

ON THE EFFECT OF FORCING ON FOLD BIFURCATIONS AND EARLY-WARNING SIGNALS IN POPULATION DYNAMICS

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ABSTRACT. The classical fold bifurcation is a paradigmatic example of a critical transition. It has been used in a variety of contexts, including in particular ecology and climate science, to motivate the role of slow recovery rates and increased autocorrelations as early-warning signals of such transitions.

We study the influence of external forcing on fold bifurcations and the respective early-warning signals. Thereby, our prime examples are single-species population dynamical models with Allee effect under the influence of either quasiperiodic forcing or bounded random noise. We show that the presence of these external factors may lead to so-called *non-smooth* fold bifurcations, and thereby has a significant impact on the behaviour of the Lyapunov exponents (and hence the recovery rates). In particular, it may lead to the absence of critical slowing down prior to population collapse. More precisely, unlike in the unforced case, the question whether slow recovery rates can be observed prior to the transition crucially depends on the chosen time-scales and the size of the considered data set.

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1. INTRODUCTION

In recent years, the notions of *tipping points* and *critical transitions* have received widespread attention throughout a broad scope of sciences. These terms usually refer to abrupt and drastic changes in a system's behaviour upon a small and slow variation of the system parameters [vNS12, Sch09, K11, SCL+12]. In this context, an important issue of immediate practical interest is that of early warning signals, that is, indicators which allow to anticipate an oncoming transition in a systems qualitative behaviour before it actually occurs. A concept that has led to widely recognised advances in this direction are *slow recovery rates (critical slowing down)*, which often come along with an *increase in autocorrelation* [vNS12, SBB+09, Sch09, SCL+12, VFD+12]. Both have been described as possible early warning signals in a variety of contexts, in theoretical as well as in experimental settings [DSvN+08, CCP+11, GJT13, vLWC+14, KGB+14, RDB+16].

The aim of our work here is to understand which of the above-mentioned features of critical transitions persist when the considered system is subject to the influence of external forcing, as it happens for many real-world processes, and also to obtain a better idea on how the above notions may best be captured in mathematical terms. Here it should be pointed out that when it comes to providing a sound mathematical framework for critical transitions, it is an eminent problem that all the above notions comprise a variety of different phenomena and still lack a precise and comprehensive mathematical definition – an issue which is well-known in non-linear dynamics and comes up in similar form for key concepts like ‘chaos’, ‘fractals’ or ‘strange attractors’ in dynamical systems theory.

A forced Allee model. We will concentrate on the classical fold bifurcation, which comprises key features of critical transitions and has emerged as a paradigmatic example in this context [Sch09]. In this bifurcation, a stable and an unstable equilibrium point of a parameter-dependent scalar ODE approach each other and eventually merge to form a single neutral equilibrium point, which then vanishes. As this leads to the disappearance of all equilibria in a certain region, it presents the abrupt change in the system's qualitative behaviour that is characteristic of critical transitions. In order to fix ideas, we consider as a specific example the single-species population model with Allee effect given by the scalar

ODE

$$\begin{aligned} x' &= rx \cdot \left(1 - \frac{x}{K}\right) \cdot \left(\frac{x}{K} - \frac{S}{K}\right) - \beta x \\ &= \frac{r}{K^2} \cdot x \cdot (K - x) \cdot (x - S) - \beta x =: v_\beta(x) \end{aligned} \quad (1)$$

Here $r > 0$ denotes the intrinsic growth factor of the population, $K > 0$ is the maximal carrying capacity and $S \in (0, 1)$ is the threshold value below which the population dies out due to an Allee effect. The term βx represents an external stress factor that puts additional pressure on the population. An increase of the parameter β leads to a fold bifurcation and the subsequent collapse of the population at some critical value $\beta_c > 0$.¹ The bifurcation pattern is drawn in Figure 1(a), whereas Figure 1(b) shows the behaviour of the Lyapunov exponents of the attracting and repelling equilibria during the bifurcation.

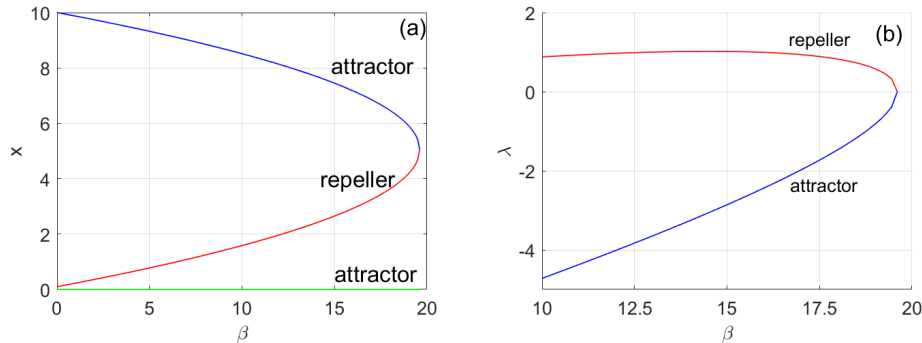


FIGURE 1. (a) Bifurcation diagram of a fold bifurcation in the Allee model (1) with parameters $r = 80$, $K = 10$, $S = 0.1$ and a bifurcation at a critical parameter $\beta_c \simeq 19.978$. The stable equilibrium at $x = 0$ is shown in green. (b) Behaviour of the Lyapunov exponents of the upper stable and the unstable equilibrium during the fold bifurcation.

In this setting, one obvious possible mathematical interpretation of *recovery rates* is to identify them with the Lyapunov exponents of the stable or neutral equilibria, so that *slow recovery rates* and *critical slowing down* correspond exactly to the fact that the two lines in Figure 1(b) meet at zero when the bifurcation parameter is reached.

As mentioned before, our main goal will be to investigate the same bifurcation pattern in forced versions of the above model (1), which are given by non-autonomous scalar ODEs of the form

$$\boxed{x'(t) = \frac{r}{K^2} \cdot x(t) \cdot (K - x(t)) \cdot (x(t) - S) - (\beta + \kappa \cdot F(t)) \cdot x =: V_{\kappa, \beta}(t, x)} \quad (2)$$

Hence, time-dependence (or external forcing) of the system (2) is introduced via a *forcing term* $\kappa \cdot F(t)$, with *coupling constant* $\kappa > 0$ and a forcing function $F : \mathbb{R} \rightarrow [0, 1]$. Thereby, we consider two different types of forcing processes.

Quasiperiodic forcing: corresponds to the influence of several periodic external factors with incommensurate frequencies $\nu_1, \dots, \nu_d \in \mathbb{R}$. As a specific example, we choose the forcing function as

$$F(t) = \kappa \cdot \prod_{i=1}^d \left(\frac{1 + \sin(2\pi(\theta_i + t \cdot \nu_i))}{2} \right)^q, \quad (3)$$

with arbitrary initial conditions $\theta_1, \dots, \theta_d \in \mathbb{R}$ and $q \in \mathbb{N}$. We already note here that due to the periodicity of the sine function, these initial values may also be viewed as elements of the circle $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, so that $\theta = (\theta_1, \dots, \theta_d)$ becomes an element of the d -torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$.

¹In fact, a simple calculation yields $\beta_c = \frac{(K-S)^2 \cdot r}{4K^2}$.

Bounded random noise: Secondly, we consider the effect of external random perturbations on the system, given for instance by a forcing function

$$F(t) = \frac{1 + \sin(\theta_0 + W_t)}{2}. \quad (4)$$

where $\theta_0 \in \mathbb{T}^1$ is again an arbitrary initial condition and W_t denotes a one-dimensional Brownian motion (but higher-dimensional analogues could be considered as well).

We thus arrive at the forced scalar differential equation (2), with forcing function F given either by (3) or (4), as basic models to which we mainly refer to in this introduction. Some of the the statements we actually prove below hold in greater generality, both with respect to the model (2) and to the employed forcing processes (3) and (4). In other cases we will need to replace the Allee model by discrete time systems with qualitatively similar behaviour in order to obtain rigorous results. These systems should then be considered as simplified models for the time-one-maps of the flow incuded by (2). However, we refer to the respective later sections for details in order to avoid too many technicalities at this point.

Lyapunov gap for fold bifurcations in forced systems. In order to discuss what happens with the fold bifurcation pattern and the corresponding early-warning signals in the forced Allee model (2), we will first concentrate on the behaviour of the Lyapunov exponents. Figure 2 shows the behaviour of the Lyapunov exponents of the attractor and the repeller of (2) throughout the bifurcation, with two different choices of the parameters κ and q in the case of quasiperiodic forcing in (a) and (b) and the random case in (c) and (d).

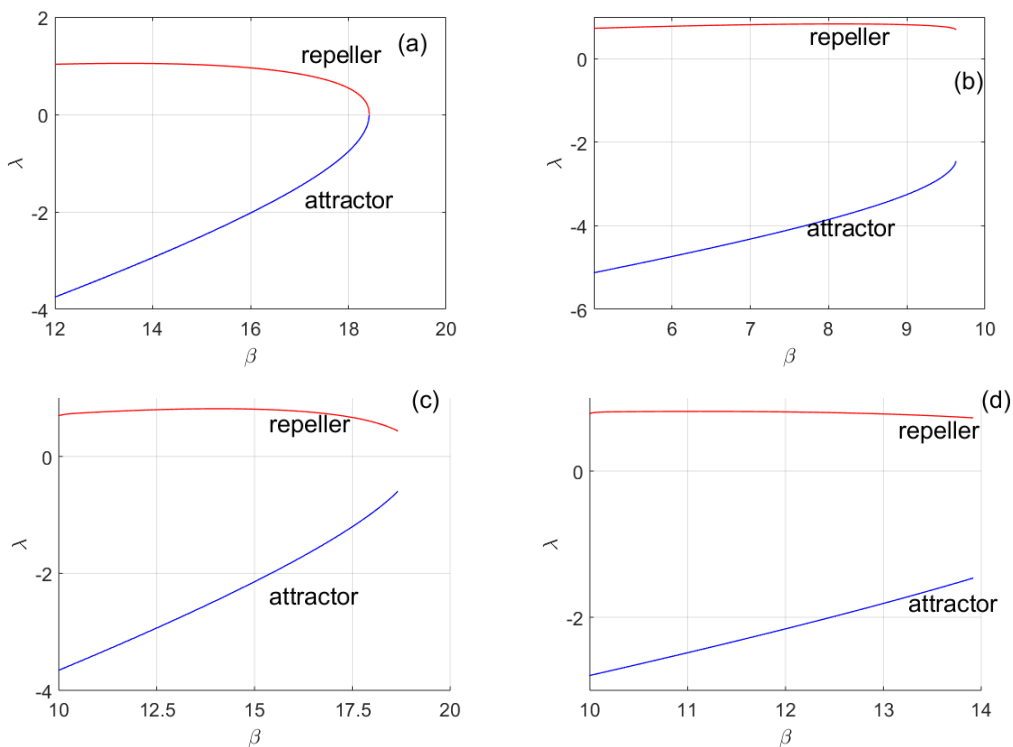


FIGURE 2. **(a) and (b):** Lyapunov exponents during fold bifurcations in the qpf Allee model; (a) smooth bifurcation with $r = 80$, $K = 10$, $S = 0.1$, $q = 1$, $\nu = (2\omega, 2\pi)$ (where ω is the irrational part of the golden mean) and $\kappa = 4$ (bifurcation at $\beta_c \simeq 18.4269$); (b) non-smooth bifurcation with $r = 80$, $K = 10$, $S = 0.1$, $q = 5$, $\nu = (2\omega, 2\pi)$ and $\kappa = 51.2$. The bifurcation occurs at $\beta_c \simeq 9.628$.

(c) and (d): Lyapunov exponents during the fold bifurcation in the randomly forced Allee model; (c) with $r = 80$, $K = 10$, $S = 0.1$, $\kappa = 1$ and bifurcation parameter $\beta_c = 18.978$; (d) with $r = 80$, $K = 10$, $S = 0.1$, $\kappa = 6$ and bifurcation parameter $\beta_c = 13.978$;

While the behaviour in (a) is in perfect analogy with the unforced case in Figure 1(b), the situation in (b)–(d) is clearly different. Although the Lyapunov exponents of the attractor and the repeller do approach each other, there remains a clear gap at the bifurcation

point, and in particular the Lyapunov exponents of the attractor (which are the ‘visible’ or ‘physically relevant’ ones) stay strictly away from zero. Given the significance of zero exponents for the observation of critical slowing down and slow recovery rates, this is certainly noteworthy and deserves a closer examination. Moreover, while in Figure 2(b)–(d) the Lyapunov exponents do at least move towards each other as the bifurcation is approached, this actually turns out to depend just on the precise form of the parameter family. In the above cases, we have just varied the bifurcation parameter β , while leaving all other constants in (2) invariant. In contrast to this, one may easily imagine that in real-world situations other system parameters, such as the intrinsic growth rate r in (2) or the noise amplitude in (4), vary as well as the pressure on the population increases. The result of such couplings is shown in Figure 3. It can be seen that in this case the Lyapunov exponents may move away from each other all throughout the bifurcation process, and hence there is no chance at all to anticipate the oncoming transition based only on their behaviour.

Remark 1.1. We should note that the phenomenon that we describe here is probably known by folklore in the field of stochastic processes and stochastic differential equations, where the presence of noise equally prevents the recovery rates from going down all the way to zero before a transition happens. However, in this context it is much harder to pin this observation down mathematically, since the presence of unbounded noise immediately ‘destroys’ the fold bifurcation and leads to the existence of a unique stationary measure in stochastic versions of (2) and similar models. Moreover, the forcing with bounded noise is arguably more reasonable from the biological viewpoint.

Mathematical analysis: skew product flows and non-smooth bifurcations. In order to understand and explain these phenomena, however, it is indispensable to have a look at the mathematical framework that is used to describe fold bifurcations in forced systems. To that end, we first concentrate on the case of quasiperiodic forcing. The rigorous analysis of non-autonomous ODE’s, such as the one given by (2) and (3), hinges on the fact that the family of equations (2), with all possible initial conditions $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$, defines a skew product flow

$$\Xi : \mathbb{R} \times \Theta \times \mathbb{R} \rightarrow \Theta \times \mathbb{R} \quad , \quad (t, \theta, x) \mapsto \Xi^t(\theta, x) = (\omega_t(\theta), \xi^t(\theta, x)) . \quad (5)$$

Here the *driving space* Θ is the d -torus, $\Theta = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (corresponding to the set of possible initial conditions) and the *driving flow* $\omega : \mathbb{R} \times \Theta \rightarrow \Theta$ is given by the irrational Kronecker flow $\omega_t(\theta) = \theta + t \cdot v$ with translation vector $v = (\nu_1, \dots, \nu_d)$,² and models the quasiperiodic dynamics of the external driving factors. The flow Ξ is uniquely determined by the fact that the mapping $t \mapsto \xi^t(\theta, x)$ is the solution to (2) with forcing function (3). A similar flow representation can be given in the case of random forcing. We will describe this passage from non-autonomous equations to skew product flows in more detail in Section 2.1, but also refer the mathematically interested reader to standard references such as [Arn98, HY09] for further reading.

The advantage of the skew product setting lies in the fact that the classical notion of equilibrium points – which does not make sense anymore for time-dependent systems as (2) – can be replaced by that of *random* or *non-autonomous equilibria*. These are defined as measurable functions $x : \Theta \rightarrow \mathbb{R}$, $\theta \mapsto x(\theta)$ that satisfy $\Xi^t(\theta, x(\theta)) = x(\omega_t(\theta))$. Hence, a non-autonomous equilibrium can be thought of as a curve, surface or higher-dimensional submanifold of the product space $\Theta \times \mathbb{R}$ that can be represented as a graph over the base space Θ , is invariant under the skew product flow Ξ and is composed of solutions of (2) with varying initial conditions. With this new notion of an equilibrium, fold bifurcations in forced systems can be described, in perfect analogy to the classical case, as the collision and subsequent extinction of a stable and an unstable equilibrium [NO07, AJ12]. This process is shown in Figure 4(a)–(d) where two such equilibrium manifolds approach each other and then merge to form a neutral equilibrium.

