin{aligned}
B_{2r_{n_j}}(\tau_{n_j}) \setminus \bigcup_{l \geq 1} B_{r_l}(\tau_l) \;&=\; B_{2r_{n_j}}(\tau_{n_j}) \setminus \bigcup_{l \geq n_j} B_{r_l}(\tau_l) \\
&=\; \left(B_{2r_{n_j}}(\tau_{n_j}) \setminus B_{r_{n_j}}(\tau_{n_j}) \right) \setminus \bigcup_{l \geq a^{(n_j-1)/md}+n_j} B_{r_l}(\tau_l) \; .
\end{aligned}
$$

Note that for $n \in \mathbb{N}$

$$
\begin{aligned}
\mathrm{Leb} \left(\bigcup_{l \geq a^{(n-1)/md}+n} B_{r_l}(\tau_l) \right) \;&\leq\; b \sum_{l \geq a^{(n-1)/md}+n} a^{-(l-1)/m} \\
&=\; \frac{b}{1 - a^{-1/m}} a^{-\left(a^{(n-1)/m}+n-1\right)/m} \;=:\; \varepsilon_n \; .
\end{aligned}
$$

By means of equation (50), this proves the statement since $\varphi_k \to \varphi^+$ as $k \to \infty$. $\qquad\square$

Proof of Proposition 6.3. Suppose $\delta < \frac{\delta_0}{2}$, where δ_0 is chosen as in Lemma 6.7 and set $\eta_j := \delta/(2\pi\alpha^{n_j})$. Further, let Ω be a given set of full measure. As $(\varphi_n)_{n \in \mathbb{N}}$ is monotonously decreasing, Proposition 6.5 (i) shows that

$$
\varphi^+(\theta) < \delta \quad \text{for all} \quad \theta \in B_{2\eta_j}(\tau_{n_j}) \; . \tag{51}
$$

By possibly going over to a subsequence (such that n_1 is big enough), Lemma 6.7 yields that for each $j \in \mathbb{N}$ there is $\theta^{n_j} \in B_{2r_{n_j}}(\tau_{n_j})$ with

$$
\mathrm{Leb} \left(\left\{ \theta \in B_{\eta_j}(\theta^{n_j}) \mid \varphi^+(\theta) > 2\delta \right\} \right) \;>\; \eta_j \; . \tag{52}
$$

Set $\Delta_j := \tau_{n_j} - \theta^{n_j}$. By (51) and (52), we have that

$$
\mathrm{Leb} \left(\left\{ \theta \in \mathbb{T}^1 : \left| \varphi^+(\theta) - \varphi^+(\theta+\Delta) \right| > \delta \text{ for all } \Delta \in B_{\eta_j}(\Delta_j) \right\} \right) \;\geq\; \eta_j \; . \tag{53}
$$

We denote by Ω_j the set of such θ which visit the set $\{ \theta \in \mathbb{T}^1 : |\varphi^+(\theta) - \varphi^+(\theta+\Delta)| > \delta \text{ for all } \Delta \in B_{\eta_j}(\Delta_j) \}$ with a frequency η_j. Note that by (53) and Birkhoff's Ergodic Theorem, $\mathrm{Leb}(\Omega_j) = 1$ such that $\tilde{\Omega} := \Omega \cap \bigcap_{j \in \mathbb{N}} \Omega_j$ has full measure.

Next, we choose 2^j points in $\Phi^+ \cap \tilde{\Omega} \times [0,1]$ which are mutually (f, δ, η_j)-separated from each other. Let $\theta \in \bigcap_{x \in \{0,1\}^j} \tilde{\Omega} - \sum_{k=1}^{j} x_k \Delta_k$ and define $\theta_x = \theta + \sum_{k=1}^{j} x_k \Delta_k$ where $x = (x_1, \ldots, x_j) \in \{0,1\}^j$. By possibly going over to a subsequence of $(n_j)_{j \in \mathbb{N}}$ (still of positive density), we may assume without loss of generality that $\sum_{k=i+1}^{\infty} (\eta_k + |\Delta_k|) < \eta_i$ for all $i \in \mathbb{N}$ such that $d(\theta_x, \theta_y) \in B_{\eta_\ell}(\Delta_\ell)$ for distinct $x, y \in \{0,1\}^j$ and $\ell := \min\{k \mid x_k \neq y_k\} \leq j$. By definition, we have for all $x \in \{0,1\}^j$ that $\theta_x \in \tilde{\Omega}$ and hence, the set $\{(\theta_x, \varphi^+(\theta_x)) \mid x \in \{0,1\}^j\}$ is (f, δ, η_j)-separated. We have thus shown

