Irrational rotation factors for conservative torus homeomorphisms

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Abstract

English. We provide an equivalent characterisation for the existence of one-dimensional irrational rotation factors of conservative torus homeomorphisms that are not eventually annular. It states that an area-preserving non-annular torus homeomorphism f is semi-conjugate to an irrational rotation R_α of the circle if and only if there exists a well-defined speed of rotation in some rational direction on the torus, and the deviations from the constant rotation in this direction are uniformly bounded. By means of a counterexample, we also demonstrate that a similar characterisation does not hold for eventually annular torus homeomorphisms.

1 Introduction

The question of irrational rotation factors, that is, the existence of a semiconjugacy or conjugacy to an irrational rotation of the circle, is a classical problem in dynamical systems theory. An equivalent formulation is to ask for the existence of continuous eigenfunctions of the associated Koopman operator with irrational eigenvalues. Typically, this issue is addressed by using the powerful tools of KAM-theory, which even yield smooth conjugacies ('smooth linearisation'). The price to pay for this, however, is the requirement of strong assumptions concerning the smoothness of the considered systems and the arithmetic properties of the involved rotation numbers. Moreover, this approach is mostly restricted to systems which are small perturbations of the underlying rotations.

At the same time, the very first result relating non-linear dynamics to irrational circle rotations is the celebrated Poincaré Classification Theorem, which states that an orientation-preserving homeomorphism of the circle is semiconjugate to an irrational rotation if and only if its rotation number is irrational [1]. It is remarkable that this statement draws strong conclusions from purely topological assumptions, and no restrictions on the rotation number other than its irrationality are needed. The fact that the existence of full conjugacies cannot generally be expected in a topological setting is well-known and demonstrated by classical examples of Denjoy [2].

Even 130 years after Poincaré's contribution, similar results in this direction are quite rare. In recent years, however, there has been substantial progress on 'topological linearisation' in a number of situations and system classes, including skew-products over irrational rotations [3], mathematical quasicrystals [4], reparametrisations of irrational flows [5], area-preserving torus homeomorphisms [6] and dynamics on circle-like continua [7]. Most importantly, some general pattern and methods start to emerge. In particular, a common element in most of the proofs is the identification of a suitable dynamically defined partition of the phase space carrying a circular order structure.

Our aim here is to push forward this line of research by providing a more or less complete description of the situation concerning the existence of one-dimensional irrational rotation factors of area-preserving homeomorphisms of the two-torus. For the non-annular case we provide an equivalent characterisation in terms of rotational behaviour, whereas in the eventually annular case we show that an analogous statement is not valid. This complements a previous result in [6], which treats the existence of two-dimensional rotation

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factors. Here, we note that a periodic point free area-preserving torus homeomorphism is annular if there exists an essential (but not fully essential) invariant continuum or, equivalently, an essential invariant open annulus. It is eventually annular if it has an annular iterate. The general definitions are given in Section 2 below. In the eventually annular case, the dynamics of the respective iterate of f can be embedded in a compact annulus. This situation is quite different from the non-annular case,[1] where the dynamics truly involve the full topology of the torus. Annular homeomorphisms have been extensively studied in own right (see, for example, [8, 9, 10]) and significant information concerning their dynamical behaviour is available. However, as we exemplify in Section 4, the question of irrational rotation factors is more intricate than in the non-annular case and does not have a similar solution.

Given a homeomorphisms f of the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we denote its lift to $\mathbb{R}^2$ by F. We say $v = (p, q) \in \mathbb{Z}^2 \backslash \{0\}$ is *reduced* if $\gcd(p, q) = 1$ and call $w \in \mathbb{Z}^2$ *complementary* if $\det(w, v) = 1$. Suppose that for some reduced integer vector v we have $F(z+v) = F(z)+v$ for all $z \in \mathbb{R}^2$. Note that this property is independent of the choice of the lift. Moreover, when f is homotopic to the identity, this holds for all reduced integer vectors. The *rotation interval of F in the direction of v* is given by

$$\rho_v(F) \;=\; \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m\geqslant n} K(F,m)} \;,$$

where $K(F, m) = \{\langle F^m(z) - z, v\rangle/m \mid z \in \mathbb{R}^2\}$. It is always a compact interval [11]. If $\rho_v(F) = \{\alpha\}$ for some $\alpha \in \mathbb{R}$ and moreover there exists a constant $C > 0$ such that

$$|\langle F^n(z) - z, v\rangle - n\alpha| \;\leqslant\; C$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{R}^2$, then we say f has *bounded deviations* (from the constant rotation) *in the direction of v*. Denote the rotation by $\alpha \in \mathbb{R}$ on $\mathbb{T}^1$ by R_α. Then our main result reads as follows.

