

IRREGULAR MODEL SETS AND TAME DYNAMICS

G. FUHRMANN¹, E. GLASNER², AND T. JÄGER³ AND C. OERTEL³

ABSTRACT. We study the dynamical properties of irregular model sets and show that the translation action on their hull always admits an infinite independence set. The dynamics can therefore not be tame and the topological sequence entropy is strictly positive. Extending the proof to a more general setting, we further obtain that tame implies regular for almost automorphic group actions on compact spaces.

In the converse direction, we show that even in the restrictive case of Euclidean cut and project schemes irregular model sets may be uniquely ergodic and have zero topological entropy. This provides negative answers to questions by Schlottmann and Moody in the Euclidean setting.

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1. INTRODUCTION

In the mathematical theory of quasicrystals and aperiodic order, one of the major constructions of aperiodic structures is the *cut and project method*, introduced by Meyer in the context of algebraic number theory [1]. The aim of this work is to contribute to a better understanding of the relations between the different ingredients in this construction and the dynamical properties of the resulting Delone dynamical systems. More precisely, we study irregular model sets which are obtained when the compact window in the cut and project construction has a positive measure boundary. In contrast to regular model sets, whose dynamics and diffraction theory are rather well-understood [2–5], the description of their irregular counterparts is still far from being satisfactory. As a byproduct, it turns out that the cut and project method also provides an alternative approach to problems in symbolic and topological dynamics outside the classical focus of aperiodic order. Our main results can be stated as follows. We refer to Section 2 for definitions and background.

Theorem 1.1. *Suppose that $\lambda(W)$ is an irregular model set, arising from a cut and project scheme with locally compact and second countable abelian groups. Then there exists an infinite independence set for the dynamical hull, and consequently the translation action on the hull is not tame.*

We note that the above conclusions also hold for regular model sets whose internal group is the circle and whose window has a Cantor set boundary, see Theorem 3.1. The question whether tame implies regular has actually been asked first in the more general context of topological group actions [6]. By modifying the proof of Theorem 1.1, the above result can be extended to this setting. Hence, we obtain the following positive answer to [6, Problem 5.7].

Theorem 1.2. *Suppose that (X, T) is an almost automorphic topological group action. If (X, T) is tame, then it is a regular extension of its maximal equicontinuous factor.*

Due to Theorem 1.1, non-tameness can be seen as the minimal dynamical complexity an irregular model set must exhibit. As a direct consequence, one also obtains positive topological sequence entropy. It is natural to ask if there are further or stronger dynamical implications of irregularity. In particular, Moody has raised the question whether irregular

¹ *Corresponding author.* DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, 180 QUEEN'S GATE, LONDON SW7 2AZ, UK.

²DEPARTMENT OF MATHEMATICS, TEL-AVIV UNIVERSITY, RAMAT AVIV, ISRAEL.

³INSTITUTE OF MATHEMATICS, FRIEDRICH SCHILLER UNIVERSITY JENA, GERMANY.

E-mail addresses: gabriel.fuhrmann@imperial.ac.uk, glasner@math.tau.ac.il, tobias.jaeger@uni-jena.de, christian.oertel@uni-jena.de.

model sets need to have positive topological entropy (see [7]), and Schlottmann suggested that they cannot be uniquely ergodic [2].

In the general setting of cut and project schemes, however, it is not too difficult to give negative answers to these questions. The reason is that any Toeplitz sequence can be interpreted as a model set [8], and examples of uniquely ergodic and zero entropy irregular Toeplitz flows have long been known [9, 10]. The situation is different in the more restrictive setting of Euclidean cut and project schemes, where both questions are still completely open. Using methods from low-dimensional dynamics, we construct counterexamples and obtain the following.

Theorem 1.3. *There exist irregular model sets arising from Euclidean cut and project schemes such that the translation action on the dynamical hull is uniquely ergodic and has zero topological entropy.*

In fact, we obtain two different types of examples: in the first case, the translation action is an at most two-to-one extension of its maximal equicontinuous factor and has zero entropy, but exhibits two distinct ergodic invariant measures. In the second case, the translation action is mean equicontinuous and hence, in particular, uniquely ergodic with zero entropy. In this case, the fibres of the factor map onto the maximal equicontinuous factor are almost surely countably infinite.

The paper is organised as follows. The required preliminaries are provided in Section 2. Theorems 1.1 and 1.2 are proven in separate subsections of Section 3. Readers who are mainly interested in the result on minimal group actions may directly start with Section 3.3 and continue afterwards with Proposition 3.3 which plays a crucial role in the proof of Theorem 1.2.

The remaining sections are devoted to the construction and study of uniquely ergodic and zero entropy examples in Euclidean CPS. General criteria for these dynamical properties in terms of the window structure are provided in Section 4, whereas the actual construction of windows with the required properties is carried out in Section 5. Finally, we discuss some implications of these constructions for the (non-continuous) dependence of entropy on the window and the diffraction spectra of our examples in Sections 5 and 6, respectively.

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2. PRELIMINARIES

2.1. Some topological dynamics. Let (X, T, ϕ) be a *topological dynamical system*, that is, T is a topological group, X a Hausdorff topological space and ϕ a continuous left action of T on X by homeomorphisms on X . We write tx for the image $\phi(t, x)$ of the action of $t \in T$ on $x \in X$. Most of the time, we keep the action ϕ implicit and simply refer to (X, T) as a topological dynamical system. In all of the following, X is assumed to be compact metric. (X, T) is called *minimal* if the orbit of every point $x \in X$ is dense in X , that is, $\overline{Tx} = X$. We say that (X, T) is *equicontinuous* when the action of T (considered as a collection of self-maps on X) is equicontinuous. It is well-known that this is the case if and only if the metric on X can be chosen invariant under the action of T . Some of the examples constructed in this work show a closely related but less rigid dynamical behaviour referred to as *mean equicontinuity*. We refer the reader to the literature (e.g., [11–15]) for more information on mean equicontinuous systems.

Given a topological dynamical system (X, T) , a Borel probability measure μ on X is called *T -invariant* if $\mu(\cdot) = \mu(t \cdot)$ for all $t \in T$. Recall that two T -invariant measures μ_1 and μ_2 on X coincide if and only if $\int_X f d\mu_1 = \int_X f d\mu_2$ for every f from the set $\mathcal{C}(X)$ of continuous real-valued functions on X . An invariant measure is called *ergodic* if for all measurable sets $A \subseteq X$ with $tA = A$ ($t \in T$) we have $\mu(A) \in \{0, 1\}$. (X, T) is called *uniquely ergodic* if there

exists exactly one invariant measure μ . Note that in this case, the unique invariant measure μ is ergodic.

Suppose (X, T) and (H, T) are topological dynamical systems. Then (H, T) is called a *factor* of (X, T) if there exists a continuous onto map $\beta : X \rightarrow H$ such that $\beta(tx) = t\beta(x)$ for all $t \in T$. The map β is called a *factor map* in this situation and the preimages of singletons under β are referred to as its *fibres*. If β is bijective, it is called an *isomorphism* and (X, T) and (H, T) are said to be *isomorphic*. It is well-known that factor maps preserve minimality and unique ergodicity.

Given a topological dynamical system (X, T) , the *Ellis semigroup* $E(X)$ associated to (X, T) is defined as the closure of $\{x \mapsto tx \mid t \in T\} \subseteq X^X$ in the product topology, where the (semi-)group operation is given by the composition. On $E(X)$, we may consider the T -action given by $E(X) \ni \tau \mapsto t\tau$ for each element $t \in T$.

Theorem 2.1 ([16, pp. 52–53]). *Suppose H is a compact metric space and (H, T) is minimal and equicontinuous. Then $E(H)$ is a compact metrisable topological group. Further, we have the following.*

- (a) *If T is abelian, then $E(H)$ is abelian and (H, T) is isomorphic to $(E(H), T)$.*
- (b) *For general T , (H, T) is a factor of $(E(H), T)$, where the factor map π is given by*

$$\pi : E(H) \rightarrow H, \quad \tau \mapsto \tau h$$

for some fixed $h \in H$. In particular, π is open.

Remark 2.2. Throughout this article, abelian groups are always denoted as additive groups, whereas general (possibly non-commutative) groups are multiplicative.

A topological dynamical system (H, T) is called a *maximal equicontinuous factor (MEF)* of (X, T) if it is an equicontinuous factor of (X, T) with the additional property that every other equicontinuous factor of (X, T) is also a factor of (H, T) .

Lemma 2.3 ([16, p. 125, Theorem 1]). *Suppose (X, T) is a topological dynamical system with X compact metric. Then (X, T) has a unique (up to conjugacy) MEF (H, T) with H compact metric.*

Given metric spaces X and H , a continuous map $\beta : X \rightarrow H$ is called *almost one-to-one* if

$$X_0 = \{x \in X \mid \beta^{-1}(\{\beta(x)\}) = \{x\}\}$$

is dense in X . Points in X_0 are called *injectivity points* of β . If β is an almost one-to-one factor map between topological dynamical systems (X, T) and (H, T) , then (X, T) is called an *almost one-to-one extension* of (H, T) . It is easy to see that the sets X_0 and $\beta(X_0)$ are residual subsets of X and H , respectively. Moreover, observe that if (H, T) is minimal, then (X, T) is also minimal.

The following elementary fact about almost one-to-one maps will be useful. Recall that a compact set $W \subseteq X$ is called *proper* if $\text{int}(W) = W$.

Lemma 2.4. *Suppose that X and H are metric spaces and $\beta : X \rightarrow H$ is almost one-to-one. Then images of proper subsets of X under β are proper subsets of H .*

Suppose now that (H, T) is equicontinuous and minimal. Then (H, T) is uniquely ergodic with a unique T -invariant measure μ . If T is abelian, we may assume $H = E(H)$ and obtain $\mu = \Theta_H$ where Θ_H denotes the Haar measure on H . In general, μ equals $\Theta_{E(H)} \circ \pi^{-1}$, where π is as in Theorem 2.1 and $\Theta_{E(H)}$ denotes the (left) Haar measure on $E(H)$ (which, since $E(H)$ is compact, coincides with the right Haar measure [17, Theorem 15.13]). In both cases, an almost one-to-one extension (X, T) of (H, T) is called *regular*, if the projection $H_0 = \beta(X_0)$ of the set of injectivity points of β has positive μ -measure. Otherwise, it is called *irregular*. In the regular case, (X, T) is uniquely ergodic and measure-theoretically isomorphic to (H, T) (with respect to the unique invariant measure). Clearly, by ergodicity of μ , the set H_0 has full measure in this case. We call a group action (X, T) *almost automorphic* if it is an almost one-to-one extension of its MEF and the MEF is minimal which, by the above observations, implies that (X, T) is minimal, too.

A system (X, T) is called *tame* if the cardinality of its Ellis semigroup $E(X)$ is at most 2^{\aleph_0} , and *non-tame* or *wild* otherwise [18, 19]. A structure theorem for minimal tame systems has

been established in [6] (see also [20–22]). If (X, T) allows for an invariant measure, it simplifies to the following statement.

Theorem 2.5 ([6, Corollary 5.4]). *Suppose (X, T) is a minimal and tame group action which has an invariant probability measure. Then (X, T) is an almost one-to-one extension of its maximal equicontinuous factor.*

The natural question we will focus on is whether this extension is regular or not [6, Problem 5.7]. (It is, as stated in Theorem 1.2 and proved in Section 3.3 below.) To that end, the following equivalent characterisation of tameness will be useful. We call a pair of closed and disjoint subsets $U_0, U_1 \subseteq X$ an *independence pair* if there exists an infinite set $S \subseteq T$ such that for all $a \in \{0, 1\}^S$ there is some $\xi \in X$ with

$$t\xi \in U_{a_t} \quad (t \in S).$$

Theorem 2.6 ([22, Proposition 6.4]). *A topological dynamical system (X, T) is non-tame if and only if there exists an independence pair.*

In the case of symbolic dynamics (with the left shift denoted by $\sigma : \{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}$), there is the following immediate

Corollary 2.7. *Suppose $\Sigma \subseteq \{0, 1\}^{\mathbb{Z}}$ is a subshift (that is, closed and shift-invariant) and there exists an infinite set $S \subseteq \mathbb{Z}$ such that for every $a \in \{0, 1\}^S$ there is some $\xi \in \Sigma$ with $\xi_s = a_s$ for all $s \in S$. Then (Σ, σ) is non-tame.*

Proof. Let $U_0 = [0]$ and $U_1 = [1]$ be the cylinder sets of length one (at position 0) in $\{0, 1\}^{\mathbb{Z}}$. By the assumptions, (U_0, U_1) forms an independence pair. \square

A very similar statement holds in the case of model sets (see Corollary 2.10 below).

