

1 **Theorem 1.1.** Denote by \mathbb{P}_κ the unique physical measure of (1) with forcing function (2). Let
 2 $p_{\kappa,N} = \mathbb{P}_\kappa(\{(\theta, x) \mid \lambda_N(\theta, x) > 0\})$. Then there exists $\kappa_1 > \kappa_0$ such that for all $\kappa \geq \kappa_1$, there
 3 are constants $\gamma_+ \geq \gamma_- > 0$ (depending on κ) such that

$$\exp(-\gamma_+ N) \leq p_{\kappa,N} \leq \exp(-\gamma_- N)$$

4 holds for all $N \in \mathbb{N}$.

5 Apart from its intrinsic interest, motivation for this result stems from the study of critical
 6 transitions. One major problem in this field is the identification of suitable (that is, observable
 7 and reliable) early warning signals [16–21] for such transitions. A commonly proposed and
 8 utilized early warning signal for fold bifurcations—which are often cited as a paradigmatic
 9 example of critical transitions—are *slow recovery rates* (also referred to as a *critical slowing*
 10 *down*) [16–18, 20]. Since this notion has been coined in an interdisciplinary context and is used
 11 in a wide variety of situations, there is no comprehensive and rigorous mathematical definition
 12 of this term and we refrain from attempting to give one here. However, in the classical case of
 13 an autonomous fold bifurcation, recovery rates can be identified with the Lyapunov exponents
 14 of the stable equilibria. Thus, in this situation, critical slowing down simply refers to the fact
 15 that when the stable and unstable equilibria involved in the bifurcation approach each other
 16 and eventually merge at the critical parameter, the resulting single fixed point is neutral, that
 17 is, it has exponent zero.

18 This picture changes significantly when a fold bifurcation takes place under the influence of
 19 external quasiperiodic forcing. First of all, the resulting *non-autonomous* systems generally do
 20 not allow for fixed points. Therefore, when carrying over ideas from an autonomous to a non-
 21 autonomous setting, one needs an appropriate replacement. In the present context, this part is
 22 played by so-called *invariant graphs* (see Section 2) and accordingly, non-autonomous fold bifur-
 23 cations occur as invariant graphs approach each other upon a change of system parameters. In
 24 stark contrast to autonomous fold bifurcations, this does not necessarily yield neutral invariant
 25 graphs but may instead lead to a strange non-chaotic attractor-repeller-pair [14, 15] created at
 26 the bifurcation point. This alternative pattern is referred to as a *non-smooth saddle-node bifur-*
 27 *cation*. Moreover, just as for pinched systems, under suitable conditions, there exists a unique
 28 physical measure \mathbb{P} which is supported on the attractor and has a negative Lyapunov exponent
 29 (see [3, 12]). However, this means that Lyapunov exponents remain \mathbb{P} -almost surely negative
 30 and bounded away from zero during a non-smooth saddle-node bifurcation (see Section 2 for
 31 more details).

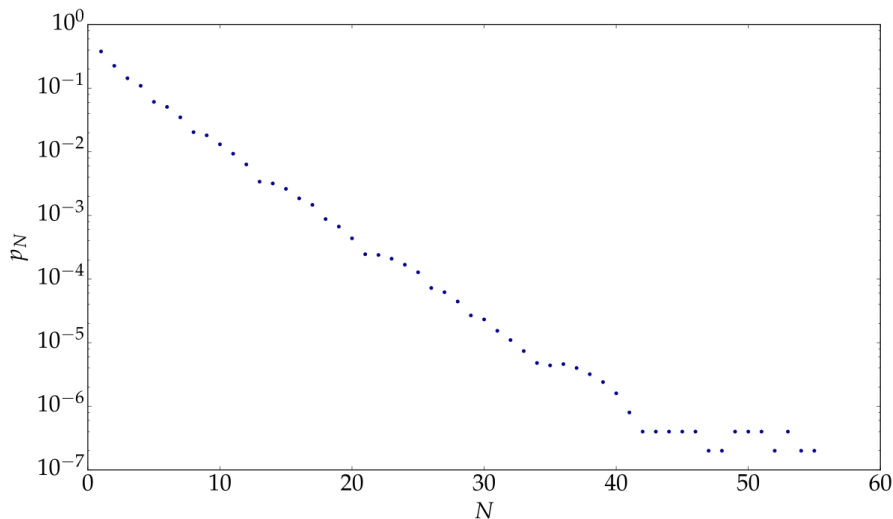


FIGURE 1. A logarithmic plot of the numerically obtained probability $p_N = p_{\kappa,N}$ over N for the system (1) with $D = 1$, $\kappa = 3$ and v the golden mean. The graph shows the relative frequency of non-negative finite-time Lyapunov exponents among a grid of $5 \cdot 10^6$ initial conditions on the SNA (see also Figure 2). Consistent with the statement of Theorem 1.1, the plot indicates an exponential decay.

