# MEAN EQUICONTINUITY, ALMOST AUTOMORPHY AND REGULARITY

FELIPE GARCÍA-RAMOS, TOBIAS JÄGER AND XIANGDONG YE

ABSTRACT. The aim of this article is to obtain a better understanding and classification of strictly ergodic topological dynamical systems with (measurable) discrete spectrum. To that end, we first determine when an isomorphic maximal equicontinuous factor map of a minimal topological dynamical system has trivial (one point) fibres. In other words, we characterize when minimal mean equicontinuous systems are almost automorphic. Furthermore, we investigate another natural subclass of mean equicontinuous systems, so-called diam-mean equicontinuous systems, and show that a minimal system is diam-mean equicontinuous if and only if the maximal equicontinuous factor is regular (the points with trivial fibres have full Haar measure). Combined with previous results in the field, this provides a natural characterization for every step of a natural hierarchy for strictly ergodic topological models of ergodic systems with discrete spectrum. We also construct an example of a transitive almost diam-mean equicontinuous system with positive topological entropy, and we give a partial answer to a question of Furstenberg related to multiple recurrence.

## 1. INTRODUCTION

The family of ergodic systems with discrete spectrum was one of the early objects of study of formal Ergodic Theory (by von Neumann in 1932 [47]). It is one of the few families where the isomorphism problem is well understood (using spectral isomorphism): as stated by the Halmos-von Neumann theorem [28], every isomorphism class can be represented with a simple object, a group rotation on a compact abelian group. More generally, the celebrated Jewett-Krieger theorem states that every ergodic system is isomorphic to a strictly ergodic (uniquely ergodic and minimal) model.

However, even in the case of discrete spectrum, strictly ergodic systems may be very different from a group rotation. Surprisingly, recent work has shown that the family of *all* topological dynamical systems that are strictly ergodic models of discrete spectrum systems may exhibit a rich range of behaviours (from a topological point of view) [39, 19, 8, 13, 23, 31]. It turns out that these properties can be classified in a natural hierarchy, in which the best-understood systems with topological discrete spectrum – namely the equicontinuous systems – just present the simplest subclass and only a small fraction of the whole family. Apart from the intrinsic interest, the aim for a better understanding of this class of systems is also motivated by mathematical studies of quasicrystals, whose associated dynamical models often fall into this category (that is, they combine strict ergodicity and discrete spectrum [6]).

*Date*: April 29, 2020.

2010 *Mathematics Subject Classification.* 54H20,

*Key words and phrases.* Maximal equicontinuous factor, discrete spectrum, mean equicontinuity, frequent stability, diam-mean equicontinuity, quasicrystals, tame, topological models.

For simplicity, we restrict to the case of $\mathbb{Z}_+$-actions and say a *topological dynamical system* (t.d.s) is a pair $(X,T)$ consisting of a compact metric space $X$ and a continuous transformation $T : X \to X$. When it comes to classifying strictly ergodic systems (with or without discrete spectrum), an important and well-known concept is that of the *maximal equicontinuous factor* $(X_{eq}, T_{eq})$ and its factor map $\pi_{eq} : X \to X_{eq}$. The latter exists for every topological dynamical system (e.g. see [11]) and can be obtained in a constructive way via the regionally proximal equivalence relation [46] (also see [2]). If $(X,T)$ is minimal, then $(X_{eq}, T_{eq})$ is both minimal and uniquely ergodic, and we denote its unique invariant measure by $\nu_{eq}$. The following is a natural hierarchy for strictly ergodic systems with discrete spectrum, based on the invertibility properties of the map $\pi_{eq}$:

$\pi_{eq}$ is a conjugacy (1-1)
$\Rightarrow \pi_{eq}$ is regular (almost surely 1-1, that is, 1-1 on a set of full measure)
$\Rightarrow \pi_{eq}$ is isomorphic and almost 1-1 (that is, 1-1 on a residual subset)
$\Rightarrow \pi_{eq}$ is isomorphic
$\Rightarrow (X,\mu,T)$ has discrete spectrum (where $\mu$ is the unique invariant Borel probability measure of $(X,T)$).

Note that if an ergodic system has discrete spectrum, then it has to be isomorphic to an equicontinuous system, but this isomorphism is not necessarily given by the maximal equicontinuous factor map (actually every ergodic system has a uniquely ergodic mixing topological model [38]). In other words, the Kronecker factor in the Halmos-von Neumann theorem may be strictly bigger than the maximal equicontinuous factor, and this is the case if and only if there exist $L^2(\mu)$-eigenfunctions that are not continuous (see [25] for an example which is a uniquely ergodic minimal distal system). Hence, the last property is strictly weaker than the previous one. Also, it is not difficult to check that systems with discrete spectrum always have zero metric entropy and are never weakly mixing (a non-uniquely ergodic t.d.s. may have fully supported measures with discrete spectrum and positive topological entropy [43]).

An important notion for this theory is *mean equicontinuity* (or mean-L-stability), which is a weakening of equicontinuity. It was introduced by Fomin already in 1951 [12], but in the next 60 years only a few papers studying this property appeared [3, 45]. In particular, it was left as an open question if minimal mean equicontinuous systems (equipped with their unique ergodic measure) have discrete spectrum. Recently, this question was answered independently by Li, Tu and Ye [39], and by García-Ramos [19] using different methods (also see [30, 22]). García-Ramos characterized when topological representations of an ergodic system have discrete spectrum, using a weaker notion called *$\mu$-mean equicontinuity* (which also relies on the measure). Li, Tu and Ye proved that mean equicontinuity is stronger than just discrete spectrum because the isomorphism to the group rotation can be achieved using the maximal equicontinuous factor. Downarowicz and Glasner [8] proved the converse, that is, if the maximal equicontinuous factor of a minimal systems yields an isomorphism then the system must be mean equicontinuous. Furthermore, they showed that some minimal mean equicontinuous systems are not almost automorphic ( $\pi_{eq}$ is not almost 1-1). Altogether, this means that the last two steps in the above hierarchy can be characterized using ($\mu$-)mean equicontinuity. For a survey on mean equicontinuity see [40].

In this paper, we aim to characterize the second and third step (the first is just equicontinuity) using weak forms of equicontinuity. To that end, we use the notions of *frequent stability* (Definition 3.3) and *diam mean equicontinuity* (Definition 4.1), closely related to Lyapunov stable sets and mean equicontinuity. Under the assumption of minimality, we show that for mean equicontinous systems the maximal equicontinuous factor map $\pi_{eq}$ is almost 1-1 if and only if $(X,T)$ is frequently stable (Theorem 3.4). Moreover, we show that $\pi_{eq}$ is regular if and only if $(X,T)$ is diam-mean equicontinuous (Theorem 4.12).

Other known families of systems with discrete spectrum are null systems (zero topological sequence entropy [36, 26, 31]) and tame systems (Glasner [23, 34]). It is now understood that these properties are very closely related to the hierarchy. It is not difficult to see that every equicontinuous system is null. Kerr and Li proved that these notions can be characterized using combinatorial independence and that every null system is tame [34]. Using a result from Glasner [24] and a result from Fuhrmann, Glasner, Jäger and Oertel [13], we obtain that for every minimal tame system, $\pi_{eq}$ is regular. See [31, 19] for previous weaker results.

In summary, we obtain the following hierarchy for minimal systems:

Equicontinuity ($\Leftrightarrow$ topological discrete spectrum)
$\Rightarrow$null ($\Leftrightarrow$no unbounded independence)
$\Rightarrow$tame ($\Leftrightarrow$ no infinite independence)
$\Rightarrow$diam-mean equicontinuous ($\Leftrightarrow$ $\pi_{eq}$ regular)
$\Rightarrow$mean equicontinous and frequently stable ($\Leftrightarrow$ $\pi_{eq}$ almost 1-1 and isomorphic)
$\Rightarrow$mean equicontinuous ($\Leftrightarrow$ $\pi_{eq}$ isomorphic)
$\Rightarrow$ $\mu-$mean equicontinuous. ($\Leftrightarrow$ discrete spectrum).

Note that there exist counter-examples showing that each of these implications is strict [26, Section 5], [35, Section 11] or [15], [24, Remark 5.8], [9, Example 5.1], and [8, Theorem 3.1]. Recently, García-Ramos and Kwietniak introduced a weakining of mean equicontinuity using the Feldman-Katok pseudometric, and characterized models of zero entropy loosely Bernoulli systems [20], this could be considered a next step in the previous hierarchy.

We should mention that some work has been done for non-minimal systems and non-ergodic measures [14, 44, 30], but further work will be required to extend our results in this direction. In contrast to this, with the use of [14], the extension to locally compact amenable group actions should be straightforward (except for Section 6). In particular for $\mathbb{Z}$-actions, $\mathbb{Z}$ (or bilateral) equicontinuity is equivalent to $\mathbb{Z}_+$ (or forward) equicontinuity, and the same is true for mean equicontinuity (e.g. see [14, Theorem 1.3]). Also, we are confident that our results and proofs remain valid for first-countable Hausdorff spaces. However, when first countability is dropped (as for uncountable products of compact metric spaces), and sequential compactness cannot be used any further, it is much less clear to us if the statements remain true. In any case, the proofs would require substantial modifications.

In addition to the above-mentioned results, we also study almost diam-mean equicontinuity, a weakening of diam-mean equicontinuity. We show that transitive almost diam-mean equicontinuous systems may have positive topological entropy (note that mean equicontinuous systems always have zero topological entropy). So, at least in this sense, almost diam-mean equicontinuous systems do not exhibit different properties than almost mean equicontinuous systems (in contrast to the minimal case). Finally, we give a partial answer to a question of Furstenberg related to multiple topological recurrence.