In contrast to the unforced case, however, there is a second possibility in which such a collision can happen. As the value of the non-autonomous equilibria depend on the forcing variable θ , the two curves or surfaces can also collide only for some values of θ , without

²Composed of the d incommensurate frequencies ν_i in (3).

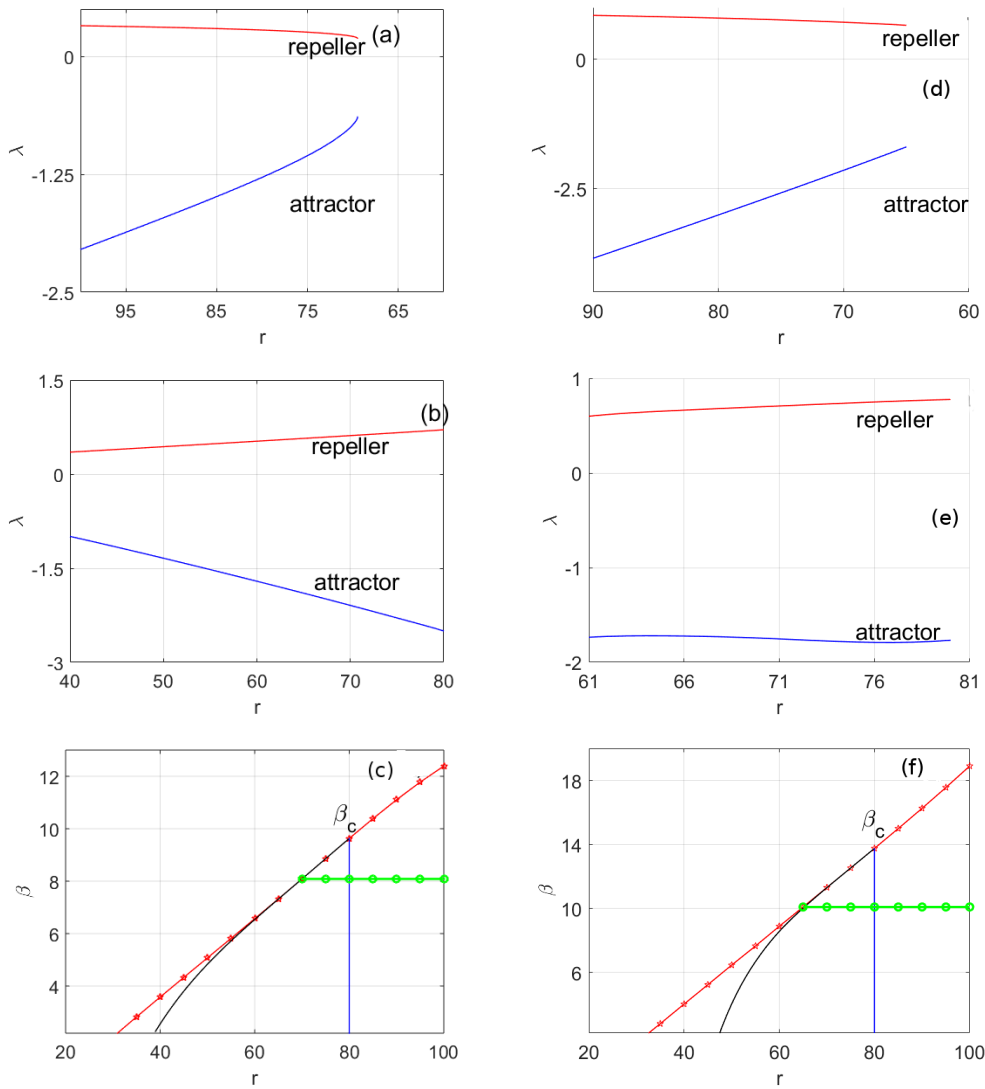


FIGURE 3. Lyapunov exponents during fold bifurcations in the forced Allee model with different variations of parameters. (a) shows the behaviour in the qpf case as the parameter r is decreased, corresponding to the horizontal green line in the two-parameter space depicted in (c). In contrast, (b) shows the behaviour when β and r are varied simultaneously along the black curve in (c). In this case, the Lyapunov exponents move apart throughout the entire bifurcation process. In (d) and (e), similar plots are shown for the randomly forced case. In (d), again only the parameter r is varied and decreases along the horizontal green line in (f). In (e), both parameters β and r are again varied at the same time, along the black curve in (f). In both (c) and (f), the red line is an interpolation of the numerically determined critical parameters for different values of β and r .

merging together uniformly. This pattern is shown in Figure 4(e)–(h). In this case, one speaks of a *non-smooth fold bifurcation*, in which the neutral equilibrium at the bifurcation point is replaced by an attractor-repeller pair. Moreover, in the case of quasiperiodic forcing the stable and unstable non-autonomous equilibria are called *strange non-chaotic attractors (SNA)* and *strange non-chaotic repellers (SNR)*, due to their unusual combination of a fractal geometry and non-chaotic dynamics [GOPY84, FKP06, Jäg09, Fuh14, FGJ14].

It is this dichotomy between smooth and non-smooth fold bifurcations which explains the different behaviours of the Lyapunov exponents observed in Figure 2. In order to make

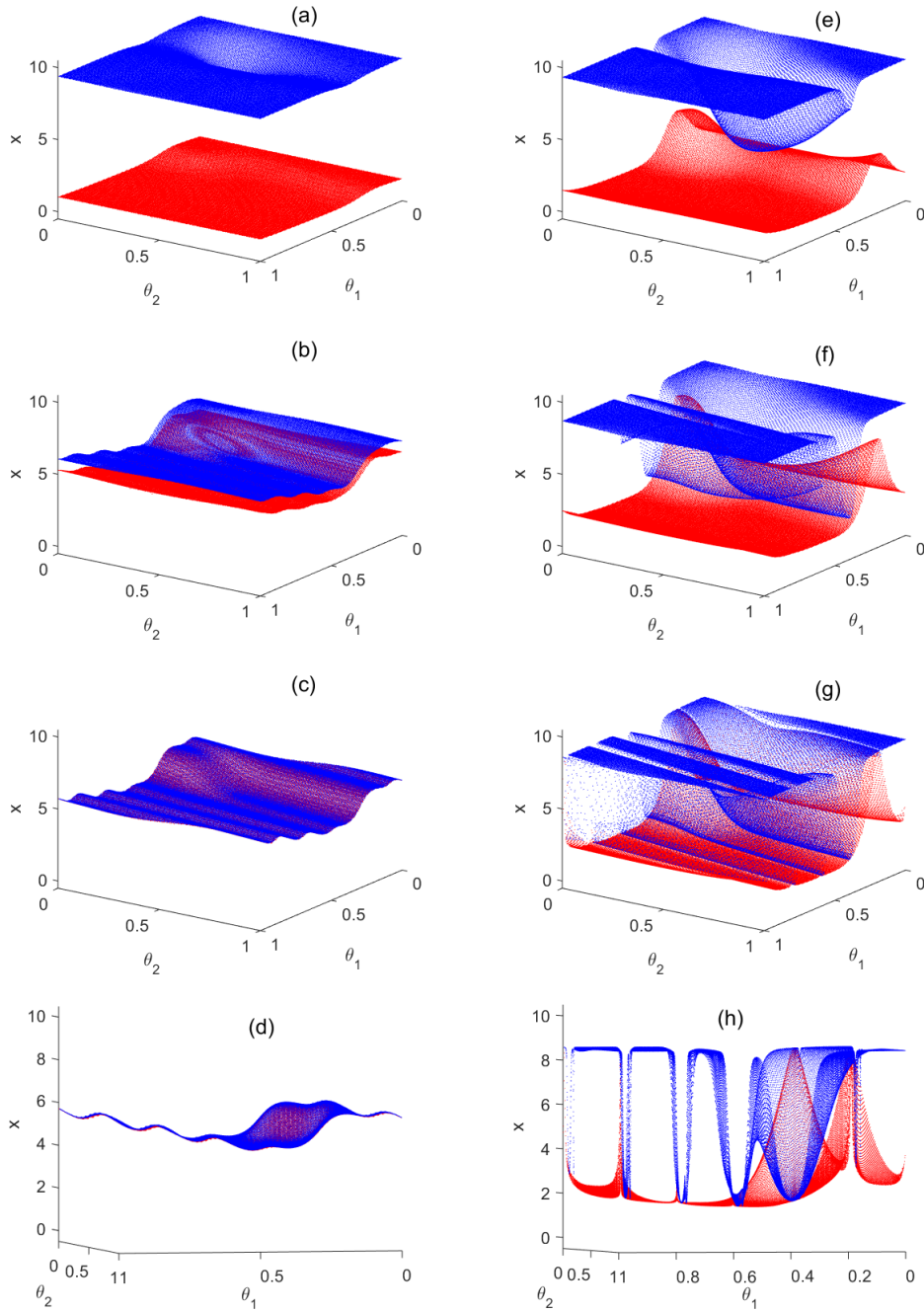


FIGURE 4. **(a)–(d)**: Smooth fold bifurcation in the qpf Allee model with parameters $r = 35$, $K = 10$, $S = 0.1$, $q = 3$ and $\kappa = 41$ and $\beta = 2$ in (a), $\beta = 7.8$ in (b) and $\beta = 7.8455$ in (c) and (d). These last two figures show the attractor (blue) and the repeller (red) at the bifurcation point from two different angles. The rotation vector ρ is $\rho = (5\omega, 5\pi)$ in all cases, where ω is the irrational part of the golden mean.

(e)–(h): Non-smooth fold bifurcation in the qpf Allee model with parameters $r = 80$, $K = 10$, $S = 0.1$, $q = 5$ and $\kappa = 51.2$ and $\beta = 2$ in (e), $\beta = 9$ in (f) and $\beta = 9.6282$ in (g) and (h). Again the last two figures show the attractor (blue) and the repeller (red) at the bifurcation point from two different angles. The rotation vector ρ is $\rho = (2\omega, 2\pi)$.

this precise, we denote by x_β^s the non-zero stable equilibrium of (2) at parameter β ,³ and by

³Note here that there is always an equilibrium at zero, which is a natural requirement for any population dynamical model and ensured by the multiplicative form of the forcing in (2). Due to the Allee effect, the zero equilibrium is stable as well and presents the unique global attractor of the system after the bifurcation.

$\lambda(x_\beta^s)$ its associated Lyapunov exponent. We refer to Section 2 for the precise definitions. Moreover, in order to obtain rigorous results we will need to apply a general framework for non-autonomous fold and saddle-node bifurcations that has been established in [AJ12]. An important condition that is required there is the concavity of the fibre maps in the considered region, which is a consequence of the concavity of the right side of the respective non-autonomous ODE. In order to ensure this concavity in (2), we need to restrict to suitable parameter ranges, as specified in the following.

Remark 1.2. We let

$$b(r, K, S) = \frac{r}{K^2} \cdot \left(\frac{K - S}{2} \right)^2 \quad \text{and} \quad \gamma(K, S) = \frac{1}{9} \cdot \left(\frac{K + S}{K - S} \right)^2 \quad (6)$$

and assume that

$$\kappa < b(r, K, S) \cdot (1 - \gamma(K, S)). \quad (7)$$

Then, as we explain in detail in Section 2.3, the family (2) with forcing term (3) or (4) undergoes a fold bifurcation in the parameter interval

$$J(r, K, S) = [b(r, K, S) \cdot (1 - \gamma(K, S)), b(r, K, S) + 1]. \quad (8)$$

It should be mentioned, however, that this restriction in the parameter ranges is merely a technical condition and could easily be improved, in particular by using numerical methods, in order to include a broader range of parameters. The crucial condition is that the time-one-maps of skew product flow induced by (2) are concave for some $t > 0$.

Theorem 1. *Suppose that (7) holds. If the Allee model (2) with forcing term given by (3) or (4) undergoes a non-smooth fold bifurcation at critical parameter $\beta_c \in J(r, K, S)$, then we have that*

$$\lim_{\beta \nearrow \beta_c} \lambda(x_\beta^s) = \lambda(x_{\beta_c}^s) < 0. \quad (9)$$

If the fold bifurcation is smooth, then we have $\lim_{\beta \nearrow \beta_c} \lambda(x_\beta^s) = 0$. The analogous results hold for the unstable equilibrium x_β^u .

Relevance of non-smooth bifurcations. A immediate question that can be asked in the context of the above observations is whether non-smooth fold bifurcations present a very relevant phenomenon, or if they are rather ‘pathological’ and may not play an important role for the description of real-world processes. However, in the case of quasiperiodic forcing, the wide-spread occurrence of non-smooth bifurcations and the related existence of SNA has been observed in a large number of numerical and experimental studies and in a variety of different contexts, ranging from classical and electronic oscillators to quantum mechanics, conceptual climate models and astrophysics (e.g. [RBO+87, DRC+89, WFP97, VLPR00, HP06, MCA15, RRM15, Zha13, LKK+15]). In addition, the simulations in Figures 2(b) and Figure 4(e)–(h) provide similar numerical evidence for the existence of nonsmooth bifurcations in the qpf Allee model (2) and (3) below. These findings are backed up by rigorous results in [Fuh14, Fuh16], showing that non-smooth fold bifurcations occur for open sets of parameter families of quasiperiodically forced scalar ODE’s. They can therefore be robust and persistent under small perturbations of the system. In the light of these results, one may say that fold bifurcations in quasiperiodically forced models may be either smooth or non-smooth, depending on the precise form of the model and the shape and strength of the forcing, and both of the cases are sufficiently widespread and persistent to be relevant in practical considerations and applications.

In the case of bounded random forcing, the situation is different in that this balance even swings completely towards the side of non-smooth bifurcations. Roughly speaking, any forcing by a sufficiently random external process inevitably leads to the non-smoothness of the bifurcation. For the case of our model system, this is established by the following result.

Theorem 2. *Suppose that (7) holds. Then any fold bifurcation that occurs in the forced Allee model (2) with random forcing term (4) at a critical parameter $\beta_c \in J(r, K, S)$ is non-smooth.*

Altogether, the possible non-smoothness of bifurcations is an issue that should arguably be dealt with if one aims at a comprehensive understanding of critical transitions.

Critical slowing down and finite-time Lyapunov exponents. The interpretation of the Lyapunov gap in a non-smooth fold bifurcation depends on the precise meaning given to the notion of recovery rates. If these are identified with the Lyapunov exponents, then it follows that, unlike in classical fold bifurcations, there are no slow recovery rates in a non-smooth fold bifurcation. However, it seems reasonable to say that the intuitive meaning of recovery rates, as used in experimental studies like [SBB+09], is better captured by the mathematical notion of *finite time Lyapunov exponents*. Instead of measuring the asymptotic stability of an orbit, these only take into account the expansion or contraction around an orbit at some finite time. Given $T > 0$, we denote the Lyapunov exponent at time T of the flow generated by (2) and starting at an initial condition $(\theta, x) \in \Theta \times \mathbb{R}$ by $\lambda_T(\theta, x)$.

In a smooth fold bifurcation, it is known that all finite time Lyapunov exponents in the basin of attraction of the stable equilibrium x_β^s will be very close to $\lambda(x_\beta^s)$, provided the time T is sufficiently large [SS00]. In contrast to this, the non-smooth case shows a characteristic spreading of these quantities, which can be observed in Figure 5.