Theorem 1.1. *Suppose f is a non-annular area-preserving homeomorphism of $\mathbb{T}^2$. Then f is semiconjugate to an irrational rotation R_α on $\mathbb{T}^1$ if and only if there exists a reduced integer vector v and a positive integer k such that $\rho_v(F) = \{\alpha/k\|v\|^2\}$ and f has bounded deviations in the direction of v.*

Suppose that f is homotopic to the identity and there exist a vector $\rho = (\alpha, \beta) \in \mathbb{R}^2$ and a constant $C > 0$ such that

$$|F^n(z) - z - n\rho| \;\leqslant\; C$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{R}^2$. In this case, we say f is a *pseudorotation* (has a unique rotation vector) *with uniformly bounded deviations*. If ρ is totally irrational, then Theorem 1.1 can be applied twice to obtain a semiconjugacy to the irrational rotation by ρ on the two-torus. This is precisely the statement of Theorem C in [6]. However, we actually use [6, Theorem C] in the proof of Theorem 1.1, in order to treat exactly the above-mentioned situation. The new ingredient we provide is a complementary argument that addresses the cases where deviations are only bounded in one direction, or where f is homotopic to a Dehn twist. This part also works under the weaker assumption of nonwandering dynamics and is stated as Theorem 3.3. The proof is based on the concepts of strictly toral dynamics and dynamically essential points developed in [12].

Section 2 collects all the required preliminaries and provides an elementary, but crucial lemma on strictly toral dynamics. The proof of Theorem 1.1 is given in Section 3, and in Section 4 we provide a simple counterexample showing that the statement of Theorem 1.1 is false in the eventually annular case.

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[1]In the context of this note, we use the term *'non-annular'* is the sense of 'not eventually annular'.

2 Definitions and preliminaries

2.1 Circloids. Let S be a two-dimensional manifold. A continuum $A \subseteq S$ is called *annular*, if it is the intersection of a nested sequence of annuli A_n such that each A_{n+1} is essential in A_n (not contained in a topological disk $D \subseteq A_n$.) An equivalent definition is to require that A has an annular neighbourhood $\mathcal{A}$ which it separates into exactly two connected components, both of which are again homeomorphic to the open annulus. We say A is *essential* if this annular neighbourhood $\mathcal{A}$ is essential in S. We call an annular continuum $C \subseteq S$ with annular neighbourhood $\mathcal{A}$ a *circloid*, if it does not contain a strictly smaller annular continuum which is also essential in $\mathcal{A}$. Note that C may contain a non-essential annular continuum as a subset, for example when it has non-empty interior.

A compact set $A \subseteq \mathbb{A} = \mathbb{R} \times \mathbb{T}^1$ is essential if $\mathbb{A} \backslash A$ has two unbounded connected components. In this case, we denote the component which is unbounded to the right by $\mathcal{U}^+(A)$ and the one unbounded to the left by $\mathcal{U}^-(A)$. Note that if $A \subseteq \mathbb{A}$ is an essential annular continuum, then $\mathbb{A} \backslash C = \mathcal{U}^-(C) \cup \mathcal{U}^+(C)$. Further, we let $\mathcal{U}^{+-}(A) = \mathcal{U}^-(\partial \mathcal{U}^+(A))$ and $\mathcal{U}^{-+}(A) = \mathcal{U}^+(\partial \mathcal{U}^-(A))$, and define the same notions for longer alternating sequences of the symbols $-$ and $+$ in the analogous way. This yields a simple procedure to obtain essential circloids from arbitrary essential compact sets.

Lemma 2.1 ([6, Lemma 3.2]). *Suppose $A \subseteq \mathbb{A}$ is compact and essential. Then*

$$\mathcal{C}^+(A) = \mathbb{A} \backslash (\mathcal{U}^{+-}(A) \cup \mathcal{U}^{+-+}(A)) \quad and \quad \mathcal{C}^-(A) = \mathbb{A} \backslash (\mathcal{U}^{-+}(A) \cup \mathcal{U}^{-+-}(A))$$

are both essential circloids.