1 While this seems to rule out the viability of slow-recovery rates as early warning signals
 2 for non-smooth fold bifurcations, one should bear in mind that experiments never measure
 3 the actual Lyapunov exponent but rather approximations of it. In other words, one rather
 4 measures finite-time Lyapunov exponents instead of asymptotic ones. Since the presence of an
 5 SNA implies that positive finite-time exponents occur with positive probability for any time
 6 $N \in \mathbb{N}$ [2, 22], one may hence wonder whether the observation of non-negative finite-time Lyapunov
 7 exponents can help to detect an SNA in practice. However, if N is chosen too small, then
 8 positive time- N -exponents can be observed already far from a bifurcation. Conversely, for large
 9 N , the probability of observing positive exponents on this time-scale converges to zero since the
 10 unique physical measure has a negative exponent. It is in this context that the scaling behaviour
 11 of the probabilities of time- N -exponents with $N \rightarrow \infty$ becomes important. Numerical studies
 12 for the quasiperiodically forced Allee model performed in [22] remained somewhat inconclusive,
 13 which is partly explained by the fact that the simulation of continuous-time systems is consider-
 14 ably more time-consuming than that of discrete-time systems. The exponential decay obtained
 15 in Theorem 1.1 is an indication that very large data sets may be required to detect this kind
 16 of early-warning signals in practice. As mentioned before, this interpretation relies on the hy-
 17 pothesis that quasiperiodically forced systems undergoing a saddle-node bifurcation—as studied
 18 in [22]—show a behaviour comparable to that of pinched systems treated here. We expect that
 19 using techniques from [8, 10, 12], similar statements can be established for non-pinched sys-
 20 tems but this would require a considerably more involved analysis due to the inherent technical
 21 difficulties.

22 This article is organised as follows. In the next section, we introduce some technical back-
 23 ground on forced monotone interval maps and their invariant graphs. There, we also describe
 24 the physical measure \mathbb{P} from above in more detail. In Section 3, we specify the class of pinched
 25 skew-products for which we prove (a more general version of) the above theorem. This proof
 26 and the full statement—Theorem 4.4 and Theorem 4.8 (which gives the upper bound and is
 27 the harder part)—are given in the final section, Section 4.

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32 2. FORCED MONOTONE INTERVAL MAPS AND INVARIANT GRAPHS

33 Throughout this note, we deal with *quasiperiodically forced (qpf) monotone interval maps*,
 34 that is, skew products of the form

$$F : \mathbb{T}^D \times [0, 1] \rightarrow \mathbb{T}^D \times [0, 1], \quad (\theta, x) \mapsto (\rho(\theta), F_\theta(x)), \quad (3)$$

35 where $\mathbb{T}^D = \mathbb{R}^D / \mathbb{Z}^D$ is the D -dimensional torus (for some $D \geq 1$),

$$\rho : \mathbb{T}^D \rightarrow \mathbb{T}^D, \quad \theta \mapsto \theta + v$$

36 is a minimal rotation with a *rotation vector* v and for each $\theta \in \mathbb{T}^D$, F_θ is a continuously differ-
 37 entiable non-decreasing map on $[0, 1]$ such that $(\theta, x) \mapsto F'_\theta(x)$ is continuous. It is customary
 38 to refer to (\mathbb{T}^D, ρ) as the *forcing system* (defined on the *base* \mathbb{T}^D); the maps F_θ ($\theta \in \mathbb{T}^D$) are
 39 also referred to as *fibre maps* (defined on the *fibres* $\{\theta\} \times [0, 1]$).

An *invariant graph* of (3) is a measurable function $\phi : \mathbb{T}^D \rightarrow [0, 1]$ which satisfies

$$F_\theta(\phi(\theta)) = \phi(\theta + \rho) \quad \text{for all } \theta \in \mathbb{T}^D.$$

40 From an intuitive perspective, invariant graphs are to be seen as non-autonomous fixed points
 41 of (3).[†] This idea is the basis for a bifurcation theory of invariant graphs, see [14, 15]. Independ-
 42 ently of this analogy, invariant graphs of qpf monotone interval maps are key to understanding
 43 the dynamics of (3) due to their intimate relationship with the invariant sets and ergodic mea-
 44 sures.

Every invariant graph ϕ comes with an ergodic measure μ_ϕ where $\mu_\phi(A) = \text{Leb}_{\mathbb{T}^D}(\phi^{-1}A)$
 for each measurable $A \subseteq \mathbb{T}^D \times [0, 1]$ and likewise, to each ergodic measure μ of (3) there is an

[†]Observe that due to the minimality of ρ , (3) does not allow for actual fixed points.

invariant graph ϕ with $\mu = \mu_\phi$ [25, 26]. Moreover, given an *invariant set* $A \subseteq \mathbb{T}^D \times [0, 1]$ (that is, A is closed and $F(A) = A$), let

$$\phi_A^+(\theta) = \sup\{x \in [0, 1] : (\theta, x) \in A\} \quad \text{and} \quad \phi_A^-(\theta) = \inf\{x \in [0, 1] : (\theta, x) \in A\}$$

1 for each $\theta \in \mathbb{T}^D$. Then ϕ_A^+ and ϕ_A^- define the so-called *upper* and *lower boundary graphs* of
 2 A which are invariant and—due to the compactness of A —upper and lower semi-continuous,
 3 respectively [5]. Of particular relevance for us will be the upper boundary graph of the *global*
 4 *attractor*

$$\bigcap_{n \in \mathbb{N}} F^n(\mathbb{T}^D \times [0, 1]),$$

5 which we simply denote by ϕ^+ .

The long-term behaviour of orbits near an invariant graph ϕ is largely characterized by its *Lyapunov exponent*

$$\lambda(\phi) = \int_{\mathbb{T}^D} \log F'_\theta(\phi(\theta)) d\theta,$$

6 provided this integral exists.