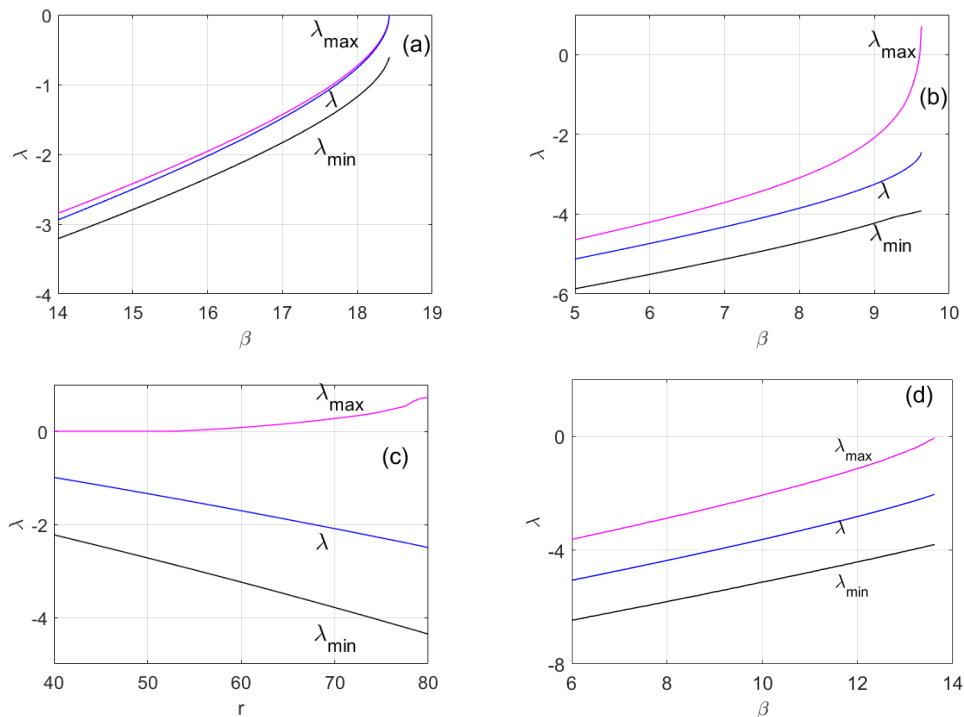


FIGURE 5. The above plots (a)-(d) show the behaviour of the finite-time Lyapunov exponents during fold bifurcations in the forced Allee model. The middle curve is always the time 2000 Lyapunov exponent (as an approximation of the asymptotic Lyapunov exponent), whereas the upper and the lower curves correspond to the maximal and minimal time $4/3$ Lyapunov exponents, respectively. (a) shows the case of a smooth fold bifurcation in the qpf Allee model with parameter values $r = 80$, $K = 10$, $S = 0.1$, $\kappa = 4$ and $q = 3$. (b) shows the case of a nonsmooth fold bifurcation in the same model with $r = 80$, $K = 10$, $S = 0.1$, $\kappa = 51.2$ and $q = 5$. (c) shows a quasiperiodic case again, but this time with the simultaneous variation of parameters as in Figure 3(b). Finally, (d) shows the case of a non-smooth fold bifurcation in the randomly forced Allee model with parameters $r = 80$, $K = 10$, $S = 0.1$ and $\kappa = 6$.

In order to translate this observation into a rigorous statement, we denote the largest time T Lyapunov exponents that is ‘observable’ on the attractor x_β^s by $\lambda_t^{\max}(x_\beta^s)$, the minimal one by $\lambda_t^{\min}(x_\beta^s)$. (We refer to Section 6 for the precise definition.) The behaviour differs according to whether the forcing is quasiperiodic or random.

Theorem 3. *Suppose that (7) holds. If the forced Allee model (2) with quasiperiodic forcing term (3) undergoes a non-smooth fold bifurcation at some critical parameter $\beta \in J(r, K, S)$,*

then we have that

$$\lim_{\beta \nearrow \beta_c} \lambda_T^{\max}(x_\beta^s) \geq \lambda(x_{\beta_c}^s) > 0, \quad (10)$$

$$\lim_{\beta \nearrow \beta_c} \lambda_T^{\min}(x_\beta^u) \leq \lambda(x_{\beta_c}^s) < 0. \quad (11)$$

In the case of random forcing (4), we have that

$$\lim_{\beta \nearrow \beta_c} \lambda_T^{\max}(x_\beta^s) \geq 0. \quad (12)$$

Both the statement and the numerical results imply that at least in theory non-smooth fold bifurcations can be anticipated and detected beforehand via a spread in the distribution of finite-time Lyapunov exponents, which reaches into the positive region. However, at the same time this highlights a variety of practical problems that may arise when trying to establish early-warning signals for forced systems. Unlike for Lyapunov exponents, which are asymptotic quantities and usually show a very uniform behaviour, the use of finite-time Lyapunov exponents requires to make a number of choices. First of all, there is the question of the time-scale (the choice of T) for which these quantities should be measured. When T is too small, it is likely that positive finite-time exponent will be observed already far from any bifurcation (depending on the geometry of the system). Conversely, if T is chosen too large, positive finite-time exponents may exist, but may only be observed with very small probabilities (thus requiring many measurements for a reliable signal). In any case, even with the right choice of time-scale and sufficient data, examples as the one in shown in Figure 5(c) will remain difficult to treat.

Distribution of finite-time Lyapunov exponents. Finally, we take a brief look at the distribution of finite-time Lyapunov exponents on different timescales, which are shown in Figure (for the qpf case, results in the random case are similar). The probability of observing exponents above or close to zero in plotted in Figure and decreases quickly. However, our simulations are somewhat inconclusive concerning the rate of decay, which seems to be somewhere between polynomial and exponential.

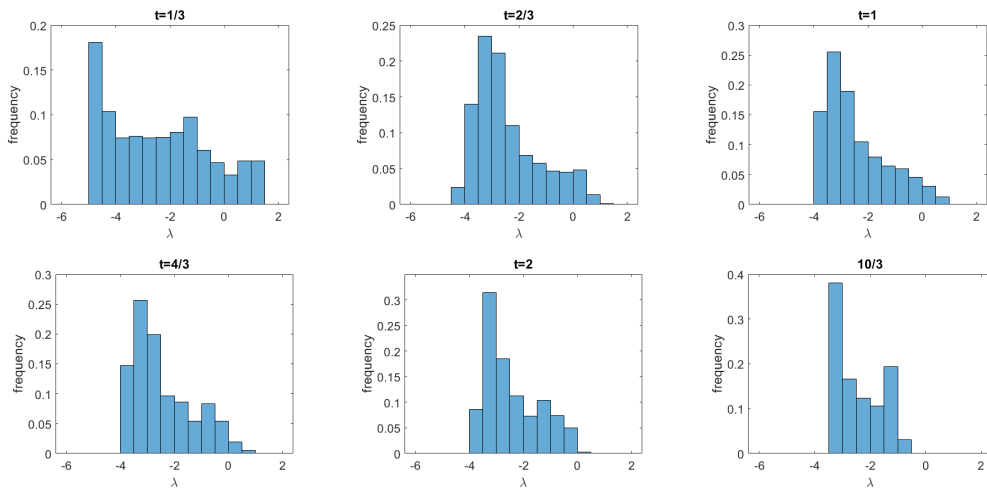


FIGURE 6. Distributions of the finite-time Lyapunov exponents in the qpf Allee model (2) with parameters $r = 80$, $K = 10$, $S = 0.1$, $\kappa =$ and $\beta =$ on different timescales, computed with sliding windows over a trajectory of length $t = 2000$.

Concluding remarks. In the above context, it should also be pointed out that although finite-time Lyapunov exponents are – by definition – observable in finite time and may therefore in principle be accessible to experimental measurements, it is a difficult task to achieve and implement this for any real-life system. At the same time, the alternative use of autocorrelation does not make sense in a forced system with moving random equilibria. Hence, the practical implementation of early-warning signals for critical transitions in forced systems remains a wide open problem, even in the simplest case of fold bifurcations.

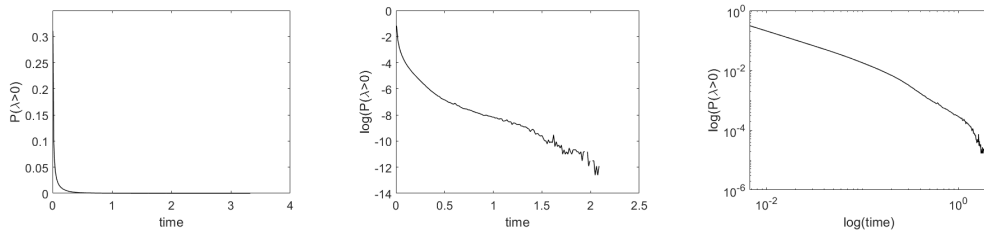


FIGURE 7. A plot of the relative frequency of positive exponents (as observed in Figure 6) on an (a) standard, (b) logarithmic and (c) log-log-scale.

On the theoretical side, an imminent problem that we tried to highlight by the above discussion is to give a precise mathematical meaning to terms like recovery rates, critical slowing down early warning signals and other notions that come up in the context of critical transitions. If theory and applications are supposed to go hand in hand, this will be an indispensable basis for further progress. The results and findings presented here should be understood as a contribution to that discussion.

Structure of the article. In Section 2, we collect the required preliminary facts concerning the mathematical theory of non-autonomous dynamics and skew product systems, with a particular emphasis on invariant graphs and fold bifurcations in this setting. The application to our forced Allee model (2) is discussed in Section 2.3. In Section 3, we also introduce some discrete-time skew product systems, which may be thought of as simplified models for the time-one maps of the skew product flow induced by the forced Allee model. Section 4 is then devoted to the discussion of non-smooth fold bifurcation and also contains the proof of (a more general version of) Theorem 2. The existence of the Lyapunov gap, stated in Theorem 1, is then proven in Section 5, which also contains a result on the slope of the Lyapunov exponents at the bifurcation point (in the setting of the discrete-time systems from Section 3). Finite-time Lyapunov exponents are then defined and discussed in Section 6, including the proof of (a more general version of) Theorem 3.

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2. PRELIMINARIES

2.1. Skew product flows and invariant graphs. In order to treat discrete-time dynamics alongside continuous-time dynamics, we refer by \mathbb{T} to the set of times which either equals \mathbb{Z} (discrete-time) or \mathbb{R} (continuous-time). In both cases, a dynamical system is a pair (Y, Ξ) of a set Y and a flow Ξ on Y , that is, a mapping

$$\Xi : \mathbb{T} \times Y \rightarrow Y \quad , \quad (t, y) \mapsto \Xi^t(y) \quad (13)$$

which satisfies the flow properties

$$\Xi^0(y) = y \quad \text{and} \quad \Xi^{t+s}(y) = \Xi^t(\Xi^s(y)) \quad . \quad (14)$$

In the discrete-time case, this implies that $\Xi^t(y) = f^t(y)$, where $f : Y \rightarrow Y$ is the bijective map given by $f(y) = \Xi^1(y)$.

We always assume that Y is equipped with a σ -algebra \mathcal{B} . A probability measure μ on Y is called Ξ -invariant if $\mu \circ \Xi^t = \mu$ for all $t \in \mathbb{T}$. The set of all μ -invariant probability measures on (Y, \mathcal{B}) is denoted by $\mathcal{M}(\Xi)$. Given $\mu \in \mathcal{M}(\Xi)$, we call the quadruple $(Y, \mathcal{B}, \mu, \Xi)$ a *measure-preserving dynamical system (mpds)*. We refer to [Arn98] and references therein for details and background.

If Y is a metric space and Ξ is continuous on the product space $\mathbb{T} \times Y$, we call the pair (Y, Ξ) a *topological dynamical system (tds)*. In this case, we throughout assume \mathcal{B} to be given by the Borel σ -algebra on Y .

Non-autonomous dynamics are often modeled by skew product systems. Given a tds (Θ, ω) or a mpds $(\Theta, \mathcal{B}, \mu, \omega)$, a *skew product flow with base Θ and phase space X* is a flow

on $Y = \Theta \times X$ of the form

$$\Xi : \mathbb{T} \times \Theta \times X \rightarrow \Theta \times X \quad , \quad (t, \theta, x) \mapsto \Xi^t(\theta, x) = (\omega^t(\theta), \xi^t(\theta, x)) . \quad (15)$$

Hence, if $\pi_\Theta : \Theta \times X$ is the canonical projection to Θ , then $\pi_\Theta \circ \Xi^t(\theta, x) = \omega^t(\theta)$. The maps $X \ni x \mapsto \xi^t(\theta, x) \in X$, with $\theta \in \Theta$ fixed, are called *fibre maps*. If X is a smooth manifold and all the fibre maps $\xi^t(\theta, \cdot)$ are r times differentiable, we call Ξ an ω -forced \mathcal{C}^r flow. If $X = \mathbb{R}$ and the fibre maps are all monotonically increasing, we say Ξ is an ω -forced *monotone flow*.

As mentioned above, in this context the notion of an equilibrium point has to be replaced by that of a ‘moving equilibrium’, to which we refer as an *invariant graph*, whose position depends on the forcing variable θ . We say a measurable function $\varphi : \Theta \rightarrow X$ is an *invariant graph* of the flow Ξ , if it satisfies the condition

$$\Xi_\theta^t(\varphi(\theta)) = \varphi(\omega^t(\theta)) \quad (16)$$

for all $\theta \in \Theta$ and $t \in \mathbb{T}$.⁴ Here, we usually do not distinguish between invariant graphs that coincide almost everywhere with respect to the given reference measure in the base, which in the qpf case is just the Lebesgue measure. Hence, whenever we speak of invariant graphs, we implicitly mean equivalence classes of functions. This is very natural when Ξ is forced by an mpds, but may become a subtle issue as soon as Θ is a metric space and topological properties of invariant graphs come into play. For instance, by saying that an invariant graph is continuous, we mean that there exists a (uniquely determined) continuous representative of the respective equivalence class. It is worth mentioning that in the case of semi-continuous graphs, there possibly exist several different semi-continuous representatives in the same equivalence class – an issue that we will come back to in Section 5 below. In the random case, we may not require that (16) is satisfied pointwise, but only almost surely. More precisely, if μ is an ω -invariant measure and (16) is satisfied μ -almost surely, then φ is called a (Ξ, μ) -invariant graph.

It turns out that there is an intimate relation between invariant graphs and the invariant ergodic measures of the system. Any (Ξ, μ) -invariant graph φ clearly defines a Ξ -invariant measure μ_φ given by

$$\mu_\varphi(A) = \mu(\{\theta \in \Theta \mid (\theta, \varphi(\theta)) \in A\}) . \quad (17)$$

A partial converse to this statement for forced monotone flows is provided by the following result, which essentially goes back to Furstenberg [Fur61] (see also [Arn98]) and highlights the significance of invariant graphs from an ergodic-theoretical viewpoint. Given $\mu \in \mathcal{M}(\omega)$, we denote by $\mathcal{M}_\mu(\Xi)$ the set of Ξ -invariant probability measures on $\Theta \times X$ which project to μ in the first coordinate.

Theorem 2.1 (see [Arn98, Theorem 1.8.4] and [Fur61, Theorem 4.1]). *Suppose Ξ is an ω -forced monotone flow, $\mu \in \mathcal{M}(\omega)$ and $\nu \in \mathcal{M}_\mu(\Xi)$. Then there exists a (Ξ, μ) -invariant graph φ such that $\nu = \mu_\varphi$.*

Hence, for monotone skew product flows there is a one-to-one correspondence between invariant ergodic measures and the invariant graphs of the system.

Similar to the autonomous case, the stability of an invariant graph φ can be characterised in terms of its *Lyapunov exponent*. This is given by

$$\lambda_\mu(\varphi) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_\Theta \log \|\partial_x \xi^t(\theta, \varphi(\theta))\| d\mu(\theta) , \quad (18)$$

where ∂_x denotes the derivative with respect to x and $\mu \in \mathcal{M}(\omega)$ is a given reference measure in the base. Note that the limit in (18) always exists due to Kingman’s Ergodic theorem [Arn98] or, in the case of forced one-dimensional flows ($X = \mathbb{R}$), by Birkhoff’s Ergodic Theorem. It is known that, under some mild assumptions, an invariant graph with negative Lyapunov exponent attracts a set of initial conditions of positive measure (with respect to the product measure $\mu \times \text{Leb}$ if $X = \mathbb{R}^d$, where Leb denotes the Lebesgue measure on \mathbb{R}^d) [Jäg03]. Hence, the graph φ is called an *attractor* in this case, and a *repeller* if $\lambda(\varphi) > 0$ [Fuh14].