Given two essential circloids $C, C' \subseteq \mathbb{A}$, we write $C \prec C'$ if $C' \subseteq \mathcal{U}^+(C)$ and $C \preccurlyeq C'$ if $C' \subseteq \mathcal{U}^+(C) \cup C$. Further, we write $[C, C'] = \mathbb{A} \backslash (\mathcal{U}^-(C) \cup \mathcal{U}^+(C'))$ and $(C, C') = \mathcal{U}^+(C) \cap \mathcal{U}^-(C')$. We will need the following elementary observation.

Lemma 2.2. *If $C \preccurlyeq C'$ and $C \neq C'$, then (C, C') is non-empty.*

Proof. Suppose for a contradiction that (C, C') is empty. Then, since $\mathcal{U}^-(C') \cap \mathcal{U}^+(C) = \varnothing$, the set $C^* = \mathbb{A} \backslash (\mathcal{U}^-(C') \cup \mathcal{U}^+(C))$ is an essential annular continuum. However, $C \preccurlyeq C'$ implies that $\mathcal{U}^-(C) \subseteq \mathcal{U}^-(C')$ and $\mathcal{U}^+(C') \subseteq \mathcal{U}^+(C)$. Therefore C^* is contained both in C and C', and by minimality of circloids amongst annular continua, with respect to inclusion, we obtain $C^* = C = C'$. $\qquad\square$

2.2 Strictly toral dynamics. An open set $U \subset S$ is called *inessential* if every loop whose image lies in U is homotopically trivial in S, otherwise U is *essential*. A general set $E \subset S$ is called inessential if it has an inessential neighborhood, otherwise E is said to be essential. Note that for subcontinua of $\mathbb{A}$, this coincides with the terminology used above. A set $E \subseteq \mathbb{T}^2$ whose complement is inessential is called *fully essential*. An open set $U \subseteq \mathbb{T}^2$ is called *annular* if it is neither inessential, nor fully essential. This is equivalent to saying that the union of U with the inessential components of its complement is a topological open annulus. If U is open and $i : \Pi_1(U) \to \Pi_1(\mathbb{T}^2)$ is the natural inclusion of its first homotopy group into the first homotopy group of $\mathbb{T}^2$, then U is inessential if and only if the image of i is trivial, U is fully essential if and only if i is onto, and U is annular if and only if the image of i is homomorphic to $\mathbb{Z}$. Note that, if U is a connected fully essential set, then $\pi^{-1}(U)$ is connected, and that any two open fully essential subsets of $\mathbb{T}^2$ must intersect.

Given a homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, we say that $x \in \mathbb{T}^2$ is an *inessential point* for f or that x is a dynamically inessential point if there exists a neighborhood U of x such that $\bigcup_{i \in \mathbb{Z}} f^i(U)$ is inessential, otherwise x is called a dynamically essential point. The set of dynamically inessential points is denoted by $\text{Ine}(f)$ and is open. Its complement is denoted by $\text{Ess}(f)$. If f is nonwandering, then x is an inessential point if and only if it is contained in some periodic open topological disk. By Brouwer's theory, it follows that

Lemma 2.3. *If f is nonwandering and $Ine(f)$ is not empty, then f has a periodic point.*

We say a homeomorphism f of the two-torus is *annular* if there exists $M > 0$, an integer vector $v \in \mathbb{Z}^2$ and a lift F of f such that, for any $x \in \mathbb{R}^2$ and any $n \in \mathbb{N}$, $|\langle F^n(x) - x, v \rangle| < M$. It is *eventually annular* if it has an annular iterate. Recall that we say f is *non-annular* if it is not eventually annular. As stated above, if f is area-preserving and has no periodic points, then f is annular (eventually annular) if and only if f has an invariant (periodic) essential open annulus [13, Proposition 3.9]. A homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is said to

be *strictly toral* or to have strictly toral dynamics if, for all $n \in \mathbb{N}$, f^n is not annular and $\mathrm{Fix}(f^n)$ is an inessential set. The following is immediate.