2.2. Topological entropy. In the following, T denotes a non-compact, locally compact second countable abelian group with Haar measure Θ_T . Let $(A_n)_{n \in \mathbb{N}}$ be a van Hove sequence in T , that is, $(A_n)_{n \in \mathbb{N}}$ is an exhausting sequence of relatively compact subsets of T such that $\overline{A_n} \subseteq A_{n+1}$ and for every compact $K \subseteq T$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{\Theta_T(A_n)} \Theta_T(\partial^K(A_n)) = 0,$$

where $\partial^K(A_n) = ((K + A_n) \setminus \text{int}(A_n)) \cup ((\overline{T \setminus A_n} + K) \cap A_n)$ is the K -boundary of A_n .¹ We say the van Hove sequence is *tempered* if there exists $C \geq 1$ such that for all $n \in \mathbb{N}$ the estimate $\Theta_T\left(\bigcup_{k=1}^{n-1} A_k^{-1} A_n\right) \leq C \Theta_T(A_n)$ holds. It is worth mentioning that every van Hove sequence admits a tempered subsequence (see [23, Proposition 1.4]).

In the following definitions, we keep the dependence on $(A_n)_{n \in \mathbb{N}}$ implicit. Given a topological dynamical system (X, T, ϕ) , with (X, d) a compact metric space, and $C \subseteq X$, we say that a set $S \subseteq X$ is (ε, n) -spanning for C if for every $\zeta \in C$ there is some $\xi \in S$ such that

$$\max_{s \in A_n} d(\phi(s, \zeta), \phi(s, \xi)) < \varepsilon.$$

We denote the minimal cardinality of a set which (ε, n) -spans C by $S^C(\phi, \varepsilon, n)$. The *topological entropy of ϕ on C* is defined as

$$h_{\text{top}}^C(\phi) = \lim_{\varepsilon \rightarrow 0} h_{\varepsilon}^C(\phi),$$

where

$$h_{\varepsilon}^C(\phi) = \limsup_{n \rightarrow \infty} \frac{1}{\Theta_T(A_n)} \log S^C(\phi, \varepsilon, n).$$

We set $h_{\text{top}}(\phi) = h_{\text{top}}^X(\phi)$. Suppose ψ is another continuous T -action on some compact metric space H which is a factor of (X, ϕ) with a factor map β . Then $h_{\text{top}}(\phi) \geq h_{\text{top}}(\psi)$. For $\xi \in H$, we let $h_{\text{top}}^{\xi}(\phi) = h_{\text{top}}^{\beta^{-1}(\xi)}(\phi)$. Clearly, we obtain $h_{\text{top}}^{\xi}(\phi) \leq h_{\text{top}}(\phi)$ for any $\xi \in H$. In case of $T = \mathbb{R}$ we have

Theorem 2.8 ([24, Theorem 17]). *If ϕ is an \mathbb{R} -action, then $h_{\text{top}}(\phi) \leq h_{\text{top}}(\psi) + \sup_{\xi \in H} h_{\text{top}}^{\xi}(\phi)$.*

¹Observe that every van Hove sequence is also a *Følner sequence* (as defined in [23], for example).

If $h_{\text{top}}(\psi) = 0$, the preceding inequalities yield $h_{\text{top}}(\phi) = \sup_{\xi \in H} h_{\text{top}}^\xi(\phi)$, that is, positive entropy of ϕ must be realised in single fibres of β already. Note also that, in the Euclidean case, vanishing entropy with respect to one van Hove sequence implies vanishing entropy with respect to all van Hove sequences (compare [25]).

Remark 2.9. Extensions of Theorem 2.8 to \mathbb{R}^N -actions and actions of more general groups will be provided in [26]. Even for higher dimensional Euclidean CPS (see Section 2.4 for the definitions), we can therefore restrict to showing zero topological fibre entropy, in order to reduce the technicalities (see Section 5.4).

2.3. Delone dynamical systems. Let G be a non-compact, locally compact second countable abelian group with Haar measure Θ_G . Note that, by the Birkhoff-Kakutani Theorem, G is metrisable with a metric d which can be chosen to be invariant under translations on the group. Furthermore, open balls with respect d are relatively compact. A set $\Gamma \subseteq G$ is called *(r)-uniformly discrete* if there exists $r > 0$ such that $d(g, h) > r$ for all $g \neq h \in \Gamma$. Further, Γ is called *(R)-relatively dense* (or *syndetic*) if there exists $R > 0$ such that $\Gamma \cap B_R(g) \neq \emptyset$ for all $g \in G$, where $B_R(g)$ denotes the R -ball centred at g . We call Γ a *Delone set* if it is uniformly discrete and relatively dense.

Given $\rho > 0$ and $g \in \Gamma$, the tuple $(B_\rho(0) \cap (\Gamma - g), \rho)$ is called a (ρ) -patch of Γ . The set of all patches of Γ is denoted by $\mathcal{P}(\Gamma)$. A Delone set Γ is said to have *finite local complexity* (FLC) if for all $\rho > 0$ the number of ρ -patches that occur is finite. Let $\mathcal{D}(G)$ denote the space of Delone subsets of a given metrisable group G . Given $\Gamma, \Gamma' \in \mathcal{D}(G)$, set

$$\text{dist}(\Gamma, \Gamma') = \inf \{ \varepsilon > 0 \mid \exists \delta \in B_\varepsilon(0) : \Gamma - \delta \cap B_{1/\varepsilon}(0) = \Gamma' \cap B_{1/\varepsilon}(0) \}.$$

Then $d(\Gamma, \Gamma') = \min\{1/\sqrt{2}, \text{dist}(\Gamma, \Gamma')\}$ defines a metric on $\mathcal{D}(G)$ (see [3, Section 2] as well as [27, Remark 2.10 (ii)]).

If $\mathcal{D}_{r,R}(G) \subseteq \mathcal{D}(G)$ denotes the subset of Delone sets that are r -uniformly discrete and R -relatively dense with fixed $r, R > 0$, then $(\mathcal{D}_{r,R}(G), d)$ is a compact metric space. Clearly, $\mathcal{D}_{r,R}(G)$ is invariant under the translation action $\varphi : (g, \Gamma) \mapsto \Gamma - g$. It follows that for any Delone set $\Gamma \subseteq G$ with FLC, the dynamical hull $\Omega(\Gamma) = \text{cl}(\{\Gamma - g \mid g \in G\})$ is compact ([2, Proposition 2.3]). The G -action $(\Omega(\Gamma), G)$ given by the translation φ is called a *Delone dynamical system*.

Finally, we will need a criterion for non-tameness analogous to Corollary 2.7. Suppose that Ω is a translation-invariant subset of $\mathcal{D}(G)$. We say Ω *admits an (infinite) free set* or *(infinite) independence set* $S \subseteq G$ if there exists a uniformly discrete set $\Lambda \subseteq G$ with $S \subseteq \Lambda$ such that for all $P \subseteq S$ there exists some $\Gamma \in \Omega$ with $\Gamma \subseteq \Lambda$ and $\Gamma \cap P = \emptyset$.

Corollary 2.10. *Suppose G is a locally compact second countable group, $\Omega \subseteq \mathcal{D}(G)$ is compact and translation-invariant and (Ω, G) denotes the action of G on Ω by translations. If Ω admits an infinite free set, then (Ω, G) is non-tame.*

Proof. Suppose $S \subseteq \Lambda$ are as above and Λ is r -uniformly discrete for some $r > 0$. Set $U_0 = \{\Gamma \in \Omega \mid \Gamma \cap B_r(0) = \emptyset\}$ and $U_1 = \{\Gamma \in \Omega \mid \Gamma \cap \overline{B_{r/2}(0)} \neq \emptyset\}$. By the assumptions, U_0 and U_1 form an independence pair. \square

2.4. Cut and project schemes and the torus parametrisation. We refer to standard references such as [1, 2, 5, 28–30] for the following basic facts. A *cut and project scheme* (CPS) consists of a triple (G, H, \mathcal{L}) of two locally compact abelian groups G (called *external group*) and H (*internal group*) and a co-compact discrete subgroup $\mathcal{L} \subseteq G \times H$ such that the natural projections $\pi_G : G \times H \rightarrow G$ and $\pi_H : G \times H \rightarrow H$ satisfy

- (i) the restriction $\pi_G|_{\mathcal{L}}$ is injective;
- (ii) the image $\pi_H(\mathcal{L})$ is dense.

If (i) and (ii) hold, we call \mathcal{L} an *irrational lattice*. As a consequence of (i), if we let $L = \pi_G(\mathcal{L})$ and $L^* = \pi_H(\mathcal{L})$, the *star map*

$$* : L \rightarrow L^* : l \mapsto l^* = \pi_H \circ \pi_G|_{\mathcal{L}}^{-1}(l)$$

is well-defined and surjective. Given a compact set $W \subseteq H$ (referred to as *window*), we define the point set

$$\lambda(W) = \pi_G(\mathcal{L} \cap (G \times W)) = \{l \in L \mid l^* \in W\}.$$

Since W is compact, $\lambda(W)$ is uniformly discrete and has FLC, and if W has non-empty interior, then $\lambda(W)$ is relatively dense. Hence, if W is *proper*, $\lambda(W)$ is Delone and has FLC. In this case, we call $\lambda(W)$ a *model set*. The window (and also the resulting model set) is called *regular* if $\Theta_H(\partial W) = 0$, otherwise it is called *irregular*. We say a subset $W \subseteq H$ is *irredundant*² if $\{h \in H \mid h + W = W\} = \{0\}$. Note that if ∂W is irredundant, then W is irredundant, too. A CPS is called *Euclidean* if $G = \mathbb{R}^N$ and $H = \mathbb{R}^M$ for some $M, N \in \mathbb{N}$, and *planar* if $N = M = 1$.

Since \mathcal{L} is a lattice in $G \times H$, the quotient $\mathbb{T} = (G \times H)/\mathcal{L}$ is a compact abelian group. A natural G -action on \mathbb{T} is given by $\omega : (u, [s, t]_{\mathcal{L}}) \mapsto [s + u, t]_{\mathcal{L}}$. Here, $[s, t]_{\mathcal{L}}$ denotes the equivalence class of $(s, t) \in G \times H$ in \mathbb{T} . Observe that due to the assumptions on (G, H, \mathcal{L}) , this action is minimal. Further, if the window $W \subseteq H$ is irredundant, (\mathbb{T}, G) is the maximal equicontinuous factor of the Delone dynamical system $(\Omega(\lambda(W)), G)$. The respective factor map β is also referred to as *torus parametrisation*.

Given an irredundant window W , the fibres of the torus parametrisation are characterised as follows: For $\Gamma \in \Omega(\lambda(W))$, we have

$$\Gamma \in \beta^{-1}([s, t]_{\mathcal{L}}) \Leftrightarrow \lambda(\text{int}(W) + t) - s \subseteq \Gamma \subseteq \lambda(W + t) - s \quad (2.1)$$

as well as

$$\Gamma \in \beta^{-1}([0, t]_{\mathcal{L}}) \Leftrightarrow \exists (t_j) \in L^{*\mathbb{N}} \text{ with } \lim_{j \rightarrow \infty} t_j = t \text{ and } \lim_{j \rightarrow \infty} \lambda(W + t_j) = \Gamma. \quad (2.2)$$

In particular, if $(\partial W + t) \cap L^* = \emptyset$, we have $\lambda(\text{int}(W) + t) - s = \lambda(W + t) - s$ so that $\beta^{-1}([s, t]_{\mathcal{L}})$ is a singleton for each s . We denote the set of such $t \in H$ by \mathcal{G}_W , that is,

$$\mathcal{G}_W = - \left(H \setminus \bigcup_{\ell^* \in L^*} \partial W - \ell^* \right).$$

Note that \mathcal{G}_W is residual. Hence, $(\Omega(\lambda(W + t) - s), G)$ is an almost automorphic system if $t \in \mathcal{G}_W$. In this case, an (ir)regular model set $(\Omega(\lambda(W + t) - s), G)$ is an (ir)regular extension of (\mathbb{T}, G) . Thus, the present notion of (ir)regularity is consistent with that of Section 2.1.