Another important generalization of equicontinuous systems are the distal systems, which are the subjects of the well-known Furstenberg structure theorem [16]. The mean version of distal systems was studied by Ornstein and Weiss [41]. Mean equicontinuity and mean distality are perpendicular notions, in the sense that a mean equicontinuous system is mean distal if and only if it is equicontinuous [19, Proposition 69].

The paper is organized as follows. In Section 2, we give some basic notions of t.d.s. In Section 3 we show that for mean equicontinous systems $\pi_{eq}$ is almost 1-1 if and only if $(X,T)$ is frequently stable. In Section 4 we study the basic properties of diam-mean equicontinuous systems and we prove that $\pi_{eq}$ is regular if and only if $(X,T)$ is diam-mean equicontinuous. In Section 5 we consider diam-mean sensitivity and almost diam-mean equicontinuity. Furstenberg asked if for every t.d.s. $(X,T)$ and $d\in\mathbb{N}$, there is $x\in X$ such that $(x,x,\dots,x)$ is a minimal point for $T\times T^2\times\dots\times T^d$. We give a positive answer for the class of mean equicontinuous systems in Section 6.

**Acknowledgments.** We thank Eli Glasner for useful discussion in the early stage of the paper, Gabriel Fuhrmann for pointing out a way to simplify the proof of Theorem 3.4 and further helpful remarks, and Jian Li for pertinent comments and references. X. Ye was supported by NNSF of China (11431012), T. Jäger by a Heisenberg grant of the German Research Council (DFG-grant OE 538/6-1) and F. García-Ramos by CONACyT (287764).

## 2. Preliminaries

Throughout this paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of non-negative integers and natural numbers, respectively. We denote the cardinality of a set $A$ by $|A|$.

2.1. **Subsets of $\mathbb{Z}_+$.** Let $F$ be a subset of $\mathbb{Z}_+$. The **upper density** and **upper Banach density** of $F$ are defined by

$$\overline{D}(F)=\limsup_{n\to\infty}\frac{|\{F\cap[0,n-1]\}|}{n}$$

and

$$BD^*(F)=\limsup_{N-M\to\infty}\frac{|\{F\cap[M,N-1]\}|}{N-M}=\limsup_{n\to\infty}\left\{\sup_{N-M=n}\frac{|\{F\cap[M,N-1]\}|}{n}\right\},$$

respectively. It is clear that $\overline{D}(F)\le BD^*(F)$ for any $F\subset\mathbb{Z}_+$. When a set is denoted with braces, for example {A}, we simply write $\overline{D}\{A\}=\overline{D}(\{A\})$ and $BD^*\{A\}=BD^*(\{A\})$.

2.2. **Topological dynamics.** We say $(X,T)$ is a **topological dynamical system (t.d.s.)** if $X$ is a compact metric space (with metric $d$) and $T:X\to X$ is a continuous function. We denote the forward orbit of $x\in X$ by $orb(x,T)=\{x,Tx,\dots\}$ and its orbit closure by $\overline{orb(x,T)}$. We say $x\in X$ is a **transitive point** if $\overline{orb(x,T)}=X$. On the other hand, a t.d.s. $(X,T)$ is **transitive** if for any non-empty open sets $U,V\subset X$ there exists $n\in\mathbb{N}$ such that $T^nU\cap V\neq\emptyset$ and is **minimal** if every point of $X$ is transitive. We say $(X,T)$

is **uniquely ergodic** if there is only one $T$-invariant Borel probability measure on $X$. A t.d.s. is **stricly ergodic** if it is minimal and uniquely ergodic.

We call $x \in X$ an **equicontinuity point** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in X$ with $d(x,y) < \delta$, we have $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$. A t.d.s. is **equicontinuous** if every $x \in X$ is an equicontinuity point (note that by compactness every equicontinuous t.d.s. is uniformly equicontinuous, so a t.d.s. is equicontinuous if and only if the family $\{T^n\}$ is equicontinuous).

Let $(X,T)$ and $(X',T')$ be two t.d.s. We say $(X',T')$ is a **factor** of $(X,T)$ if there exists a surjective continuous map $\pi : X \to X'$ such that $\pi \circ T = T' \circ \pi$ (we refer to $\pi$ as a **factor map**). A factor map $\pi : X \to X'$ is **almost 1-1** if $\{x \in X : \pi^{-1}\pi(x) = \{x\}\}$ is **residual** (that is, it is the countable intersection of dense open sets) and **almost finite to one** if $\{x \in X : |\pi^{-1}\pi(x)| < \infty\}$ is residual. If $(X',T')$ is minimal then $\pi$ is almost 1-1 if and only if $\{x \in X : \pi^{-1}\pi(x) = \{x\}\}$ is non-empty. Every t.d.s. $(X,T)$ has a unique (up to conjugacy) **maximal equicontinuous factor (m.e.f.)** $(X_{eq}, T_{eq})$ [11], that is, an equicontinuous factor such that every other equicontinuous factor of $(X,T)$ is a factor of $(X_{eq}, T_{eq})$. We will denote the maximal equicontinuous factor map by $\pi_{eq}$. Every transitive equicontinuous system is strictly ergodic and we denote the invariant Borel probability measure by $\nu_{eq}$.

Given a t.d.s. the **Besicovitch pseudometric** is given by

$$\rho_b(x,y) := \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d(T^i x, T^i y).$$

A t.d.s. $(X,T)$ is a **subshift** if $X \subset \Sigma_k = \{0,1,\dots,k-1\}^{\mathbb{N}}$ is a closed (with respect to the product topology) shift invariant subset. The $i$th coordinate of $x \in \Sigma_k$ is denoted by $x_i$ and the **shift map** $\sigma : \Sigma_k \to \Sigma_k$ is defined by the condition that $\sigma(x)_n = x_{n+1}$ for $n \in \mathbb{N}$. *Toeplitz subshifts* are almost 1-1 extensions of odometers (i.e. $(X_{eq}, T_{eq})$ is an odometer and $\pi_{eq}$ is almost 1-1). Although we will not work directly with Toeplitz subshifts, some examples will be relevant in the paper. For a survey on Toeplitz subshifts see [7].

## 3. Characterization of almost automorphic mean equicontinuous systems

In this section we characterize when the maximal equicontinuous factor map of a minimal mean equicontinuous t.d.s. is almost 1-1.

**Definition 3.1.** A t.d.s. $(X,T)$ is **mean equicontinuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then $\rho_b(x,y) \leq \varepsilon$.

**Theorem 3.2** ([39, Theorem 3.8], [8, Theorem 2.1])**.** *A minimal t.d.s. $(X,T)$ is mean equicontinuous if and only if $(X,T)$ is uniquely ergodic (with a measure $\mu$) and $\pi_{eq}$ is a measurable isomorphism between $(X,T,\mu)$ and $(X_{eq}, T_{eq}, \nu_{eq})$.*

Every equicontinuous t.d.s. is mean equicontinuous. The previous theorem implies that the Sturmian subshift (isomorphic to an irrational rotation) or any regular Toeplitz subshift (isomorphic to an odometer [33, Theorem 6]) is mean equicontinuous. Nonetheless, simple arguments indicate that no aperiodic subshift can be equicontinuous.

These previously mentioned examples (as most known minimal mean equicontinuous examples) are **almost automorphic**, i.e. $\pi_{eq}$ is almost 1-1. These systems can be constructed with concrete methods using a compact abelian group and a semi-cocycle (see [7, 42]). Nonetheless, there exist (more complicated) minimal mean equicontinuous systems that are not almost automorphic [8, Theorem 3.1]. In this section we characterize exactly when this happens using a simple condition which we call *frequent stability*.

By $B_\delta(x)$ we denote the open ball of radius $\delta$ centered at $x$. For $A \subset X$, we denote the diameter of $A$ by $\mathrm{diam}(A)$.

**Definition 3.3.** Let $(X,T)$ be a t.d.s. We say that $x \in X$ is a **frequently stable** point, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\overline{D}\left\{i \in \mathbb{Z}_+ : \mathrm{diam}(T^i B_\delta(x)) > \varepsilon\right\} < 1.$$

**Theorem 3.4.** *Let $(X,T)$ be a minimal mean equicontinuous t.d.s. Then the following are equivalent:*

(i) $\pi_{eq} : (X,T) \to (X_{eq}, T_{eq})$ *is almost 1-1;*
(ii) *every $x \in X$ is frequently stable;*
(iii) *there exists at least one frequently stable point of $x$;*
(iv) $\pi_{eq} : (X,T) \to (X_{eq}, T_{eq})$ *is almost finite to one.*

To prove this theorem we will need the following lemmas.

**Lemma 3.5** ([39, Theorem 3.5], [19, Proposition 49])**.** *Let $(X,T)$ be a t.d.s. If $(X,T)$ is mean equicontinuous then $\rho_b(x,y) = 0$ if and only if $\pi_{eq}(x) = \pi_{eq}(y)$.*

**Lemma 3.6** ([8, Lemma 2.4])**.** *Let $(X,T)$ and $(X',T')$ be minimal t.d.s., $(Y,S)$ invertible, and $\pi : X \to X'$ a factor map. Then either $\pi$ is almost 1-1 or there exists $\varepsilon > 0$ such that* $\mathrm{diam}(\pi^{-1}(y)) > \varepsilon$ *for every $y \in X'$.*

It is well-known that factor maps between minimal systems are always semi-open. We provide a proof for the convenience of the reader.