Lemma 2.4. *If f is nonwandering and non-annular for all $n \in \mathbb{N}$, then every essential periodic open set is fully essential.*

An important piece in the understanding the behavior of essential points is to understand the boundedness properties of the dynamics. We have

Lemma 2.5. *Suppose f is a torus homeomorpihsm for which no power of f is a pseudo-rotation with uniformly bounded deviations. Then for any fundamental domain $D \subset \mathbb{R}^2$, any lift F of f, and any $K > 0$, there exists $n > 0$ such that $F^n(D)$ has diameter larger than K.*

Proof. Fix a lift F of f, and first consider the case where f is not homotopic to a periodic homeomorphism. Then there exists some simple closed curve $\alpha : [0,1] \to \mathbb{T}^2$ such that, if $\tilde{\alpha}$ is a lift of α to $\mathbb{R}^2$, then $\lim_{n\to\infty} \|F^n(\tilde{\alpha}(0)) - F^n(\tilde{\alpha}(1))\| = \infty$, and so for any fundamental domain D the diameter of $F^n(D)$ is not uniformly bounded.

Now consider the case where f is homotopic to a periodic homeomorphism, in which case there exists some n_0 such that $g = f^{n_0}$ is homotopic to the identity, and let $G = F^{n_0}$. If g is not a pseudo-rotation, then there exists z_1, z_2 in $\mathbb{R}^2$ and rotation vectors ρ_1, ρ_2 with $\rho_1 \neq \rho_2$ such that $\lim n \to \infty \frac{G^n(z_i) - z_i}{n} = \rho_i$, $i \in \{1, 2\}$. Since both points can be taken in the same fundamental domain but $\|G^n(z_1) - G^n(z_2)\| = \|F^{nn_0}(z_1) - F^{nn_0}(z_2)\|$ diverges, the result holds.

Now assume g is a pseudo-rotation. Then there exists ρ such that $\lim_{n\to\infty} \frac{G^n(z) - z}{n} = \rho$ for all $z \in \mathbb{R}^2$. Given $w \in \mathbb{R}^2$, let $D_w(n, z) = \langle G^n(z) - z - nv, w \rangle$. In other words, D_w denotes the deviation from the rigid rotation in the w direction. [13, Lemma 3.2] shows that, for every $w \in \mathbb{R}^2 \backslash \{0\}$, there exists points $y_{w,+}$ and $y_{w,-}$ such that, for all $n \in \mathbb{N}$, $D_w(n, y_{w,+}) > -1$, $D_w(n, y_{w,-}) < 1$, and we may assume that both points belong to D. If D is a fundamental domain with diameter R, and if g does not have uniformly bounded deviations, there exists a unit vector w_0, $n_0 > 0$ and $z \in D$ such that $|D_{w_0}(n_0, z)| > K + R + 2$, and so either $\langle G^n(z) - G^n(y_{w_0,+}), w_0 \rangle = \langle F^{nn_0}(z) - F^{nn_0}(y_{w_0,+}), w_0 \rangle < -K$ or $\langle G^n(z) - G^n(y_{w_0,-}), w_0 \rangle = \langle F^{nn_0}(z) - F^{nn_0}(y_{w_0,-}), w_0 \rangle > K$. $\qquad\square$

The following proposition is the main new result of this subsection. Its proof is inspired by those of Theorem 8 and Proposition 9 of [14].

Proposition 2.6. *Suppose f is a strictly toral nonwandering torus homeomorphism and that no power of f is a pseudo-rotation with bounded deviations. Then for any neighbourhood U of an essential point x there exists $n \in \mathbb{N}$ such that $U \cup f^n(U)$ is essential.*

Proof. By maybe taking a subneighborhood, we can assume that U is inessential and contained in $B_{1/2}(x)$. Let us fix a connected component $\tilde{U}$ of $\pi^{-1}(U)$. Since f is strictly toral, every invariant or periodic essential set is fully essential and invariant, and since x is an essential point, this implies that $O = \bigcup_{i \in \mathbb{Z}} f(U)$ is fully essential and invariant. Therefore there exists simple closed curves $\alpha_1, \alpha_2 : \mathbb{T}^1 \to \mathbb{T}^2$ whose images lie in O and which generate the fundamental group of $\mathbb{T}^2$. This implies that the connected components of the complement of $\pi^{-1}(\alpha_1 \cup \alpha_2)$ are uniformly bounded. Note also that for each point z in $\pi^{-1}(\alpha_1 \cup \alpha_2)$, there exists some integer i_z and some $v_z \in \mathbb{Z}^2$ such that $z \in F^{i_z}(\tilde{U}) + v_z$.