Remark 2.11. Let $\Gamma, \Gamma' \in \beta^{-1}([s, t]_{\mathcal{L}})$ and $\varepsilon > 0$. Recall that $d(\Gamma, \Gamma') < \varepsilon$ if and only if Γ and Γ' coincide on $B_{1/\varepsilon}(0)$ up to a small translation $\delta \in B_\varepsilon(0)$. Observe that $\lambda(W + t) - s$ is r -uniformly discrete with $r = \min_{l \neq l' \in \lambda(W + t)} d_G(l, l')$. Hence, if $\varepsilon < \frac{r}{2}$, equation (2.1) immediately yields $\delta = 0$, i.e., $d(\Gamma, \Gamma') < \varepsilon'$ if and only if $\Gamma \cap B_{1/\varepsilon'}(0) = \Gamma' \cap B_{1/\varepsilon'}(0)$.

Remark 2.12. Note that if W is not irredundant, it is possible to construct a CPS (G, H', \mathcal{L}') with irredundant window $W' \subseteq H'$ such that for each $\Lambda \in \Omega(\lambda(W))$ with $\lambda(\text{int}(W)) \subseteq \Lambda \subseteq \lambda(W)$ we have $\lambda(\text{int}(W')) \subseteq \Lambda \subseteq \lambda(W')$ (compare [30, Section 5] and [5, Lemma 7]). Thus, in the following, we may assume that all occurring windows are irredundant.

3. TAME IMPLIES REGULAR

The aim of this section is to prove that tame implies regular for model sets, minimal subshifts and general minimal topological actions. The result on subshifts will be obtained as a special case of the one for model sets, whereas the case of general group actions requires some modifications to adapt the proof to the more general setting. We should also point out that the information obtained for irregular model sets is slightly stronger than in the general case, since the existence of an infinite free set –as defined in Section 2.3– is stronger than the existence of an independence pair.

3.1. The case of model sets.

Theorem 3.1. *Suppose that (G, H, \mathcal{L}) is a CPS with locally compact and second countable abelian groups G, H . Denote by Θ_H the Haar measure on H . If W is a proper window with $\Theta_H(\partial W) > 0$, or if $H = \mathbb{T}^1 (= \mathbb{R}/\mathbb{Z})$ and ∂W is a Cantor set, then $\Omega(\lambda(W))$ admits an infinite free set $S \subseteq G$.*

²This notion corresponds to that of *separating covers* in the context of almost automorphic symbolic dynamical systems [31] or, in the non-symbolic case, to the concept of *being invariant under no rotation* in the setting of semi-cocycle extensions (see, for example, [32, Section 5]).

The proof will be based on the following criterion for the existence of infinite free sets, which translates the dynamical problem into a purely geometric question about the structure of the window.

Lemma 3.2. *Suppose that (G, H, \mathcal{L}) is a CPS that satisfies the above assumptions and there exists a relatively compact set $S^* \subseteq L^*$ such that for each $P^* \subseteq S^*$ we have*

$$H(S^*, P^*) = \bigcap_{s^* \in P^*} (W - s^*) \cap \bigcap_{s^* \in S^* \setminus P^*} (W^c - s^*) \cap (-\mathcal{G}_W) \neq \emptyset. \quad (3.1)$$

Then the set $S = \{s \in G \mid s^* \in S^*\}$ is free.

Proof. Let $P \subseteq S$ and choose $h \in H(S^*, P^*)$. Then $\lambda(W - h) \in \Omega(\lambda(W))$ since $-h \in \mathcal{G}_W$ (see Section 2.4). Further, for any $s \in S$, we have

$$s \in P \Leftrightarrow h \in W - s^* \Leftrightarrow s^* \in W - h \Leftrightarrow s \in \lambda(W - h).$$

Therefore, $\lambda(W - h) \cap S = P$. Note that if $P^* \neq \emptyset$, then $H(S^*, P^*) \subseteq W - P^* \subseteq W - S^*$. Hence, we may assume without loss of generality that the points h from above belong to the compact set $V := \overline{W - S^*} \cup \{h_0\} \subseteq H$, where h_0 is some point in $(W^c - S^*) \cap (-\mathcal{G}_W)$. Clearly, $\lambda(W - h) \subseteq \lambda(W - V) =: \Lambda$ for $h \in V$. Since $W - V$ is a compact window, the set Λ is uniformly discrete. As $P \subseteq S$ was arbitrary, we obtain that $S \subseteq \Lambda$ is a free set. \square

Hence, in order to prove Theorem 3.1, it suffices to prove the existence of an infinite set S^* that satisfies the assumptions of the previous lemma. With a view towards the later extension of the proof to topological group actions in Section 3.3, we reformulate the required statement in a slightly more abstract form.

Proposition 3.3. *Suppose that H is a locally compact second countable Hausdorff topological group with left Haar measure Θ_H and $V_0, V_1 \subseteq H$ are closed subsets that satisfy*

- (i) $\text{int}(\overline{V_0}) = V_0$ and $\text{int}(\overline{V_1}) = V_1$,
- (ii) $\text{int}(V_0) \cap \text{int}(V_1) = \emptyset$,
- (iii) $\Theta_H(V_0 \cap V_1) > 0$.

Further, assume that $T \subseteq H$ is a dense subgroup and $\mathcal{G} \subseteq H$ is a residual set. Then there exists an infinite set $I \subseteq T$ such that for all $a \in \{0, 1\}^I$ there exists $h \in \mathcal{G}$ with the property that

$$th \in \text{int}(V_{a_t}) \quad (t \in I). \quad (3.2)$$

The same result holds if $H = \mathbb{T}^1$ and (iii) is replaced by the assumption that $V_0 \cap V_1$ is a Cantor set.

Remark 3.4. In the situation of Theorem 3.1, we can apply this statement with $V_1 = W$, $V_0 = \overline{W^c}$, $T = L^*$ and $\mathcal{G} = -\mathcal{G}_W$. This yields an infinite set I that satisfies the assertions of the proposition. Moreover, for $a_t = 1$ ($t \in I$), equation (3.2) yields $h \in \mathcal{G}$ with $h + I \subseteq V_1 = W$ which, by the compactness of W , gives that I is relatively compact. Hence, $S^* = I$ satisfies the assumptions of Lemma 3.2, and this proves Theorem 3.1. (Note that h in (3.2) is contained in the respective intersection in (3.1) with $P^* = \{s^* \in S^* \mid a_{s^*} = 1\}$.)

For the proof of the proposition, we need some measure-theoretic estimates concerning intersections of translates of $V_0 \cap V_1$. We denote the right Haar-measure on H by Θ_H^r . Recall that Θ_H^r (as well as the left Haar measure Θ_H) on a locally compact second countable group H is outer regular. Hence, if $C \subseteq H$ is a Borel set of positive measure and we set

$$\eta^C(\varepsilon) = \frac{\Theta_H^r(B_\varepsilon(C))}{\Theta_H^r(C)} - 1,$$

then $\lim_{\varepsilon \rightarrow 0} \eta^C(\varepsilon) = 0$.

Let $\Sigma_n = \{0, 1\}^n$ (with $n \in \mathbb{N}$) and $\Sigma_* = \bigcup_{n \in \mathbb{N}} \Sigma_n$. Denote by $|a|$ the length of a word $a \in \Sigma_*$ (so that $a \in \Sigma_{|a|}$ for all $a \in \Sigma_*$). In the following, we assume the metric d on the group H to be invariant under multiplication from the left. We denote the neutral element of H by e .

Lemma 3.5. *Suppose that $C \subseteq H$ is a Borel set with $\Theta_H^r(C) > 0$ and $(\xi_a)_{a \in \Sigma_*}$ is a family of elements $\xi_a \in H$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers such that*

$$\varepsilon_n \geq \sup_{a \in \Sigma_n} d(e, \xi_a).$$

For $j \in \mathbb{N}$, $n \in \mathbb{N} \cup \{\infty\}$, let $\delta_j^n = \sum_{\ell=j}^n \varepsilon_\ell$. Further, given $n \in \mathbb{N}$ and $a \in \Sigma_n$, let $\gamma_a = \prod_{j=1}^n \xi_{a_1, \dots, a_j} = \xi_{a_1} \xi_{a_1, a_2} \dots \xi_{a_1, \dots, a_n}$. Then for each $n \in \mathbb{N}$, we have

$$\Theta_H^r \left(\bigcap_{a \in \Sigma_n} C\gamma_a^{-1} \right) \geq \Theta_H^r(C) \cdot \left(1 - \sum_{j=1}^n 2^{j-1} \eta^C(\delta_j^n) \right). \quad (3.3)$$

Proof. We proceed by induction on n .

Base case ($n = 1$): Note that $\Sigma_1 = \{0, 1\}$, $\gamma_0 = \xi_0$, $\gamma_1 = \xi_1$, $\delta_1^1 = \varepsilon_1$ and $d(e, \xi_0^{-1}), d(e, \xi_1^{-1}) < \varepsilon_1$. Both $C\gamma_0^{-1}$ and $C\gamma_1^{-1}$ are sets of measure $\Theta_H^r(C)$ that are contained in $B_{\delta_1^1}(C)$ which is of measure $(1 + \eta^C(\delta_1^1)) \cdot \Theta_H^r(C)$. This implies that $\Theta_H^r((C\gamma_0^{-1}) \cap (C\gamma_1^{-1})) \geq \Theta_H^r(C) \cdot (1 - \eta^C(\delta_1^1))$ as required.

Inductive step ($n \rightarrow n+1$): Suppose the statement holds for some $n \in \mathbb{N}$, all sets $C \subseteq H$, and all collections $(\xi_a)_{a \in \Sigma_n}$ as well as all sequences (ε_n) as above. Given $a \in \Sigma_{n+1}$, let $\xi'_a = \xi_{0a}$ and $\xi''_a = \xi_{1a}$ and define γ'_a, γ''_a accordingly. Then

$$\bigcap_{a \in \Sigma_{n+1}} C\gamma_a^{-1} = \left(\underbrace{\left(\bigcap_{a' \in \Sigma_n} C\gamma_{a'}^{-1} \right) \xi_0^{-1}}_{=: I'} \right) \cap \left(\underbrace{\left(\bigcap_{a'' \in \Sigma_n} C\gamma_{a''}^{-1} \right) \xi_1^{-1}}_{=: I''} \right).$$

By the induction hypothesis, both $I' \xi_0^{-1}$ and $I'' \xi_1^{-1}$ are sets of measure

$$\Theta_H^r(I'), \Theta_H^r(I'') \geq \Theta_H^r(C) \cdot \left(1 - \sum_{j=1}^n 2^{j-1} \eta^C(\delta_{j+1}^{n+1}) \right)$$

that are contained in $B_{\delta_1^{n+1}}(C)$. Hence, we obtain that

$$\begin{aligned} \Theta_H^r((I' \xi_0^{-1}) \cap (I'' \xi_1^{-1})) &\geq \Theta_H^r(C) \cdot \left(1 - \eta^C(\delta_1^{n+1}) - 2 \cdot \sum_{j=1}^n 2^{j-1} \eta^C(\delta_{j+1}^{n+1}) \right) \\ &= \Theta_H^r(C) \cdot \left(1 - \sum_{j=1}^{n+1} 2^{j-1} \eta^C(\delta_j^{n+1}) \right). \end{aligned}$$

This completes the proof. \square

We can now turn to the

Proof of Proposition 3.3. Let $\mathcal{G} = \bigcap_{n \in \mathbb{N}} G_n$, where each G_n is an open and dense subset of H . We will construct a sequence $(t_n)_{n \in \mathbb{N}}$ of points in T and a collection $(U_a)_{a \in \Sigma_n}$ of compact subsets of H with the following properties for all $n \in \mathbb{N}$ and $a \in \Sigma_n$

- (I1) $U_a \subseteq (t_n^{-1} \text{int}(V_0)) \cap G_n$ if $a_n = 0$ and $U_a \subseteq (t_n^{-1} \text{int}(V_1)) \cap G_n$ if $a_n = 1$;
- (I2) $U_{a_0} \cup U_{a_1} \subseteq U_a$.

This will prove the statement: if we define $I = \{t_n \mid n \in \mathbb{N}\}$, then for any given $a \in \{0, 1\}^{\mathbb{N}}$ we let $a^{(n)} = (a_{t_1}, \dots, a_{t_n})$ and obtain from (I2) that $\bigcap_{n \in \mathbb{N}} U_{a^{(n)}}$ is a nested intersection of compact sets and therefore non-empty. By (I1), any $h \in \bigcap_{n \in \mathbb{N}} U_{a^{(n)}}$ has the property that $h \in \mathcal{G}$ and $t_n h \in \text{int}(V_{a_{t_n}})$ for all $n \in \mathbb{N}$, as required by (3.2).