**Lemma 3.7.** *Let $\pi : (X,T) \to (X',T')$ be a factor map between two minimal t.d.s. with $(X',T')$ invertible. Then for each $\delta > 0$ there is $\eta > 0$ such that the image under $\pi$ of any $\delta$-ball in $X$ contains an $\eta$-ball in $X'$.*

*Proof.* Fix $\delta > 0$ and $x \in X$. By minimality of $(X,T)$, there exists $n \in \mathbb{N}$ such that $\bigcup_{i=0}^n T^i B_\delta(x)$ covers $X$. Therefore, $\pi(B_\delta(x))$ needs to have non-empty interior, otherwise $Y = \bigcup_{i=0}^n \pi(T^i B_\delta(x))$ would be a finite union of meager sets (complement of a residual set), contradicting Baire's category theorem. Using compactness again, one obtains that there exists $\eta > 0$ such that the image under $\pi$ of any $\delta$-ball in $X$ contains an $\eta$-ball in $X'$. □

We can now turn to the proof of Theorem 3.4. The crucial part in this is the implication from (iii) to (i). Since the argument includes a number of technicalities, we want to give a heuristic description of the main idea before we turn to the rigorous implementation.

*Strategy for the proof of Theorem 3.4.* We will proceed by contradiction and assume that $(X,T)$ is not almost 1-1, so that all fibres are non-trivial and the diameter of the fibres is uniformly bounded from below by some constant $\varepsilon > 0$ (Lemma 3.6). As $\pi$ is isomorphic (Theorem 3.2), there exists a measurable inverse $\rho : X_{eq}^0 \to X^0$, where

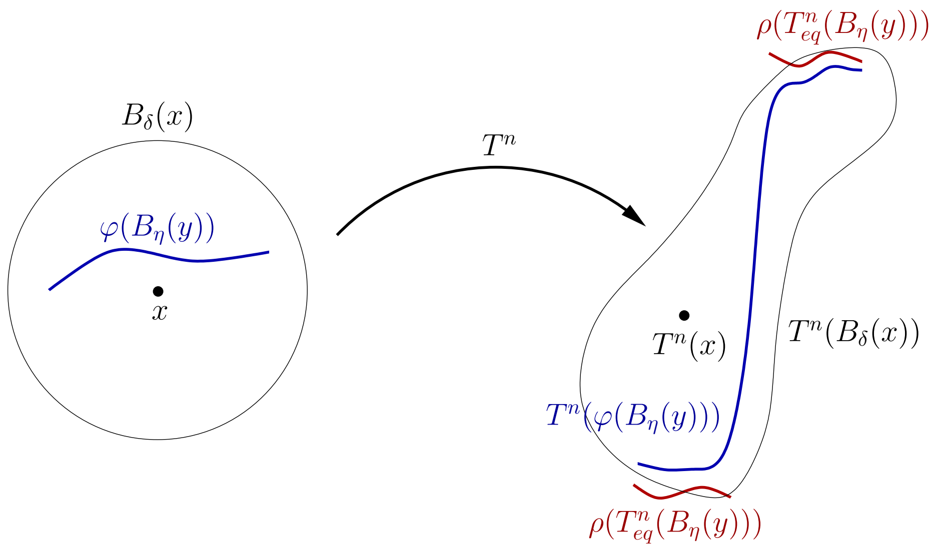


FIGURE 1. Illustration of the proof of implication (iii) $\Rightarrow$ (i) in Theorem 3.4. In order to show that no point is frequently stable, we start with a $\delta$-ball $B_\delta(x)$ and a measurable selection $\varphi : B_\eta(y) \to B_\delta(x)$, where $B_\eta(y) \subseteq \pi(B_\delta(x))$. Under iteration, for most $n \in \mathbb{N}$, the image $T^n(\varphi(B_\eta(y)))$ will be close to $\rho(B_\eta(T^n_{eq}(y))$. As the range of the latter set is at least $\varepsilon$, one obtains a uniform lower bound for the range of $T^n(\varphi(B_\eta(y)))$, and hence for $T^n(B_\delta(x))$.

$X^0_{eq}$ and $X^0$ are full measure subsets of $X_{eq}$ and $X$, respectively. The function $\rho$ maps $\mu_{eq}$ to $\mu$, that is $\rho_*\mu_{eq} = \mu$, so that the latter measure is supported on the image of $\rho$.

As the fibres of $\pi$ are $\varepsilon$-large and $\mu$ has full topological support, the image of every $\eta$-ball $B_\eta(x)$ under $\rho$ will have diameter $\varepsilon$. Furthermore, it is possible to show that $\pi$ maps to two $\varepsilon/4$-separated regions in $X$ with a positive probability that only depends on $\eta$.

Now, the aim is to prove that no point in $X$ is frequently stable. Using the fact that $\pi$ is semi-open, for any $\delta > 0$ the projection $\pi_{eq}(B_\delta(x)$ contains some $\eta$-ball $B_\eta(y)$, with $\eta$ only dependent on $\delta$. We then consider a mapping $\varphi : B_\eta(y) \to B_\delta(x)$ which is a local measurable left inverse of $\pi$, that is, $\pi \circ \varphi(y') = y'$ for all $y' \in B_\eta(y)$.

The crucial observation is the fact that the image of $\varphi$ will converge to that of $\rho$ (in a suitable sense, to be made precise below) under forward iteration. Hence, for most sufficiently large $n$, the diameter of $T^n(\varphi(B_\eta(y)))$ will be close to the diameter of $\rho(T^n_{eq}(B_\eta(y)))$, and in particular larger than $\varepsilon/8$. As $\varphi(B_\eta(y)) \subseteq B_\delta(x)$, this will show that $\mathrm{diam}(T^n(B_\delta(x))) \geq \varepsilon/8$ for most $n$. As $x \in X$ and $\delta > 0$ are chosen arbitrary, this will prove the non-existence of frequently stable points and thus conclude the contradiction argument. An illustration of the last step is given in Figure 1.

*Proof of Theorem 3.4.* The facts that (ii) implies (iii), and (i) implies (iv) are obvious.

First we will prove (iv) implies (i). Suppose that there exists $y \in X_{eq}$ such that $\pi^{-1}_{eq}(y)$ is finite. Using Lemma 3.5, we have that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(T^n x, T^n x') \leq \varepsilon$ for every $x, x' \in \pi^{-1}_{eq}(y)$. Note that $T^n x, T^n x' \in \pi^{-1}_{eq}(T^n_{eq} y)$.

Since $(X,T)$ is minimal, we have that $T : X \to X$ is surjective and hence, for each $z \in \pi_{eq}^{-1}(T_{eq}^n y)$ there is $x \in \pi_{eq}^{-1}(y)$ such that $T^n x = z$. Thus, $d(z,z') < \varepsilon$ for all $z,z' \in \pi_{eq}^{-1}(T_{eq}^n y)$. Using Lemma 3.6, we conclude that $\pi_{eq}$ is almost 1-1.

The fact that (i) implies (ii) is an application of Theorem 3.1 and Theorem 3.4 of [29]. Nonetheless, we can provide a short direct proof. By hypothesis, there exists $y_0 \in X_{eq}$ such that $|\pi_{eq}^{-1}(y_0)| = 1$. Let $\varepsilon > 0$. There exists $\eta > 0$ such that

$$\operatorname{diam}(\pi_{eq}^{-1}(B_\eta(y))) \leq \varepsilon$$

for every $y \in B_\eta(y_0)$. Since $\nu_{eq}$ is fully supported, $B_\eta(y_0)$ has positive measure and (using standard arguments) must contain a smaller ball with positive measure and $\nu_{eq}$-null boundary. Thus, by strict ergodicity, we have that

$$\overline{D}\left\{n \in \mathbb{N} : T_{eq}^n y \in B_\eta(y_0)\right\} > 0$$

for every $y \in X_{eq}$.

Let $x \in X$. Since $\pi_{eq}$ is continuous, there exists $\delta > 0$ such that

$$\operatorname{diam}(\pi_{eq}(B_\delta(x))) < \eta.$$

Without loss of generality, we may assume $T_{eq}$ is an isometry. This implies that

$$B_\eta(T_{eq}^n \pi_{eq}(x)) = T_{eq}^n B_\eta(\pi_{eq}(x))$$

for every $n \in \mathbb{N}$. Consequently

$$T^n(B_\delta(x)) \subseteq \pi_{eq}^{-1}(B_\eta(T_{eq}^n \pi_{eq}(x)))$$

for all $n \in \mathbb{N}$. So, if $T_{eq}^n \pi_{eq}(x) \in B_\eta(y_0)$ then $\operatorname{diam}(T^n B_\delta(x)) < \varepsilon$; since this happens with positive density, we conclude $(X,T)$ is frequently stable.

Hence, it remains to show that (iii) implies (i). Without loss of generality, suppose that $\operatorname{diam}(X) = 1$. Assume that $\pi_{eq}$ is not almost 1-1. We will show that no point in $X$ is frequently stable. Using Theorem 3.2, there exists $X_0 \subset X$, $Y_0 \subset X_{eq}$ such that $\mu(X_0) = 1$ and $\pi_{eq} : X_0 \to Y_0 = \pi_{eq}(X_0)$ is bijective. Let $\rho : Y_0 \to X_0$ be the inverse of $\pi_{eq}$. This implies that $\mu(A) = \nu_{eq}(\{y \in Y \mid \rho(y) \in A\})$ for every measurable set $A \subseteq X$.