⁴We use φ instead of x to denote invariant graphs from now on (unlike in the introduction) to stress the fact that these are functions.

An important notion in the context of forced systems is that of pinched sets and pinched invariant graphs [Gle02, Sta03, FJJK05, JS06, Jäg07]. Suppose that Θ is a compact metric space, $X = [a, b] \subseteq \mathbb{R}$ and $\varphi^-, \varphi^+ : \Theta \rightarrow X$. Further, assume that φ^- is lower semi-continuous and φ^+ is upper semi-continuous and $\varphi^- \leq \varphi^+$. Then φ^- and φ^+ are called *pinched* if there exists a point $\theta \in \Theta$ with $\varphi^-(\theta) = \varphi^+(\theta)$. If we only have that for any $\varepsilon > 0$ there exists θ_ε with $|\varphi^+(\theta_\varepsilon) - \varphi^-(\theta_\varepsilon)| < \varepsilon$, then we call φ^+ and φ^- *weakly pinched*. In the case of random forcing, we have the following measure-theoretic analogue. Suppose $(\Theta, \mathcal{B}, \mu)$ is a measure space, $X = [a, b] \subseteq \mathbb{R}$ and $\varphi^- \leq \varphi^+ : \Theta \rightarrow X$ are measurable. Then φ^- and φ^+ are called *measurably pinched* if the set $A_\delta := \{\theta \in \Theta \mid \varphi^+(\theta) - \varphi^-(\theta) < \delta\}$ has positive measure for all $\delta > 0$. Otherwise, we call φ^- and φ^+ μ -uniformly separated. For strictly ergodic (that is, minimal and uniquely ergodic) forcing processes, all three notions of pinching coincide, see [AJ12, Lemma 3.5]. In this case, two pinched invariant graphs always coincide on a dense subset of Θ .

2.2. Fold bifurcation scenario. With the above notions, we can now formulate the bifurcation scenario – both in a deterministic and a random setting – which will provide the general framework for all our further studies. We start with the deterministic case. Given $A \subseteq \Theta \times X$ and $\theta \in \Theta$, we let $A_\theta = \{x \in X \mid (\theta, x) \in A\}$.

Theorem 2.2 ([AJ12, Theorem 6.1]). *Let ω be a flow on a compact metric space Θ and suppose $(\Xi_\beta)_{\beta \in [0,1]}$ is a parameter family of ω -forced monotone C^2 flows. Further assume that there exist continuous functions $\gamma^-, \gamma^+ : \Theta \rightarrow X$ with $\gamma^- < \gamma^+$ such that the following conditions hold for all $\beta \in [0, 1]$, $\theta \in \Theta$ and all positive $t \in \mathbb{T}$, where applicable.*

- (i) *There exist two distinct continuous Ξ_0 -invariant graphs and no Ξ_1 -invariant graph in $\Gamma = \{(\theta, x) \mid \gamma^-(\theta) < x < \gamma^+(\theta)\}$;*
- (ii) *$\xi_\beta^t(\theta, \gamma^\pm(\theta)) \leq \gamma^\pm(\omega^t(\theta))$;*
- (iii) *the maps $(\beta, \theta, x) \mapsto \partial_x^i \xi_\beta^t(\theta, x)$ with $i = 0, 1, 2$ and $(\beta, \theta, x) \mapsto \partial_\beta \xi_\beta^t(\theta, x)$ are continuous;*
- (iv) *$\partial_x \xi_\beta^t(\theta, x) > 0 \forall x \in \Gamma_\theta$*
- (v) *$\partial_\beta \xi_\beta^t(\theta, x) < 0 \forall x \in \Gamma_\theta$*
- (vi) *$\partial_x^2 \xi_\beta^t(\theta, x) < 0 \forall x \in \Gamma_\theta$.*

Then there exists a unique critical parameter $\beta_c \in [0, 1]$ such that

- *If $\beta < \beta_c$, then there exist two continuous Ξ_β -invariant graphs $\varphi_\beta^- < \varphi_\beta^+$ in Γ . For any ω -invariant measure μ we have $\lambda_\mu(\varphi_\beta^-) > 0$ and $\lambda_\mu(\varphi_\beta^+) < 0$.*
- *If $\beta = \beta_c$, then either there exists exactly one continuous Ξ_β -invariant graph φ_β in Γ (smooth bifurcation), or there exists a pair of weakly pinched f_β -invariant graphs $\varphi_\beta^- \leq \varphi_\beta^+$ in Γ with φ_β^- lower and φ_β^+ upper semi-continuous (non-smooth bifurcation). If μ is an ω -invariant measure, then in the first case $\lambda_\mu(\varphi_\beta) = 0$. In the second case $\varphi_\beta^-(\theta) = \varphi_\beta^+(\theta)$ μ -almost surely implies $\lambda_\mu(\varphi_\beta^\pm) = 0$, whereas $\varphi_\beta^-(\theta) < \varphi_\beta^+(\theta)$ μ -almost surely implies $\lambda_\mu(\varphi_\beta^-) > 0$ and $\lambda_\mu(\varphi_\beta^+) < 0$.*
- *If $\beta > \beta_c$, then no Ξ_β -invariant graphs exist in Γ .*

Remark 2.3. (a) We note that the result in [AJ12] is stated for convex fibre maps but the above version for concave fibre maps is equivalent and discussed in [AJ12, Remark 6.2 (c)].

(b) Likewise, the statement in [AJ12] is given for the closed region $\bar{\Gamma}$ instead of the open set Γ that we use here (for convenience later on), but the proof in [AJ12] can be adjusted with minor modifications.

(c) Non-continuous invariant graphs with negative Lyapunov exponents, as they appear in non-smooth fold bifurcation of quasiperiodically forced systems, are called *strange non-chaotic attractors* (SNA) [GOPY84, Kel96, Sta03, FKP06, NO07, AJ12].

Remark 2.4. Continuous-time skew product flows are typically defined via non-autonomous ODE's of the form

$$x'(t) = F(\omega^t(\theta), x) . \quad (19)$$

In fact, (19) a priori only yields a *local* flow where trajectories may diverge and hence not be defined for all times $t \in \mathbb{R}$. As we will only deal with bounded solutions (see also Lemma 4.5),

this issue is not of further importance. We refer the interested reader to [Fuh16] for more details.

Now, in order to apply the above statements to flows defined by equations of the form (19), it is crucial that the validity of the assumptions can be read off directly from the differential equations. Fortunately, there is a rather immediate translation between the properties of parameter families of non-autonomous vector fields F_β and the relevant properties of the resulting skew product flow.

First, the curves γ^\pm can usually be chosen constant, in which case (i) may be checked by hand and (ii) follows from $F_\beta(\theta, \gamma^\pm(\theta)) < 0$ for all $\theta \in \Theta$. Second, by standard results on the regularity of the dependence of solutions of an ODE on parameters, it suffices to assume that for each $\theta \in \Theta$ the mapping $[0, 1] \times \mathbb{R} \times \mathbb{R} \ni (\beta, t, x) \mapsto F_\beta(\omega^t(\theta), x)$ is continuous, \mathcal{C}^1 with respect to β , and \mathcal{C}^2 with respect to x in order to ensure that Ξ_β is indeed \mathcal{C}^2 and continuously differentiable with respect to β . Hence, the above expressions are well-defined and (iii) is verified, too. The monotonicity in (iv) follows immediately from the uniqueness of the solutions to (19). The monotonicity condition (v) always holds if $\beta \mapsto F_\beta(\omega^t(\theta), x)$ is monotonically decreasing. Finally, the convexity of the fibre maps required in (vi) is a consequence of the concavity of F_β in the considered region. We refer to [AJ12, Fuh16] for further details, as well as to the discussion of the application to the forced Allee model in Section 2.3.

The above remarks equally apply to the following measure-theoretic version of the bifurcation scenario as long as the paths $t \mapsto \omega^t(\theta)$ are continuous for every $\theta \in \Theta$. Note that this assumption will be verified by the examples considered in this work.

Theorem 2.5 ([AJ12, Theorem 4.1]). *Let $(\Theta, \mathcal{B}, \mu, \omega)$ be a measure preserving dynamical system and suppose $(\Xi_\beta)_{\beta \in [0, 1]}$ is a parameter family of ω -forced monotone \mathcal{C}^2 flows. Further assume that there exist measurable functions $\gamma^-, \gamma^+ : \Theta \rightarrow X$ with $\gamma^- < \gamma^+$ such that the following conditions hold for all $\beta \in [0, 1]$, $\theta \in \Theta$ and positive $t \in \mathbb{T}$, where applicable.*

- (i) *There exist two μ -uniformly separated (Ξ_0, ν) -invariant graphs but no (Ξ_1, ν) -invariant graph in Γ ;*
- (ii) $\xi_\beta^t(\theta, \gamma^\pm(\theta)) \leq \gamma^\pm(\omega^t(\theta))$;
- (iii) *the maps $(\beta, t, x) \mapsto \xi_\beta^t(\theta, x)$ and $(\beta, t, x) \mapsto \partial_x \xi_\beta^t(\theta, x)$ are continuous;*
- (iv) $\partial_x \xi_{\beta, \theta}^t(x) > 0 \forall x \in \Gamma_\theta$;
- (v) *for some $t > 0$ there exist constants $C < c_1 \leq 0$ such that $C \leq \partial_\beta \xi_\beta^t(\theta, x) \leq c_1 \forall x \in \Gamma_\theta$;*
- (vi) $\partial_x^2 \xi_\beta^t(\theta, x) < 0 \forall x \in \Gamma_\theta$.
- (vii) *the function $\eta(\theta) = \sup \left\{ |\log \partial_x \xi_\beta^t(\theta, x)| \mid x \in \Gamma_\theta, \beta \in [0, 1] \right\}$ is integrable with respect to μ ;*

Then there exists a unique critical parameter $\beta_c \in [0, 1]$ such that:

- *If $\beta < \beta_c$, then there exist exactly two (Ξ_β, ν) -invariant graphs $\varphi_\beta^- < \varphi_\beta^+$ in Γ which are ν -uniformly separated and satisfy $\lambda(\varphi_\beta^-) > 0$ and $\lambda(\varphi_\beta^+) < 0$.*
- *If $\beta = \beta_c$, then either there exists exactly one (f_β, μ) -invariant graph φ_β in Γ , or there exist two measurably pinched invariant graphs $\varphi_\beta^- \leq \varphi_\beta^+$ in Γ . In the first case, $\lambda_\mu(\varphi_\beta) = 0$; in the second case, $\lambda_\mu(\varphi_\beta^-) > 0$ and $\lambda_\mu(\varphi_\beta^+) < 0$.*
- *If $\beta > \beta_c$, then no f_β -invariant graphs exist in Γ .*

In analogy to the deterministic setting, we again speak of a smooth bifurcation if there exists a unique neutral invariant graph at the bifurcation point, and of a non-smooth bifurcation if there exists an attractor-repeller pair.

2.3. Application to the forced Allee model. We now aim to verify that the forced Allee model (2) satisfies the assumptions of Theorems 2.2 and 2.5, respectively. More precisely, we specify the admissible parameter ranges stated in Remark 1.2 in the introduction and show that the respective conditions are met for all admissible parameters. As pointed out in Remark 2.4, some of the conditions in Theorems 2.2 and 2.5 follow directly from the specific form of the scalar field (2). However, it remains to specify a suitable parameter range and appropriate functions γ^\pm so that (i), (ii) and (vi) hold.

It is easy to check that the fold bifurcation of the unforced equation (1) takes place at

$$\beta = b(r, K, S) := \frac{r}{K^2} \cdot \left(\frac{K - S}{2} \right)^2. \quad (20)$$

Moreover, the neutral equilibrium point at the bifurcation is $x_0 = \frac{K+S}{2}$. If $\kappa < b(r, K, S)$ and $\beta \leq b(r, K, S) - \kappa$, then we have that $V_\beta(\theta, x_0) > 0$ for all $\theta \in \Theta$ and both forcing terms (3) and (4) (note here that $F \leq 1$). At the same time, given $\beta < b(r, K, S)$, the unforced Allee model (1) has equilibrium points $x = 0$ and

$$x_\beta^\pm = \frac{K + S}{2} \pm \frac{1}{2} \sqrt{(K - S)^2 - \frac{4\beta K^2}{r}} = \frac{K + S}{2} \pm \frac{K - S}{2} \cdot \sqrt{1 - \bar{\beta}},$$

where

$$\bar{\beta} = \frac{4\beta K^2}{r(K - S)^2}.$$

As the forcing is always downwards (recall that the forcing term reads $-\kappa F$ with $F \geq 0$), this implies in particular that $V_\beta(\theta, x_{\beta_0}^\pm) < 0$ for all $\theta \in \Theta$ and $\beta \geq \beta_0$. Hence, we obtain a forward invariant region $\Theta \times [x_0, x_{\beta_0}^+]$ and a backward invariant region $\Theta \times [x_{\beta_0}^-, x_0]$, where β_0 will be specified below. Using the concavity of V_β , equally shown below, this implies the existence of two invariant graphs in $[\gamma^-, \gamma^+] = [x_{\beta_0}^-, x_{\beta_0}^+]$. Similarly, if $\beta > b(r, K, S)$, then the bifurcation has happened and there will not be any invariant graphs above the equilibrium at 0. Hence, conditions (i) and (ii) are satisfied.

It remains to ensure the concavity of $V_\beta(\theta, \cdot)$ in the considered region $\Theta \times (\gamma^-, \gamma^+)$, where γ^\pm still need to be specified. The second derivative of $V_{\kappa, \beta}$ with respect to x is given by

$$\partial_x^2 V_{\kappa, \beta}(\theta, x) = \frac{r}{K^2} \cdot (-6x + 2(K + S)),$$

and thus independent of β and κ . We have

$$\partial_x^2 V_{\kappa, \beta}(\theta, x) < 0 \quad \Leftrightarrow \quad x > \frac{K + S}{3}.$$

Hence, we simply need to choose β_0 such that $x_{\beta_0}^- \geq \frac{K+S}{3}$. By the above, this means that we require

$$x_\beta^- = \frac{K + S}{2} - \frac{K - S}{2} \cdot \sqrt{1 - \bar{\beta}} \geq \frac{K + S}{3},$$

which is equivalent to

$$\bar{\beta} \geq 1 - \gamma(K, S),$$

where $\gamma(K, S) = \frac{1}{9} \left(\frac{K+S}{K-S} \right)^2$, and hence to

$$\beta \geq b(r, K, S) \cdot (1 - \gamma(K, S)).$$

This means that if $\kappa > 0$ satisfies

$$\kappa < b(r, K, S) \cdot \gamma(K, S)$$

and we let

$$J(r, K, S) = [b(r, K, S) \cdot (1 - \gamma(K, S)), b(r, K, S) + 1],$$

then the parameter family $(V_{\kappa, \beta})_{\beta \in J(r, K, S)}$ satisfies all the assertions of Theorem 2.2 (modulo rescaling the parameter interval $J(r, K, S)$) and therefore undergoes a non-autonomous fold bifurcation.