We are going to construct $(t_n)_{n \in \mathbb{N}}$ and $(U_a)_{a \in \Sigma_n}$ by induction on $n = |a|$. Let us first specify some details. We will choose U_a as closed balls of the form $U_a = B_{r(|a|)}(\gamma_a)$ which we can ensure to be compact by choosing $r(n)$ sufficiently small. We set $\xi_0 = \gamma_0$, $\xi_1 = \gamma_1$ and $\xi_{a_0} = \gamma_a^{-1} \gamma_{a_0}$, $\xi_{a_1} = \gamma_a^{-1} \gamma_{a_1}$ for $a \in \Sigma_n$, $n \geq 1$. By definition, we hence have $\gamma_a = \prod_{j=1}^n \xi_{a_1, \dots, a_j}$ which is consistent with the notation of Lemma 3.5. Further, we let $C = V_0 \cap V_1$. Observe that if $\Theta_H(C) > 0$, then $\Theta_H^r(C) > 0$ since Θ_H and Θ_H^r are mutually absolutely continuous [17, Theorem 15.15 & Comment 15.27]. We fix a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that

$$\sum_{j=1}^{\infty} 2^{j-1} \eta^C(\delta_j^\infty) < 1,$$

where the δ_j^n are defined as in Lemma 3.5. We moreover include the condition

$$(I3) \sup_{a \in \Sigma_n} d(e, \xi_a) \leq \varepsilon_n$$

in the inductive assumption. Note that this boils down to choosing γ_{a0} and γ_{a1} ε_{n+1} -close to γ_a in each step of the construction.

In the case that $H = \mathbb{T}^1$ and $C = V_0 \cap V_1$ is a Cantor set, the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ and condition (I3) will not be needed. Instead, we will use the assumption

$$(I3') \text{ for all } a \in \Sigma_n \text{ we have } \partial U_a \subseteq t_{n+1}^{-1} C$$

in this case.

Let us first consider the case that $\Theta_H(C) > 0$.

Base case ($n = 1$): We choose $t_1 \in T$ and two open balls $U'_0 = \overline{B_{r(1)}(\xi'_0)} \subseteq \text{int}(V_0) \cap t_1(G_1 \cap B_{\varepsilon_1}(e))$ and $U'_1 = \overline{B_{r(1)}(\xi'_1)} \subseteq \text{int}(V_1) \cap t_1(G_1 \cap B_{\varepsilon_1}(e))$. If we let $U_0 = t_1^{-1} U'_0$ and $U_1 = t_1^{-1} U'_1$, then (I1) and (I3) are satisfied for $n = 1$, and (I2) is still void.

Inductive step ($n \rightarrow n + 1$): Suppose now that t_1, \dots, t_n and U_a for $a \in \bigcup_{j=1}^n \Sigma_j$ have been chosen and satisfy (I1)–(I3). Then Lemma 3.5 gives

$$\Theta_H^r \left(\bigcap_{a \in \Sigma_n} C \gamma_a^{-1} \right) \geq \Theta_H^r(C) \cdot \left(1 - \sum_{j=1}^n 2^{j-1} \eta^C(\delta_j^\infty) \right) > 0.$$

In particular, the set on the left is non-empty and we can choose $h \in \bigcap_{a \in \Sigma_n} C \gamma_a^{-1}$. Clearly, $\gamma_a \in h^{-1} C$ for all $a \in \Sigma_n$. Now, we choose $t_{n+1} \in T$ close enough to h to guarantee that $t_{n+1}^{-1} C$ intersects $B_{r'(n+1)/2}(\gamma_a) \subseteq U_a$ for all $a \in \Sigma_n$, where $r'(n+1) = \min\{\varepsilon_{n+1}, r(n)\}$. However, since points in C lie in the closure of the interior of both V_0 and V_1 , this allows to find $r(n+1) > 0$ as well as closed balls $U_{a0} = B_{r(n+1)}(\gamma_{a0})$ and $U_{a1} = B_{r(n+1)}(\gamma_{a1})$ with midpoints γ_{a0} and γ_{a1} ε_{n+1} -close to γ_a for all $a \in \Sigma_n$ such that (I1)–(I3) are satisfied for $n + 1$.

If $H = \mathbb{T}^1$ and C is a Cantor set, the statement follows in a similar way without invoking Lemma 3.5. The crucial observation here is that if we choose some Δt_{n+1} sufficiently close to zero, then the rotation by $t_{n+1} = \Delta t_{n+1} t_n$ will send one of the endpoints of each U_a , $a \in \Sigma_n$, into $\text{int}(U_a)$ (the left endpoints if Δt_{n+1} is locally to the right of zero and vice versa). Hence, we arrive at a similar situation as in the first case. \square

Altogether, we have now completed the proof of Theorem 3.1.

3.2. Application to symbolic systems. In the following, given a subshift (Σ, \mathbb{Z}) , we denote by $\beta: \Sigma \rightarrow H$ the factor map onto its MEF. Note that (H, \mathbb{Z}) is completely characterised by a self-map on H which we denote by ρ , that is, $n h = \rho^n(h)$ ($h \in H$, $n \in \mathbb{N}$).

The basis for the direct application of the results from the last section to symbolic systems is provided by the following fact.

Proposition 3.6 (Compare [8, 31]). *An almost automorphic subshift (Σ, \mathbb{Z}) is isomorphic to the system $(\Omega(\wedge(W)), \mathbb{Z})$ obtained from the CPS $(\mathbb{Z}, H, \mathcal{L})$ with lattice $\mathcal{L} = \{(n, \rho^n(h_0)) : n \in \mathbb{Z}\}$, where*

- $h_0 \in H$ has unique preimage under the factor map β ;
- $W = \beta([1])$, where $[1] = \{\xi \in \{0, 1\}^{\mathbb{Z}} : \xi_0 = 1\}$.

Moreover, the window W is proper, that is, $\overline{\text{int}(W)} = W$.

As a consequence of Theorem 3.1 and Corollary 2.7, we obtain

Corollary 3.7. *If an almost automorphic subshift (Σ, \mathbb{Z}) is irregular, then it has an infinite free set. In particular, it is non-tame. The same result holds if the maximal equicontinuous factor is an irrational circle rotation and $\beta([0]) \cap \beta([1])$ is a Cantor set.*

For the special case of Toeplitz flows (where (H, \mathbb{Z}) is an adding machine), a similar result has been established previously by Downarowicz [33]. Further, note that the existence of an infinite free set also implies positive sequence entropy.

3.3. The case of minimal group actions. Theorem 1.2 provides an analogue to Theorem 3.1 for the case of general automorphic systems. However, it is worth noting that it does not imply Theorem 3.1 as a corollary, since the existence of an infinite free set –as defined in Section 2.3– does not follow directly from non-tameness.

Proof of Theorem 1.2. Let us denote the maximal equicontinuous factor of (X, T) by (H, T) . As (H, T) is minimal and equicontinuous, Theorem 2.1 implies that (H, T) is a factor of $(E(H), T)$. We denote the corresponding factor map by π and the unique T -invariant measure on H by μ . Recall that $\mu = \Theta_{E(H)} \circ \pi^{-1}$ and π is open.

Assume for a contradiction that β is not almost surely one-to-one with respect to the measure μ on H so that (X, T) is an irregular extension of (H, T) . We aim to show the existence of an independence pair (U_0, U_1) for (X, T) which implies non-tameness by Theorem 2.6.

To that end, denote by $\mathcal{K}(X)$ the space of compact subsets of X , equipped with the Hausdorff metric d_H , and consider the mapping

$$F : H \rightarrow \mathcal{K}(X), \quad \xi \mapsto \beta^{-1}(\xi).$$

By compactness of X and continuity of β , the map F is upper semicontinuous and hence measurable. By Lusin's theorem, we may therefore choose a compact set $K \subseteq H$ of positive measure such that $F|_K$ is continuous. Let $K_0 \subseteq K$ denote the topological support of the measure $\mu|_K$ (that is, the essential closure of K). Then, $\mu(K_0) = \mu(K) > 0$. Hence, by irregularity, we can find $h_0 \in K_0$ such that $\sharp\beta^{-1}(h_0) > 1$. Moreover, we have that $\mu(V \cap K) > 0$ for any neighbourhood V of h_0 .

Choose $\xi_0 \neq \xi_1 \in \beta^{-1}(h_0)$ and let $\varepsilon = d(\xi_0, \xi_1)/4$ and $U_0 = \overline{B_\varepsilon(\xi_0)}$, $U_1 = \overline{B_\varepsilon(\xi_1)}$. We aim to show that (U_0, U_1) is an independence pair for (X, T) , that is, there is an infinite set $I \subseteq T$ such that for any $a \in \{0, 1\}^I$ there exists $\xi \in X$ with the property that

$$t\xi \in U_{a_t} \quad (t \in I). \quad (3.4)$$

Let $V_0 = \beta(U_0)$ and $V_1 = \beta(U_1)$. By Lemma 2.4, both these sets are proper, that is, $V_0 = \overline{\text{int}(V_0)}$ and $V_1 = \overline{\text{int}(V_1)}$. Moreover, they have disjoint interiors since points with singleton fibres are dense.

Due to the continuity of F on K , we can choose $\delta > 0$ such that for any $h \in B_\delta(h_0) \cap K$ we have $d_H(F(h), F(h_0)) < \varepsilon$. This yields that the fibre $F(h) = \beta^{-1}(h)$ intersects both U_0 and U_1 , so that $h \in V_0 \cap V_1$. Therefore, $B_\delta(h_0) \cap K \subseteq V_0 \cap V_1$ so that $\mu(V_0 \cap V_1) \geq \mu(B_\delta(h_0) \cap K) > 0$. Set $V'_0 = \pi^{-1}(V_0)$ and $V'_1 = \pi^{-1}(V_1)$. Since π is open, V'_0 and V'_1 are proper. Moreover, $\Theta_{E(H)}(V'_0 \cap V'_1) = \mu(V_0 \cap V_1) > 0$. Thus, the assertions of Proposition 3.3 are met by $V'_0, V'_1 \subseteq E(H)$ with $\mathcal{G} = \pi^{-1}(\beta(X_0))$, where X_0 denotes the set of injectivity points of β (observe that \mathcal{G} is residual, since π is open).

Hence, we obtain an infinite set $I \subseteq T$, and for each $a \in \{0, 1\}^I$ a point $h' \in \mathcal{G}$ such that $th' \in \text{int}(V'_{a_s})$ ($t \in I$) and hence

$$th \in \text{int}(V_{a_s}) \quad (t \in I) \quad (3.5)$$

for $h = \pi(h') \in \beta(X_0)$. However, since h has a unique preimage under β (and the same is true for all points in its orbit), (3.5) directly implies (3.4) so that (U_0, U_1) is an independence pair as claimed. \square

4. SELF-SIMILARITY AND LOCALLY DISJOINT COMPLEMENTS: TWO CRITERIA FOR ZERO ENTROPY

Let G and H be locally compact abelian second countable groups and let G be non-compact. Consider a CPS (G, H, \mathcal{L}) with proper window $W \subseteq H$ and torus parametrisation $\beta : \Omega(\lambda(W)) \rightarrow \mathbb{T}$. In this section, we provide sufficient criteria for zero entropy of $(\Omega(\lambda(W)), G)$ in terms of the local structure of W . We note that all points $[s, t]_{\mathcal{L}} \in \mathbb{T}$ are translates of $[0, t]_{\mathcal{L}}$ and hence $\sharp\beta^{-1}([s, t]_{\mathcal{L}}) = \sharp\beta^{-1}([0, t]_{\mathcal{L}})$. Therefore, in the following, it is sufficient to consider points $[0, t]_{\mathcal{L}} \in \mathbb{T}$.

4.1. Self similar windows. As a direct consequence of the discussions in Section 2.2 and Theorem 2.8, we obtain

Lemma 4.1. (i) *If $\sharp\beta^{-1}(\xi) < \infty$ for $\xi \in \mathbb{T}$, then $h_{\text{top}}^\xi(\varphi) = 0$.*
(ii) *If $G = \mathbb{R}$ and $\sharp\beta^{-1}(\xi) < \infty$ for all $\xi \in \mathbb{T}$, then $h_{\text{top}}(\varphi) = 0$.*

In view of the above lemma, it is our first goal to control the number of elements in the fibres of β . Note that the above assumptions are quite strong, since *all* fibres are assumed to be finite. Even if the measure of ∂W vanishes (so that $\sharp\beta^{-1}(\xi) = 1$ for $\Theta_{\mathbb{T}}$ -a.e. $\xi \in \mathbb{T}$), there may still exist fibres with infinite cardinality (compare the constructions in [34], [10]).