Since $\pi_{eq}$ is not almost 1-1, we have that $\left|\pi_{eq}^{-1}(y)\right| > 1$ for all $y \in X_{eq}$. By Lemma 3.6, there exists $\varepsilon > 0$ such that $\operatorname{diam}(\pi_{eq}^{-1}(y)) > \varepsilon$ for every $y \in X_{eq}$.

*Claim*: for every $\eta > 0$ there exists $\kappa = \kappa(\eta) > 0$ such that for any $y \in Y$ and any $x \in \pi_{eq}^{-1}(y)$ we have that $\mu(B_{\varepsilon/8}(x) \cap \pi_{eq}^{-1}(B_\eta(y)) \geq \kappa$.

Suppose for a contradiction that this is not the case. Then, there exist $y_n \in Y_0$ and $x_n \in \pi_{eq}^{-1}(B_\eta(y_n))$ such that $\lim_{n\to\infty} \mu(B_{\varepsilon/8}(x_n) \cap \pi_{eq}^{-1}(B_\eta(y_n))) = 0$. By minimality, the topological support of $\mu$ is all of $X$; hence, every non-empty open subset of $X$ has positive measure. In particular, if $y = \lim_{n\to\infty} y_n$ and $x = \lim_{n\to\infty} x_n$ (where we go over to convergent subsequences if necessary to ensure existence of the limits), the open set $U = B_{\varepsilon/16}(x) \cap \pi_{eq}^{-1}(B_{\eta/2}(y))$ has positive measure. However, for $n$ large enough, $U$ is contained in $B_{\varepsilon/8}(x_n) \cap \pi_{eq}^{-1}(B_\eta(y_n))$, leading to a contradiction. This proves the claim.

If we apply the above claim to any $y \in Y_0$ and two points $x_1, x_2 \in \pi_{eq}^{-1}(y)$ with $d(x_1,x_2) > \varepsilon/2$, and define $A_j = \pi_{eq}\left(B_{\varepsilon/8}(x_j) \cap \pi_{eq}^{-1}(B_\eta(y) \cap Y_0)\right)$ for $j \in \{1,2\}$, we obtain the following statement.

(1) *For every $\eta > 0$ there exists $\kappa > 0$ such that for all $y \in Y_0$ there are $A_1(y), A_2(y) \subset B_\eta(y)$ such that $\nu_{eq}(A_1(y)), \nu_{eq}(A_2(y)) > \kappa$ and $d(\rho(A_1(y)), \rho(A_2(y))) > \varepsilon/4$.*

Let $x \in X$ and $\delta > 0$. Due to Lemma 3.7, there exist $\eta > 0$ and $y \in \pi_{eq}(B_\delta(x))$ such that $B_\eta(y) \subseteq \pi_{eq}(B_\delta(x))$. Using the Jankov-von Neumann selection theorem (e.g. see [27, Corollary 2.6]), there exists a measurable mapping $\varphi : B_\eta(y) \to B_\delta(x)$ that satisfies $\pi_{eq} \circ \varphi(y') = y'$ for all $y' \in B_\eta(y)$. Then Lemma 3.5 implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} d(T^i\varphi(y'), T^i\rho(y')) \;=\; 0$$

for all $y' \in B_\eta(y)$. Hence, the dominated convergence theorem yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \int_{B_\eta(y)} d(T^i\varphi(y'), T^i\rho(y'))\, d\nu_{eq}(y') \;= 0.$$

Furthermore, if we let

$$E_i \;=\; \{\, y' \in B_\eta(y) \cap Y_0 : d(T^i\varphi(y'), T^i\rho(y')) \le \varepsilon/8\}\,,$$

then this implies

$$\overline{D}\left\{i \in \mathbb{Z}_+ : \nu_{eq}(X \setminus E_i) \ge \kappa/2\right\} =\; 0\,.$$

Consequently, we obtain that

$$\overline{D}\left\{i \in \mathbb{Z}_+ : \nu_{eq}(E_i) > \nu_{eq}(B_\eta(y)) - \kappa/2\right\} =\; 1. \tag{2}$$

For any $i \in \mathbb{Z}_+$, let $A_1' := A_1(T_{eq}^i(y))$ and $A_2' := A_2(T_{eq}^i(y))$ be given by (1). Then,

$$\nu_{eq}(A_1'), \nu_{eq}(A_2') > \kappa\,.$$

Hence, if

$$\nu_{eq}(E_i) > \nu_{eq}(B_\eta(y)) - \kappa/2, \tag{3}$$

then

$$A_j' \cap E_i \ne \emptyset$$

for $j = 1,2$. Since (3) happens for a set of $i$ of full density and

$$d(\rho(A_1'), \rho(A_2')) > \varepsilon/4,$$

we obtain that

$$\overline{D}\left\{i \in \mathbb{N} : \exists y_1^i, y_2^i \in B_\eta(y) \text{ with } d(\varphi(T_{eq}^i y_1^i)), \varphi(T_{eq}^i y_2^i)) \ge \varepsilon/8\right\} =\; 1.$$

Considering that $\pi_{eq}$ is a factor map and $\pi_{eq} \circ \varphi(y') = y'$ for all $y' \in B_\eta(y)$, then

$$\overline{D}\left\{i \in \mathbb{N} : \exists y_1^i, y_2^i \in B_\eta(y) \text{ with } d(T^i\varphi(y_1^i), T^i\varphi(y_2^i)) \ge \varepsilon/8\right\} = 1.$$

Consequently, as $\varphi(y_1^i), \varphi(y_2^i) \in B_\delta(x)$, we obtain

$$\overline{D}\left\{i \in \mathbb{N} : \operatorname{diam}(T^i B_\delta(x)) \ge \varepsilon/8\right\} =\; 1.$$

As $x \in X$ and $\delta > 0$ were arbitrary, we conclude that there are no frequently stable points of $(X,T)$. □

This result implies that the mean equicontinuous systems from [8] cannot be frequently stable and the maximal equicontinuous factors, of those systems, have to be infinite to one at every point.

**Corollary 3.8.** *Let $(X,T)$ be a minimal t.d.s. Then $\pi_{eq}$ is isomorphic and almost 1-1 if and only if $(X,T)$ is mean equicontinuous and frequently stable.*

**Question 1.** Does there exist a minimal t.d.s. that is frequently stable but not almost automorphic (obviously non-mean equicontinuous)?

## 4. Characterization of diam-mean equicontinuity

As we noted in the previous section, an isomorphic $\pi_{eq}$ does not imply $\pi_{eq}$ is almost 1-1. Actually, the converse also does not hold; for every Toeplitz subshift $\pi_{eq}$ is almost 1-1 but some of them even have positive topological entropy [7]. Nonetheless, if

$$\mu(\{x \in X : \pi_{eq}^{-1}\pi_{eq}(x) = \{x\}\}) = 1,$$

then $\pi_{eq}$ must automatically be an isomorphism. Using a stronger form of mean equicontinuity, namely *diam-mean equicontinuity*, we will characterize when this happens (Theorem 4.12). Note that there exist Toeplitz subshifts (hence almost automorphic) that are mean equicontinuous but not regular, that is, the above equation does not hold ([9, Example 5.1]). An improved understanding of these kind of examples can be obtained using the cut and project formalism [5].

### 4.1. The basic properties of diam-mean equicontinuous systems.

Diam-mean equicontinuity was introduced in [19].

**Definition 4.1.** Let $(X,T)$ be a t.d.s. We say $x \in X$ is a **diam-mean equicontinuity point** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \operatorname{diam}(T^i B_\delta(x)) < \varepsilon.$$

We say $(X,T)$ is **diam-mean equicontinuous** if every $x \in X$ is a diam-mean equicontinuity point. We say $(X,T)$ is **almost diam-mean equicontinuous** if the set of diam-mean equicontinuity points is residual.

Since $X$ is compact, a t.d.s. is diam-mean equicontinuous if and only if for every $\varepsilon > 0$ and $x \in X$ there exists $\delta > 0$ such that $\limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \operatorname{diam}(T^i B_\delta(x)) < \varepsilon$. It is not difficult to check that a transitive t.d.s. is almost diam-mean equicontinuous if and only if there exists a diam-mean equicontinuity point and that a minimal almost diam-mean equicontinuous system is always diam-mean equicontinuous.

A t.d.s $(X,T)$ is mean equicontinuous if and only if for every $\varepsilon > 0$ there exist $\delta > 0$ and $N \in \mathbb{N}$ such that if $d(x,y) \le \delta$ then $\frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) < \varepsilon$ for any $n \ge N$ [44]. We will now give a similar characterization.