2.4. Forcing processes. For later use, we introduce forcing processes both discrete and continuous time. Quasiperiodic motion in discrete time is given by a rotation $\omega : \mathbb{T}^d \rightarrow \mathbb{T}^d$, $\theta \mapsto \theta + \rho \bmod 1$ which is *irrational*, in the sense that its rotation vector $\rho = (\rho_1, \dots, \rho_d)$ has incommensurate entries.⁵ In this case, the transformation ω is minimal and uniquely ergodic, with the Lebesgue measure on \mathbb{T}^d as the unique invariant probability measure. The continuous time analogue is an irrational Kronecker flow $\omega : \mathbb{R} \times \mathbb{T}^d \rightarrow \mathbb{T}^d$, $\omega^t(\theta) = \theta + t\rho$, where ρ (or some scalar multiple thereof) is again incommensurate.

In order to model random forcing in discrete time, we will simply use Bernoulli processes as examples. Hence, we let $\Sigma = \{0, 1\}^{\mathbb{Z}}$ and equip this space with the measure μ given by

⁵Here ρ_1, \dots, ρ_d are called *incommensurate* if $n_0 + \sum_{j=1}^d n_j \rho_j = 0$ implies $n_0 = n_1 = \dots = n_d = 0$.

the infinite product of that measure on $\{0, 1\}$ which assigns equal probability $1/2$ to either value. Actually, we could likewise set Σ to be $[0, 1]^{\mathbb{Z}}$ and μ to be the infinite product $\text{Leb}_{[0,1]}^{\mathbb{Z}}$ of the Lebesgue measure on $[0, 1]$, or even replace $\text{Leb}_{[0,1]}$ by any measure λ on $[0, 1]$ whose topological support is not a singleton. In any case, the dynamics on Σ are given by the shift map $\sigma : \Sigma \ni (\theta_n)_{n \in \mathbb{Z}} \mapsto (\theta_{n+1})_{n \in \mathbb{Z}}$ which is ergodic with respect to each such measure.

A slight complication occurs in the case of continuous-time random forcing. As mentioned in the introduction, we would like to use $\sin(W_t)$ as a forcing term in (2). Hence, it is natural to consider the Wiener space, that is, the space of continuous real-valued functions $\mathcal{C}(\mathbb{R}, \mathbb{R})$ equipped with the Borel σ -algebra generated by uniform topology and the classical Wiener measure \mathbb{P} . However, in order to obtain a skew product flow we need a measure-preserving transformation on our probability space. If $\theta \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is a path of Brownian motion, the standard measure-preserving shift on Wiener space is given by $\omega^t(\theta)(s) = \theta(s+t) - \theta(t)$. The problem that occurs is the fact that if we now want to define a forcing term f on $\mathcal{C}(\mathbb{R}, \mathbb{R})$ by evaluating the sinus at $\theta(0)$, that is, $f(\theta) = \sin(2\pi\theta(0))$, then $f(\omega^t(\theta)) = 0$ for all $t \in \mathbb{R}$ (the standard Brownian motion starts in zero, and the classical shift respects this property). We therefore use a slightly modified version of this process to model bounded random forcing for our purposes. To that end, we let $p : \mathcal{C}(\mathbb{R}, \mathbb{R}) \rightarrow \mathcal{C}(\mathbb{R}, \mathbb{T}^1) = \Theta$ be the projection of real-valued to circle-valued functions (induced by the canonical projection $\pi : \mathbb{R} \rightarrow \mathbb{T}^1$) and let $\mathbb{P}_0 = p_*\mathbb{P}$ be the push-forward of \mathbb{P} . Further, we let $S : \mathbb{T}^1 \times \Theta \rightarrow \Theta$, $(x, \theta) \mapsto \theta + x$ and equip Θ with the measure $\nu = S_*(\text{Leb}_{\mathbb{T}^1} \times \mathbb{P}_0)$. By definition, ν has equidistributed marginals and can therefore be seen to be invariant under the shift $\omega : \mathbb{R} \times \Theta \rightarrow \Theta$ defined by $\omega^t(\theta)(s) = \theta(t+s)$. This construction will allow us to define a forcing term simply by evaluating the sinus (viewed as a function on \mathbb{T}^1) at $\theta(0)$.

3. A SIMPLIFIED DISCRETE-TIME MODEL

As a basic model for the discrete-time case, we will consider the parameter families of skew product maps

$$f_\beta : \Theta \times \mathbb{R} \rightarrow \Theta \times \mathbb{R} \quad , \quad (\theta, x) \mapsto (\omega(\theta), \arctan(\alpha x) - \kappa \cdot F(\theta) - \beta) \quad (21)$$

with real parameters $\alpha > \pi/2$, $\kappa \in (0, 1)$ and $\beta \in [0, 1]$ and forcing processes given either by $\Theta = \mathbb{T}^d$ and a rotation $\omega : \theta \mapsto \theta + \rho$ with rotation vector $\rho \in \mathbb{T}^d$ (quasiperiodic forcing) or by $\Theta = \Sigma$, where $\Sigma = \{0, 1\}^{\mathbb{Z}}$ and ω coincides with the shift σ on Σ (random forcing), all as in Section 2.4 above. For the forcing function F , we use

$$F(\theta) = \frac{\sin(2\pi\theta) + 1}{2} \quad (22)$$

in the qpf case and

$$F(\theta) = \theta_0 \quad (23)$$

in the random case.

The behaviour of the attractors and repellers during the smooth and non-smooth bifurcations in the qpf case are shown in Figure 9. This figure also illustrates some key features of non-smooth bifurcations in qpf systems and allows us to give a heuristic description of the mechanism that causes the non-smoothness. The rigorous description of this mechanism is the basis for the mathematical analysis of non-smooth bifurcations in [Jäg09, Fuh14]. As can be seen in Figure 9, when the attracting and repelling graphs approach each other in a non-smooth way, they develop a sequence of ‘peaks’. These appear in an ordered way, and the next peak is always the image of the previous one and is generated as soon as the latter reaches into the region with large derivatives, which is centred around the 0-line $\mathbb{T}^1 \times \{0\}$. The first peak is located around the minimum of the blue curve in (d). The second peak emerges in (e) and is fully developed in (f), where a number of further peaks can be seen as well. Thereby, the movement of each peak is amplified by the large derivatives close to zero (of magnitude α , see (21)). For this reason, as β is increased, the speed by which the peaks move as β is varied increases exponentially with the order of the peak, whereas its width decreases exponentially (since each peak is stretched vertically due to the expansion around 0). In the limit, the two curves touch each other with the tips of the peaks. Note that only a finite number of peaks can be observed at the bifurcation point in (f), since these quickly

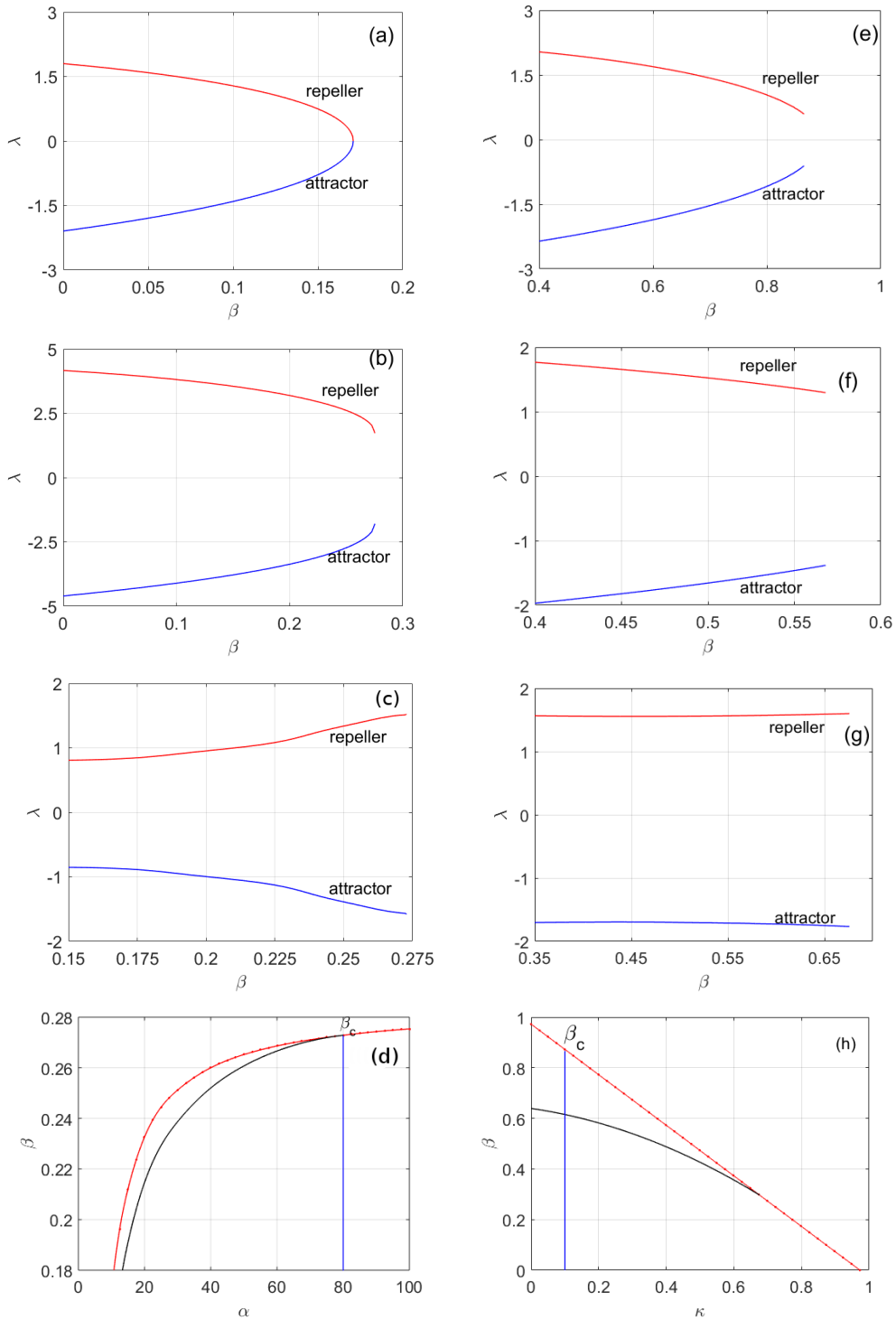


FIGURE 8. Lyapunov exponents during saddle-node bifurcations in the family (21). (a) smooth bifurcation in the qpf case, with parameters $\alpha = 10$ and $\kappa = 1$. The bifurcation occurs at $\beta_c = 0.341502$. (b) a non-smooth bifurcation in the same model, with parameters $\alpha = 100$ and $\kappa = 1$. The bifurcation occurs at $\beta_c = 0.5507468$. (c) non-smooth bifurcation with simultaneous variation of parameters $\alpha =$ and $\kappa =$ along the black curve in (d). (e) and (f) Lyapunov exponents in the randomly forced case, with parameters $\alpha = 10$ and $\kappa = 0.1$ and bifurcation parameter $\beta = 0.866$ in (e) and $\alpha = 10$ and $\kappa = 0.4$ and bifurcation parameter $\beta = 0.566$ in (f). In (g), the parameters κ and β are varied again at the same time along the black parameter curve shown in (h).

become too thin to be visible in numerical simulations. However, it is known that the region between the two graphs in (f) is actually filled densely by further peaks [GJ13, FGJ14]. We refer to the introduction of [Jäg09] for a more detailed discussion.

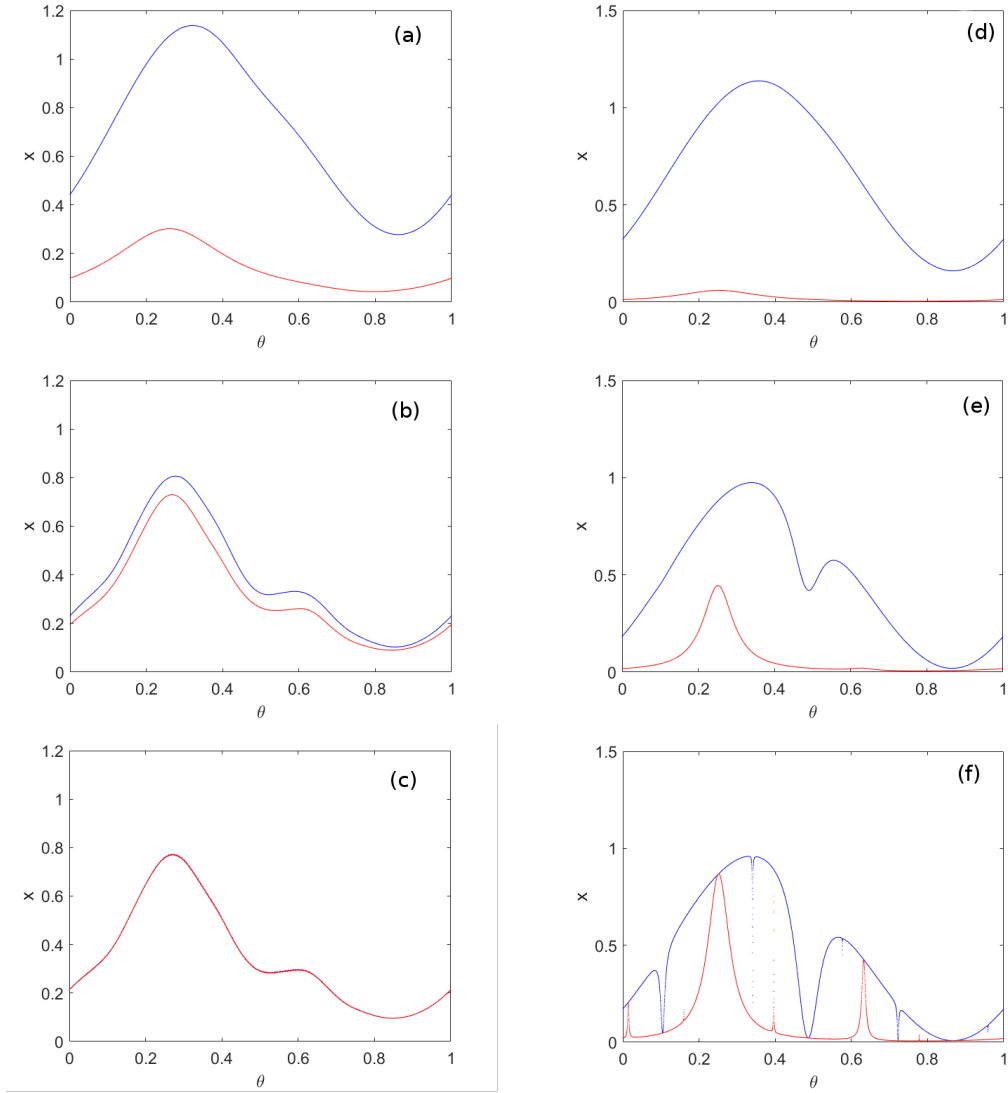


FIGURE 9. **(a)–(c)**: Smooth saddle-node bifurcation in (21) with quasiperiodic forcing. Parameter values are $\alpha = 10$, $\kappa = 1$, $\rho = \omega$ (golden mean) and (a) $\beta = 0.2$ (b) $\beta = 0.34$ and (c) $\beta = 0.341502$.
(d)–(f): Non-smooth saddle-node bifurcation in (21) with quasiperiodic forcing. Parameter values are $\alpha = 100$, $\kappa = 1$, $\rho = \omega$ and (a) $\beta = 0.2$ (b) $\beta = 0.6$ and (c) $\beta = 0.0866$.

Finally, the range of the finite time Lyapunov exponents for the same parameter families as in Figure 8 is shown in Figure 10.