7 If $\lambda(\phi) > 0$, then ϕ is repelling; if $\lambda(\phi) < 0$, then ϕ is attracting and μ_ϕ is a physical measure,
 8 see [23] for the details. Here, by *physical measure*, we refer to an F -invariant ergodic measure \mathbb{P}
 9 for which there is a positive Lebesgue measure set $V \subseteq \mathbb{T}^D \times [0, 1]$ such that for every continuous
 10 observable $f : \mathbb{T}^D \times [0, 1] \rightarrow \mathbb{R}$ and all $(\theta, x) \in V$

$$1/n \cdot \sum_{\ell=0}^{n-1} f(F^\ell(\theta, x)) = \int f d\mathbb{P},$$

11 see also [27]. It is noteworthy that under suitable concavity assumptions on F_θ (which are
 12 verified by (1)), (3) has a unique physical measure given by the measure μ_{ϕ^+} on the upper
 13 boundary graph ϕ^+ of the global attractor, see [3, 15].

Observe that due to Birkhoff's Ergodic Theorem, $\lambda(\phi)$ equals the Lyapunov exponent of the
 point $(\theta, \phi(\theta))$ for $\text{Leb}_{\mathbb{T}^D}$ -almost every θ (equivalently: for μ_ϕ -almost every (θ, x)) since

$$\lambda(\theta, \phi(\theta)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(F_\theta^n)'(\phi(\theta)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \log F'_{\rho^\ell(\theta)}(\phi(\rho^\ell(\theta)))$$

for $\text{Leb}_{\mathbb{T}^D}$ -almost every $\theta \in \mathbb{T}^D$. Note that on the left-hand side of the above equation, we
 made use of the customary notation

$$F_\theta^n(x) = \pi_2 \circ F^n(\theta, x) = F_{\theta+(n-1)\rho} \circ \dots \circ F_{\theta+\rho} \circ F_\theta(x), \quad (4)$$

14 where $\pi_2 : \mathbb{T}^D \times [0, 1] \rightarrow [0, 1]$ denotes the projection to the second coordinate.

15 **Remark 2.1.** Note that for the model (1) and the *a priori* invariant graph $\psi = 0$ given by
 16 zero line, we have that $F'_\theta(\psi(\theta)) = F'_\theta(0) = \kappa \cdot g(\theta)$, so that

$$\lambda(\psi) = \log \kappa 0 \int_{\mathbb{T}^D} \log g(\theta) d\theta = \log \kappa - \log \kappa_0$$

17 in this case. Hence, the zero line is repelling for all $\kappa > \kappa_0$, and pointwise Lyapunov exponents
 18 on this line are positive almost surely (with respect to the Lebesgue measure on $\mathbb{T}^D \times \{0\}$).

As we will discuss below, the unique physical measure of (1) is given by μ_{ϕ^+} , where ϕ^+ is the
 upper boundary graph of the global attractor. Therefore, a big part of the proof of Theorem 1.1
 boils down to analysing ϕ^+ in considerable detail. In that context, we will utilize the obvious
 fact that ϕ^+ is the pointwise limit of the sequence of *iterated upper boundary lines* $(\phi_n)_{n \in \mathbb{N}_{\geq 0}}$,
 where

$$\phi_n : \mathbb{T}^D \rightarrow [0, 1], \quad \theta \mapsto F_{\theta-n\rho}^n(1). \quad (5)$$

19 Note that the graph of ϕ_n coincides with $F^n(\mathbb{T}^D \times \{1\})$ (recall the notation from (4)). It is
 20 further easy to see (and important to note) that the monotonicity of the fibre maps F_θ implies
 21 $\phi_{n+1} \leq \phi_n$ for all $n \in \mathbb{N}$.

22 For the convenience of the reader, we close this section with a brief description of the invariant
 23 graphs of (1). While this description will help to develop an intuition for the dynamics of (1)

1 and, more broadly, for the results discussed in this note, it is strictly speaking not a prerequisite
 2 for the discussion in Section 3 and Section 4. For simplicity, we may assume that $D = 1$ in the
 3 remainder of this section.

4 It is immediate that independently of the value of κ , one invariant graph of (1) is given by
 5 the 0-line (which just happens to be the lower boundary graph of the global attractor). Let us
 6 denote this graph by ϕ^- . By direct computation, one can obtain that $\lambda(\phi^-) = \log \kappa - \log 2$.

7 Clearly, if ϕ^- equals the upper boundary graph ϕ^+ of the global attractor, then ϕ^- is the
 8 only invariant graph of F_κ . However, with help of the iterated upper boundary lines, one can
 9 show that $\lambda(\phi^+) \leq 0$, see [23]. Accordingly, if $\kappa > 2$, the 0-line ϕ^- is $\text{Leb}_{\mathbb{T}^1}$ -almost surely
 10 distinct from ϕ^+ .

11 In other words, F_κ has at least two invariant graphs if $\kappa > 2$. Moreover, just as concavity of
 12 interval maps implies the existence of at most two fixed points (one of which is attracting and
 13 one of which is repelling), one can show that the concavity of the fibre maps of F_κ implies that
 14 ϕ^- and ϕ^+ are the only invariant graphs (and further, $\lambda(\phi^-) > 0 > \lambda(\phi^+)$), see [3, 15].[‡]

15 Now, since $F(0, x) = 0$, we have that ϕ^+ necessarily intersects the 0-line along the orbit
 16 of $(\rho, 0)$ which is, by minimality of ρ , dense in $\mathbb{T}^D \times \{0\}$. Therefore, while ϕ^+ is upper-
 17 semicontinuous (as the upper boundary graph of the global attractor) it clearly is not continuous
 and ϕ^+ is referred to as a *strange non-chaotic attractor*, see Figure 2 for a plot of ϕ^+ .