Let $D \subset \mathbb{R}^2$ be a fundamental domain, and consider the connected set D' which is the union of D with all connected components of the complement of $\pi^{-1}(\alpha_1 \cup \alpha_2)$ intersecting D. Then D' is a bounded set, and its boundary is contained in $\pi^{-1}(\alpha_1 \cup \alpha_2)$. Let R be larger than the diameter of D'. By compactness, one can find integers $i_j, 1 \leqslant j \leqslant k$, and integer vectors $v_j, 1 \leqslant j \leqslant k$ such that $\partial D' \subset \bigcup_{j=1}^{k} F^{i_j}(\tilde{U}) + v_j$, and $\bigcup_{j=1}^{k} F^{i_j}(\tilde{U}) + v_j$ is connected. Since no power of f is a pseudo-rotation with uniformly bounded deviations, by Lemma 2.5 there exists some $n > 0$ such that the diameter of $F^n(D)$ is larger than $k(k+1)R$. This implies that the diameter of $\bigcup_{j=1}^{k} F^{i_j+n}(\tilde{U}) + v_j$ is greater than $k(k+1)R$ and thus, for some $j_0 \leqslant k$, the diameter of $F^{i_{j_0}+n}(\tilde{U}) + v_{j_0}$ is larger than $(k+1)R$.

But this implies that $F^{i_{j_0}+n}(\tilde{U}) + v_{j_0}$ intersect at least $k+2$ integer translates of D', and so $F^{i_{j_0}+n}(\tilde{U}) + v_{j_0}$ must intersect at least $k+1$ integer translates of $\partial D'$. Let $w_l, 1 \leqslant l \leqslant k+1$ be vectors in $\mathbb{Z}^2$ such that $F^{i_{j_0}+n}(\tilde{U}) + v_{j_0}$ intersects $\partial D' + w_l$ for each l. As each copy $\partial D' + w_l$ is covered by the k sets $F^{i_j}(\tilde{U}) + v_j + w_l$, there exists

$w_{l_1} \neq w_{l_2}$ and some $j_1 \leqslant k$ such that $F^{i_{j_0}+n}(\widetilde{U}) + v_{j_0}$ intersects both $F^{i_{j_1}}(\widetilde{U}) + v_{j_1} + w_{l_1}$ and $F^{i_{j_1}}(\widetilde{U}) + v_{j_1} + w_{l_2}$.

This implies that $F^{i_{j_0}+n-i_{j_1}}(\widetilde{U}) + v_{j_0} - v_{j_1}$ intersects both $\widetilde{U} + w_{l_1}$ and $\widetilde{U} + w_{l_2}$. Therefore, the connected components of $\pi^{-1}\left(f^{i_{j_0}+n-i_{j_1}}(U) \cup U\right)$ do not project injectively on $\mathbb{T}^2$ and so $f^{i_{j_0}+n-i_{j_1}}(U) \cup U$ is an essential set. $\qquad\square$

3   Proof of Theorem 1.1

We start with the "only if"-direction. We identify the fundamental group $\Pi_1(\mathbb{T}^d)$ of the d-dimensional torus with $\mathbb{Z}^d$ and denote the action of a continuous map $\psi : \mathbb{T}^d \to \mathbb{T}^k$ on the fundamental groups by $\psi^* : \Pi_1(\mathbb{T}^d) \to \Pi_1(T^k)$. In the following, if $h : \mathbb{T}^2 \to \mathbb{T}^1$ semiconjugates a torus homeomorphism f to R_α and H and F are lifts, then we implicitly assume these to be chosen such that H semiconjugates F to $x \mapsto x + \alpha$ on $\mathbb{R}$.

Lemma 3.1. *Suppose a homeomorphism f of the two-torus is semiconjugate to an irrational rotation R_α on $\mathbb{T}^1$ via the factor map $h : \mathbb{T}^2 \to \mathbb{T}^1$. Then $h^*\Pi_1(\mathbb{T}^2) \to \Pi_1(\mathbb{T}^1)$ is given by $u \to k\langle u, v\rangle$ for some reduced integer vector v and a positive integer k. Moreover, we have $f^*(v^\perp) \in \{v^\perp, -v^\perp\}$, $\rho_v(F) = \{\alpha/k\|v\|^2\}$ and f has bounded deviations in the direction of v. In addition, if w is complementary to $v^\perp$, then $f^*(w) = w + mv^\perp$ for some $m \in \mathbb{Z}$.*