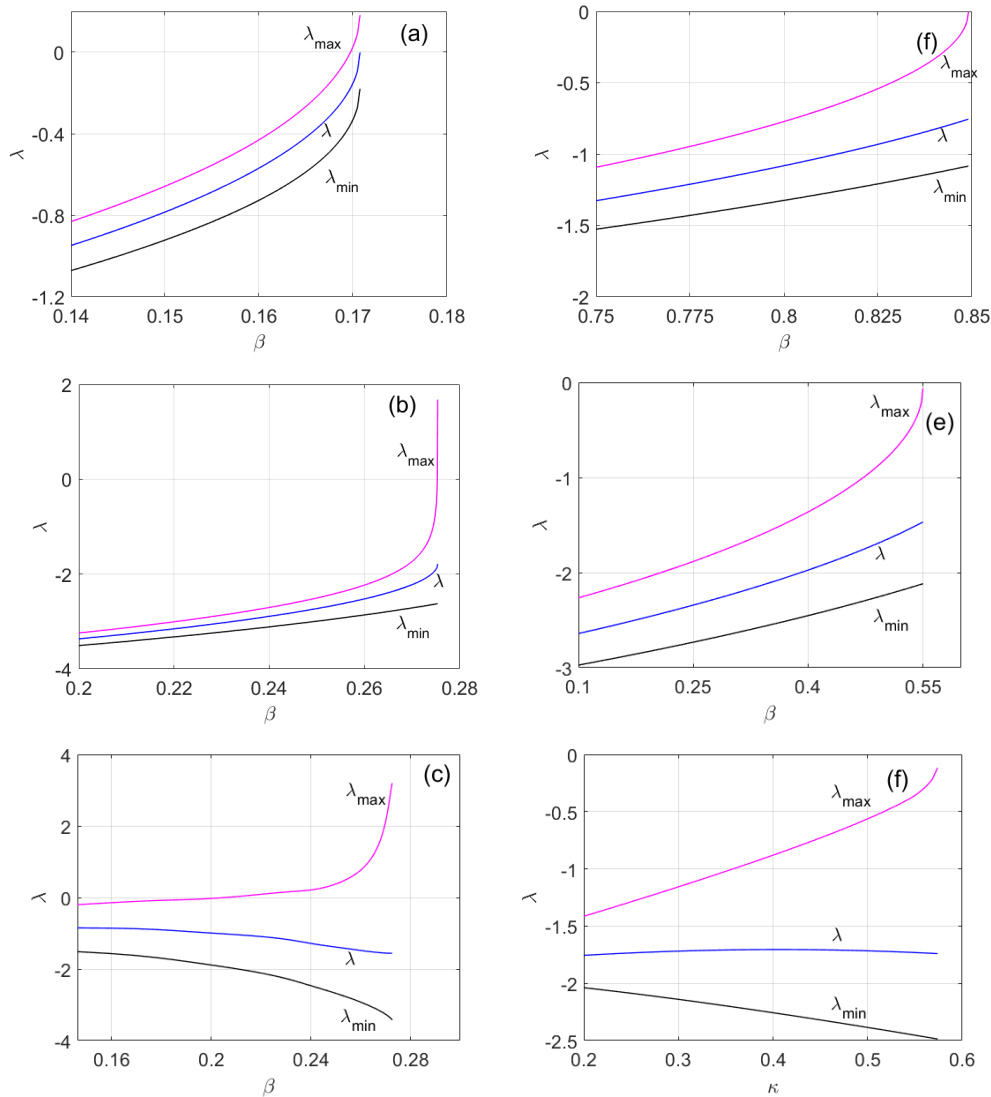


FIGURE 10. Minimal and maximal finite-time Lyapunov exponents (with the asymptotic one plotted in the middle) for the parameter families used in Figure 8 (in the same order). The time is $n = 5$ in all cases.

4. ABUNDANCE OF NONSMOOTH FOLD BIFURCATIONS

4.1. Quasiperiodic forcing. The results discussed in the previous section provide a general setting for non-autonomous saddle-node bifurcations. We shall now take a closer look at non-smooth bifurcations and discuss their widespread occurrence in forced systems. In the case of quasiperiodic forcing, the latter is well-established by a number of rigorous results both in the discrete- and continuous-time case. Thereby, arithmetic properties of the rotation numbers or vectors in the base play a crucial role. Given $\tau, \kappa > 0$, we say $\rho \in \mathbb{T}^d$ is *Diophantine (of type (τ, κ))* if

$$\forall k \in \mathbb{Z}^d \setminus \{0\} : \inf_{p \in \mathbb{Z}} \left| p + \sum_{i=1}^d \rho_i k_i \right| \geq \tau |k|^{-\kappa}.$$

Generalising the example (21), consider discrete-time flows given by quasiperiodically forced monotone interval maps of the form

$$f : \Theta \times \mathbb{R} \rightarrow \Theta \times \mathbb{R}, \quad (\theta, x) \mapsto (\omega(\theta), f_\theta(x)), \quad (24)$$

where $\Theta = \mathbb{T}^d$, $\omega : \Theta \rightarrow \Theta, \theta \mapsto \theta + \rho$ is again an irrational rotation with rotation vector ρ and $f_\theta(\cdot)$ is \mathcal{C}^2 and strictly increasing on X . For a given rotation vector $\rho \in \mathbb{T}^d$, we further consider the space of one-parameter families

$$\mathcal{F}_\rho = \{(f_\beta)_{\beta \in [0,1]} : f_\beta \text{ is of form (24) and } (\beta, \theta, x) \mapsto f_{\beta,\theta}(x) \text{ is } \mathcal{C}^2\}$$

equipped with the metric

$$d((f_\beta)_{\beta \in [0,1]}, (g_\beta)_{\beta \in [0,1]}) = \sup_{\beta \in [0,1]} (\|f_\beta - g_\beta\|_2 + \|\partial_\beta f_\beta - \partial_\beta g_\beta\|_0).$$

Then the following result is established in [Fuh14], with precursors in [Bje05, Jäg09].

Theorem 4.1 ([Fuh14]). *Suppose that $\rho \in \mathbb{T}^d$ is Diophantine. Then there exists a non-empty open set $\mathcal{U} \subseteq \mathcal{F}_\omega$ such that each $(f_\beta)_{\beta \in [0,1]} \in \mathcal{U}$ satisfies the assertions of Theorem 2.2 and undergoes a non-smooth saddle-node bifurcation.*

This confirms that the set of parameter families with a non-smooth saddle-node bifurcation is large in a certain sense, and that the phenomenon can occur in a robust way (both corresponding to the openness of the set \mathcal{U}). Thereby, it is important to note that in [Fuh14] the set \mathcal{U} in this result is characterised by explicit \mathcal{C}^2 -estimates. This makes it possible to check if it contains a given parameter family and therefore provides explicit examples of non-smooth saddle-node bifurcations.

Corollary 4.1.1 ([Fuh14]). *If ρ is Diophantine and α is sufficiently large, then the parameter family $(f_\beta)_{\beta \in [0,1]}$ defined in (21) belongs to the set \mathcal{U} and hence undergoes a non-smooth saddle-node bifurcation.*

Continuous-time analogues of these results have been established in [Fuh16]. In this case, one considers non-autonomous vector fields, given by differentiable functions of the form

$$V : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}. \quad (25)$$

which induce quasiperiodically forced flows via the corresponding differential equation

$$x'(t) = V(\omega_t(\theta_0), x(t)), \quad (26)$$

where $\omega : \mathbb{R} \times \mathbb{T}^d \rightarrow \mathbb{T}^d$, $(t, \theta) \mapsto \theta + t\rho$ is an irrational Kronecker flow with rotation vector $\rho \in \mathbb{T}^d$, as described above. We let

$$\mathcal{V} = \{(V_\beta)_{\beta \in [0,1]} \mid V \text{ is of the form (25) and } (\beta, \theta, x) \mapsto V_\beta(\theta, x) \text{ is } \mathcal{C}^2\} \quad (27)$$

and equip \mathcal{V} with the metric

$$d((V_\beta)_{\beta \in [0,1]}, (W_\beta)_{\beta \in [0,1]}) = \sup_{\beta \in [0,1]} (\|V_\beta - W_\beta\|_2 + \|\partial_\beta V_\beta - \partial_\beta W_\beta\|_0).$$

Theorem 4.2 ([Fuh16]). *For any Diophantine $\rho \in \mathbb{T}^d$, there exists an open set $\mathcal{U}_\rho \subseteq \mathcal{V}$ such that for any $(V_\beta)_{\beta \in [0,1]} \in \mathcal{V}$ the flow induced by (25) satisfies the assertions of Theorem 2.2 and undergoes a non-smooth fold bifurcation.*

Again, the explicit characterisation of the set \mathcal{U}_ρ given in [Fuh16] makes it in principle possible to check for non-smooth bifurcations in specific examples. However, this is considerably more technical than in the discrete-time case. Moreover, the application to the forced Allee model (2) with quasiperiodic forcing (3) would require a number of highly non-trivial and technical modifications, so that we refrain from going into details here. However, Figures 2(b) and 4(e)-(h) provide numerical evidence for the occurrence of non-smooth fold bifurcations in this case.

4.2. Random forcing. In contrast to the quasiperiodic case, the influence of bounded random noise on saddle-node bifurcations has not been studied systematically so far. Our aim for the remainder of this section is to establish the occurrence of non-smooth bifurcations in a broad class of randomly forced monotone flows and maps. To that end, we need the notion of an autonomous reference system. Let $\gamma^- < \gamma^+ \in \mathbb{R}$ and suppose $(g_\beta)_{\beta \in [0,1]}$ is a one-parameter family of differentiable flows $g_\beta : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ with the following properties (which are supposed to hold for all $\beta \in [0,1], t \in \mathbb{T}$ and $x \in \mathbb{R}$, where applicable).

- (g1) g_0 has two fixed points in the interval $[\gamma^-, \gamma^+]$, whereas g_1 has none;
- (g2) $g_\beta^t(\gamma^\pm) \leq \gamma^\pm$;

- (g3) $\partial_x g_\beta^t(x) > 0$;
- (g4) the mapping $(\beta, t, x) \mapsto g_\beta^t(x)$ is continuous;
- (g5) the mapping $\beta \mapsto g_\beta^t(x)$ is differentiable and $\partial_\beta g_\beta^t(x) < 0$;
- (g6) $\partial_x^2 g_\beta^t(x) < 0$ for all $x \in [\gamma^-, \gamma^+]$ (convexity).

We call such a family $(g_\beta)_{\beta \in [0,1]}$ an (autonomous) reference family.

- Remark 4.3.** (a) Properties (g1)–(g6) imply that the family $(g_\beta)_{\beta \in [0,1]}$ undergoes a fold bifurcation in the interval $[\gamma^-, \gamma^+]$: Due to the concavity in (g6), g_β can have at most two fixed points in this region, with the upper one attracting and the lower one repelling. By (g1), the map g_0 has two such fixed points. Due to the monotone dependence on the parameter assumed in (g5), these two fixed points have to move towards each other as β is increased. They have to vanish before $\beta = 1$, as g_1 has no fixed points, and the only possibility to do so is to collide at a unique bifurcation parameter β_c .
- (b) If $\mathbb{T} = \mathbb{Z}$ (that is, in the discrete time case), the simplest way to obtain a reference family of this kind is to fix some strictly increasing map $g : \mathbb{R} \rightarrow \mathbb{R}$ such that g maps the points γ^\pm below themselves, is strictly concave on $[\gamma^-, \gamma^+]$ and has two fixed points in $[\gamma^-, \gamma^+]$, but $g - 1$ does not have any fixed points in this interval. Then $g_\beta = g - \beta$ satisfies the above properties.

Our main result of this section now states that under some mild conditions, any random perturbation of such a reference family will undergo a non-smooth fold bifurcation.

Theorem 4.4. *Suppose that $(\Theta, \mathcal{B}, \nu, \omega)$ is an mpds and $(\Xi_\beta)_{\beta \in [0,1]}$ is a parameter family of ω -forced flows that satisfies the assumptions of Theorem 2.5, with constant curves γ^\pm . Further, assume that $(g_\beta)_{\beta \in [0,1]}$ is an autonomous reference family such that the following conditions hold.*

- (i) For all $(\theta, x) \in \Theta \times X$ and $t > 0$ we have $g_\beta^t(x) \leq \xi_\beta^t(\theta, x)$. (Lower bound)
- (ii) For all $\varepsilon, T > 0$ there exists a set $A_{\varepsilon, T} \subseteq \Theta$ of positive measure $\nu(A_{\varepsilon, T}) > 0$ such that

$$|\xi_\beta^t(\theta, x) - g_\beta(x)| \leq \varepsilon \quad (28)$$

for all $\theta \in A_{\varepsilon, T}$, $|t| \leq T$ and $x \in [\gamma^-, \gamma^+]$. (Shadowing)

- (iii) For ν -almost every $\theta \in \Theta$ there exists $t \in \mathbb{T}$ and $\delta > 0$ such that $\xi^t(\omega^{-t}(\theta), x) \geq g^t(x) + \delta$ for all $x \in [\gamma^-, \gamma^+]$. (Separation)

Then $(\Xi_\beta)_{\beta \in [0,1]}$ undergoes a non-smooth fold bifurcation, and the bifurcation parameter β_c is the same as in the reference family $(g_\beta)_{\beta \in [0,1]}$.

In order to prove this result, we first provide the following auxiliary statement about the equivalence of the existence of invariant graphs and the existence of orbits that remain bounded in the region $\Gamma = \Theta \times [-\gamma, \gamma]$ at all times.

Lemma 4.5. *Suppose Ξ is a monotone skew product flow of the form (5) with an mpds $(\Omega, \mathcal{B}, \nu, \omega)$ in the base. Further, assume that there exist measurable curves $\gamma^- \leq \gamma^+ : \Omega \rightarrow X$ that satisfy*

$$\xi^t(\theta, \gamma^\pm(\theta)) \leq \gamma^\pm(\omega(\theta)) \quad (29)$$

for ν -almost every $\theta \in \Theta$. Let

$$\Gamma = \{(\theta, x) \mid \theta \in \Theta, \gamma^-(\theta) \leq x \leq \gamma^+(\theta)\}. \quad (30)$$

Then there exists a (Ξ, ν) -invariant graph φ in Γ if and only if

$$\xi^t(\theta, \gamma^+(\theta)) \geq \gamma^-(\omega^t(\theta)) \quad (31)$$

holds for all $t \geq 0$ and ν -almost every $\theta \in \Omega$.

Proof. An important ingredient for the proof are the graph transforms $\Xi_*^t \gamma$ of a measurable function $\gamma : \Theta \rightarrow X$, which are defined for any $t \in \mathbb{T}$ by

$$\Xi_*^t \gamma(\theta) = \xi^t(\omega^{-t}(\theta), \gamma(\omega^{-t}(\theta))). \quad (32)$$

If $t \geq 0$, one usually speaks of a *forwards transform* and if $t \leq 0$, of a *backwards transform*. We define

$$\gamma_t^+ = \Xi_*^t \gamma^+ \text{ and } \gamma_t^- = \Xi_*^{-t} \gamma^-. \quad (33)$$

Then (29) together with the monotonicity of the fibre maps implies that the family of functions γ_t^+ is decreasing in t . Similarly, the family γ_t^- is increasing (note here that $\xi^t(\theta, \gamma^-(\theta)) \leq \gamma^-(\theta)$ for $t > 0$ implies $\xi^t(\theta, \gamma^-(\theta)) > \gamma^-(\theta)$ for $t < 0$).

Suppose now that (31) holds for all $t > 0$ and ν -almost every $\theta \in \Theta$. Then γ_t^+ is bounded below by γ^- for all $t > 0$ and thus converges ν -almost everywhere to a function

$$\varphi^+(\theta) = \lim_{t \rightarrow \infty} \gamma_t^+(\theta). \quad (34)$$

Due to the continuity of the fibre maps, we have that

$$\begin{aligned} \xi^s(\theta, \varphi^+(\theta)) &= \xi^s\left(\theta, \lim_{t \rightarrow \infty} \gamma_t^+(\theta)\right) = \lim_{t \rightarrow \infty} \xi^s(\theta, \gamma_t^+(\theta)) \\ &= \lim_{t \rightarrow \infty} \gamma_{t+s}^+(\omega^s(\theta)) = \varphi^+(\omega^s(\theta)) \end{aligned} \quad (35)$$

ν -almost everywhere so that φ^+ is the required invariant graph.