$$
\begin{aligned}
\varliminf_{\nu \to 0} \frac{\log \mathrm{Sep}_{\Phi^+ \cap (\Omega \times [0,1])}(f, \delta, \nu)}{\log \nu^{-1}} \;&\geq\; \varliminf_{j \to \infty} \frac{\log \mathrm{Sep}_{\Phi^+ \cap (\tilde{\Omega} \times [0,1])}(f, \delta, \eta_j)}{\log \eta_{j+1}^{-1}} \\
&\geq\; \varliminf_{j \to \infty} \frac{\log 2^j}{n_{j+1} \log \alpha - \log \delta/2\pi} \;=\; \frac{\log 2}{\log \alpha} \varliminf_{j \to \infty} j/n_{j+1} \;>\; 0
\end{aligned}
$$

by Lemma 6.6. As $\varliminf_{j \to \infty} j/n_{j+1}$ is independent of the set Ω, this proves the statement. $\quad\square$

We now turn to the upper bound of the amorphic complexity. Let $\Omega \subseteq \mathbb{T}^1$ be the set of all $\theta \in \mathbb{T}^1$ such that for all $q \in \mathbb{N}$

$$\lim_{n \to \infty} \frac{\# \left\{ 0 \leq i \leq n-1 \mid \theta + i\omega \in \bigcup_{j=q}^{\infty} B_{r_j}(\tau_j) \right\}}{n} = \text{Leb} \left(\bigcup_{j=q}^{\infty} B_{r_j}(\tau_j) \right).$$

Note that Birkhoff's Ergodic Theorem yields that $\mu_{\varphi^+}(\Phi^+ \cap (\Omega \times [0,1])) = \text{Leb}(\Omega) = 1$ (see Remark 6.2). The upper bound on $\overline{\text{ac}}_{\Phi^+ \cap (\Omega \times [0,1])}(f)$ will follow easily from the following assertion.

Lemma 6.8. *There exist $\kappa, c_0 > 0$ such that for all positive δ and small enough ν, we have that for each $\theta, \theta' \in \Omega$ with $d(\theta, \theta') < \varepsilon = \varepsilon(\delta, \nu) = c_0 \delta \nu^{\kappa m}$ the points $(\theta, \varphi^+(\theta))$ and $(\theta', \varphi^+(\theta'))$ are not (f, δ, ν)-separated.*

Proof. Observe that there is a constant $C > 0$, independent of both δ and ν, such that for $q(\nu) := \lceil -C \log \nu \rceil$ we have

$$\text{Leb} \left(\bigcup_{j=q(\nu)}^{\infty} B_{r_j}(\tau_j) \right) \leq \sum_{j=q(\nu)}^{\infty} a^{-(j-1)/m} < \nu/2 . \tag{54}$$

Set $n(\nu) := mq(\nu) + 1$ and assume that $n(\nu)$ is large enough (i.e. ν is small) to guarantee

$$\left| \varphi^+(\theta) - \varphi_{n(\nu)}(\theta) \right| < \delta/4 \tag{55}$$

for all $\theta \notin \bigcup_{j=q(\nu)}^{\infty} B_{r_j}(\tau_j)$ (cf. Proposition 6.5(ii)). Note that if $d(\theta, \theta') < \varepsilon(\delta, \nu) := \delta/(4\pi\alpha^{m(-C \log \nu + 1)+1})$, then

$$\left| \varphi_{n(\nu)}(\theta) - \varphi_{n(\nu)}(\theta') \right| \leq \pi\alpha^{n(\nu)} d(\theta, \theta') < \delta/4 \tag{56}$$

for all θ, θ' (cf. Proposition 6.5(i)). Now, assume $\theta, \theta' \in \Omega$ verify $d(\theta, \theta') < \varepsilon(\delta, \nu) < \delta/4$. By (54) and definition of Ω, we know that

$$\lim_{n \to \infty} \frac{\# \left\{ 0 \leq i \leq n-1 \mid \theta + i\omega, \theta' + i\omega \notin \bigcup_{j=q(\nu)}^{\infty} B_{r_j}(\tau_j) \right\}}{n} > 1 - \nu.$$

Further, (55) and (56) yield that $|\varphi^+(\theta + i\omega) - \varphi^+(\theta' + i\omega)| < \frac{3}{4}\delta$ whenever both $\theta + i\omega$ and $\theta' + i\omega$ are not in $\bigcup_{j=q(\nu)}^{\infty} B_{r_j}(\tau_j)$. Hence,

$$\overline{\lim_{n \to \infty}} \frac{S_n(f, \delta, (\theta, \varphi(\theta)), (\theta', \varphi(\theta')))}{n} < \nu,$$

so that $(\theta, \varphi(\theta))$ and $(\theta', \varphi(\theta'))$ are not (f, δ, ν)-separated. $\square$

We thus have

Proposition 6.9. *Suppose α in (36) is sufficiently large and Ω is as in the previous lemma. Then*

$$\overline{\text{ac}}_{\Phi^+ \cap (\Omega \times [0,1])}(f) \leq \kappa m,$$

with κ as in Lemma 6.8.