Proof. First, suppose for a contradiction that $h^* = 0$ and let $H : \mathbb{R}^2 \to \mathbb{R}^1$ be a lift of h. Then $\sup_{z \in \mathbb{R}^2} |H(z)| < \infty$. However, since $H(F^n(z)) = H(z) + n\alpha$, this contradicts the fact that $\alpha \neq 0$. Hence, the kernel of h^* is homomorphic to $\mathbb{Z}$ and generated by some reduced integer vector $\tilde{v}$, so if we let $v = \tilde{v}^\perp$ we have $h^*(u) = k\langle u, v\rangle$ for some $k \in \mathbb{Z}\setminus\{0\}$. By replacing v with $-v$ if necessary, we may assume $k > 0$. Let w be a reduced integer vector complementary to $v^\perp$ and $A = (w, v^\perp) \in \mathrm{SL}(2, \mathbb{Z})$. Then A induces a linear torus homeomorphism ϕ_A with lift $\Phi_A : z \mapsto Az$, and $\tilde{h} = h \circ \phi_A$ semiconjugates $\tilde{f} = \phi_A^{-1} \circ f \circ \phi_A$ to R_α. Since $\tilde{h}^* = h^* \circ A$ and A sends $(1, 0)$ to w and $(0, 1)$ to $v^\perp$, $\tilde{h}^*$ sends $(1, 0)$ to k and $(0, 1)$ to zero. This implies that $\tilde{h}$ is homotopic to the map $z \mapsto k\pi_1(z) \bmod 1$. By replacing the circle by a k-fold covering and R_α by $R_{\alpha/k}$, we may assume that $\tilde{h}$ is homotopic to π_1 and semiconjugates $\tilde{f}$ to $R_{\alpha/k}$.

Let $\tilde{F}$ and $\tilde{H}$ be lifts of $\tilde{f}, \tilde{h}$, again chosen such that $\tilde{H}$ semiconjugates $\tilde{F}$ to the translation by α/k on $\mathbb{R}$. The fact that $\tilde{h}$ is homotopic to π_1 implies that $C = \sup_{z \in \mathbb{R}^2} |\tilde{H}(z) - \pi_1(z)| < \infty$. This further yields

$$
\begin{aligned}
|\pi_1(\tilde{F}^n(z) - z) - n\alpha/k| &= \\
(3.1)\qquad &= |\pi_1(\tilde{F}^n(z)) - \tilde{H}(\tilde{F}^n(z)) + \underbrace{\tilde{H}(\tilde{F}^n(z)) - \tilde{H}(z) - n\alpha/k}_{=0} + \tilde{H}(z) - \pi_1(z)| \\
&\leqslant |\pi_1(\tilde{F}^n(z)) - \tilde{H}(\tilde{F}^n(z))| + |\tilde{H}(z) - \pi_1(z)| \leqslant 2C \ .
\end{aligned}
$$

Applied to $n = 1$, this means in particular that $|\pi_1(\tilde{F}(0, 0) - \tilde{F}(0, l))| = l|\pi_1(\tilde{f}^*(0, 1))| \leqslant 4C$ for all $l \in \mathbb{N}$, so that $\pi_1(f^*(0, 1)) = 0$. Hence, $(0, 1)$ is an eigenvector for the action of $\tilde{f}^*$. Since $\tilde{f}$ is a homeomorphisms and thus $\tilde{f}^*$ has to permute reduced integer vectors, it must send $(0, 1)$ either to itself or to $(0, -1)$. Thus $f^*(v^\perp) \in \{v^\perp, -v^\perp\}$. Similarly, we have $|\pi_1(\tilde{F}(l, 0) - l - \tilde{F}(0, 0))| \leqslant 4C$ for all $l \in \mathbb{N}$, which implies that $\pi_1(\tilde{f}^*(1, 0)) = 1$, so $\tilde{f}^*(1, 0) = (1, m)$ for some $m \in \mathbb{Z}$. Therefore $f^*(w) = w + mv^\perp$.

Finally, (3.1) directly implies that $\tilde{f}$ has rotation number α/k and bounded deviations in the direction of $(1, 0)$. Going back to the original coordinates, we obtain that $F = \Phi_A \circ F \circ \Phi_A^{-1}$ has a well-defined speed of rotation and bounded deviations orthogonal to $A(0, 1)^t = u^t$, and the rotation number in this direction is $\alpha/k \cdot \frac{\langle w, u^\perp \rangle}{\|u\|^2} = \alpha/k\|u\|^2$ (see also [6, Proof of Proposition 2.1]). This proves the statement. $\qquad\square$

By choosing $v^\perp$ and a complementary vector w as a basis of $\mathbb{Z}^2$, we can thus determine the possible homotopy types of f.