Consider a point $[0, t]_{\mathcal{L}} \in \mathbb{T}$. As a consequence of equation (2.1), Delone sets contained in $\beta^{-1}([0, t]_{\mathcal{L}})$ basically differ from each other in points l whose conjugates l^* are contained in $\partial W + t$. Hence, in case of $(\partial W + t) \cap L^* = \emptyset$, the fibre $\beta^{-1}([0, t]_{\mathcal{L}})$ is a singleton, carrying no entropy. In case $(\partial W + t) \cap L^* \neq \emptyset$, the cardinality of $\{l^* \mid l^* \in (\partial W + t) \cap L^*\}$ may be finite or not. In the first case, the cardinality of the respective fibre of $\xi = [0, t]_{\mathcal{L}}$ will be finite, so that we immediately obtain zero entropy by Lemma 4.1. Hence, we just have to investigate the latter case. Note that by Birkhoff's Ergodic Theorem, positive measure of ∂W ensures the existence of points $[0, t]_{\mathcal{L}} \in \mathbb{T}$ such that $\#\{l^* \mid l^* \in (\partial W + t) \cap L^*\} = \infty$.

We call $[s, t]_{\mathcal{L}}$ *critical* if $(\partial W + t) \cap L^* \neq \emptyset$. For a given critical point $[0, t]_{\mathcal{L}}$, we say that $l_1^*, l_2^* \in \partial W + t \cap L^*$ are *similar with respect to t* if there exists some $\varepsilon > 0$ such that

$$(B_\varepsilon(l_1^*) \cap (W + t)) - l_1^* = (B_\varepsilon(l_2^*) \cap (W + t)) - l_2^*.$$

Clearly, being similar with respect to some fixed t is an equivalence relation. We call the corresponding equivalence classes *similarity classes with respect to t* . If for each t there are only finitely many similarity classes, we call W *self similar*. If the maximal number of similarity classes for any t is k , we call W *k -self similar*. Finally, if $k = 1$, then we call W *perfectly self similar*.

Lemma 4.2. *Let $[0, t]_{\mathcal{L}} \in \mathbb{T}$. Suppose $l_1^*, l_2^* \in (\partial W + t) \cap L^*$ are similar with respect to t . Then for each $\Gamma \in \beta^{-1}([0, t]_{\mathcal{L}})$, we have $l_1 \in \Gamma$ if and only if $l_2 \in \Gamma$.*

Proof. Let $\Gamma \in \beta^{-1}([0, t]_{\mathcal{L}})$ and suppose $l_1 \in \Gamma$. Due to Equation 2.2 and FLC, there exists a sequence $t_j \in L^*$ with $t_j \rightarrow t$ such that $l_1 \in \lambda(W + t_j)$ for all j . Thus, $l_1^* \in W + t_j$. By the assumptions, we also obtain $l_2^* \in W + t_j$ for large enough j . Hence, $l_2 \in \lim_{j \rightarrow \infty} \lambda(W + t_j) = \Gamma$. By symmetry, the statement follows. \square

Lemma 4.3. *Let (G, H, \mathcal{L}) be a CPS with proper window $W \subseteq H$. If there exist k similarity classes with respect to t , then $\beta^{-1}([0, t]_{\mathcal{L}}) \leq 2^k$. Hence, if W is self similar, we have $\#\beta^{-1}(\xi) < \infty$ and therefore $h_{\text{top}}^\xi(\varphi) = 0$ for all $\xi \in \mathbb{T}$, and if W is k -self similar, then $\beta^{-1}(\xi) \leq 2^k$ for all $\xi \in \mathbb{T}$.*

Note that in particular this means that if W is perfectly self similar, then all fibres contain at most two elements, so that $(\Omega(\Lambda(W)), G)$ is a 2-1-extension of (\mathbb{T}, G) .

Proof. Fix an arbitrary $\xi = [0, t]_{\mathcal{L}} \in \mathbb{T}$. Without loss of generality, we may assume ξ to be critical. By the self similarity of W , there are finitely many equivalence classes E_1^*, \dots, E_p^* such that $(\partial W + t) \cap L^* = \bigcup_{i=1}^p E_i^*$. Due to Lemma 4.2, $\beta^{-1}(\xi) < 2^p$. By Lemma 4.1, we obtain $h_{\text{top}}^\xi(\varphi) = 0$. \square

Remark 4.4. A perfectly self similar window $W \subseteq \mathbb{R}$ (for arbitrary planar CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$) will be constructed in Section 5.3 (see Lemma 5.9).

4.2. Windows with locally disjoint complements. In the above considerations, we obtained zero entropy by ensuring that β has finite fibres. However, under certain assumptions on W , the entropy vanishes although the fibres contain infinitely many elements. We say W has *locally disjoint complements* if for all critical $[0, t]_{\mathcal{L}} \in \mathbb{T}$ and $l_1^*, l_2^* \in (\partial W + t) \cap L^*$ there exists $\varepsilon > 0$ such that

$$((B_\varepsilon(l_1^*) \cap (W + t)^c) - l_1^*) \cap ((B_\varepsilon(l_2^*) \cap (W + t)^c) - l_2^*) = \emptyset.$$

Lemma 4.5. *Suppose (G, H, \mathcal{L}) is a CPS with proper window $W \subseteq H$ and W has locally disjoint complements. Then for all critical $\xi = [0, t]_{\mathcal{L}} \in \mathbb{T}$ there is $\Gamma_+ \in \beta^{-1}(\xi)$ such that for all $\Gamma \in \beta^{-1}(\xi)$ we have that*

$$(i) \quad \Gamma \subseteq \Gamma_+,$$

$$(ii) \quad \Gamma \text{ differs from } \Gamma_+ \text{ in at most one point.}$$

Proof. Fix a critical $\xi = [0, t]_{\mathcal{L}} \in \mathbb{T}$ and let $l_0^* \in \partial W + t \cap L^*$. Due to equations (2.1) and (2.2), there exists $\Gamma' \in \beta^{-1}(\xi)$ such that $l_0 \notin \Gamma'$ and a sequence $t'_j \rightarrow t$ such that $\Gamma' = \lim_{j \rightarrow \infty} \lambda(W + t'_j)$ and $l_0^* \in (W + t'_j)^c$. Now, let $l^* \in (\partial W + t \cap L^*) \setminus \{l_0^*\}$. Since W has locally disjoint complements, there exists $\varepsilon > 0$ such that

$$0 \in (B_\varepsilon(l_0^*) \cap (W + t'_j)^c) - l_0^* \implies 0 \notin (B_\varepsilon(l^*) \cap (W + t'_j)^c) - l^*$$

for large enough j . Hence, $l^* \in W + t'_j$ for sufficiently large j which implies $l \in \Gamma'$.

As l_0^* was arbitrary, the above yields the existence of a sequence (Γ_n) in $\beta^{-1}(\xi)$ such that $\{l \mid l^* \in \partial W + t \cap L^*\} \cap B_n \subseteq \Gamma_n$. Compactness of $\beta^{-1}(\xi)$ gives a convergent subsequence with limit Γ_+ which verifies (i). (ii) follows immediately. \square

Lemma 4.6. *Let (G, H, \mathcal{L}) be a CPS with proper window $W \subseteq H$. If W has locally disjoint complements, we have $h_{\text{top}}^\xi(\varphi) = 0$ for all $\xi \in \mathbb{T}$.*

Proof. Let $\xi = [0, t]_{\mathcal{L}} \in \mathbb{T}$ be critical and (A_n) a van Hove sequence in G . By Lemma 4.5, there is $\Gamma_+ \in \beta^{-1}(\xi)$ such that every other set $\Gamma \in \beta^{-1}(\xi)$ differs from Γ_+ in one point. We denote this point by $l(\Gamma)$. For $\varepsilon > 0$ and $n \in \mathbb{N}$, define

$$\begin{aligned} S(\varphi, \varepsilon, n) &= \{\Gamma \in \beta^{-1}(\xi) \mid l(\Gamma) \in K_{1/\varepsilon} + A_n\} \cup \{\Gamma_+\} \\ &= \{\Gamma \in \beta^{-1}(\xi) \mid l(\Gamma) \in \partial^{K_{1/\varepsilon}}(A_n) \cup A_n\} \cup \{\Gamma_+\}, \end{aligned}$$

where K_r denotes the closed r -ball around 0. Observe that $\beta^{-1}(\xi) = \bigcup_{\Gamma \in S(\varphi, \varepsilon, n)} \{\Gamma' \in \beta^{-1}(\xi) \mid \max_{s \in A_n} d(\varphi_s(\Gamma), \varphi_s(\Gamma')) < \varepsilon\}$. Thus, $S(\varphi, \varepsilon, n)$ is an (ε, n) -spanning set for $\beta^{-1}(\xi)$. Further,

$$\#S(\varphi, \varepsilon, n) \leq \frac{1}{\Theta_G(K_r)} \Theta_G(\partial^{K_{1/\varepsilon+r}}(A_n) \cup A_n),$$

where $r = \min_{l \neq l' \in \Gamma_+} d_G(l, l')/2$. This yields

$$\begin{aligned} h_\varepsilon^\xi(\varphi) &\leq \limsup_{n \rightarrow \infty} \frac{1}{\Theta_G(A_n)} \log \left(\frac{1}{\Theta_G(K_r)} \Theta_G(\partial^{K_{1/\varepsilon+r}}(A_n) \cup A_n) \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{\Theta_G(A_n)} \log \left(\frac{1}{\Theta_G(K_r)} \right) + \frac{1}{\Theta_G(A_n)} (\log \Theta_G(\partial^{K_{1/\varepsilon+r}}(A_n)) + \log \Theta_G(A_n)). \end{aligned}$$

Since A_n is a van Hove sequence, we clearly have $\lim_{n \rightarrow \infty} \frac{1}{\Theta_G(A_n)} \log \Theta_G(\partial^{K_{1/\varepsilon+r}}(A_n)) = 0$ which yields $h_\varepsilon^\xi(\varphi) = 0$. Hence, $h_{\text{top}}^\xi(\varphi) = 0$. \square

Let us point out that, in fact, Lemma 4.6 readily follows from the next statement if the variational principle holds for continuous actions of G on compact metric spaces.

Lemma 4.7. *Let (G, H, \mathcal{L}) be a CPS with proper window $W \subseteq H$ and suppose W has locally disjoint complements. Then $(\Omega(\lambda(W)), G)$ is uniquely ergodic.*

Proof. Let $(A_n)_{n \in \mathbb{N}}$ be a tempered van Hove sequence in G and suppose there exist two invariant ergodic measures μ_1, μ_2 on $\Omega(\lambda(W))$. Given $f \in \mathcal{C}(\Omega(\lambda(W)))$ and $i \in \{1, 2\}$, Lindenstrauss' Pointwise Ergodic Theorem [23, Theorem 1.2] yields a subset $\Omega_i^f \subseteq \Omega(\lambda(W))$ of full μ_i -measure such that for all $\Gamma \in \Omega_i^f$

$$A_n(f, \Gamma) := \frac{1}{\Theta_G(A_n)} \int_{A_n} f(\Gamma - s) d\Theta_G(s) \xrightarrow{n \rightarrow \infty} \mu_i(f) := \int_{\Omega(\lambda(W))} f d\mu_i. \quad (4.1)$$

We want to show that (4.1) holds for all $\Gamma \in \beta^{-1}(M_i^f)$, where $M_i^f = \beta(\Omega_i^f)$. To that end, given $g_0 \in G$ and $\varepsilon > 0$, consider

$$f_{g_0; \varepsilon}: \Omega(\lambda(W)) \rightarrow \mathbb{R}, \quad f_{g_0; \varepsilon}(\Gamma) = \max \left\{ 0, 1 - 1/\varepsilon \cdot \min_{g \in \Gamma} d(g, B_\varepsilon(g_0)) \right\}.$$

Clearly, $f_{g_0; \varepsilon}$ is continuous and the set $\mathcal{F} = \{f_{g_0; \varepsilon} : g_0 \in G, \varepsilon > 0\}$ separates points and contains the constant function equal to 1. Its algebraic closure \mathcal{F}' is hence dense in $\mathcal{C}(\Omega(\lambda(W)))$, due to the Stone-Weierstrass Theorem.

Now, given $f \in \mathcal{F}$, Lemma 4.5 immediately yields

$$\lim_{n \rightarrow \infty} A_n(f, \Gamma') = \lim_{n \rightarrow \infty} A_n(f, \Gamma) \quad (\Gamma \in \Omega_i^f(\lambda(W)), \Gamma' \in \beta^{-1}(\beta(\Gamma))). \quad (4.2)$$

Observe that (4.2) straightforwardly extends to all $f \in \mathcal{F}'$ and thereby, in fact, to all $f \in \overline{\mathcal{F}'} = \mathcal{C}(\Omega(\lambda(W)))$. This shows (4.1) for all $\Gamma \in \beta^{-1}(M_i^f)$ with $f \in \mathcal{C}(\Omega(\lambda(W)))$.