Conversely, assume that there exists an invariant graph φ in Γ . Then the monotonicity of the fibre maps gives

$$\xi^t(\theta, \gamma^+(\theta)) \geq \xi^t(\theta, \varphi(\theta)) = \varphi(\omega^t(\theta)) \geq \gamma^-(\omega^t(\theta)) \quad (36)$$

for ν -almost every $\theta \in \Theta$. \square

Remark 4.6. As we have seen in the proof above, if Ξ satisfies the assumptions of Lemma 4.5 and has at least one invariant graph, then the formula

$$\varphi^+(\theta) = \lim_{t \rightarrow \infty} \gamma_t^+(\theta) = \lim_{t \rightarrow \infty} \xi^t(\omega^{-t}(\theta), \gamma^+) \quad (37)$$

defines one such graph. This way of defining an invariant graph is called a *pullback construction* and generally works if the graph is an attractor. In a similar fashion, it is possible to show that another invariant graph may be defined by a *pushforward construction*

$$\varphi^-(\theta) = \lim_{t \rightarrow -\infty} \gamma_t^-(\theta) = \lim_{t \rightarrow \infty} \xi^{-t}(\omega^t(\theta), \gamma^-), \quad (38)$$

which usually yields a repeller. It is possible, however, that both graphs φ^- and φ^+ coincide, as in the case of a smooth fold bifurcation in Theorem 2.5.

We can now turn to the

Proof of Theorem 4.4. Let β_c be the bifurcation parameter for the family $(\Xi_\beta)_{\beta \in [0,1]}$ and $\tilde{\beta}_c$ the one for the reference family $(g_\beta)_{\beta \in [0,1]}$. Then $g_{\tilde{\beta}_c}$ has a unique fixed point $x_0 \in [\gamma^-, \gamma^+]$. Now, for any $\theta \in \Theta$ and $\beta < \tilde{\beta}_c$, we obtain

$$\xi_\beta^t(\theta, \gamma^+) \geq \xi_{\tilde{\beta}_c}^t(\theta, \gamma^+) \geq g_{\tilde{\beta}_c}^t(\gamma^+) \geq g_{\tilde{\beta}_c}^t(x_0) = x_0 > \gamma^-.$$

Hence, Lemma 4.5 implies that Ξ_β has at least one invariant graph in Γ for all $\beta \leq \tilde{\beta}_c$, and thus $\beta_c \geq \tilde{\beta}_c$.

Conversely, suppose that $\beta > \tilde{\beta}_c$. As g_β has no fixed points in $[\gamma^-, \gamma^+]$ anymore and $g_\beta(\gamma^+) < \gamma^+$, we obtain that $g_\beta^T(\gamma^+) < \gamma^-$ for some $T > 0$. Let $\varepsilon > 0$ be such that $g_\beta^T(\gamma^+) < \gamma^- - \varepsilon$. By assumption, the set $A_{\varepsilon, T}$ in the statement of the theorem has positive measure. For any $\theta \in A_{\varepsilon, T}$, we obtain

$$\xi^T(\theta, \gamma^+) \leq g_\beta^T(\gamma^+) + \varepsilon < \gamma^-.$$

Due to Lemma 4.5, this excludes the existence of an invariant graph in Γ for $\beta > \beta_c$. We therefore obtain $\beta_c \leq \tilde{\beta}_c$ and conclude $\beta_c = \tilde{\beta}_c$.

It remains to show the non-smoothness of the bifurcation. To that end, note that condition (i) in Theorem 4.4 together with the monotonicity implies $\xi_{\tilde{\beta}_c}^t(\theta, x) \leq g_{\tilde{\beta}_c}^t(x)$ for all $x \in [\gamma^-, \gamma^+]$ and $t < 0$. As a consequence, we have that the limit

$$\varphi^-(\theta) = \lim_{t \rightarrow \infty} \xi^{-t}(\omega^t(\theta), \gamma^-) \leq x_0$$

exists (note that condition (ii) in Theorem 2.5 together with the monotonicity of the fibre maps implies that $t \mapsto \xi^{-t}(\omega^t(\theta), \gamma^-)$ is non-decreasing so that the convergence is monotone) and defines an invariant graph of Ξ_{β_c} (compare Remark 4.6). Likewise,

$$\varphi^+(\theta) = \lim_{t \rightarrow \infty} \xi_{\beta_c}^t(\omega^{-t}(\theta), \gamma^+)$$

defines an invariant graph of Ξ_{β_c} . Hence, to finish the proof, it suffices to show that $\varphi^-(\theta) < \varphi^+(\theta)$ ν -almost surely.

Now, by assumption (iii) we have that for ν -almost every $\theta \in \Theta$ there exists $\delta > 0$ and $s \in \mathbb{T}$ such that $\xi_{\beta_c}^s(\omega^{-s}(\theta), x_0) > g_{\beta_c}^s(x_0) + \delta = x_0 + \delta$. Thus, we obtain

$$\begin{aligned} \varphi^+(\theta) &= \lim_{t \rightarrow \infty} \xi_{\beta_c}^t(\omega^{-t}(\theta), \gamma^+) \\ &= \lim_{t \rightarrow \infty} \xi_{\beta_c}^s(\omega^{-s}(\theta), \xi_{\beta_c}^{t-s}(\omega^{-t}(\theta), \gamma^+)) \\ &\geq \xi_{\beta_c}^s(\omega^{-s}(\theta), x_0) \geq x_0 + \delta \end{aligned}$$

and hence, in particular, $\varphi^+(\theta) > x_0 \geq \varphi^-(\theta)$. This implies that Ξ_{β_c} has two distinct invariant graphs. We conclude that the bifurcation is non-smooth. \square

5. LYAPUNOV EXPONENTS IN NONSMOOTH FOLD BIFURCATIONS

5.1. Lyapunov gap in nonsmooth fold bifurcations. The aim of this section is to provide a proof of Theorem 1, which we restate here in a more general form.

Theorem 5.1. *Suppose that $(\Xi_\beta)_{\beta \in [0,1]}$ is a parameter family of forced monotone \mathcal{C}^2 -flows that satisfies the assumptions of Theorem 2.2 (for deterministic forcing) or Theorem 2.5 (for random forcing). Further, assume that the fold bifurcation that occurs in this family at the critical parameter β_c is non-smooth. Then*

$$\lim_{\beta \nearrow \beta_c} \lambda(\varphi_\beta^+) = \lambda(\varphi_{\beta_c}^+) < 0. \quad (39)$$

If the fold bifurcation is smooth, then $\lim_{\beta \nearrow \beta_c} \lambda(\varphi_\beta^+) = 0$. The analogous results hold for the unstable equilibrium φ_β^- .

Proof. We only consider the deterministic case. The random case can be dealt with similarly. Let $(\Xi_\beta)_{\beta \in [0,1]}$ satisfy the assumptions of Theorem 2.2. We claim that for each $\theta \in \Theta$ we have that $\varphi_{\beta_c}^+(\theta)$ coincides with

$$\varphi(\theta) = \lim_{\beta \nearrow \beta_c} \varphi_\beta^+(\theta).$$

Note that φ is well-defined due to the monotone dependence of ξ_β^t on β (see assumption (v) in Theorem 2.2) which results in $\varphi_\beta \geq \varphi_{\beta'}$ whenever $\beta < \beta' \leq \beta_c$.

In order to see that $\varphi(\theta) = \varphi_{\beta_c}^+(\theta)$, fix $\theta \in \Theta$, $t > 0$ and $\varepsilon > 0$. Choose $\delta > 0$ such that $|\xi_{\beta_c}^t(\varphi(\theta)) - \xi_{\beta_c}^t(x)| < \varepsilon$ for all $x \in B_\delta(\varphi(\theta))$ and at the same time $|\xi_\beta^t(x) - \xi_{\beta'}^t(x)| < \varepsilon$ whenever $|\beta - \beta'| < \delta$. Note that such δ exists due to the uniform continuity of $(\beta, x) \mapsto \xi_\beta^t(\theta, x)$. Let $\beta < \beta_c^+$ be such that $\beta_c^+ - \beta < \delta$, $\varphi_\beta^+(\theta) - \varphi(\theta) < \delta$ and $\varphi_\beta^+(\omega^t(\theta)) - \varphi(\omega^t(\theta)) < \varepsilon$. We obtain

$$\begin{aligned} |\xi_{\beta_c}^t(\varphi(\theta)) - \varphi(\omega^t(\theta))| &\leq |\xi_\beta^t(\varphi(\theta)) - \varphi_\beta^+(\omega^t(\theta))| + 2\varepsilon \\ &= |\xi_\beta^t(\varphi(\theta)) - \xi_\beta^t(\varphi_\beta^+(\theta))| + 2\varepsilon \leq 3\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, this proves $\xi_{\beta_c}^t(\varphi(\theta)) = \varphi(\omega^t(\theta))$ and hence the invariance of φ under Ξ_{β_c} . Now, since the graphs φ_β^+ are monotonically decreasing in β , we have $\varphi \geq \varphi_{\beta_c}^+$. As there is no Ξ_{β_c} -invariant graph above $\varphi_{\beta_c}^+$ in the considered region Γ , we obtain $\varphi_{\beta_c}^+ = \varphi$. Using dominated convergence, this proves the statement. \square

5.2. Slope at the bifurcation point. Although this is not in our main focus, we want to comment in this section on a particular qualitative difference between non-smooth fold bifurcations in the quasiperiodically forced and the randomly forced case. As it can be seen from Figure 2(b)–(c), the slope of the Lyapunov exponents of the attractors increases more strongly towards the bifurcation in the quasiperiodically forced case, whereas it only increases slightly (Figure 2(c)) or even remains constant (Figure 2(d)) in the random case.

In fact, the heuristic description of non-smooth fold bifurcations in qpf system given in Section 3 suggests that $\partial_\beta \lambda(\varphi_\beta^+)$ should actually increase to infinity as $\beta \nearrow \beta_c^+$. The reason is that due to the concavity of the right side of the vector field, the Lyapunov exponent increases whenever the graph decreases. Thereby, the quantitative contribution of each peak that develops should be the product between its width and its speed, which is more or less constant since both decrease, respectively increase, with the same exponential rate. Hence, every peak contributes a similar amount to the slope of the Lyapunov exponent, and as there are infinitely many peaks, this slope grows to infinity as the bifurcation is approached. In principle, we believe that this heuristic explanation could be made precise by using the machinery for the proof of non-smooth fold bifurcations in [Fuh14, Fuh16] which, however, goes beyond our current scope.

For the case of random forcing, we provide a proof for the boundedness of the slope of the Lyapunov exponent of φ_β^+ as $\beta \nearrow \beta_c$. In order to avoid too many technicalities and to not obstruct the view on the underlying mechanism, we restrict to the case of the discrete-time example (21). We note, however, that the proof can be generalised to broader classes of monotone skew product maps and, with some more work required, to continuous-time systems.

Theorem 5.2. *Suppose $(f_\beta)_{\beta \in [0,1]}$ is the family of skew product maps given by (21) with $\Theta = \{0,1\}^{\mathbb{Z}}$ the Bernoulli space equipped with the shift map σ and the Bernoulli measure μ . Let $\beta_c = \frac{\sqrt{2\alpha-\pi}}{\alpha\sqrt{\pi}}$. Then $(f_\beta)_{\beta \in [0,1]}$ satisfies the hypothesis of Theorem 2.5 (with $\gamma^- = 0$ and $\gamma^+ = 1$) and undergoes a non-smooth saddle-node bifurcation with critical parameter $\beta_c \in [0,1]$. Moreover, there exists a constant $C > 0$ such that*

$$|\partial_\beta \lambda_\mu(\varphi_\beta^\pm)| \leq C$$

for all $\beta \in [0, \beta_c]$.

Before we turn to the proof of the Theorem 5.2, we first need the following preliminary result.

Lemma 5.3. *In the situation of Theorem 5.2 we let $\gamma_{n,\beta}^\pm = f_{\beta_*}^{\pm n} \gamma^\pm$, where the graph transform $f_{\beta_*}^n$ is defined as in the proof of Theorem 4.5. Then $\gamma_{n,\beta}^\pm(\theta)$ converges to $\varphi_\beta^\pm(\theta)$ uniformly in β and θ (with $\beta \in [0, \beta_c]$ and $\theta \in \Sigma$) as $n \rightarrow \infty$.*

Moreover, for all $\theta \in \Sigma$, the map $[0, \beta_c) \ni \beta \mapsto \varphi_\beta^\pm(\theta)$ is differentiable and for $\beta' \in [0, \beta_c)$, we have that

$$\partial_\beta \gamma_{n,\beta}^\pm(\theta) = - \sum_{i=1}^n \prod_{\ell=1}^{i-1} \partial_x f_{\beta, \sigma^{-\ell}\theta}^{\pm 1}(f_{\beta, \sigma^{-n}\theta}^{\pm(n-\ell)}(1)) \xrightarrow{n \rightarrow \infty} \partial_\beta \varphi_\beta^\pm(\theta) \quad (40)$$

uniformly in β and θ for all $\beta \in [0, \beta']$ and all $\theta \in \Sigma$.

Proof. We only consider $\gamma_{n,\beta}^+$ and φ_β^+ , the statements for $\gamma_{n,\beta}^-$ and φ_β^- follow analogously. Recall that

$$f_\beta : \Sigma \times \mathbb{R} \rightarrow \Sigma \times \mathbb{R} \quad , \quad (\theta, x) \mapsto (\omega(\theta), g(x) + \kappa \cdot \theta_0 - \beta) \quad ,$$

with $g(x) = \frac{2}{\pi} \arctan(\alpha x)$.

First, observe that $\partial_x^2 f_{\beta,\theta}^n(x) < 0$ for all $n \in \mathbb{N}$ as long as $x, f_{\beta,\theta}(x), \dots, f_{\beta,\omega^{n-1}(\theta)}(x) > 0$ since the composition of concave increasing functions is again concave. Second, note that with $\theta^* = \dots 0, 0, 0 \dots \in \Sigma$, we have

$$f_{\beta_c, \theta^*}(x) = f_{\beta,\theta}(x) - (\beta_c - \beta) - \kappa \theta_0 \quad (41)$$

for all $\beta \in [0, \beta_c]$ and all $\theta \in \Sigma, x \in X$.

Hence, we obtain that for all $\beta \in [0, \beta_c]$, $\theta \in \Sigma$ and all $n, n' \in \mathbb{N}$ with $n \geq n'$ we have

$$\begin{aligned}
|\gamma_{n',\beta}^+(\theta) - \gamma_{n,\beta}^+(\theta)| &= f_{\beta,\sigma^{-n'}\theta}^{n'}(1) - f_{\beta,\sigma^{-n'}\theta}^{n'}(f_{\beta,\sigma^{-n}\theta}^{n-n'}(1)) \\
&\leq f_{\beta,\sigma^{-n'}\theta}^{n'}(1) - f_{\beta,\sigma^{-n'}\theta}^{n'}(f_{\beta_c,\theta^*}^{n-n'}(1)) \\
&= f_{\beta,\sigma^{1-n'}\theta}^{n'-1}(f_{\beta,\sigma^{-n'}\theta}(1)) - f_{\beta,\sigma^{1-n'}\theta}^{n'-1}(f_{\beta,\sigma^{-n'}\theta}(f_{\beta_c,\theta^*}^{n-n'}(1))) \\
&\leq f_{\beta,\sigma^{1-n'}\theta}^{n'-1}(f_{\beta_c,\theta^*}(1)) - f_{\beta,\sigma^{1-n'}\theta}^{n'-1}(f_{\beta_c,\theta^*}(f_{\beta_c,\theta^*}^{n-n'}(1))) \\
&\leq f_{\beta,\sigma^{2-n'}\theta}^{n'-2}(f_{\beta_c,\theta^*}^2(1)) - f_{\beta,\sigma^{2-n'}\theta}^{n'-2}(f_{\beta_c,\theta^*}^2(f_{\beta_c,\theta^*}^{n-n'}(1))) \\
&\leq \dots \leq f_{\beta_c,\theta^*}^{n'}(1) - f_{\beta_c,\theta^*}^{n'}(f_{\beta_c,\theta^*}^{n-n'}(1)) \\
&= |\gamma_{n',\beta_c}^+(\theta^*) - \gamma_{n,\beta_c}^+(\theta^*)|,
\end{aligned}$$

where we used the above mentioned concavity together with (41) in the steps to the fourth, fifth and sixth line. This proves the first part.