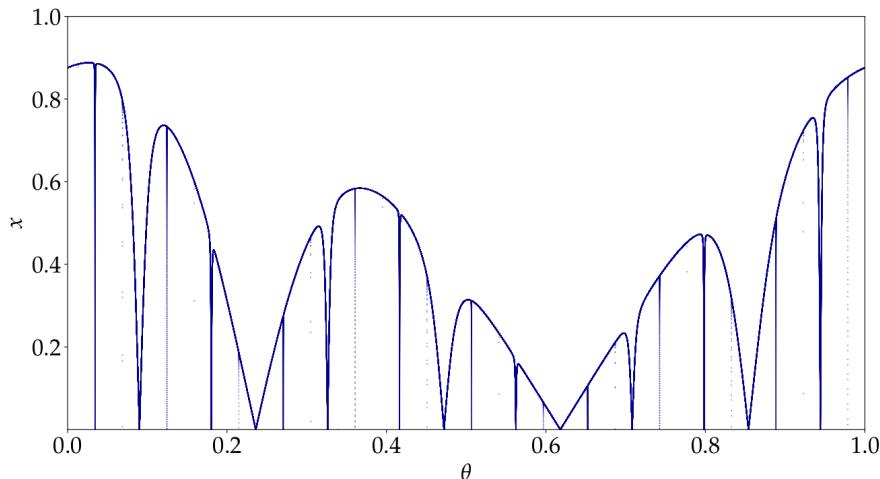


FIGURE 2. The SNA ϕ^+ of the parameter family $(\theta, x) \mapsto (\theta + \rho, \tanh(\kappa x) \cdot \sin(\pi\theta))$ with $\kappa = 3$ and ρ the golden mean. The points in the above plot are exactly the initial conditions used to estimate $p_{\kappa, N}$ in Figure 1.

18

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3. PINCHED SKEW-PRODUCT SYSTEMS

In this section, we specify the class of skew products within which we derive asymptotic estimates on the probability of positive finite-time Lyapunov exponents. For later reference, we repeat some of the assumptions from the previous section. By \mathcal{F} , we refer to the class of quasiperiodically forced monotone interval maps of the form

$$F : \mathbb{T}^D \times [0, 1] \rightarrow \mathbb{T}^D \times [0, 1], \quad (\theta, x) \mapsto (\rho(\theta), F_\theta(x)),$$

20 which satisfy

- 21 (\mathcal{F}_1) : the fibre maps F_θ are non-decreasing;
- 22 (\mathcal{F}_2) : the fibre maps F_θ are differentiable and $(\theta, x) \mapsto F'_\theta(x)$ is continuous on $\mathbb{T}^D \times I$;
- 23 (\mathcal{F}_3) : F is *pinched*, that is, there is $\theta_* \in \mathbb{T}^D$ with $F_{\theta_*}(x) = 0$ for all $x \in [0, 1]$;
- 24 (\mathcal{F}_4) : $F_\theta(0) = 0$ for all $\theta \in \mathbb{T}^D$ (invariance of the 0-line).

[‡]Note that accordingly, the physical measure \mathbb{P} in the introduction has to coincide with μ_{ϕ^+} .

Besides the qualitative assumptions (\mathcal{F}_1) – (\mathcal{F}_4) , we need a number of quantitative assumptions. Let $F \in \mathcal{F}$ and assume that there exist parameters $\alpha > 2$, $\beta > 0$, $\gamma > 0$ and $L_0 \in (0, 1)$ such that for all $\theta \in \mathbb{T}^D$, the following holds.

$$|F_\theta(x) - F_\theta(y)| \leq \alpha |x - y| \quad \text{for all } x, y \in [0, 1], \quad (\text{F1})$$

$$|F_\theta(x) - F_\theta(y)| \leq \alpha^{-\gamma} |x - y| \quad \text{for all } x, y \in [L_0, 1], \quad (\text{F2})$$

$$|F_\theta(x) - F_{\theta'}(x)| \leq \beta d(\theta, \theta') \quad \text{for all } x \in [0, 1]. \quad (\text{F3})$$

- 1 In particular, (F2) implies that the fibre maps F_θ are contracting in $[L_0, 1]$.

While the above assumptions determine F in the fibres, we need some additional control over the forcing on \mathbb{T}^D . To that end, we assume that the rotation vector $\rho \in \mathbb{T}^D$ is *Diophantine*. More specifically, setting $\tau_n = \rho^n(\theta_*) = \theta_* + n\rho$ (the n^{th} -iterate of the *pinched point* θ_*), we assume that there are constants $c > 0$ and $d > 1$ such that

$$d(\tau_n, \theta_*) \geq c \cdot n^{-d} \quad \text{for all } n \in \mathbb{N}. \quad (\text{F4})$$

Finally, bringing the behaviour along the fibres and the dynamics on the base \mathbb{T}^D together, we assume that there are constants $m \in \mathbb{N}$, $a > 1$ and $0 < b < 1$ with

$$m > 22 \left(1 + \frac{1}{\gamma}\right), \quad (\text{F5})$$

$$a \geq (m+1)^d, \quad (\text{F6})$$

$$b \leq c, \quad (\text{F7})$$

$$d(\tau_n, \theta_*) > b \quad \text{for all } n \in \{1, \dots, m-1\} \quad (\text{F8})$$

such that

$$F_\theta(x) \geq \min\{2L_0, ax\} \cdot \min\left\{1, \frac{2}{b}d(\theta, \theta_*)\right\} \quad \text{for all } (\theta, x) \in \mathbb{T}^D \times [0, 1]. \quad (\text{F9})$$