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Proof. By the previous lemma, we know that for small enough ν

$$\mathrm{Span}_{\Phi^+ \cap (\Omega \times [0,1])}(f, \delta, \nu) \ \leq \ \left\lceil \frac{1}{\varepsilon(\delta, \nu)} \right\rceil + 1 \ = \ \left\lceil \frac{\nu^{-\kappa m}}{c_0 \delta} \right\rceil + 1 \ < \ 2 \frac{\nu^{-\kappa m}}{c_0 \delta} \ . \qquad \square$$

Proof of Theorem 6.1. It is left to show the upper bound. To that end, we show that there is an invariant set of full measure $\tilde{\Omega} \subseteq \Omega$ (Ω as above) such that

$$\lim_{n \to \infty} \varphi^+(\theta + n\omega) - f_\theta^n(x) \ = \ 0 \tag{57}$$

for all $\theta \in \tilde{\Omega}$ and $x \in (0, 1]$. In other words, we show that $f|_{\tilde{\Omega} \times (0,1]}$ has the unique target property with respect to $\Phi^+ \cap (\tilde{\Omega} \times (0,1])$. By means of Lemma 3.11 and Proposition 6.9 this yields that $\overline{\mathrm{ac}}(f|_{\tilde{\Omega} \times (0,1]}) \leq \kappa m$ and it is easy to see that κm is in fact an upper bound for $\overline{\mathrm{ac}}(f|_{\tilde{\Omega} \times [0,1]})$.

Note that since φ^+ is the upper bounding graph of the global attractor, we have (57) for all θ and $x \in [\varphi^+(\theta), 1]$. Now, define $\psi(\theta) := \sup\{x \in [0, \varphi^+(\theta)] \mid (57) \text{ does not hold}\}$. Due to monotonicity, ψ is an invariant graph. By [Jäg03, Proposition 3.3], we have that for Leb-a.e. θ there is $\delta(\theta) > 0$ such that (57) holds for $x \in (\varphi^+(\theta) - \delta(\theta), \varphi^+(\theta)]$. Hence, ψ is distinct from φ^+. Since concavity of the fibre maps f_θ only allows for two invariant graphs (equivalence up to sets of measure zero, cf. [AJ12, Theorem 2.1]), this shows that $\psi = 0$ Lebesgue almost surely. Set $\tilde{\Omega} := \bigcap_{n \in \mathbb{Z}} n\omega + \{\theta \in \Omega \mid \psi(\theta) = 0\}$. This ends the proof. $\qquad \square$

References

[AA77]   S. Aljančić and D. Aranđelović. O-Regularly Varying Functions. *Publ. Inst. Math., Nouv. Sér.*, 22(36):5 − 22, 1977.

[AB92]   J.-P. Allouche and R. Bacher. Toeplitz sequences, paperfolding, towers of hanoi and progression-free sequences of integers. *Enseign. Math. (2)*, 38(3–4):315–327, 1992.

[AJ12]   V. Anagnostopoulou and T. Jäger. Nonautomous saddle-node bifurcations: Random and deterministic forcing. *J. Differ. Equations*, 253(2):379–399, 2012.

[AS99]   J.-P. Allouche and and J. Shallit. The Ubiquitous Prouhet-Thue-Morse Sequence. In C. Ding, T. Helleseth and H. Niederreiter, editors, *Sequences and their Applications*, pages 1–16. Springer London, 1999.

[Aus88]   J. Auslander. *Minimal flows and their extensions*. Elsevier Science Ltd, 1988.

[BKS06]   V. V. Buldygin, O.I. Klesov, and J.G. Steinebach. On some extensions of Karamata's theory and their applications. *Publ. Inst. Math., Nouv. Sér.*, 80(94):59–96, 2006.

[BLM07]   M. Baake, D. Lenz, and R.V. Moody. Characterization of model sets by dynamical systems. *Ergodic Theory Dyn. Syst.*, 27(2):341–382, 2007.

[Bje05]   K. Bjerklöv. Positive Lyapunov exponent and minimality for a class of one-dimensional quasi-periodic Schrödinger equations. *Ergodic Theory Dyn. Syst.*, 25:1015–1045, 2005.