**Definition 4.2.** Let $(X,T)$ be a t.d.s. We say a set $U \subset X$ is $\varepsilon-$**stable in the mean** if

$$\sup_{N\in\mathbb{N}}\{\frac{1}{N}\sum_{i=1}^{N}\mathrm{diam}(T^iU)\} < \varepsilon. \tag{4}$$

**Lemma 4.3.** *Let $(X,T)$ be a t.d.s. Then, $(X,T)$ is diam mean equicontinuous if and only if for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $x \in X$, $B_\delta(x)$ is $\varepsilon$-stable in the mean.*

*Proof.* It is clear that the late condition implies diam mean equicontinuity. Now assume that $(X,T)$ is diam mean equicontinuous. For each $\varepsilon > 0$ there is $\delta > 0$ such that for each $x \in X$

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathrm{diam}(T^iB_\delta(x)) < \varepsilon. \tag{5}$$

Assume that (4) does not hold. Then, there are $x_i \in X$, $\delta_i \to 0$, $\varepsilon_0 > 0$ and $N_i \to \infty$ such that

$$\frac{1}{N_i}\sum_{j=1}^{N_i}\mathrm{diam}(T^jB_{\delta_i}(x_i)) \geq \varepsilon_0,\ \forall i \in \mathbb{N}.$$

Without loss of generality, assume that $x_i \to x$ and $\delta = \delta(\varepsilon_0)$. When $i$ is large enough we have that $B_{\delta_i}(x_i) \subset B_\delta(x)$; a contradiction to (5). □

**Lemma 4.4.** *Let $(X,T)$ be a t.d.s. Then, $x \in X$ is a diam-mean equicontinuity point if and only if for every $\eta > 0$ there exists $\delta > 0$ such that*

$$\overline{D}\{i \in \mathbb{Z}_+ : \mathrm{diam}(T^iB_\delta(x)) > \eta\} < \eta.$$

*Proof.* We assume, without loss of generality, that the diameter of $X$ is bounded by 1.

($\Rightarrow$) Let $x \in X$ be a diam-mean equicontinuity point. Assume that there exists $\eta > 0$ such that for every $\delta > 0$ we have that

$$\overline{D}\{j \in \mathbb{Z}_+ : \mathrm{diam}(T^iB_\delta(x)) > \eta\} \geq \eta.$$

Let $\varepsilon = \eta^2$. Since $x$ is a diam-mean equicontinuity point, we may choose $\delta \in (0,\varepsilon)$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{diam}(T^iB_\delta(x)) < \varepsilon.$$

At the same time, we have that

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{diam}(T^iB_\delta(x))\\ &\geq \eta\overline{D}\{j \in \mathbb{Z}_+ : \mathrm{diam}(T^iB_\delta(x)) > \eta\}\\ &\geq \eta^2 = \varepsilon,\end{aligned}$$

a contradiction.

($\Leftarrow$) Now assume that for every $\eta > 0$ there exists $\delta > 0$ such that

$$\overline{D}\{i \in \mathbb{Z}_+ : \mathrm{diam}(T^iB_\delta(x)) > \eta\} < \eta.$$

Given $\varepsilon > 0$, let $\eta = \varepsilon/2$ and choose $\delta > 0$ such that the previous inequality holds. Thus,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \operatorname{diam}(T^i B_\delta(x)) \\
&\leq \overline{D}\left\{j \in \mathbb{Z}_+ : \operatorname{diam}(T^i B_\delta(x)) > \eta\right\} + \overline{D}\left\{j \in \mathbb{Z}_+ : \operatorname{diam}(T^i B_\delta(x)) \leq \eta\right\} \cdot \eta \\
&\leq 2\eta \ = \ \varepsilon.
\end{aligned}$$

Hence, $x$ is a diam-mean equicontinuity point. □

In summary, we have the following.

**Proposition 4.5.** *Let $(X,T)$ be a t.d.s. The following are equivalent:*

(1) *$(X,T)$ is diam-mean equicontinuous*
(2) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$*

$$\sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{j=1}^{n} \operatorname{diam}(T^j B_\delta(x)) < \varepsilon.$$

(3) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$*

$$\overline{D}\left\{j \in \mathbb{Z}_+ : \operatorname{diam}(T^j B_\delta(x)) > \varepsilon\right\} \leq \varepsilon.$$

4.2. **Banach diam-mean equicontinuity and regularity.** Banach mean equicontinuity was introduced in [39] and has been studied in [44, 8, 14] (on the last two papers under the name Weyl equicontinuity). In this paper we introduce the diam version.

**Definition 4.6.** Let $(X,T)$ be a t.d.s. We say $x \in X$ is a **Banach diam-mean equicontinuity point** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{N-M\to\infty} \frac{1}{N-M}\sum_{i=M+1}^{N} \operatorname{diam}(T^i B_\delta(x)) < \varepsilon.$$

We say $(X,T)$ is **Banach diam-mean equicontinuous** if every $x \in X$ is a diam-mean equicontinuity point. We say $(X,T)$ is **almost Banach diam-mean equicontinuous** if the set of Banach diam-mean equicontinuity points is residual.

A t.d.s. is Banach diam-mean equicontinuous if and only if for every $\varepsilon > 0$ and $x \in X$ there exists $\delta > 0$ such that $\limsup_{N-M\to\infty} \frac{1}{N-M}\sum_{i=M+1}^{N} \operatorname{diam}(T^i B_\delta(x)) < \varepsilon$. A transitive t.d.s. is almost Banach diam-mean equicontinuous if and only if there exists a Banach diam-mean equicontinuity point and a minimal almost Banach diam-mean equicontinuous system is always Banach diam-mean equicontinuous.

Every Banach diam-mean equicontinuity point is a diam-mean equicontinuity point but the converse does not hold (see Section 5). The proof of the following lemma is similar to the proof of Lemma 4.4 and will be omitted.

**Lemma 4.7.** *Let $(X,T)$ be a t.d.s. Then $x \in X$ is a Banach diam-mean equicontinuity point if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$BD^*\left\{i \in \mathbb{Z}_+ : \operatorname{diam}(T^i B_\delta(x)) > \varepsilon\right\} < \varepsilon.$$

**Definition 4.8.** Let $(X,T)$ be a t.d.s. We say $\pi_{eq}$ is **regular (or almost surely 1-1)** if

$$\nu_{eq}(\left\{y \in X_{eq} : \left|\pi_{eq}^{-1}(y)\right| = 1\right\}) = 1.$$

The next result can be considered a measurable version of (i) implies (ii) in Theorem 3.4.

**Proposition 4.9.** *Let $(X,T)$ be a minimal t.d.s. and suppose $\pi : X \to X_{eq}$ is regular. Then $(X,T)$ is Banach diam-mean equicontinuous.*

*Proof.* Without loss of generality, we may assume $T_{eq}$ is an isometry. Let

$$F = \{y \in X_{eq} : \pi_{eq}^{-1}(y) = \{x_y\}\}.$$

Since $\pi_{eq}$ is regular we have that $\nu_{eq}(F) = 1$. Let $\varepsilon > 0$. There is a compact $F' \subset F$ such that $\nu_{eq}(F') > 1-\varepsilon$.

For any $y \in F'$, there is $\delta_y > 0$ such that $\pi_{eq}^{-1}(B_{2\delta_y}(y)) \subset B_\varepsilon(x_y)$. Consider the open cover $\{B_{\delta_y}(y) : y \in F'\}$ of $F'$. Since $F'$ is compact, there are $n \in \mathbb{N}$ and $y_1,\ldots,y_n \in F'$ such that $\cup_{i=1}^n B_{\delta_{y_i}}(y_i) \supset F'$. Let

$$\eta = \min\{\delta_{y_i} : 1 \le i \le n\} \text{ and } G_\varepsilon = \cup_{i=1}^n B_{\delta_{y_i}}(y_i).$$

It is clear that $G_\varepsilon$ is open and $\nu_{eq}(G_\varepsilon) > 1-\varepsilon$. Moreover, for all $y \in G_\varepsilon$, there is $y_i$ such that $y \in B_{\delta_{y_i}}(y_i)$. This implies that $B_\eta(y) \subset B_{\delta_{y_i}}(y) \subset B_{2\delta_{y_i}}(y_i)$. By uniform ergodicity, there exists $L > 0$ such that

$$\frac{1}{K}\sum_{i=j}^{K+j-1} 1_{G_\varepsilon} \circ T_{eq}^i(y) \ge 1-\varepsilon$$

for all $j \in \mathbb{N}$, $K \ge L$ and $y \in X_{eq}$.
Now, choose $\delta > 0$ such that if $U \subset X$ and $\operatorname{diam}(U) < \delta$ then $\operatorname{diam}(\pi_{eq}(U)) < \eta/2$. Let $U$ be a non-empty open subset of $X$ with $\operatorname{diam}(U) < \delta$. Using an equivalent metric we may assume $T_{eq}$ is an isometry. This implies that $\operatorname{diam}(T_{eq}^i \pi_{eq}(U)) < \eta/2$ for every $i \in \mathbb{N}$. Let $y \in \pi_{eq}(U)$. For every $i \in \mathbb{N}$ such that $T_{eq}^i y \in G_\varepsilon$, we have that $T_{eq}^i \pi_{eq}(U) \subset B_\eta(T_{eq}^i y) \subset B_{2\delta_{y_l}}(y_l)$ for some $1 \le l \le n$. Thus,

$$T^i(U) \subset \pi_{eq}^{-1}\pi_{eq}(T^iU) = \pi_{eq}^{-1} T_{eq}^i \pi_{eq}(U) \subset \pi_{eq}^{-1}(B_{2\delta_{y_l}}(y_l)) \subset B_\varepsilon(x_{y_l}).$$

We conclude that when $K \ge L$,

$$\begin{aligned}\frac{1}{K}\sum_{i=j}^{K+j-1} \operatorname{diam}(T^iU) &= \frac{1}{K}\sum_{\substack{i=j\\ T_{eq}^i y \in G_\varepsilon}}^{K+j-1} \operatorname{diam}(T^iU) + \frac{1}{K}\sum_{\substack{i=j\\ T_{eq}^i y \notin G_\varepsilon}}^{K+j-1} \operatorname{diam}(T^iU)\\ &\le 2\varepsilon + \operatorname{diam}(X)\left(\frac{1}{K}\sum_{i=j}^{K+j-1} 1_{X\setminus G_\varepsilon} \circ T_{eq}^i(y)\right)\\ &< 2(1+\operatorname{diam}(X))\varepsilon.\end{aligned}$$

□

A t.d.s. is *null* if it has zero topological sequence entropy with respect to every subsequence [26]. In [19, Corollary 66], it was shown that every minimal null system is diam-mean equicontinuous. A t.d.s. is *tame* if the cardinality of the Ellis semigroup [10] is smaller or equal than $\aleph_1$ [23]. Every null t.d.s. is tame [35, Proposition 5.4 and 6.4]. Using Proposition 4.9 and the recent result that the m.e.f. map, $\pi_{eq}$, of every minimal tame t.d.s. is regular ([13, Theorem 1.2] and [24, Corollary 5.4]), we obtain the following corollary.