Our analysis of positive finite-time Lyapunov exponents will take place within the class

$$\mathcal{F}^* = \{F \in \mathcal{F} : F \text{ satisfies (F1)–(F9)}\}.$$

- 2 Instead of the abstract description of \mathcal{F}^* given above, readers may simply think of the system
 3 given in (1) (for large κ) in all of the following. This is justified by the next statement.
 4 **Lemma 3.1** (see [7, Lemma 4.2]). *Consider F_κ as in (1) and let ρ satisfy the Diophan-*
 5 *tine condition (F4) for some $c > 0$ and $d > 1$. There exists a constant $\kappa_0 = \kappa_0(c, d, D)$*
 6 *such that for all $\kappa \geq \kappa_0$, the map F_κ satisfies (F1)–(F9) (with appropriately chosen constants*
 7 *$\alpha, \gamma, \beta, L_0, m, a, b, c$).*

Note that as $[0, 1] \ni x \mapsto F'_\theta(x)$ is continuous for each $\theta \in \mathbb{T}^D$ (due to (\mathcal{F}_2)), the mean value theorem and (F9) imply

$$F'_\theta(0) \geq a \cdot \min\left\{1, \frac{2}{b}d(\theta, \theta_*)\right\} \quad \text{for all } \theta \in \mathbb{T}^D. \quad (\text{F10})$$

While (F10) yields the existence of positive finite-time Lyapunov exponents on the zero line (see Lemma 4.3 below), in order to ensure big enough lower bounds on the probability of positive finite-time Lyapunov exponents outside the zero line, we additionally assume that for all $\delta > 0$ there is $x_\delta > 0$ with

$$F'_\theta(x) \geq (1 - \delta) \cdot F'_\theta(0) \quad \text{for all } x \in [0, x_\delta] \text{ and all } \theta \in \mathbb{T}^D. \quad (\text{F11})$$

- 8 Clearly, this additional assumption is satisfied by (1) (for all κ, D and ρ).

9 4. RIGOROUS BOUNDS ON THE PROBABILITY OF POSITIVE FINITE-TIME LYAPUNOV 10 EXPONENTS

- 11 In this section, we show that within the class of pinched skew-products, the μ_{ϕ^+} -measure of
 12 points (θ, x) with $\lambda_N(\theta, x) \geq 0$ decays exponentially as $N \rightarrow \infty$.[§]

[§]Recall that ϕ^+ refers to the upper boundary graph of the global attractor, see Section 2.

1 We start by deriving a lower bound for this probability. To that end, we first need to study
 2 the occurrence of positive finite-time Lyapunov exponents on the zero line. Due to (F10), this
 3 essentially amounts to analysing the frequency of visits of points $\theta \in \mathbb{T}^D$ to the vicinity of θ_* .

For $j \in \mathbb{N}$, set

$$r_j = \frac{b}{2} a^{-(j-1)} \quad \text{and} \quad R_j = \frac{b}{2} a^{-\frac{(j-1)}{m}}. \quad (6)$$

4 **Proposition 4.1.** *Suppose (F4)–(F8) are satisfied. Then, for $n \in \{1, \dots, m-1\}$, we have*

$$B_{r_1}(\theta_*) \cap (B_{r_1}(\theta_*) + n\rho) = \emptyset.$$

5 *Similarly, for $j \geq 2$ and $n \in \{1, \dots, (m+1)^{(j-1)}\}$, we have*

$$B_{r_j}(\theta_*) \cap (B_{r_j}(\theta_*) + n\rho) = \emptyset.$$

6 **Proof.** We only discuss $j \geq 2$. With (F8), the other case is obvious.

7 Suppose $B_{r_j}(\theta_*) \cap (B_{r_j}(\theta_*) + n\rho) \neq \emptyset$ for some n , that is, $d(\theta_*, \tau_n) < 2r_j = ba^{-(j-1)}$. Note
 8 that (F4) gives $d(\theta_*, \tau_n) \geq c \cdot n^{-d}$. Therefore, $ba^{-(j-1)} > c \cdot n^{-d}$ and thus, $n > (c/b)^{1/d} \cdot$
 9 $a^{(j-1)/d} \geq (m+1)^{j-1}$, where we used (F6) and (F7) in the last step. \square

10 This immediately gives

11 **Corollary 4.2.** *Assume (F4)–(F8) and let $\theta \in \mathbb{T}^D$. Suppose $n_1 < n_2 \in \mathbb{N}$ are such that
 12 $\theta + n_1\rho \in B_{r_j}(\theta_*)$ and $\theta + n_2\rho \in B_{r_j}(\theta_*)$. If $j = 1$, then $n_2 - n_1 \geq m$ and if $j \geq 2$, then
 13 $n_2 - n_1 \geq (m+1)^{j-1}$.*

14 **Lemma 4.3.** *Suppose $F \in \mathcal{F}^*$ and $N \in \mathbb{N}$. For each $\theta \in B_{r_N}(\theta_*)$, we have*

$$\lambda_N(\theta + \rho, 0) \geq 1/2 \cdot \log a.$$

Proof. Set $\Delta_j = (m+1)^{j-1}$. By the previous corollary, given θ as in the assumptions and
 $2 \leq j \leq N$, we have

$$\#\{\ell \in \{1, \dots, N\}: \theta + \ell\rho \in B_{r_j}(\theta_*)\} \leq \lfloor N/\Delta_j \rfloor$$

and

$$\#\{\ell \in \{1, \dots, N\}: \theta + \ell\rho \in B_{r_1}(\theta_*)\} \leq \lfloor N/m \rfloor.$$