[Bje07]   K. Bjerklöv. Dynamics of the Quasi-Periodic Schrödinger Cocycle at the Lowest Energy in the Spectrum. *Commun. Math. Phys.*, 272(2):397–442, 2007.

[BFK97]   F. Blanchard, E. Formenti and P. Kůrka Cellular Automata in the Cantor, Besicovitch, and Weyl Topological Spaces. *Complex Syst.*, 11(2):107–123, 1997.

[BMN00]   F. Blanchard, A. Maass, and A. Nogueira, editors. *Topics in Symbolic Dynamics and Applications*, volume 279 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2000.

[Bow71]   R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Am. Math. Soc.*, 153:401–414, 1971.

[BS02]   M. Brin and G. Stuck. *Introduction to Dynamical Systems*. Cambridge University Press, 2002.

[CK97]   J. Cassaigne and J. Karhumäki. Toeplitz Words, Generalized Periodicity and Periodically Iterated Morphisms. *Eur. J. Comb.*, 18(5):497–510, 1997.

[CFMM97] G. Cattaneo, E. Formenti, L. Margara and J. Mazoyer. A shift-invariant metric on $S^{\mathbb{Z}}$ inducing a non-trivial topology. In I. Prívara and P. Ružička, editors, *Mathematical Foundations of Computer Science 1997*, volume 1295, pages 179–188. Springer Berlin Heidelberg, 1997.

[CD05] J.-R. Chazottes and F. Durand. Local rates of Poincaré recurrence for rotations and weak mixing. *Discrete Contin. Dyn. Syst., Ser. A*, 12(1):175–183, 2005.

[Den32] A. Denjoy. Sur les courbes définies par les équations différentielles à la surface du tore. *J. Math. Pures Appl. (9)*, 11:333–376, 1932.

[dMvS93] W. de Melo and S. van Strien. *One-dimensional dynamics*. Springer, 1993.

[Dow05] T. Downarowicz. Survey of odometers and Toeplitz flows. In S.F. Kolyada, Y.I. Manin, and T. Ward, editors, *Algebraic and Topological Dynamics*, volume 385, pages 7–37. American Mathematical Society, 2005.

[Dow14] T. Downarowicz. Positive topological entropy implies chaos DC2. *Proc. Am. Math. Soc.*, 142(1):137–149, 2014.

[DG15] T. Downarowicz and E. Glasner. Isomorphic extensions and applications. *arXiv:1502.06999*, 2015.

[Ebe71] E. Eberlein. Toeplitz-Folgen und Gruppentranslationen. *Arch. Math.*, 22(1):291–301, 1971.

[Edg98] G. A. Edgar. *Integral, probability, and fractal measures*. Springer, 1998.

[EW11] M. Einsiedler and T. Ward. *Ergodic Theory: With a View Towards Number Theory*, volume 259 of *Graduate Texts in Mathematics*. Springer, 2011.

[Fal07] K. Falconer. *Fractal geometry: mathematical foundations and applications*. Wiley, 2007.

[Fuh14] G. Fuhrmann. Non-smooth saddle-node bifurcations I: Existence of an SNA. To appear in *Ergodic Theory Dyn. Syst.*

[FG] G. Fuhrmann and M. Gröger. Amorphic complexity and symbolic dynamics. In preparation.

[FGJ] G. Fuhrmann, M. Gröger, and T. Jäger. Amorphic complexity of cut and project quasicrystals. In preparation.

[FGJ14] G. Fuhrmann, M. Gröger, and T. Jäger. Non-smooth saddle-node bifurcations II: dimensions of strange attractors. *arXiv:1412.6054*, 2014.

[Gle02] P. Glendinning. Global attractors of pinched skew products. *Dyn. Syst.*, 17:287–294, 2002.

[GJK06] P. Glendinning, T. Jäger, and G. Keller. How chaotic are strange non-chaotic attractors. *Nonlinearity*, 19(9):2005–2022, 2006.

[GOPY84] C. Grebogi, E. Ott, S. Pelikan, and J.A. Yorke. Strange attractors that are not chaotic. *Physica D*, 13:261–268, 1984.

[GJ] M. Gröger and T. Jäger. Some remarks on modified power entropy. In preparation.

[GJ13] M. Gröger and T. Jäger. Dimensions of Attractors in Pinched Skew Products. *Commun. Math. Phys.*, 320(1):101–119, 2013.

[HK97] B. Hasselblatt and A. Katok. *Introduction to the Modern Theory of Dynamical Systems*, volume 54 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 1997. 8th printing 2006.