**Corollary 4.10.** *Every minimal tame t.d.s. is Banach diam-mean equicontinuous.*

**Proposition 4.11.** *Let $(X,T)$ be a diam-mean equicontinuous minimal t.d.s. Then $\pi_{eq}$ is regular.*

*Proof.* We have that $(X,T)$ is mean equicontinuous and frequently stable, hence $\pi_{eq}:(X,T)\to(X_{eq},T_{eq})$ is almost 1-1. Assume that $\pi_{eq}$ is not regular.

Thus,

$$\nu_{eq}(\{y\in X_{eq}:\operatorname{diam}(\pi_{eq}^{-1}(y))>0\})=1.$$

This implies that there is $\varepsilon>0$ such that $\delta:=\nu_{eq}(A_\varepsilon)>0$, where

$$A_\varepsilon=\{y\in X_{eq}:\operatorname{diam}(\pi_{eq}^{-1}(y))>\varepsilon\}.$$

By Birkhoff's ergodic theorem, there is $y\in A_\varepsilon$ such that

$$\frac{1}{N}\left|\{1\le i\le N:T_{eq}^i y\in A_\varepsilon\}\right|=\frac{1}{N}\sum_{i=1}^{N}1_{A_\varepsilon}(T_{eq}^i y)\to\nu_{eq}(A_\varepsilon)>0.$$

Now, let $U$ be an non-empty open subset of $X$. Since $\pi_{eq}(U)$ contains an open non-empty subset of $X_{eq}$, there is $y_0\in X_{eq}$ such that $\pi_{eq}^{-1}(y_0)\subset U$ is a singleton (because $\pi_{eq}$ is almost 1-1). Since $(X,T)$ is minimal, there is a sequence $(n_j)_{j\in\mathbb{N}}$ such that $T_{eq}^{n_j}y\to y_0$. This implies that there is $m=m(U)$ such that $T^m(\pi_{eq}^{-1}(y))\subset U$. It follows that

$$\lim_{N\to\infty}\frac{1}{N}\left|\{1\le n\le N:\operatorname{diam}(T^nU)>\varepsilon\}\right|\ge\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}1_{A_\varepsilon}(T_{eq}^{i+m}y)=\nu_{eq}(A_\varepsilon)=\delta.$$

A contradiction to the diam-mean equicontinuity of $(X,T)$, since $\delta$ is independent of $U$. □

Using a much stronger hypothesis, a similar result was obtained in Theorem 54 of [19].

Combining the above two propositions, we have obtain the main result of the section.

**Theorem 4.12.** *Let $(X,T)$ be a minimal t.d.s. The following statements are equivalent:*

(1) *$(X,T)$ is diam-mean equicontinuous.*
(2) *$(X,T)$ is Banach diam-mean equicontinuous.*
(3) *$\pi_{eq}:X\to X_{eq}$ is regular.*

A t.d.s. (even non-minimal) is mean equicontinuous if and only if it is Banach mean equicontinuous [44, 14]. So we ask.

**Question 2.** Do any of the equivalences of Theorem 4.12 hold for non-minimal systems?

In the following section we will see that locally (1) and (2) are not equivalent.

## 5. DIAM-MEAN SENSITIVITY AND ALMOST DIAM-MEAN EQUICONTINUITY

In this section, first we present a counterpart of diam-mean equicontinuity, diam-mean sensitivity. Then, we investigate the local version of diam-mean equicontinuity, that is, almost diam-mean equicontinuity. The main result of this section is the construction of an almost diam-mean equicontinuous systems with positive topological entropy (Theorem 5.6).

### 5.1. Diam-mean sensitivity.

**Definition 5.1.** A t.d.s. $(X,T)$ is **diam-mean sensitive** if there exists $\varepsilon > 0$ such that for every non-empty open set $U$ we have

$$\overline{D}\left\{i \in \mathbb{Z}_+ : diam(T^i U) > \varepsilon\right\} > \varepsilon.$$

The following result (the diam mean version of Akin/Auslander/Berg dichotomy [1]) was proved in [19].

**Proposition 5.2.** *A minimal t.d.s. is either diam mean equicontinuous or diam-mean sensitive. A transitive t.d.s. is either almost diam mean equicontinuous or diam-mean sensitive.*

**Definition 5.3.** A t.d.s. $(X,T)$ is **Banach diam-mean sensitive** if there exists $\varepsilon > 0$ such that for every open set $U$ we have

$$BD^*\left\{i \in \mathbb{Z}_+ : diam(T^i U) > \varepsilon\right\} > \varepsilon.$$

The proof of the following proposition is similar to the proof of Proposition 5.2 in [19], and will be omitted.

**Proposition 5.4.** *A minimal t.d.s. is either Banach diam mean equicontinuous or Banach diam-mean sensitive. A transitive t.d.s. is either almost Banach diam mean equicontinuous or Banach diam-mean sensitive.*

### 5.2. Almost diam-mean equicontinuity.

If $(X,T)$ is minimal, then $x \in X$ is a diam-mean equicontinuity point if and only if it is a Banach diam-mean equicontinuity point (Theorem 4.12). We will see this is not true for transitive systems.
The *topological entropy* is a conjugate invariant number that can be assigned to any t.d.s. For an introduction to the topic see [48, Chapter 7]. Almost Banach mean equicontinuous systems always have zero topological entropy but transitive almost mean equicontinuous systems may have positive topological entropy [39, Corollary 6.7 and 4.8] (for an example with dense periodic points see see [21]). In this section, we will see that even transitive almost diam-mean equicontinuous systems may have positive topological entropy.

We will construct symbolic dynamical systems. Let $\Sigma_k = \{0,1,\dots,k-1\}^{\mathbb{N}}$ endowed with the product topology. For $x \in \Sigma_k$, we denote $x_i = x(i)$. A metric inducing the topology is given by $d(x,y) = 0$, if $x = y$; otherwise, $d(x,y) = \frac{1}{i}$, with $i = \min\{i : x_i \neq y_i\}$. For $n \in \mathbb{N}$, we call $A \in \{0,1,\dots,k-1\}^n$ a **word** and denote the length with $\lambda(A) = n$. For $x \in \Sigma_k$ and $i < j$, $x_{[i,j]}$ stands for the finite word $x_i x_{i+1} \dots x_j$. Given a word $A$, we define $[A] = \left\{x \in \Sigma_k : x_{[1,\lambda(A)]} = A\right\}$. Given two non-empty words $A = x_1 \dots x_n$ and $B = y_1 \dots y_m$, we denote the concatenation of the two words with $AB = x_1 \dots x_n y_1 \dots y_m$; also, we say $A$ **appears in** $B$, if there exist words $C$ and $D$, such that $CAD = B$.

Recall that the shift map $\sigma : \Sigma_k \to \Sigma_k$ is defined by the condition that $\sigma(x)_n = x_{n+1}$ for $n \in \mathbb{N}$. It is easy to see that $\sigma$ is a continuous surjection. We say $X \subset \Sigma_k$ is a **subshift** if $X$ is non-empty, closed, and $\sigma$-invariant (i.e. $\sigma(X) \subset X$); in this case $(X, \sigma)$ is a topological dynamical system.

**Example 5.5.** Let $y \in \{2,3\}^{\mathbb{N}}$ and $\mathscr{K} = \{k_n\}_{n\in\mathbb{N}}$ a sequence of positive integers. We will construct a subshift $X_y^{\mathscr{K}} \subset \Sigma_4 = \{0,1,2,3\}^{\mathbb{N}}$. In order to do this, we recursively define words $A_n$. Let $B_n = y_{[1,n]}$, and $A_1 = 11$. Assume that $A_n$ is defined. We set

$$A_{n+1} = A_n 0^{k_n} B_n A_n.$$

Let $x = (x_1, x_2, \dots) = \lim_{n\to\infty} A_n$. Then $x$ can be written as

$$A_1 0^{k_1} B_1 A_1 0^{k_3} B_3 A_1 0^{k_1} B_1 A_1 0^{k_2} B_2 A_1 0^{k_1} B_1 A_1 0^{k_4} B_4 \dots$$

Let $X_y^{\mathscr{K}}$ be the orbit closure of $x$. It is clear that $x$ is a recurrent point, so $(X_y^{\mathscr{K}}, \sigma)$ is transitive, and that $\overline{orb(y)} \subset X_y^{\mathscr{K}}$.