Note further that for $j \geq 0$, (F10) gives

$$F'_{\theta+\ell\rho}(0) \geq a \cdot \frac{2}{b} r_{j+1} = a^{-(j-1)} \quad \text{whenever } \theta + \ell\rho \in B_{r_j}(\theta_*) \setminus B_{r_{j+1}}(\theta_*),$$

where—for notational convenience— $r_0 = \sqrt{D}$ and hence $B_{r_0}(\theta_*) = \mathbb{T}^D$. We therefore have

$$\begin{aligned} \lambda_N(\theta + \rho, 0) &= 1/N \cdot \sum_{\ell=1}^N \log F'_{\theta+\ell\rho}(0) \geq 1/N \cdot \sum_{\ell=1}^N \sum_{j \geq 0} \log a^{-(j-1)} \cdot \mathbf{1}_{B_{r_j}(\theta_*) \setminus B_{r_{j+1}}(\theta_*)}(\theta + \ell\rho) \\ &= 1/N \cdot \sum_{\ell=1}^N \sum_{j \geq 0} (1-j) \log a \cdot \mathbf{1}_{B_{r_j}(\theta_*) \setminus B_{r_{j+1}}(\theta_*)}(\theta + \ell\rho) \\ &= \log a - \log a \cdot 1/N \cdot \sum_{j \geq 1} j \cdot \sum_{\ell=1}^N \mathbf{1}_{B_{r_j}(\theta_*) \setminus B_{r_{j+1}}(\theta_*)}(\theta + \ell\rho) \\ &\geq \log a - \log a \cdot 1/N \cdot (\lfloor N/m \rfloor + \sum_{j \geq 2} j \cdot \lfloor N/\Delta_j \rfloor) \geq \log a - \log a \cdot (1/m + \sum_{j \geq 2} j/\Delta_j) \\ &= \log a - \log a \cdot (1/m + \sum_{j \geq 2} j/(m+1)^{j-1}) \geq 1/2 \cdot \log a, \end{aligned}$$

15 where we used (F5) in the last step. \square

16 In order to prove the lower bound in Theorem 1.1, it remains to show that the positive
 17 finite-time Lyapunov exponents on the zero line are observable not only *on* but already *close*
 18 *to* the zero line. This is what the proof of the next statement is about.

19 **Theorem 4.4.** *Suppose $F \in \mathcal{F}^*$ satisfies (F11). Then there is $\gamma_+ > 0$ such that for all $N \in \mathbb{N}$*

$$\mu_{\phi^+} \{(\theta, x) \in \mathbb{T}^D \times [0, 1]: \lambda_N(\theta, x) \geq 0\} \geq e^{-\gamma_+ N}.$$

Proof. Choose $\delta > 0$ small enough such that

$$\log(1 - \delta) > -(\log a)/4 \quad (7)$$

1 and let x_δ be such that (F11) holds true. Without loss of generality, we may assume that
 2 $x_\delta \leq \beta b/2$ (with β from (F3)). For $N \in \mathbb{N}$, set $\tilde{r}_N = x_\delta/\beta \cdot \alpha^{-(N-1)}$. Observe that $\alpha \geq a$
 3 (because of (F1) and (F9)) so that $\tilde{r}_N \leq r_N$ for all N . We first show that for $\theta \in B_{\tilde{r}_N}(\theta_*)$ and
 4 $j = 1, \dots, N$, we have $\phi^+(\theta + j\rho) \leq x_\delta$.

5 To that end, observe that the monotonicity of the sequence of the iterated upper boundary
 6 lines $(\phi_n)_{n \in \mathbb{N}}$ (recall (5)) and (F3) yield

$$\phi^+(\theta + \rho) = \lim_{n \rightarrow \infty} \phi_n(\theta + \rho) \leq F_\theta(1) \leq \beta \cdot d(\theta, \theta_*).$$

7 Therefore, given $\theta \in B_{\tilde{r}_N}(\theta_*)$ and $j = 1, \dots, N$, we have—due to \mathcal{F}_1 and (F1)—that

$$\phi^+(\theta + j\rho) = F_{\theta+\rho}^{j-1}(\phi^+(\theta + \rho)) \leq F_{\theta+\rho}^{j-1}(\beta \cdot d(\theta, \theta_*)) \leq \alpha^{j-1} \beta \cdot d(\theta, \theta_*) \leq x_\delta.$$

8 As a consequence, Lemma 4.3 and (7) in conjunction with (F11) give that $\lambda_N(\theta + \rho, x) \geq$
 9 $(\log a)/4$ for all $(\theta, \phi^+(\theta))$ with $\theta \in B_{\tilde{r}_N}(\theta_*)$. The statement follows. \square

10 Having thus seen how within \mathcal{F}^* (under the additional assumption of (F11)) the probability
 11 of positive finite-time Lyapunov exponents decays at most exponentially, we next come to show
 12 that this decay is, in fact, not slower than exponential.

13 Before we turn to the rigorous analysis, we briefly explain its idea on an intuitive level. First,
 14 note that (F2) implies that above L_0 , fibres are contracted—we emphasize this fact by calling
 15 $\mathbb{T}^D \times [L_0, 1]$ the *contracting region*. In other words, visits to $\mathbb{T}^D \times [L_0, 1]$ contribute negatively
 16 to the (finite-time) Lyapunov exponent of an orbit. Second, (F9) enables us to control the
 17 number of times an orbit spends outside of the contracting region. Finally, since (F1) gives an
 18 upper bound for the possible fibre-wise expansion, the control obtained through (F9) enables
 19 us to ensure an overall contraction, that is, a negative (finite-time) Lyapunov exponent, along
 20 *most* finite orbits.