Corollary 3.2. *If a torus homeomorphisms f is semiconjugate to an irrational rotation of the circle, then the action $f^* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is conjugate to a linear transformation given by a matrix of the form $\begin{pmatrix} \pm 1 & m \\ 0 & 1 \end{pmatrix}$ with $m \in \mathbb{Z}$.*

In order to proof the converse direction, we may assume by means of a linear coordinate change that $v = (1,0)$. If we let $\mathbb{A} = \mathbb{R} \times \mathbb{T}^1$ and denote by $\widehat{F} : \mathbb{A} \to \mathbb{A}$ the respective lift of f, then due to the bounded deviations in the v-direction we have

$$(3.2) \qquad\qquad \left| \pi_1 \left(\widehat{F}^n(z) - z \right) - n\alpha \right| \leqslant C$$

for some constant $C \geqslant 0$. Suppose that some iterate f^p of f has a well-defined rotation number and bounded deviations also orthogonal to v. Since we exclude the eventually annular case, this means that $\rho_{v\perp}(F^p) = \{\beta\}$ for some $\beta \in \mathbb{R}$ which is not rationally related to α [6, Proposition 3.9]. It is easy to see that this can neither happen if f is orientation-reversing nor if it is homotopic to a Dehn-twist. Thus, in the light of Corollary 3.2, f has to be homotopic to the identity. In this case, the existence of a semiconjugacy to the two-dimensional rotation by $\rho = (\alpha, \beta)$ on $\mathbb{T}^2$ follows directly from [6, Theorem C].

Hence, it remains to treat the case no iterate of f is a pseudorotation with bounded deviations. As we have mentioned, the argument which we will apply here only requires f to be nonwandering. Note that if h semiconjugates f to $R_{\alpha/k}$ and $\tau(x) = kx \bmod 1$, then $\tau \circ h$ semiconjugates f to R_α. Hence, in order to complete the proof of Theorem 1.1, it suffices to show the following statement.

Theorem 3.3. *Suppose f is a nonwandering non-annular torus homeomorphism and $\rho(F) \subseteq \{\alpha\} \times \mathbb{R}$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Moreover, assume that f has bounded deviations in the direction of $(1,0)$, but no iterate of f is a pseudorotation with uniformly bounded deviations. Then f is semiconjugate to the irrational rotation R_α on $\mathbb{T}^1$.*

Proof. Due to (3.2), the sets $A_r = \bigcup_{n \in \mathbb{Z}} \widehat{F}^n(\{r - n\alpha\} \times \mathbb{T}^1)$ are compact, and it is easy to check that they satisfy $F(A_r) = A_{r+\alpha}$ and $A_{r+1} = T(A_r)$, where $T : \mathbb{A} \to \mathbb{A}$, $(x,y) \mapsto (x+1, y)$. Moreover, these relations carry over to the circloids $C_r = \mathcal{C}^+(A_r)$. In addition, the monotonicity of the construction implies $C_r \preccurlyeq C_s$ whenever $r \leqslant s$. We now define

$$H : \mathbb{A} \to \mathbb{R} \quad , \quad z \mapsto \sup\{r \in \mathbb{R} \mid z \in \mathcal{U}^+(C_r)\} .$$

Then $H \circ F(z) = \sup\{r \in \mathbb{R} \mid F(z) \in \mathcal{U}^+(C_r)\} = \sup\{r \in \mathbb{R} \mid z \in \mathcal{U}^+(C_{r-\alpha})\} = H(z) + \alpha$. In the same way, one can see that $H \circ T = T \circ H$. Hence, H projects to a map $h : \mathbb{T}^2 \to \mathbb{T}^2$ that satisfies $h \circ f = R_\alpha \circ h$. If h is continuous, then it follows immediately from the minimality of R_α that h is also onto. Thus, it only remains to check the continuity of h.

In order to do so, however, it suffices to prove that the circloids $C_r \subseteq \mathbb{A}$ with $r \in \mathbb{R}$ are pairwise disjoint. This fact is shown on [6, page 615] (*Construction of the semi-conjugacy*), and we refrain from repeating the argument here. Thus, in order to complete the proof, it remains to show the disjointness of the circloids.