Since β sends μ_1 and μ_2 to the unique invariant measure $\Theta_{\mathbb{T}}$ on \mathbb{T} , we clearly have $M_1^f \cap M_2^f \neq \emptyset$. Hence, $\mu_1(f) = \lim_{n \rightarrow \infty} A_n(f, \Gamma) = \mu_2(f)$ for all $\Gamma \in \beta^{-1}(M_1^f \cap M_2^f)$. Since $f \in \mathcal{C}(\Omega(\lambda(W)))$ was arbitrary, this shows $\mu_1 = \mu_2$ and thus finishes the proof. \square

Remark 4.8. In Section 5.3, we construct a planar CPS with window $V \subseteq \mathbb{R}$ with locally disjoint complements. For an (implicit) application of the criterion of locally disjoint complements outside the setting of Euclidean CPS, see [35, Example 5.1].

5. CONSTRUCTION OF A SELF-SIMILAR WINDOW FOR PLANAR CPS

The main goal of this section is to show that for each *planar* CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$, there are irredundant windows with boundaries of positive Lebesgue measure which are self-similar and which have locally disjoint complements. By means of the results of the previous section, this proves Theorem 1.3. Moreover, given any *higher dimensional* CPS $(\mathbb{R}^n, \mathbb{R}, \mathcal{L})$, we show that there are windows such that the associated Delone dynamical system has zero topological entropy.

5.1. Planar CPS and irrational rotations. For the constructions in this section, it is important to note that the set L^* is generated by an irrational circle rotation: observe that for each irrational lattice $\mathcal{L} \subseteq \mathbb{R}^2$ there exists a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{GL}(2, \mathbb{R})$ with $a_{11}/a_{12}, a_{21}/a_{22} \in \mathbb{R} \setminus \mathbb{Q}$ such that $\mathcal{L} = A(\mathbb{Z}^2)$. Put $\omega = a_{21}$. Without loss of generality, we may assume $a_{22} = 1$. Thus

$$L^* = \pi_2(\mathcal{L}) = \{n\omega + m \mid (n, m) \in \mathbb{Z}^2\} = \pi^{-1}(\{n\omega \pmod{1} \mid n \in \mathbb{Z}\}),$$

where $\pi : \mathbb{R} \rightarrow \mathbb{T}^1$ denotes the canonical projection onto $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and $\pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection to the second coordinate.

As seen in Section 4, the entropy of the Delone dynamical system $(\Omega(W), \mathbb{R})$ is related to the local structure of $W + t$ at points in $L^* \cap \partial W + t$. Given $t \in \mathbb{R}$, if $W \subseteq [0, 1]$, then a point in $L^* \cap \partial W + t$ corresponds to some $n \in \mathbb{Z}$ with $n\omega - t \pmod{1} \in \partial W$. Thus, a self-similar window $W \subseteq [0, 1]$ for the CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ can be understood as a subset $W \subseteq \mathbb{T}^1$ such that for all orbits $\mathcal{O}(x) = x + \omega\mathbb{Z}$ (of the rotation on \mathbb{T}^1 by angle ω) there are finitely many $n_1, \dots, n_N \in \mathbb{Z}$ such that for all $y = x + n\omega \in \partial W \cap \mathcal{O}(x)$ there is $i \in \{1, \dots, N\}$ and $\varepsilon > 0$ with $(B_\varepsilon(y) \cap W) + (n_i - n)\omega = B_\varepsilon(x + n_i\omega) \cap W$. Consistently with the terminology in Section 4, we call a subset of \mathbb{T}^1 with this property *self-similar*.

In the following, we fix $\omega \in \mathbb{T}^1 \setminus \mathbb{Q}$ and denote by R_ω the rotation by ω on \mathbb{T}^1 , that is, $R_\omega(x) = x + \omega$. Without loss of generality, we may assume $|\omega| < 1/2$. We set $q_1 = \min\{\ell \in \mathbb{N} : d(R_\omega^\ell(0), 0) < |\omega|\}$ and define the sequence of *closest return times* $(q_n)_{n \in \mathbb{N}}$ recursively by putting $q_{n+1} = \min\{\ell \in \mathbb{N} : d(R_\omega^\ell(0), 0) < d(R_\omega^{q_n}(0), 0)\}$. We further set I_n to be the closed interval of length $|I_n| = d(R_\omega^{q_n}(0), 0)$ with endpoints 0 and $R_\omega^{q_n}(0)$. Our construction makes use of the following well-known facts (see, e.g., [36], [37, Chapter I.1] and [38, Theorem 4.5]).

Proposition 5.1. *Given an irrational rotation R_ω on \mathbb{T}^1 , let $\mathcal{P}_n = \{R_\omega^j(I_n) : 1 \leq j \leq q_{n+1}\} \cup \{R_\omega^j(I_{n+1}) : 1 \leq j \leq q_n\}$, where q_n and I_n are defined as above. Then*

- (i) $\mathbb{T}^1 = \bigcup_{J \in \mathcal{P}_n} J$ and $J_1 \cap J_2 = \emptyset$ for each $J_1 \neq J_2 \in \mathcal{P}_n$;
- (ii) For each $J \in \mathcal{P}_n$ and each $m > n$, there is $\mathcal{Q}_{J,m} \subseteq \mathcal{P}_m$ such that $J = \bigcup_{K \in \mathcal{Q}_{J,m}} K$;
- (iii) If $J, J' \in \mathcal{P}_n$ and $J = R_\omega^\ell(J')$ for some $\ell \in \mathbb{N}$, then $\mathcal{Q}_{J,m} = R_\omega^\ell(\mathcal{Q}_{J',m}) = \{R_\omega^\ell(K) : K \in \mathcal{Q}_{J',m}\}$ for all $m > n$.

Remark 5.2. In simple terms, (i) gives that the elements of \mathcal{P}_n basically partition \mathbb{T}^1 and (ii) yields that the partition by elements of \mathcal{P}_{n+1} is a refinement of that given by \mathcal{P}_n . Point (iii) is to be understood as a self-similarity of the respective partitions.

Remark 5.3. Our goal is to construct a proper set W (which we want to be self-similar) and another proper set V (with locally disjoint complements) whose boundaries are irredundant and verify $\text{Leb}_{\mathbb{T}^1}(\partial W), \text{Leb}_{\mathbb{T}^1}(\partial V) > 1 - \varepsilon$ for $0 < \varepsilon < 1$.

To that end, we first construct an irredundant self-similar Cantor set C as the limit of a nested sequence (C_ℓ) of recursively defined compact subsets of \mathbb{T}^1 . At each step ℓ of the construction, the set C_ℓ is obtained by removing elements of \mathcal{P}_{n_ℓ} from $C_{\ell-1}$ (for some appropriately chosen increasing sequence (n_ℓ)) so that by Proposition 5.1 (ii), C_ℓ is a union of intervals from \mathcal{P}_{n_ℓ} . To establish the self-similarity of the limit set C , we treat the intervals which comprise $C_{\ell-1}$ equally. That is, roughly speaking, if $J_1, J_2 \in \mathcal{P}_{n_{\ell-1}}$ with $J_1, J_2 \subseteq C_{\ell-1}$ are translated copies of each other, say $R_\omega^n(J_1) = J_2$, then we keep $J \in \mathcal{Q}_{J_1, n_\ell}$ in C_ℓ if and

only if $R_\omega^n(J) \in \mathcal{Q}_{J_2, n_\ell}$ is kept in C_ℓ . Eventually, the limit set C will serve as the boundary of both W and V . We will obtain W by filling the gaps of C in such a way that the self-similarity of C is preserved while it has to be destroyed in a particular way in order to obtain V .

We note that the simple idea of ‘treating all partition intervals of equal length in C_ℓ equally in all subsequent steps’ rather easily leads to self similar windows with at most two similarity classes for each $t \in \mathbb{R}$ (see Remark 5.4 below), and hence at most four elements in every fibre. The construction presented below is somewhat more subtle, as it includes some refinements in order to produce *perfectly* self similar windows with a Cantor set boundary.

5.2. Construction of a self-similar Cantor set. Given $0 < \varepsilon < 1$, pick a sequence (β_ℓ) of positive numbers with $\sum_{\ell=1}^{\infty} 3\beta_\ell < \varepsilon$ and let (n_ℓ) be a sequence of positive integers with $|I_{n_{\ell+1}}|/|I_{n_\ell}| > 1/\beta_\ell$. For technical reasons, we may assume without loss of generality that $n_{\ell+1} \geq n_\ell + 6$. In particular, this yields $\#\mathcal{Q}_{J, n_{\ell+1}} \geq 8$ for each $\ell \in \mathbb{N}$ and $J \in \mathcal{P}_{n_\ell}$.

We recursively define a nested sequence of compact sets $(C_\ell) \subseteq \mathbb{T}^1$ whose limit will be a Cantor set C satisfying the desired self-similarity condition. To that end, let us introduce some terminology. Suppose we have already constructed $C_\ell \subseteq C_{\ell-1} \subseteq \dots \subseteq C_1 = \mathbb{T}^1$. We call a connected component J of the complement C_ℓ^c a *gap* of C_ℓ and we say J is of *level* k (with $k \in \{2, 3, \dots, \ell\}$) if $J \cap C_k^c \neq \emptyset$ and $J \cap C_{k-1}^c = \emptyset$. We further call an interval $J \in \mathcal{P}_{n_\ell}$ with $J \subseteq C_\ell$ *k-accessible from the left/right* if its left/right³ endpoint is at the boundary of a gap of C_ℓ which is of level k . It is worth mentioning that we will construct C_ℓ ($\ell \in \mathbb{N}$) in such a way that each $J \in \mathcal{P}_{n_\ell}$ is accessible from at most one side. Given C_ℓ , we obtain $C_{\ell+1}$ by removing from C_ℓ

- (1) the interior of the two left-most/right-most intervals as well as the interior of the right-most/left-most interval of $\mathcal{Q}_{J, n_{\ell+1}}$ if $J \in \mathcal{P}_{n_\ell}$ is k -accessible from the left/right and $\ell - k$ is even;
- (2) the interior of the left-most and the right-most interval of $\mathcal{Q}_{J, n_{\ell+1}}$ for all $J \in \mathcal{P}_{n_\ell}$ which haven’t been dealt with in (1);
- (3) all isolated points which remain after having removed intervals according to (1) & (2).

Put $C = \bigcap_\ell C_\ell$. Observe that C is a Cantor set of positive Lebesgue measure as it is compact, nowhere dense (since $\mathcal{O}^+(0) = \{R_\omega^n(0) : n \in \mathbb{N}\} \subseteq C^c$), and

$$\text{Leb}_{\mathbb{T}^1}(C) = \lim_{\ell \rightarrow \infty} \text{Leb}_{\mathbb{T}^1}(C_\ell) \geq 1 - \sum_{\ell=1}^{\infty} 3\beta_\ell > 1 - \varepsilon.$$

Moreover, it turns out that C is irredundant (see Section 5.3).

Remark 5.4. Coming back to the last paragraph of the previous section, observe that in order to provide a self-similar Cantor set, we could simply follow step (2) and (3) but this time applying (2) to *every* $J \in \mathcal{P}_{n_\ell}$. Let us denote the resulting Cantor set of this simplified construction by \tilde{C} . In principle, we could replace C by \tilde{C} in the following. As a matter of fact, this would not change the proofs of some of the next statements (in particular, Lemma 5.5 and Lemma 5.7) while the proof of Lemma 5.6 would even be shortened. However, as we point out in Remark 5.10 below, \tilde{C} can’t be the boundary of a perfectly self-similar window, that is, there are fibres of the factor map β from $(\Omega(\tilde{W}), \mathbb{R})$ onto (\mathbb{T}^1, R_ω) with more than two elements.

Before we turn to the construction of the sets W and V , let us study C locally along orbits. Given $x \in \mathbb{T}^1$, $n \in \mathbb{Z}$ and $\ell \in \mathbb{N}$, we write $x \sim_\ell R_\omega^n(x)$ whenever $x, R_\omega^n(x) \notin \bigcup_{j=1}^{|n|} R_\omega^j(I_{n_\ell} \cup I_{n_{\ell+1}})$ and if there are $1 \leq j_0, j_1 \leq q_{n_\ell+1-i}$, with i either 0 or 1, such that

- (a) $x \in \text{int } R_\omega^{j_0}(I_{n_{\ell+i}})$ and $R_\omega^n(x) \in \text{int } R_\omega^{j_1}(I_{n_{\ell+i}})$;
- (b) $R_\omega^{j_0}(I_{n_{\ell+i}})$ and $R_\omega^{j_1}(I_{n_{\ell+i}})$ are from the same side k - and k' -accessible, respectively, with $k - k'$ even or $R_\omega^{j_0}(I_{n_{\ell+i}})$ and $R_\omega^{j_1}(I_{n_{\ell+i}})$ are not accessible at all.