Let us specify this control in quantitative terms by collecting two auxiliary statements from
 [7]. Given $\theta \in \mathbb{T}^D$ and $n \in \mathbb{N}$, let $\theta_k := \rho^{k-n}(\theta)$ and $x_k := \phi_k(\theta_k)$ for $0 \leq k \leq n$. Note that
 $\phi_k(\theta_k) = F_{\theta_0}^k(1)$ and $\phi_n(\theta) = F_{\theta_k}^{n-k}(x_k)$. Let

$$s_k^n := \#\{k \leq j < n : x_j < 2L_0\}$$

21 and set $s_n^n(\theta) = 0$. Recall the definition of R_j in (6).

Lemma 4.5 ([7, Lemma 4.6]). *Let $F \in \mathcal{F}^*$ and $q, n \in \mathbb{N}$ with $n \geq mq + 1$. Suppose that
 $\theta \notin \bigcup_{j=q}^n B_{R_j}(\tau_j)$ and consider $t \geq mq$. Then*

$$s_{n-t}^n(\theta) \leq \frac{11t}{m}.$$

22 As discussed in Section 2, the iterated upper boundary lines ϕ_n approximate the graph ϕ^+
 23 whose measure μ_{ϕ^+} we are interested in. The next statement effectively provides numerical
 24 bounds for this approximation.

25 **Proposition 4.6** ([7, Proposition 4.4]). *Let $F \in \mathcal{F}^*$, $q \in \mathbb{N}$ and $\eta = \gamma - \frac{11}{m}(1 + \gamma) > 0$. Then,
 26 for $n \geq mq + 1$ and $\theta \notin \bigcup_{j=q}^n B_{R_j}(\tau_j)$, we have that $|\phi_n(\theta) - \phi_{n-1}(\theta)| \leq \alpha^{-\eta(n-1)}$.*

27 **Remark 4.7.** Observe that $\eta > 0$ due to (F5) and note that η is independent of q .

Clearly, Proposition 4.6 gives that if $k, n \in \mathbb{N}$ satisfy $mq \leq k < n$ and $\theta \notin \bigcup_{j=q}^n B_{R_j}(\tau_j)$, then

$$|\phi_n(\theta) - \phi_k(\theta)| \leq \sum_{i=k+1}^n |\phi_i(\theta) - \phi_{i-1}(\theta)| \leq \sum_{i=k+1}^n \alpha^{-\eta(i-1)} \leq \frac{\alpha^{-\eta k}}{1 - \alpha^{-\eta}}. \quad (8)$$

28 With the above statements, we are now in a position to prove the upper bound in Theorem 1.1.

29 **Theorem 4.8.** *Suppose $F \in \mathcal{F}^*$. Then there is $\gamma_- > 0$ such that for all $N \in \mathbb{N}$*

$$\mu_{\phi^+} \{(\theta, x) \in \mathbb{T}^D \times [0, 1] : \lambda_N(\theta, x) \geq 0\} \leq e^{-\gamma_- N}.$$

1 **Proof.** Note that it suffices to show the statement for sufficiently large N .

Let $N \in \mathbb{N}$ be given. As ϕ^+ is the pointwise limit of the non-increasing sequence ϕ_n and due to the continuous differentiability of the fibre maps (see (\mathcal{F}_2)), we have that for each θ

$$\lambda_N(\theta_0, \phi^+(\theta_0)) = \frac{1}{N} \sum_{k=0}^{N-1} \log |F'_{\theta_k}(\phi^+(\theta_k))| = \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \log |F'_{\theta_k}(\phi_n(\theta_k))|,$$

2 where—as above— $\theta_k = \rho^{k-N}(\theta)$.

3 In a first instance, our goal is to derive assumptions on θ which ensure that the expression
4 $\frac{1}{N} \sum_{k=0}^{N-1} \log |F'_{\theta_k}(\phi_n(\theta_k))|$ is negative and bounded away from zero for $n \geq N$ and large enough
5 N . The statement then follows by showing that these assumptions are only violated in a set of
6 exponentially small measure as $N \rightarrow \infty$.

We start by collecting a number of estimates which we will then combine to obtain an upper bound for $\frac{1}{N} \sum_{k=0}^{N-1} \log |F'_{\theta_k}(\phi_n(\theta_k))|$. First, let $\kappa \in \mathbb{N}$ be large enough such that $\kappa \geq m$ and $\frac{\alpha^{-\eta \cdot \kappa}}{1 - \alpha^{-\eta}} < L_0$. Consider $k_0 \in \mathbb{N}$ with $k_0 \geq \kappa q$ (for some $q \in \mathbb{N}$ which we may consider fixed for now). Then (8) gives that for every $n > k \geq k_0$ and $\theta \notin \bigcup_{j=q}^n B_{R_j}(\tau_j)$, we obtain $|\phi_n(\theta) - \phi_k(\theta)| < L_0$. In particular, if $\phi_k(\theta) \geq 2L_0$, then $\phi_n(\theta) \geq L_0$. Therefore, if $n \geq k$ and $\theta_k \notin \bigcup_{j=q}^n B_{R_j}(\tau_j)$ for some $k \geq k_0$ with $\phi_k(\theta_k) > 2L_0$, (F2) yields

$$|F'_{\theta_k}(\phi_n(\theta_k))| \leq \alpha^{-\gamma}. \quad (9)$$

7 Second, observe that due to (F1), we have

$$\sum_{k=0}^{N-1} \log |F'_{\theta_k}(\phi_n(\theta_k))| \leq \sum_{k=0}^{k_0+1} \log \alpha + \sum_{k=k_0}^{N-1} \log |F'_{\theta_k}(\phi_n(\theta_k))|. \quad (10)$$