**Theorem 5.6.** *There exists $\mathscr{K} = \{k_n\}_{n\in\mathbb{N}}$ such that $(X_y^{\mathscr{K}}, \sigma)$ is an almost diam-mean equicontinuous t.d.s.*

*Proof.* For $z \in X_y^{\mathscr{K}}$ and $i \in \mathbb{N}$, let $p_i^z$ be the smallest integer such that $z_{[p_i^z+1, p_i^z+k_i]} = 0^{k_i}$, and $p_i := p_i^x$. It is easy to check that $p_i = \lambda(A_i)$. Hence, $p_{i+1} = 2p_i + k_i + i$. Let $\{k_i\}$ be a sequence defined inductively with the property

$$(2p_i + p_{i+1} - k_i)/(k_i - p_i) \to 0.$$

Let $\varepsilon > 0$. First we will prove that

$$\limsup_{N\to\infty} \frac{1}{N} a_N \le \varepsilon,$$

where

$$a_N = |\{1 \le m \le N : \text{there exists } z \in X_y^{\mathscr{K}} \cap [A_1] \text{ such that } z_m \ne 0\}|.$$

For a given $i \in \mathbb{N}$, $x$ can be written as

$$x = A_i 0^{k_i} B_i A_i\ 0^{k_{i+1}} B_{i+1}\ A_i 0^{k_i} B_i A_i\ 0^{k_{i+2}} B_{i+2}\ A_i 0^{k_i} B_i A_i\ 0^{k_{i+1}} B_{i+1}\ A_i 0^{k_i} B_i A_i \dots.$$

Since $A_1$ only appears in $A_i$ in the above equality, and $A_i$ is followed by $0^{k_j}$ for a $j \ge i$, then for every $\sigma^n(x) \in [A_1]$ and $i \in \mathbb{N}$, we have $p_i^{\sigma^n(x)} \le p_i$. Thus, for every $z \in X_y^{\mathscr{K}} \cap [A_1]$ and $i \in \mathbb{N}$ we have $p_i^z \le p_i$. Hence, for every $i'$ such that $p_i + 1 \le i' \le k_i - p_i$ we have that $z_{i'} = 0$ for every $z \in X_y^{\mathscr{K}} \cap [A_1]$.

There exists $i_0$, such that $(2p_i + p_{i+1} - k_i)/(k_i - p_i) \le \varepsilon$ for $i \ge i_0 - 1$. For $i \ge i_0$, let $N \in [p_i, p_{i+1}]$. We have that

$$\begin{aligned} \{m \in [1,N] : z_m \ne 0\} \subseteq\ & \{1 \le m \le p_i : \exists\, z \in X_y^{\mathscr{K}} \cap [A_1] \text{ such that } z_m \ne 0\} \\ \cup\ & \{p_i + 1 \le m \le k_i - p_i : \exists\, z \in X_y^{\mathscr{K}} \cap [A_1] \text{ such that } z_m \ne 0\} \\ \cup\ & \{k_i - p_i + 1 \le m \le p_{i+1} : \exists\, z \in X_y^{\mathscr{K}} \cap [A_1] \text{ such that } z_m \ne 0\}\ . \end{aligned}$$

By the previous argument, we obtain

$$|\{p_i + 1 \le m \le k_i - p_i : \exists\, z \in X_y^{\mathscr{K}} \cap [A_1] \text{ such that } z_m \ne 0\}| = 0.$$

We claim that for all $N \in [k_i - p_i + 1, k_{i+1} - p_{i+1}]$, we have $\frac{1}{N} a_N \le \varepsilon$. Indeed, if $N \in [k_i - p_i + 1, p_{i+1}]$, then

$$\frac{1}{N} a_N \le \frac{1}{k_i - p_i} a_N \le \frac{1}{k_i - p_i}(p_i + p_{i+1} - (k_i - p_i)) < \varepsilon.$$

If $N \in [p_{i+1}, k_{i+1} - p_{i+1}]$, then $a_N = a_{p_{i+1}}$ and hence $a_N/N \le a_{p_{i+1}}/p_{i+1} < \varepsilon$. As the claim holds for all $i \ge i_0$ and $\varepsilon > 0$ was arbitrary, we obtain that

$$\lim_{N\to\infty} \frac{1}{N} a_N = 0.$$

To conclude the proof, we need a standard argument. There is $K_\varepsilon \in \mathbb{N}$ such that for any $a, b \in \Sigma_4$ if $a[1, K_\varepsilon] = b[1, K_\varepsilon]$ then $d(a,b) < \varepsilon$. Thus,

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \operatorname{diam}(T^i[A_1]) &\le \varepsilon \cdot \limsup_{N\to\infty} \frac{1}{N} |\{1 \le i \le N\colon (T^i z)_{[1,K_\varepsilon]} = 0^{K_\varepsilon}, \forall\, z \in [A_1]\}| \\ &\quad + \limsup_{N\to\infty} \frac{1}{N} |\{1 \le i \le N\colon (T^i z)_{[1,K_\varepsilon]} \neq 0^{K_\varepsilon}, \text{for some } z \in [A_1]\}| \\ &\le \varepsilon + \limsup_{N\to\infty} \frac{1}{N} K_\varepsilon a_N = \varepsilon\ .\end{aligned}$$

Hence $x$ is a diam-mean equicontinuity point. □

Furthermore, if we choose $y$ so that $(\overline{orb(y)}, \sigma)$ has positive topological entropy (for example a Toeplitz subshift with positive topological entropy) then we obtain the following corollary.

**Corollary 5.7.** *There exist transitive almost diam-mean equicontinuous t.d.s. with positive topological entropy.*

Almost Banach diam-mean equicontinuous systems are almost Banach mean equicontinuous and thus they always have zero topological entropy [39, Corollary 6.7]. This yields the following.

**Corollary 5.8.** *There exists diam-mean equicontinuity points on transitive systems that are not Banach diam-mean equicontinuity points.*

In summary, for local properties, we have the following diagram.

| | | |
|---|---|---|
| 1) Banach diam-mean eq. point | $\to$ | 2) diam-mean eq. point |
| $\downarrow$ | | $\downarrow$ |
| 3) Banach mean eq. point | $\to$ | 4) mean eq. point |

Every implication is strict. Furthermore, there is no relationship between 2) and 3). Because, on the one hand, the same reasoning of the previous corollary gives us diam-mean equicontinuity points that are not Banach mean equicontinuity points. On the other hand, 3) does not imply 2); for this, consider a non diam-mean equicontinuous system that is mean equicontinuous, and hence Banach mean equicontinuous.

**Remark 5.9.** Note that in the proof of Theorem 5.6, we prove that the average diameter of the orbit of $A_1$ is zero. This property is stronger than almost diam-mean equicontinuity.

## 6. Furstenberg's question

Given a t.d.s. $(X,T)$, we say a point $x$ is **minimal** if $(\overline{orb(x)},T)$ is a minimal t.d.s. In [18, p. 231], Furstenberg asked if for every t.d.s. $(X,T)$ and $d\in\mathbb{N}$, there is $x\in X$ such that $(x,x,\ldots,x)$ is a minimal point for $T\times T^2\times\ldots\times T^d$. Since each t.d.s. has a minimal subset, we only need to consider the question for a minimal t.d.s. In this section, we answer the question for the class of mean equicontinuous systems. Recall that every minimal mean equicontinuous system is uniquely ergodic (e.g. see [39, Corollary 3.4]).

**Theorem 6.1.** *Let $(X,T)$ be minimal mean equicontinuous t.d.s. There is a Borel set $X_0$ with $\mu(X_0)=1$ (where $\mu$ is the unique measure) such that for any $d\in\mathbb{N}$ and $x\in X_0$, $(x,x,\ldots,x)$ is a minimal point for $T\times T^2\times\ldots\times T^d$.*

*Proof.* Let $\{V_i\}$ be a countable base for the topology of $X$. Let $X_0$ be the set such that the pointwise multiple ergodic theorem holds, that is, the points $x\in X_0$, so that for each $1_{V_i}$ and each $d\in\mathbb{N}$ we have that

$$\frac{1}{N}\sum_{n=0}^{N-1}1_{V_i}(T^nx)1_{V_i}(T^{2n}x)\ldots1_{V_i}(T^{dn}x)$$

converges as $N\to\infty$. Theorem 3.2 implies that $(X,T,\mu)$ is measurably distal, hence using [32, Theorem C] we obtain that $\mu(X_0)=1$. Furthermore, this limit must be positive [17].

Now, fix $d\in\mathbb{N}$. Then, for $x\in X_0$, and any non-empty open neighbourhood $U=B_\varepsilon(x)$ of $x$, there is $i\in\mathbb{N}$ such that $x\in V_i\subset U$. Thus,

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}1_U(T^nx)\ldots1_U(T^{dn}x)\geq\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}1_{V_i}(T^nx)\ldots1_{V_i}(T^{dn}x)>0. \tag{6}$$

Let $y=\pi_{eq}(x)$ and $\pi_{eq}^{-1}y=X_y$. Since $(y,y,\ldots,y)$ is minimal for $T_{eq}\times T_{eq}^2\times\ldots\times T_{eq}^d$, there is a minimal point $(x_1,x_2,\ldots,x_d)\in X_y^d$ for $T\times T^2\times\ldots\times T^d$. We may assume that $x_1=x$. To see this, let $T^{n_i}x_1\to x$ and assume that $T^{2n_i}x_2\to x_2',\ldots,T^{dn_i}x_d\to x_d'$. Since $X_{eq}$ is equicontinuous, $T^{n_i}y\to y$ implies $T^{jn_i}y\to y$ for $j=2,\ldots,d$. We conclude that $\pi_{eq}(x_j')=y$ for $j=2,\ldots,d$.