To that end, let $r' < s'$ and suppose without loss of generality that $s' - r' \leqslant 1$. Let $r = (2r' + s')/3$ and $s = (r' + 2s')/3$. If $C_r = C_s$, then $C_t = C_r$ for all $t \in (r, s)$, and we may choose t of the form $t = r + p\alpha - q$ for suitable integers n, m. Then $C_r = C_{r+p\alpha-q} = T^{-q} \circ \widehat{F}^p(C_r)$, which implies that $\rho_v(F) = \{p/q\}$, contradicting the irrationality of α.

Hence, we have $C_r \neq C_s$, and due to Lemma 2.2 we can find an open disk $\widehat{U} \subseteq (C_r, C_s)$. Let $U = \pi(\widehat{U})$. According to Proposition 2.6, there exists an integer $n \in \mathbb{N}$ such that $f^n(U) \cup U$ is essential. This implies in particular that $f^n(U) \cap U \neq \varnothing$, so that we can choose $m \in \mathbb{N}$ such that $\widehat{V} = T^{-m} \circ \widehat{F}^n(\widehat{U})$ intersects $\widehat{U}$.

Since $\widehat{U} \subseteq (C_r, C_s)$ and $\widehat{V} \subseteq (C_{r+n\alpha-m}, C_{s+n\alpha-m})$, this is only possible if $|n\alpha - m| < (s' - r')/3$. This yields that $\widehat{U} \cup \widehat{V} \subseteq (C_{r'}, C_{s'})$, and as a consequence $\widehat{V}$ cannot intersect any translates of $\widehat{U}$. Since $U \cup f^n(U)$ is essential, this means $\widehat{U} \cup \widehat{V}$ is essential as well. We thus obtain an essential open set between $C_{r'}$ and $C_{s'}$ which separates the two circloids, so that these must be disjoint. Since $r' < s'$ were arbitrary, this completes the proof. $\quad\square$

4   A counterexample in the annular case

In this section we sketch the construction of an area-preserving transitive annular irrational pseudo-rotation of the 2-torus with uniformly bounded deviations that is not semi-conjugated to any irrational rotation.

To begin, Besicovitch in [15](see also [16]) has shown the existence of a transitive homeomorphism $g : \mathbb{T}^1 \times \mathbb{R}$, which is a skew-product over an irrational rotation, that is, g is of the form $g(x, y) = (x + \alpha, y + \phi(x))$ with α irrational. We claim that the only possible

semi-conjugacies between g and an irrational rotation are the trivial ones, given by the projection onto the first coordinate composed with a uniform rotation. Indeed, if (x_0, y_0) is a point with dense forward orbit, then for any (x, y) there exists a sequence $(n_k)_{k \in \mathbb{N}}$ such that $g^{n_k}(x_0, y_0)$ converges to (x, y) and, in particular, $n_k \alpha$ converges to $x - x_0$. If $h : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1$ is the map semiconjugating g to a rotation R_α, then $h(x_0, y_0) - (x - x_0) = \lim_{k \to \infty} R_\alpha^{n_k} h(x_0) = \lim_{k \to \infty} h(g^{n_k}(x_0, y_0)) = h(x, y)$ and so $h(x, y) - h(x_0, y_0) = x - x_0$.

Now, let $\psi : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times (-1, 1)$ be defined as

$$\psi(x, y) = (x + \sin(\log|y| + 1), \frac{y}{|y| + 1}).$$

One can easily verify that, if f is the homeomorphism of $\mathbb{T}^1 \times (-1, 1)$ given by $f = \psi \circ g \circ \psi^{-1}$, then f can be extended to a homeomorphism of the closed annulus $\mathbb{A} = \mathbb{T}^1 \times [-1, 1]$ by defining $f(x, 1) = (x + \alpha, 1)$ and $f(x, -1) = (x + \alpha, -1)$. It is also immediate that f has uniformly bounded deviations from the rigid rotation by α, and that the ψ image of any fiber $\{x\} \times \mathbb{R}$ accumulates on all boundary points of $\mathbb{A}$.

Now, if by contradiction there exists h that semi-conjugates f to the rotation R_α, then $h \circ \psi$ semi-conjugates g to R_α. This implies that, for each $\theta \in \mathbb{T}^1$, $h^{-1}(\theta)$ must contain $\psi(x, \mathbb{R})$ for some x, and by continuity $h^{-1}(\theta)$ must contain both boundaries of $\mathbb{A}$, which is impossible since the image of a fiber of the semiconjugation must be disjoint from its iterates.

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