8 Third, let $N_0 = N_0(k_0) \in \mathbb{N}$ be the smallest integer such that $N_0 - k_0 \geq mk_0/\kappa \geq mq$. Then,
9 Lemma 4.5 allows us to estimate the number of times for which $\phi_k(\theta_k) < 2L_0$. Specifically, if
10 $N \geq N_0$ and $k \in \{k_0, \dots, N-1\}$, we obtain for all $\theta \notin \bigcup_{j=q}^N B_{R_j}(\tau_j)$

$$s_{k_0}^N(\theta) = s_{N-(N-k_0)}^N(\theta) \leq \frac{11}{m}(N - k_0). \quad (11)$$

Observe that with (F1), (11) and (9), we get

$$\begin{aligned} & \sum_{k=k_0}^{N-1} \log |F'_{\theta_k}(\phi_n(\theta_k))| \leq s_{k_0}^N(\theta) \log \alpha - (N - k_0 - s_{k_0}^N(\theta)) \gamma \log \alpha \\ & \leq \frac{11}{m}(N - k_0) \log \alpha - \gamma \cdot \left(N - k_0 - \frac{11}{m}(N - k_0) \right) \log \alpha \\ & = \left(\left(\gamma - \frac{11}{m}(1 + \gamma) \right) k_0 + \left(\frac{11}{m}(1 + \gamma) - \gamma \right) N \right) \log \alpha = (\eta k_0 - \eta N) \log \alpha \end{aligned} \quad (12)$$

whenever $n \geq N \geq N_0$ and $\theta_k \notin \bigcup_{j=q}^n B_{R_j}(\tau_j)$ for all $k = k_0, \dots, N$ and where η is as in Proposition 4.6. Plugging (12) into (10), we obtain (under the same assumptions as above)

$$\sum_{k=0}^{N-1} \log |F'_{\theta_k}(\phi_n(\theta_k))| \leq (2 + (\eta + 1)k_0 - \eta N) \log \alpha. \quad (13)$$

11 Now, as $\eta > 0$, there is $\nu = \nu(\eta) > 0$ such that for all $N \geq k_0/\nu$, the right-hand side in (13) is
12 negative (so that for all $n \geq N$, the left-hand side is negative and bounded away from 0). Note
13 that we may assume without loss of generality that ν is small enough to ensure $k_0/\nu \geq N_0(k_0)$.

14 Set

$$B_{q, k_0, N} = \left\{ \theta \in \mathbb{T}^D : \theta_k \in \bigcup_{j=q}^{\infty} B_{R_j}(\tau_j) \text{ for some } k \in \{k_0, \dots, N\} \right\}.$$

Observe that (13) holds whenever θ is in the complement of the set $B_{q,k_0,N}$ (given $k_0 \geq \kappa q$ and $n \geq N \geq N_0(k_0)$). Note that

$$\begin{aligned} \text{Leb}_{\mathbb{T}^D}(B_{q,k_0,N}) &\leq (N - k_0 + 1) \cdot \text{Leb}_{\mathbb{T}^D} \left(\bigcup_{j=q}^{\infty} B_{R_j}(\tau_j) \right) \\ &\leq (N - k_0 + 1) \cdot \zeta_D \cdot \left(\frac{b}{2} \right)^D \sum_{j=q}^{\infty} a^{-\frac{(j-1)D}{m}} \\ &= (N - k_0 + 1) \cdot \zeta_D \cdot \left(\frac{b}{2} \right)^D a^{-\frac{(q-1)D}{m}} \sum_{j=0}^{\infty} (a^{-\frac{D}{m}})^j = (N - k_0 + 1) \cdot a^{-\frac{(q-1)D}{m}} c(D), \end{aligned}$$

- 1 where ζ_D denotes the $\text{Leb}_{\mathbb{T}^D}$ -measure of the D -dimensional unit ball and $c(D)$ simply collects
 2 all the terms in the above estimate which are independent of q , k_0 and N , that is, $c(D) =$
 3 $\zeta_D \cdot \left(\frac{b}{2} \right)^D \sum_{j=0}^{\infty} (a^{-\frac{D}{m}})^j$.

Now, set $k_0(N) = \lfloor \delta N \rfloor$ for some $\delta \in (0, \nu)$ and $q(N) = \lfloor \varepsilon N \rfloor$ for some $\varepsilon \in (0, \delta/\kappa)$. Note that for large enough N , we have $k_0(N) \geq \kappa q(N)$ and $N \geq k_0(N)/\nu \geq N_0[= N_0(k_0(N))]$. Hence, for sufficiently large N , the above gives

$$\begin{aligned} \text{Leb}_{\mathbb{T}^D}(\{\theta \in \mathbb{T}^D : (\lambda_N(\theta, \phi^+(\theta)) \geq 0\}) &\leq \text{Leb}_{\mathbb{T}^D}(B_{q(N),k_0(N),N}) \\ &\leq N \cdot a^{-\frac{(\lfloor \varepsilon N \rfloor - 1)D}{m}} c(D) \leq a^{-\frac{\varepsilon ND}{2m}}. \end{aligned}$$

- 4 The statement follows with $\gamma_- = (\log a) \cdot \varepsilon D/2M$. □

5

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