Let

$$B_1=\{n\in\mathbb{N}:T^nx\in U,T^{2n}x\in U\}\text{ and }B_j=\{n\in\mathbb{N}:d(T^{jn}x,T^{jn}x_j)<\varepsilon\},\ 2\leq j\leq d.$$

Since $B_1$ has positive upper density (see (6)) and $B_j$ has Banach density 1 (see Lemma 4.7 and consider $T^j$), there must exist $n\in\cap_{j=1}^dB_j\neq\emptyset$. Thus, we have

$$T^n(x)\in U=B_\varepsilon(x)\text{ and }T^{jn}x_j\in B_{2\varepsilon}(x),\ 2\leq j\leq d.$$

We conclude that $(x,x,\ldots,x)$ is in the orbit closure of $(x,x_2,\ldots,x_d)$ under $T\times T^2\times\ldots\times T^d$ and hence, minimal under $T\times T^2\times\ldots\times T^d$. ☐

The first part of this proof is related to Theorem 3.16 of [37].

This is not the first time that an open question is answered partially for mean equicontinuous systems. For instance, in [9] Downarowicz and Kasjan proved the Sarnak conjecture under this hypothesis.

## REFERENCES


[1] E. Akin, J. Auslander and K. Berg, *When is a transitive map chaotic?*, in Convergence in Ergodic Theory and Probability (editors: Bergelson, March, and Rosenblatt), Ohio State Univ. Math. Res. Inst. Publ. **5** (1959): 25–40.

[2] J. Auslander, *Minimal flows and their extensions*, North-Holland Mathematics Studies **153** (1988).

[3] J. Auslander, *Mean-L-stable systems*, Illinois J. Math. **3**.4 (1959): 566–579.

[4] J. Auslander and J.A. Yorke, *Interval maps, factors of maps, and chaos*, Tohoku Math. J. **32**.2 (1980): 177–579.

[5] M. Baake, T. Jäger, and D. Lenz, *Toeplitz flows and model sets*, Bull. London Math. Soc. **48**.4 (2016): 691–698.

[6] M. Baake, D. Lenz, and R.V. Moody, *Characterization of model sets by dynamical systems*, Ergodic Theory Dyn. Syst. **27**.2 (2007): 341–382.

[7] T. Downarowicz, *Survey of odometers and Toeplitz flows*, in Algebraic and Topological Dynamics (editors: Kolyada, Manin, and Ward), Cont. Math. **385** (2005): 7-38.

[8] T. Downarowicz and E. Glasner, *Isomorphic extension and applications*, Topol. Methods Nonl. An. **48**.1 (2016): 321-338.

[9] T. Downarowicz and S. Kasjan, *Odometers and Toeplitz systems revisited in the context of Sarnak's conjecture*, Stud. Math. **229**.1 (2015), 45-72.

[10] R. Ellis, *A semigroup associated with a transformation group*, Trans. Am. Math. Soc. **94**.2 (1960): 272-281.

[11] R. Ellis and W.H. Gottschalk, *Homomorphisms of transformation groups*, Trans. Am. Math. Soc. **94**.2 (1960): 258-271.

[12] S. Fomin, *On dynamical systems with a purely point spectrum*, Proc. USSR Acad. Sci. **77** (1951): 29-32 (in Russian).

[13] G. Fuhrmann, E. Glasner, T. Jäger, and C. Oertel, *Irregular model sets and tame dynamics*, arXiv: 1811.06283 (2018).

[14] G. Fuhrmann, M. Gröger, and D. Lenz, *The structure of mean equicontinuous group actions*, arXiv:1812.10219 (2018).

[15] G. Fuhrmann, and D. Kwietniak, *On tameness of almost automorphic dynamical systems for general groups*, Bull. London Math. Soc., in press.

[16] H. Furstenberg, *The structure of distal flows*, Am. J. Math. **85**.3 (1963): 477–515.

[17] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Anal. Math. **31**.1 (1977): 204–256.

[18] H. Furstenberg, *Poincare recurrence and number theory*, Bull. Am. Math. Soc. **5**.3 (1981): 211–234.

[19] F. García-Ramos, *Weak forms of topological and measure theoretical equicontinuity: relationships with discrete spectrum and sequence entropy*, Ergodic Theory Dyn. Syst. **37**.4 (2017): 1211-1237.

[20] F. García-Ramos and D. Kwietniak, *On topological models of zero entropy loosely Bernoulli systems*, preprint.

[21] F. García-Ramos, J. Li, and R. Zhang, *When is a dynamical system mean sensitive?*, Ergodic Theory Dyn. Syst. **39**.6 (2019): 1608-1636.

[22] F. García-Ramos and B. Marcus, *Mean sensitive, mean equicontinuous and almost periodic functions for dynamical systems,* Discr. Cont. Dyn. Syst. A **39**.2 (2019): 729-746.

[23] E. Glasner, *The structure of tame minimal dynamical systems*, Ergodic Theory Dyn. Syst. **27**.6 (2007):1819-1837.

[24] E. Glasner, *The structure of tame minimal dynamical systems for general groups*, Invent. Math. **211**.1 (2018): 213-244.

[25] E. Glasner, W. Huang, S. Shao and X. Ye, *Regionally proximal relation of order d along arithmetic progressions and nilsystems*, Sci. China Math., in press (arXiv:1911.04691).

[26] T.N.T. Goodman, *Topological sequence entropy*, Proc. London Math. Soc. **3**.2 (1974): 331-350.

[27] S. Graf, *Selected results on measurable selections*, in Proceedings of the 10th Winter School on Abstract Analysis, Rend. Circ. Math. Palermo **2** (1982): 87–122.

[28] P. Halmos, and J. von Neumann, *Operator methods in classical mechanics II*, Ann. Math. **43**.2 (1942): 332-350.
[29] W. Huang, S. Kolyada, G. Zhang, *Analogues of Auslander-Yorke theorems for multi-sensitivity*. Ergodic Theory Dyn. Syst. **38**.2 (2018), 651–665.
[30] W. Huang, J. Li, J.P. Thouvenot, L. Xu, and X. Ye, *Bounded complexity, mean equicontinuity and discrete spectrum*, Ergodic Theory Dyn. Syst., in press.
[31] W. Huang, S. Li, S. Shao and X. Ye, *Null systems and sequence entropy pairs*, Ergodic Theory Dyn. Syst. **23**.5 (2003): 1505–1523.
[32] W. Huang, S. Shao and X. Ye, *Pointwise convergence of multiple ergodic averages and strictly ergodic models*, J. Anal. Math., in press.
[33] K. Jacobs and M. Keane, *0-1-sequences of Toeplitz type*, Z. Wahrsch. verw. Gebiete **13**.2 (1969): 123-131.
[34] D. Kerr and H. Li, *Dynamical entropy in Banach spaces*, Invent. Math. **162**.3 (2005): 649-686.
[35] D. Kerr and H. Li, *Independence in topological and C*-dynamics*, Math. Ann. **338**.4 (2007): 869–926.
[36] A. G. Kushnirenko, *On metric invariants of entropy type*, Russian Math. Surveys **22**.5 (1967): 53-61.
[37] D. Kwietniak, J. Li, P. Oprocha, X. Ye, *Multi-recurrence and van der Waerden systems*, Sci. China Math. **60**.1 (2017), 59-82.
[38] E. Lehrer, *Topological mixing and uniquely ergodic systems*, Isr. J. Math. **57**.2 (1972): 239-255.
[39] J. Li, S. Tu and X. Ye, *Mean equicontinuity and mean sensitivity*, Ergodic Theory Dyn. Syst. **35**.8 (2015): 2587-2612.
[40] J. Li, X. Ye and T. Yu, *Mean equicontnuity, bounded complexity and applications.*, Discrete Contin. Dyn. Syst. A, 25 annularity of DCDS, in press.
[41] D. Ornstein and B. Weiss, *Mean distality and tightness*, Proc. Steklov Inst. Math., **244**.0 (2004): 312-319.
[42] M.E. Paul, *Construction of almost automorphic symbolic minimal flows*, General Topology Appl. **6**.1 (1976): 45-56.
[43] P.A.B. Pleasants and C. Huck, *Entropy and Diffraction of the k-Free Points in n-Dimensional Lattices*, Discr. Comput. Geom. **50**.1 (2013): 39–68
[44] J. Qiu and J. Zhao, *A note on mean equicontinuity*, J. Dyn. Differ. Equ., in press.
[45] B. Scarpellini, *Stability properties of flows with pure point spectrum*, J. London Math. Soc. **2**.3 (1982): 451–464.
[46] W. A. Veech, *The Equicontinuous Structure Relation for Minimal Abelian Transformation Groups*, Am. J. Math. **90**.3 (1968): 723–732.
[47] J. von Neumann, *Zur Operatorenmethode in der klassischen Mechanik*, Ann. Math. **33**.2 (1932): 587–642.
[48] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics **79** (1982).
[49] X. Ye and R. Zhang, *On sensitivity sets in topological dynamics*, Nonlinearity **21**.7 (2008): 1601–1620.

(F. García-Ramos) CONACYT - INSTITUTE OF PHYSICS, UNIVERSIDAD AUTÓNOMA DE SAN LUIS POTOSÍ - AV. MANUEL NAVA #6, ZONA UNIVERSITARIA, C.P. 78290, SAN LUIS POTOSÍ, S.L.P., MEXICO
*Email address*: fgramos@conacyt.mx

(T. Jäger) INSTITUTE OF MATHEMATICS, FRIEDRICH SCHILLER UNIVERSITY JENA, GERMANY
*Email address*: tobias.jaeger@uni-jena.de

(X. Ye) WU WEN-TSUN KEY LABORATORY OF MATHEMATICS, USTC, CHINESE ACADEMY OF SCIENCES AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, ANHUI 230026, CHINA
*Email address*: yexd@ustc.edu.cn