Next, we show that $\partial_\beta \gamma_{n,\beta}^+(\theta)$ converges uniformly in θ and β which immediately implies the second part. To that end, we first provide a uniform upper bound on

$$\sup_{\theta \in \Sigma, \beta \in [0, \beta']} \partial_x f_{\beta,\theta}(\gamma_{n,\beta}^+(\theta))$$

for all $n \in \mathbb{N}$: Similarly as in the proof of Theorem 4.4, we see that

$$x_{\min}(\beta) := \min_{\theta \in \Sigma} \varphi_\beta^+(\theta) \geq x_0(\beta')$$

for all $\beta \in [0, \beta']$, where $x_0(\beta')$ is the upper fixed point of the map $g - \beta'$. Hence, we have

$$\begin{aligned}
0 \leq \sup_{\theta \in \Sigma, \beta \in [0, \beta']} \partial_x f_{\beta,\theta}(\gamma_{n,\beta}^+(\theta)) &= \sup_{\theta \in \Sigma, \beta \in [0, \beta']} g'(\gamma_{n,\beta}^+(\theta)) \leq g'(\varphi_\beta^+(\theta)) \\
&\leq g'(x_0(\beta')) =: c < 1,
\end{aligned} \tag{42}$$

where we used the concavity of g and the monotone dependence of $\varphi_\beta^+(\theta)$ on β .

Now, observe that

$$\begin{aligned}
\partial_\beta \gamma_{n,\beta}^+(\theta) &= \partial_\beta f_{\beta,\sigma^{-n}\theta}^n(1) \\
&= \partial_\beta f_{\beta,\sigma^{-1}\theta}(f_{\beta,\sigma^{-n}\theta}^{n-1}(1)) + \partial_x f_{\beta,\sigma^{-1}\theta}(f_{\beta,\sigma^{-n}\theta}^{n-1}(1)) \cdot \partial_\beta f_{\beta,\sigma^{-n}\theta}^{n-1}(1) \\
&= \dots = \sum_{i=1}^n \partial_\beta f_{\beta,\sigma^{-i}\theta}(f_{\beta,\sigma^{-n}\theta}^{n-i}(1)) \prod_{\ell=1}^{i-1} \partial_x f_{\beta,\sigma^{-\ell}\theta}(f_{\beta,\sigma^{-n}\theta}^{n-\ell}(1)) \\
&= - \sum_{i=1}^n \prod_{\ell=1}^{i-1} \partial_x f_{\beta,\sigma^{-\ell}\theta}(f_{\beta,\sigma^{-n}\theta}^{n-\ell}(1)).
\end{aligned}$$

Together with (42), we hence obtain for $n \geq n'$

$$|\partial_\beta \gamma_{n',\beta}^+(\theta) - \partial_\beta \gamma_{n,\beta}^+(\theta)| = \sum_{i=n'+1}^n \prod_{\ell=1}^{i-1} \partial_x f_{\beta,\sigma^{-\ell}\theta}(f_{\beta,\sigma^{-n}\theta}^{n-\ell}(1)) \leq \sum_{i=n'+1}^n c^{i-1},$$

which proves the statement. \square

We can now turn to the

Proof of Theorem 5.2. We keep the notation of the previous proof. Clearly, the fact that $(f_\beta)_{\beta \in [0,1]}$ undergoes a non-smooth saddle-node bifurcation with critical parameter β_c (given by the bifurcation parameter of the family $(g - \beta)_{\beta \in [0,1]}$) is a direct consequence of Theorem 4.4. Hence, it remains to prove the existence of a uniform bound on the slope of the Lyapunov exponent. As before, we only consider φ_β^+ . Further, we show the statement for $\beta \in [0, \beta_c)$ which immediately yields the full statement by means of the mean value theorem. Let g be as in the proof of Lemma 5.3.

Given $\beta \in [0, \beta_c)$, observe that

$$\begin{aligned} \partial_\beta \lambda_\mu(\varphi_\beta^+) &= \partial_\beta \int_\Sigma \log \partial_x f_{\beta, \theta}(\varphi_\beta^+(\theta)) d\theta \\ &= \partial_\beta \int_\Sigma \log g'(\varphi_\beta^+(\theta)) d\theta = \int_\Sigma \frac{g''(\varphi_\beta^+(\theta))}{g'(\varphi_\beta^+(\theta))} \cdot \partial_\beta \varphi_\beta^+(\theta) d\theta. \end{aligned}$$

With $c \geq \sup_{x \in [0, 1]} |g''(x)/g'(x)|$, we hence obtain

$$|\partial_\beta \lambda_\mu(\varphi_\beta^+)| \leq c \int_\Sigma |\partial_\beta \varphi_\beta^+(\theta)| d\theta.$$

Now, let

$$\alpha = \sup_{\theta \in \Sigma, \theta_0=1} f'_{\beta_c, \sigma\theta}(f_{\beta_c, \theta}(x_c))$$

and note that

$$\alpha \leq g'(g(x_c) - \beta_c + \kappa/2) = g'(x_c + \kappa/2) < g'(x_c) = 1,$$

where x_c is the neutral fixed point of the map $g - \beta_c$. Then

$$\begin{aligned} \int_\Sigma |\partial_\beta \varphi_\beta^+(\theta)| d\theta &= \int_\Sigma |\partial_\beta \lim_{n \rightarrow \infty} \gamma_{n, \beta}^+(\theta)| d\theta = \lim_{n \rightarrow \infty} \int_\Sigma |\partial_\beta \gamma_{n, \beta}^+(\theta)| d\theta \\ &\stackrel{(40)}{=} \lim_{n \rightarrow \infty} \int_\Sigma \sum_{i=1}^n \prod_{\ell=1}^{i-1} \partial_x f_{\beta, \sigma^{-\ell}\theta}(f_{\beta, \sigma^{-n}\theta}^n(1)) d\theta \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_\Sigma \prod_{\ell=1}^{i-1} \partial_x f_{\beta, \sigma^{-\ell}\theta}(f_{\beta, \sigma^{-n}\theta}^n(1)) d\theta \\ &\leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_\Sigma \prod_{\ell=1}^{i-1} \partial_x f_{\beta, \sigma^{-\ell}\theta}(f_{\beta, \sigma^{-\ell-1}\theta}(x_c)) d\theta \\ &\leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{k=0}^{i-1} \alpha^k \cdot \mu(\{\theta \in \Sigma : k = \#\{1 < \ell \leq i : \theta_{-\ell} = 1\}\}) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{k=0}^{i-1} \alpha^k \cdot \binom{i-1}{k} (1/2)^{i-1} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{k=0}^{i-1} \binom{i-1}{k} (\alpha/2)^k (1/2)^{i-1-k} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (\alpha/2 + 1/2)^{i-1} < \infty. \end{aligned}$$

Since α is independent of β , the statement follows. \square

6. RANGE OF FINITE-TIME LYAPUNOV EXPONENTS

The aim of this section is to provide a proof of Theorem 3, which we restate below in a more general form. To that end, let us introduce the maximal finite-time Lyapunov exponents on the attractor. As invariant graphs only need to be defined almost surely, we only take into account exponents that can be ‘seen’ on a set of positive measure by setting

$$\lambda_k^{\max}(\varphi_\beta^+) = \sup \left\{ \lambda \in \mathbb{R} \mid \mu_{\varphi_\beta^+}(\{(\theta, x) \mid \lambda_k(f_\beta, \theta, x) \geq \lambda\}) > 0 \right\}.$$

Here, the graph measure $\mu_{\varphi_\beta^+}$ is as discussed in Section 2.1. Note that if the forcing is quasiperiodic, then the attractors prior to the bifurcation are all continuous so that we actually have $\lambda_k^{\max}(\varphi_\beta^+) = \max \left\{ \lambda_k(\theta, \varphi_\beta^+(\theta)) \mid \theta \in \mathbb{T}^1 \right\}$ whenever $\beta < \beta_c$.

We first consider the case of quasiperiodic forcing, where the general statement we aim at reads as follows.

Theorem 6.1. *Suppose $(\Xi_\beta)_{\beta \in [0,1]}$ is a parameter family of qpf monotone flows that satisfies the hypothesis of Theorem 2.2. Then for all $k \in \mathbb{N}$ we have*

$$\liminf_{\beta \nearrow \beta_c} \lambda_k^{\max}(\varphi_\beta^+) \geq \lambda(\varphi_{\beta_c}^-). \quad (43)$$

Before we turn to the proof however, we have to address some subtleties concerning the topology of pinched invariant graphs in this setting. Suppose we are in the situation of Theorem 2.2 so that there exist exactly two graphs $\varphi_{\beta_c}^- < \varphi_{\beta_c}^+$ (up to modifications on sets of measure zero), where $\varphi_{\beta_c}^-$ is lower and $\varphi_{\beta_c}^+$ is upper semicontinuous. Let $A^+ = \text{supp}(\mu_{\varphi_{\beta_c}^+})$ and $A^- = \text{supp}(\mu_{\varphi_{\beta_c}^-})$, where $\text{supp}(\nu)$ denotes the topological support of a measure ν .⁶ Then A^+ is Ξ_{β_c} -invariant, and consequently the upper and lower bounding graphs $\varphi_{A^+}^u$ and $\varphi_{A^+}^l$ given by

$$\varphi_{A^+}^u(\theta) = \sup A_\theta^+ \quad \text{and} \quad \varphi_{A^+}^l(\theta) = \inf A_\theta^+$$

are Ξ_β -invariant graphs, with $\varphi_{A^+}^u$ upper and $\varphi_{A^+}^l(\theta)$ lower semicontinuous (see [Sta03]). As $\varphi_{\beta_c}^\pm$ are the only Ξ_β -invariant graphs in the considered region Γ , we must have $\varphi_{A^+}^l(\theta) = \varphi_{\beta_c}^-$ and $\varphi_{A^+}^u = \varphi_{\beta_c}^+$ almost surely. This implies in particular that $(\theta, \varphi_{\beta_c}^-(\theta)) \in A^+$ almost surely, so that $\mu_{\varphi_{\beta_c}^-}(A^+) = 1$ and hence $A^- \subseteq A^+$. As the converse inclusion follows in the same way, we have $A^- = A^+$. One particular consequence that follows from this discussion is the following

Lemma 6.2. *For μ -almost every $\theta_0 \in \Theta$ and every $\delta > 0$ there exists a set $B \subseteq \Theta$ of positive measure such that*

$$\{(\theta, \varphi_{\beta_c}^+(\theta)) \mid \theta \in B\} \subseteq B_\delta((\theta_0, \varphi_{\beta_c}^-(\theta_0))).$$

We can now turn to the

Proof of Theorem 6.1. Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. We claim that there exists a set of positive measure of $\theta \in \Theta$ such that

$$\lambda_k(\theta, \varphi_{\beta_c}^-(\theta)) \geq \lambda(\varphi_{\beta_c}^-). \quad (44)$$

In order to see this, suppose for a contradiction that

$$\lambda_k(\theta, \varphi_{\beta_c}^-(\theta)) \leq \lambda(\varphi_{\beta_c}^-) - \delta$$

for almost every $\theta \in \Theta$ and some $\delta > 0$. This implies that the pointwise Lyapunov exponent also satisfies

$$\lambda(\theta, \varphi_{\beta_c}^-(\theta)) \leq \lambda(\varphi_{\beta_c}^-) - \delta$$

for almost every θ , contradicting Birkhoff's Ergodic Theorem.

Hence, there exists a positive measure set of θ which satisfies (44). Due to Lemma 6.2, we can therefore fix $\delta > 0$ and choose $\theta_0 \in \Theta$ which satisfies (44) and a set $B \subseteq \Theta$ of positive measure such that $(\theta, \varphi_{\beta_c}^+(\theta)) \in B_\delta(\theta_0, \varphi_{\beta_c}^-(\theta_0))$ for all $\theta \in B$. If δ is chosen small enough, then it follows by continuity that

$$\lambda_k(\theta, \varphi_{\beta_c}^+(\theta)) \geq \lambda(\varphi_{\beta_c}^-) - \varepsilon$$

for all $\theta \in B$. As $\varepsilon > 0$ was arbitrary, we obtain that $\lambda_k^{\max}(\varphi_{\beta_c}^+) \geq \lambda(\varphi_{\beta_c}^-)$. Finally, as $\lim_{\beta \nearrow \beta_c} \varphi_\beta^+(\theta) = \varphi_{\beta_c}^+(\theta)$ almost surely, we obtain (43) again by continuity. \square

We now turn to the random case. In this case, we have to restrict to the setting of Theorem 4.4 (instead of the more general situation of Theorem 2.5).

Theorem 6.3. *Suppose that $(\Xi_\beta)_{\beta \in [0,1]}$ is a parameter family of randomly forced monotone flows that satisfies the assumptions of Theorem 4.4. Then for all $k \in \mathbb{R}$*

$$\lim_{\beta \nearrow \beta_c} \lambda_k^{\max}(\varphi_\beta^+(\theta)) = 0.$$

⁶Given a Borel measure ν on some second countable metric space X , the support of ν is defined as $\text{supp}(\nu) = \{x \in X \mid \nu(B_\delta(x)) > 0 \forall \delta > 0\} = X \setminus \bigcup_{U \text{ open}, \nu(U)=0} U$. It is easy to see that $\text{supp}(\nu)$ is always closed and can be characterised as the smallest closed set $A \subseteq X$ with $\nu(X \setminus A) = 0$. Moreover, if ν is invariant under some continuous transformation f , then so is $\text{supp}(\nu)$.

Proof. Let $(g_\beta)_{\beta \in [0,1]}$ be the autonomous reference family from Theorem 4.4. Then we have that g_{β_c} has a unique fixed point $x_0 \in [\gamma^-, \gamma^+]$ and the Lyapunov exponent of x_0 vanishes, that is, $\log \partial_t g_{\beta_c}^t(x_0) = 0$ for all $t > 0$. By continuity, this means that given $k \in \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ and $|\beta - \beta_c| < \delta$ implies $|\log(g_\beta^k(x))|/k < \varepsilon$. Moreover, as $\lim_{t \rightarrow \infty} g_{\beta_c}^t(\gamma^+) = x_0$, we can further require that $g_{\beta_c}^t(\gamma^+) < x_0 + \delta/2$ for some $t > 0$ and all $\beta \in [\beta_c - \delta, \beta_c]$.

Now, by assumption (ii) of Theorem 4.4 there exists a set $A_{\delta/2,t} \subseteq \Theta$ of positive measure such that for all $\theta \in \omega^t(A_{\delta/2,t})$ we have

$$\xi_\beta^t(\omega^{-t}(\theta), \gamma^+) \leq x_0 + \delta.$$

As $\varphi_\beta^+(\theta)$ is the monotone limit of the sequence $\xi^t(\omega^{-t}(\theta), \gamma^+)$ (see the proof of Theorem 4.4) and is bounded below by x_0 , this implies that $x_0 \leq \varphi_\beta^+(\theta) \leq x_0 + \delta$ and therefore $|\lambda_k(\theta, \varphi_\beta^+(\theta))| < \varepsilon$ for all $\theta \in A_{\delta/2,k}$. As $\varepsilon > 0$ was arbitrary, this completes the proof. \square

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