In the following, the reader should keep in mind that, by construction, $\mathcal{O}^+(0) \cap C = \emptyset$ so that for each $x \in C$ we have that $x \in R_\omega^{j_0}(I_{n_{\ell+i}})$ actually means $x \in \text{int } R_\omega^{j_0}(I_{n_{\ell+i}})$.

³In any of the two, from now on fixed, orientations on \mathbb{T}^1 .

Lemma 5.5. *Consider $x \in C$. If $x \sim_\ell R_\omega^n(x)$ for some $\ell \in \mathbb{N}$ and $n \in \mathbb{Z}$ with $R_\omega^n(x) \in C_\ell$, then $R_\omega^n(x) \in C$.*

Proof. Let j_0, j_1 be as above. Observe that $j_0 + n = j_1$ because of Proposition 5.1 (i) and because $x, R_\omega^n(x) \notin \bigcup_{j=1}^{|n|} R_\omega^j(I_{n_\ell} \cup I_{n_{\ell+1}})$. Hence, the distance of $R_\omega^n(x)$ to the left (and right) endpoint of $R_\omega^{j_1}(I_{n_{\ell+i}}) = R_\omega^{j_0+n}(I_{n_{\ell+i}})$ equals the distance of x to the left (and right) endpoint of $R_\omega^{j_0}(I_{n_{\ell+i}})$. By Proposition 5.1 (iii), we further have $\mathcal{Q}_{R_\omega^{j_1}(I_{n_{\ell+i}}, n_{\ell+1})} = R_\omega^n(\mathcal{Q}_{R_\omega^{j_0}(I_{n_{\ell+i}}, n_{\ell+1})})$. By definition of $C_{\ell+1}$, this indeed shows $R_\omega^n(x) \in C_{\ell+1}$ as well as $x \sim_{\ell+1} R_\omega^n(x)$ and hence gives $R_\omega^n(x) \in C$ by induction on ℓ . \square

The next statement is crucial for establishing the self-similarity of C .

Lemma 5.6. *If $x \in C$ and $y \in \mathcal{O}(x) \cap C$, then $x \sim_\ell y$ for sufficiently large ℓ .*

Proof. Without loss of generality, we may assume $y = R_\omega^{-n}(x)$ for some $n \in \mathbb{N}$. As $\mathcal{O}^+(0) \cap C = \emptyset$ and due to Proposition 5.1 (i), there is $\ell_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$ there is $i_\ell \in \{0, 1\}$ with $x \in \text{int} \bigcup_{j=2n+1}^{q_{n_{\ell+1}-i_\ell}} R_\omega^j(I_{n_{\ell+i_\ell}})$ and hence $y \in \text{int} \bigcup_{j=n+1}^{q_{n_{\ell+1}-i_\ell}} R_\omega^j(I_{n_{\ell+i_\ell}})$. In other words, there are $1 \leq j_0^\ell, j_1^\ell \leq q_{n_{\ell+1}-i_\ell}$ with $x \in \text{int} R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ and $y \in \text{int} R_\omega^{j_1^\ell}(I_{n_{\ell+i_\ell}}) = \text{int} R_\omega^{j_0^\ell-n}(I_{n_{\ell+i_\ell}})$. As in the proof of Lemma 5.5, we see that x and y have the same distance to the endpoints of $R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ and $R_\omega^{j_1^\ell}(I_{n_{\ell+i_\ell}})$, respectively, and that $\mathcal{Q}_{R_\omega^{j_1^\ell}(I_{n_{\ell+i_\ell}}, n_{\ell+1})} = R_\omega^{-n}(\mathcal{Q}_{R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}}, n_{\ell+1})})$. It remains to show (b) for sufficiently large ℓ . To that end, pick some $\ell \geq \ell_0$. We have to consider the following cases.

Case 1: $R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ and $R_\omega^{j_1^\ell}(I_{n_{\ell+i_\ell}})$ are accessible from different sides. By the construction of $C_{\ell+1}$ and as $\#\mathcal{Q}_{R_\omega^{j_0^\ell}(I_{n_{\ell+i_0}}, n_{\ell+1})} \geq 6$, we have that either $R_\omega^{j_0^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ and $R_\omega^{j_1^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ are accessible from the same side, or at least one of the two intervals is not accessible. Hence, we have reduced the problem to one of the following cases.

Case 2: $R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ as well as $R_\omega^{j_1^\ell}(I_{n_{\ell+i_\ell}})$ are from the same side k - and k' -accessible, respectively, and $k - k'$ is odd. We may assume without loss of generality that both intervals are accessible from the right and that $\ell - k$ is even. If $R_\omega^{j_0^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is still accessible from the right, then $R_\omega^{j_1^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is not accessible anymore since $R_\omega^{j_1^\ell}(I_{n_{\ell+i_\ell}})$ has been dealt with according to (2) while $R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ has been dealt with according (1). Hence, we are in Case 4. If, however, $R_\omega^{j_0^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is not accessible from the right anymore, then the same is true for $R_\omega^{j_1^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ and hence either both are $\ell + 1$ -accessible from the left or not accessible at all. In both cases we are done.

Case 3: $R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ is k -accessible (from some side) with $\ell - k$ even while $R_\omega^{j_1^\ell}(I_{n_{\ell+i_\ell}})$ is not accessible. Without loss of generality, we may assume that $R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ is accessible from the right. If $R_\omega^{j_0^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is accessible from the left or not accessible at all, the same holds true for $R_\omega^{j_1^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ and we are done. If $R_\omega^{j_0^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is still k -accessible from the right, then $R_\omega^{j_1^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is not accessible. In this case we are in Case 4.

Case 4: $R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ is k -accessible (from some side) with $\ell - k$ odd while $R_\omega^{j_1^\ell}(I_{n_{\ell+i_\ell}})$ is not accessible. Without loss of generality, we may assume that $R_\omega^{j_0^\ell}(I_{n_{\ell+i_\ell}})$ is accessible from the right. Note that $R_\omega^{j_0^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is still k -accessible from the right if and only if $R_\omega^{j_1^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is k' -accessible from the right as well with $k' = \ell + 1$ in which case we are done since $k' - k$ is even. In the situation where $R_\omega^{j_0^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ is accessible from the left or not accessible at all, the same holds true for $R_\omega^{j_1^{\ell+1}}(I_{n_{\ell+1+i_{\ell+1}}})$ and we are done, too. \square

Lemma 5.7. *Suppose we are given $x, y \in C$ with $y = R_\omega^n(x)$ for some $n \in \mathbb{Z}$. Then there is $\varepsilon > 0$ such that $R_\omega^n(B_\varepsilon(x) \cap C) = B_\varepsilon(y) \cap C$.*

Proof. Due to Lemma 5.6, there is $\ell \in \mathbb{N}$ such that $x \sim_\ell y$. In particular, there are hence $i \in \{0, 1\}$ and $1 \leq j_0, j_1 \leq q_{n_\ell+1-i}$ with $x \in \text{int } R_\omega^{j_0}(I_{n_\ell+i}) \subseteq C_\ell$ and $R_\omega^n(x) \in \text{int } R_\omega^{j_1}(I_{n_\ell+i}) \subseteq C_\ell$. Let $\varepsilon > 0$ be such that $B_\varepsilon(x) \subseteq R_\omega^{j_0}(I_{n_\ell+i})$ (and hence, $B_\varepsilon(y) \subseteq R_\omega^{j_1}(I_{n_\ell+i})$, too). Suppose there is $z \in B_\varepsilon(x) \cap C$. Then we clearly have $R_\omega^n(z) \in C_\ell$ and $z \sim_\ell R_\omega^n(z)$. By Lemma 5.5, we hence have $R_\omega^n(z) \in C$. In other words, $R_\omega^n(B_\varepsilon(x) \cap C) \subseteq B_\varepsilon(y) \cap C$. Similarly, we get the opposite inclusion and hence obtain the desired equality. \square

5.3. Filling the gaps of the self-similar Cantor set. We now come to the construction of two windows W and V with $\partial W = \partial V = C$ which give rise to model sets which are almost automorphic extensions of (\mathbb{T}, R_ω) and have zero entropy. These model sets are at two different ends of low complexity dynamics: the fibres of $\beta: \Omega(\lambda(W)) \rightarrow \mathbb{T}$ have at most two elements and $\Omega(\lambda(W))$ allows for two distinct ergodic measures; $\beta: \Omega(\lambda(V)) \rightarrow \mathbb{T}$ is a metric isomorphism (see the discussion in Section 6) and hence $(\Omega(\lambda(V)), \mathbb{R})$ is mean equicontinuous (which, in particular, yields unique ergodicity) for the prize of fibres of infinite cardinality. It is worth mentioning that infinite fibres are, in fact, a necessary requirement for an irregular almost automorphic system to be mean equicontinuous [15].

In order to construct W and V , it remains to fill the gaps of C , that is, the connected components of C^c , appropriately. As a preparation, we first take a closer look at the accessible points of C . To that end, let us provide the following observation.

Proposition 5.8. *For all $\ell \in \mathbb{N}$, there are $J_{\ell+1}^1, J_{\ell+1}^2 \in \mathcal{P}_{n_{\ell+1}}$ such that for each $J \in \mathcal{P}_{n_\ell}$ the left-most interval (the interval second from left) of $\mathcal{Q}_{J, n_{\ell+1}}$ is a translated copy of $J_{\ell+1}^1$ ($J_{\ell+1}^2$).*

A similar statement holds if we replace left by right.

Proof. We only consider the “left case”. Without loss of generality, we may assume that $I_{n_\ell+2}$ is an interval to the right of zero (otherwise, we may proceed with $n_\ell + 3$ instead of $n_\ell + 2$). Now, recall that $q_{n+1} \geq q_n + q_{n-1}$ for each $n \in \mathbb{N}$.⁴ Given any $J \in \mathcal{P}_{n_\ell}$, this yields that the left-most interval of $\mathcal{Q}_{J, n_{\ell+2}}$ is a translated copy of $I_{n_\ell+2}$. Since we assume $n_{\ell+1} \geq n_\ell + 6$, the statement follows by means of Proposition 5.1 (iii). \square

Now, let (J_n) be an enumeration of the gaps of C and denote by $x_n \in C$ the right endpoint of the gap J_n . Similarly as in the previous section, we say that J_n is of level ℓ if $J_n \cap C_\ell^c \neq \emptyset$ while $J_n \subseteq C_{\ell-1}$. Assuming that J_n is of level ℓ , let y_n denote the isolated point in J_n which had to be removed in step (3) of the construction of C_ℓ . Now, let k be the level of J_n and k' the level of $J_{n'}$ and assume without loss of generality that $k < k'$. Suppose $k' - k$ is even. Then

$$x_n - x_{n'} = y_n - y_{n'} + \sum_{\ell=k}^{k'-1} \alpha_{-1^{\ell-k}; \ell}$$

where $\alpha_{1; \ell} = |J_\ell^1|$ and $\alpha_{-1; \ell} = |J_\ell^1| + |J_\ell^2|$ (recall that every second step of the construction of $C_{k'}$, we remove two intervals on either side of $J_n \cap C_k$). Hence, $x_n - x_{n'}$ is an integer multiple of ω . In other words, all right endpoints of the even-level gaps of C belong to one orbit and all right end-points of odd-level gaps belong to one orbit. Now, suppose $k - k'$ is odd. Then

$$x_n - x_{n'} = y_n - y_{n'} + \sum_{\ell=k}^{k'-1} \alpha_{-1^{\ell-k}; \ell} + \sum_{\ell=k'}^{\infty} (-1)^{\ell-k'} |J_\ell^2|.$$

We may assume without loss of generality that $\sum_{\ell=2}^{\infty} (-1)^\ell |J_\ell^2|$ (and thus $\sum_{\ell=k'}^{\infty} (-1)^{\ell-k'} |J_\ell^2|$) is not an integer multiple of ω .⁵ Then x_n and $x_{n'}$ belong to different orbits of R_ω . Similarly, we have that the left endpoints of even-level gaps belong to one orbit and those of odd-level gaps belong to a different one. Since two gaps are of equal length if and only if they are of the same level, this gives that C is, in fact, irredundant.