[HK02] B. Hasselblatt and A. Katok. *Handbook of dynamical systems, Principal structures*. Elsevier, 2002.

[Her79] M. R. Herman. Sur la Conjugaison Différentiable des Difféomorphismes du Cercle a des Rotations. *Publ. Math., Inst. Hautes Étud. Sci.*, 49(1):5–233, 1979.

[Her83] M. R. Herman. Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. *Comment. Math. Helv.*, 58:453–502, 1983.

[HOR12] L. Hernández-Corbato, R. Ortega, and F. R. Ruiz del Portal. Attractors with irrational rotation number. *Math. Proc. Camb. Philos. Soc.*, 153:59–77, 2012.

[JK69] K. Jacobs and M. Keane. 0-1-sequences of Toeplitz type. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 13(2):123–131, 1969.

[Jäg03] T. Jäger. Quasiperiodically forced interval maps with negative Schwarzian derivative. *Nonlinearity*, 16(4):1239–1255, 2003.

[Jäg06]   T. Jäger. The creation of strange non-chaotic attractors in non-smooth saddle-node bifurcations. *Mem. Am. Math. Soc.*, 945:1–106, 2009.

[Jäg07]   T. Jäger. On the structure of strange nonchaotic attractors in pinched skew products. *Ergodic Theory Dyn. Syst.*, 27(2):493–510, 2007.

[Kel96]   G. Keller. A note on strange nonchaotic attractors. *Fundam. Math.*, 151(2):139–148, 1996.

[Lab13]   C. Labrousse Polynomial entropy for the circle homeomorphisms and for C^1 nonvanishing vector fields on $\mathbb{T}^2$. *arXiv:1311.0213*, 2013.

[LP03]   J. C. Lagarias and P.A.B. Pleasants. Repetitive delone sets and quasicrystals. *Ergodic Theory Dyn. Syst.*, 23(3):831–867, 2003.

[LTY14]   J. Li, S. Tu and X. Ye. Mean equicontinuity and mean sensitivity. To appear in *Ergodic Theory Dyn. Syst.*

[LW00]   E. Lindenstrauss and B. Weiss. Mean topological dimension. *Israel J. Math.*, 115(1):1–24, 2000.

[Mar13]   J.-P. Marco. Polynomial entropies and integrable Hamiltonian systems. *Regul. Chaotic Dyn.*, 18:623–655, 2013.

[Mar70]   N. G. Markley. Homeomorphisms of the circle without periodic points. *Proc. Lond. Math. Soc.*, 3(4):688–698, 1970.

[Mil85]   J. Milnor. On the concept of attractor. *Commun. Math. Phys.*, 99(2):177–195, 1985.

[MH38]   M. Morse and G. A. Hedlund. Symbolic dynamics. *Am. J. Math.*, 60(4):815-866, 1938.

[MH40]   M. Morse and G. A. Hedlund. Symbolic dynamics II. Sturmian trajectories. *Am. J. Math.*, 62(1):1–42, 1940.

[Moo00]   R. V. Moody. Model sets: A survey. In *From quasicrystals to more complex systems*, pages 145–166. Springer, 2000.

[MP79]   N.G. Markley and M.E. Paul. Almost automorphic symbolic minimal sets without unique ergodicity. *Israel J. Math.*, 34(3):259–272, 1979.

[Poi85]   H. Poincaré. Mémoire sur les courbes défines par une équation différentielle. *Sur les courbes défines par les équations différentielles*, 4:167–244, 1885.

[PU09]   Yu. L. Pritykin and J. Ulyashkina. Aperiodicity Measure for Infinite Sequences. In A. Frid, A. Morozov, A. Rybalchenko and K. W. Wagner, editors, *Computer Science - Theory and Applications*, pages 274–285. Springer Berlin Heidelberg, 2009.

[Sch99]   M. Schlottmann. Generalized model sets and dynamical systems. In *CRM Monograph Series*, 1999.

[Sta03]   J. Stark. Transitive sets for quasi-periodically forced monotone maps. *Dyn. Syst.*, 18(4):351–364, 2003.

[Vee65]   W.A. Veech. Almost automorphic functions on groups. *Am. J. Math.*, 87(3):719–751, 1965.

[Wal82]   P. Walters. *An Introduction To Ergodic Theory*, volume 79 of *Graduate Texts in Mathematics*. Spinger, 1982.

[Wil84]   S. Williams. Toeplitz minimal flows which are not uniquely ergodic. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 67(1):95–107, 1984.

[You97]   L.-S. Young. Lyapunov exponents for some quasi-periodic cocycles. *Ergodic Theory Dyn. Syst.*, 17(2):483–504, 1997.