⁴In fact, if a_n is the n -th coefficient of the continued fraction expansion of ω , then $q_{n+1} = a_n q_n + q_{n-1}$ [38, Section 4.4].

⁵First, by possibly going over to a subsequence, we may assume that $2 \sum_{\ell=k+1}^{\infty} |J_\ell^2| < |J_k^2|$ for all integers $k \geq 2$. Hence, $\sum_{\ell=2}^{\infty} (-1)^{\ell_j} |J_{\ell_j}^2| \neq \sum_{\ell=2}^{\infty} (-1)^{\ell'_j} |J_{\ell'_j}^2|$ for distinct subsequences (n_{ℓ_j}) and $(n_{\ell'_j})$ of (n_ℓ) . Second, there clearly are uncountably many subsequences but only countably many integer multiples of ω .

Without loss of generality, we may assume in the following that J_{2n} is of an even level while J_{2n+1} is of an odd level for each $n \in \mathbb{N}$. We define the window W by

$$W = C \cup \bigcup_{n \in \mathbb{N}} J_{2n}.$$

Observe that between two gaps of level ℓ , there always is a gap of level $\ell+1$ so that $\partial W = C$ and $W = \overline{\text{int } W}$.

Lemma 5.9. *Suppose we are given $x, y \in C$ with $y = R_\omega^n(x)$ for some $n \in \mathbb{Z}$. Then there is $\varepsilon > 0$ such that $R_\omega^n(B_\varepsilon(x) \cap W) = B_\varepsilon(y) \cap W$.*

Proof. Lemma 5.7 yields $\varepsilon > 0$ such that $R_\omega^n(B_\varepsilon(x) \cap C) = B_\varepsilon(y) \cap C$. In particular, each left/right endpoint $x' \in B_\varepsilon(x)$ of a gap J_n which intersects $B_\varepsilon(x)$ corresponds to a left/right endpoint $y' = R_\omega^n(x') \in B_\varepsilon(y)$ of a gap $J_{n'}$ which intersects $B_\varepsilon(y)$. As J_n and $J_{n'}$ thus have endpoints of one and the same orbit, the above discussion shows that, by definition of W , $J_n \subseteq W$ if and only if $J_{n'} \subseteq W$ and hence $R_\omega^n(B_\varepsilon(x) \cap W) = B_\varepsilon(y) \cap W$. \square

Remark 5.10. Here, we see the advantage of the Cantor set C over the alternative set \tilde{C} discussed in Remark 5.4: A similar analysis as the one before shows that *all* points of \tilde{C} which are accessible from the left belong to *one* orbit as do all points which are accessible from the right. Hence, by filling some but not all gaps of \tilde{C} , we have that along those orbits which correspond to accessible points of \tilde{C} there are at least two local configurations of \tilde{W} so that \tilde{W} is not perfectly self-similar.

Of course, in order to overcome this problem, we can fill every gap partially: With the above notation, put $\tilde{W} = \tilde{C} \cup \bigcup_{n \in \mathbb{N}} [y_n, x_n]$. The window \tilde{W} would be perfectly self-similar but its boundary would contain isolated points and thus not be a Cantor set anymore.

Next, we turn to the construction of the window V . Let (J_n) be some enumeration of the gaps of C . Given a gap J_n and some level $k \in \mathbb{N}_{\geq 2}$, let $J(k; J_n)$ be a k -level gap which minimises the distance to J_n . We set

$$V = \mathbb{T}^1 \setminus \bigcup_{k \in \mathbb{N}_{\geq 2}} J(k; J_k).$$

Clearly, $\partial V = C$ and $\overline{\text{int } V} = V$. Moreover, if $x, R_\omega^n(x) \in C$, then there is $\varepsilon > 0$ such that

$$R_\omega^n(B_\varepsilon(x) \cap V^c) \cap B_\varepsilon(R_\omega^n(x)) \cap V^c = \emptyset$$

since V has exactly one gap of each level and since C is self-similar according to Lemma 5.7.

Remark 5.11. We would like to close this paragraph with a remark on the dependence of the topological entropy of a Delone dynamical system on its window. In the following, consider the CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ of this section and let (J_n) as well as C be as above. Given a sequence $x \in \{0, 1\}^{\mathbb{N}}$, we may associate to x a set $W(x) \subseteq \mathbb{T}^1$ by setting

$$W(x) = C \cup \bigcup_{n \in \mathbb{N}, x_n=1} J_n.$$

We denote by \mathbb{P} the Bernoulli measure on $\{0, 1\}^{\mathbb{N}}$ with equal probability $1/2$ for both symbols 0 and 1. In contrast to the results of this article, we have

Theorem 5.12 ([34, Theorem 1.1]; see also [10, Theorem 8]). *For \mathbb{P} -almost every $x \in \{0, 1\}^{\mathbb{N}}$, $W(x)$ is proper and $\Omega(\lambda(W(x)))$ has positive topological entropy.*

Given two sequences $x, y \in \{0, 1\}^{\mathbb{N}}$, denote by $z(n; x, y) \in \{0, 1\}^{\mathbb{N}}$ that sequence which coincides with x on the first n entries and with y on all of the remaining entries. Suppose x and y are elements of $\{0, 1\}^{\mathbb{N}}$ such that $W(x) = W$ while $W(y)$ is a proper set such that $\Omega(\lambda(W(y)))$ has positive topological entropy. Observe that for each $n \in \mathbb{N}$ we have that $W(z(n; x, y))$ and $W(z(n; y, x))$ are proper and $h_{\text{top}}(\Omega(\lambda(W(z(n; x, y)))) = h_{\text{top}}(\Omega(\lambda(W(y))))$ as well as $h_{\text{top}}(\Omega(\lambda(W(z(n; y, x)))) = 0$. This immediately yields

Corollary 5.13. *Suppose we are given a CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$. Consider the class of proper windows in \mathbb{R} equipped with the Hausdorff metric. The map which sends each window to the topological entropy of the corresponding Delone dynamical system is neither upper nor lower semicontinuous.*

5.4. Higher dimensional Euclidean CPS and zero entropy. In this section, we show how for every higher dimensional Euclidean CPS the irregular windows constructed above yield model sets whose associated Delone dynamical system has zero topological entropy in every fibre.

Consider a CPS $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$. Analogously to the discussion at the beginning of this section, the lattice \mathcal{L} can be represented as $\mathcal{L} = A(\mathbb{Z}^{N+1})$, where $A = (a_{ij}) \in \text{GL}(N+1, \mathbb{R})$ and each row $(a_{ij})_{j=1}^{N+1}$ has rationally independent entries. Let $v_i = (a_{1i}, \dots, a_{Ni})^T$ and put $\omega_i = a_{N+1,i}$. Without loss of generality, we may assume $\omega_{N+1} = 1$ and $W \subseteq [0, 1]$ for the rest of this section.

Before we come to the main results of this subsection, let us introduce some useful concepts and notation. Let $\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ denote the canonical projection and $\pi_i : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ the projection onto the i -th coordinate. We have

$$L^* = \pi_{N+1}(\mathcal{L}) = \left\{ \sum_{i=1}^N n_i \omega_i + n_{N+1} : n_i \in \mathbb{Z} \right\} = \pi^{-1} \left(\left\{ \sum_{i=1}^N n_i \omega_i \pmod{1} : n_i \in \mathbb{Z} \right\} \right).$$

In other words, L^* is the lift of an orbit of a \mathbb{Z}^N -rotation on \mathbb{R}/\mathbb{Z} with N rationally independent rotation numbers $\omega_1, \dots, \omega_N$. To each rotation number, we associate a set $L_i^* = \pi^{-1}(\{n\omega_i \pmod{1} : n \in \mathbb{Z}\})$ and put $A_i = \begin{pmatrix} a_{ii} & a_{i,N+1} \\ \omega_i & 1 \end{pmatrix}$. Then each $A_i \in \text{GL}(2, \mathbb{R})$ has rationally independent rows so that $\mathcal{L}_i = A_i(\mathbb{Z}^2) \subseteq \mathbb{R}^2$ is an irrational lattice. Note that $\pi_2(\mathcal{L}_i) = L_i^*$. In this way, we associate N planar CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L}_i)$ with window $W \subseteq \mathbb{R}$ to a given CPS $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ with exactly the same window $W \subseteq \mathbb{R}$. We denote the corresponding Delone dynamical systems by $(\Omega(\lambda_i(W)), \varphi_i)$. Observe that we have $n\omega_i \pmod{1} \in W$ if and only if $nv_i - \lfloor n\omega_i \rfloor v_{N+1} \in \lambda(W)$; likewise, we have $n\omega_i \pmod{1} \in W$ if and only if $na_{ii} - \lfloor n\omega_i \rfloor a_{i,N+1} \in \lambda_i(W)$.

Fix $t \in \mathbb{R}$. Given $p = nv_1 + kv_{N+1} + \sum_{i=2}^N m_i v_i \in \lambda(W+t)$, put $\mathbf{m}_p = (m_2, \dots, m_N) \in \mathbb{Z}^{N-1}$. Note that $nv_1 + kv_{N+1} + \sum_{i=2}^N m_i v_i \in \lambda(W+t)$ is equivalent to $n\omega_1 + k \in W+t - \sum_{i=2}^N m_i \omega_i$. For $\mathbf{m} = (m_2, \dots, m_N) \in \mathbb{Z}^{N-1}$, we define the *pseudoline*

$$G_{W+t}(\mathbf{m}) = \left\{ nv_1 + kv_{N+1} + \sum_{i=2}^N m_i v_i : n, k \in \mathbb{Z}, n\omega_1 + k \in W+t - \sum_{i=2}^N m_i \omega_i \right\}.$$

Let us mention a number of immediate and important properties of pseudolines. First, $\{G_{W+t}(\mathbf{m}) \mid \mathbf{m} \in \mathbb{Z}^{N-1}\}$ partitions $\lambda(W+t)$, i.e., $\lambda(W+t) = \bigsqcup_{\mathbf{m} \in \mathbb{Z}^{N-1}} G_{W+t}(\mathbf{m})$. Second, the restriction of π_1 to $G_{W+t}(\mathbf{m})$ is injective since a_{11} and $a_{1(N+1)}$ are rationally independent. Third, observe that for any $p \in \lambda(W+t)$ we have

$$\pi_1(G_{W+t}(\mathbf{m}_p)) = \sum_{i=2}^N m_i a_{1i} + \lambda_1 \left(W+t - \sum_{i=2}^N m_i \omega_i \right) \in \Omega(\lambda_1(W+t)). \quad (5.1)$$

Finally, notice that there exists $C > 0$ (independent of t) such that for each $G_{W+t}(\mathbf{m})$ we have $G_{W+t}(\mathbf{m}) \subseteq B_C(\ell(\mathbf{m}))$, where $\ell(\mathbf{m})$ is the line $\{\lambda \cdot (1/\omega_1 \cdot v_1 - v_{N+1}) + \sum_{i=2}^N m_i v_i : \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^N$. Since $A \in \text{GL}(N+1, \mathbb{R})$, we have $(1/\omega_1 \cdot v_1 - v_{N+1}) \notin \text{span}\{v_2, \dots, v_N\}$ and therefore immediately obtain the following statement.

Lemma 5.14. *Suppose $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ is a CPS with proper window $W \subseteq \mathbb{R}$. Then there exists $\kappa > 0$ such that for each $t \in \mathbb{R}$ we have*

$$\sharp \{G_{W+t}(\mathbf{m}) \mid \mathbf{m} \in \mathbb{Z}^{N-1}, G_{W+t}(\mathbf{m}) \cap B_M^N(0) \neq \emptyset\} \leq \kappa \cdot \text{Leb}(B_M^{N-1}(0)),$$

where $B_M^d(0) \subseteq \mathbb{R}^d$ denotes the d -dimensional M -ball centred at 0.

The next result provides a whole class of higher dimensional CPS with irregular windows whose associated Delone dynamical systems have zero entropy.

Theorem 5.15. *Let $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ be a CPS with proper window $W \subseteq \mathbb{R}$. Furthermore, assume that there exists $i \in \{1, \dots, N\}$ such that $h_{\text{top}}(\varphi_i) = 0$. Then we have $h_{\text{top}}^\xi(\varphi) = 0$ for all $\xi \in \mathbb{T}$.*

Proof. Without loss of generality, we may assume that $h_{\text{top}}(\varphi_1) = 0$. We equip \mathbb{R} as well as \mathbb{R}^N with the Euclidean metric and consider the entropy of φ_1 and φ obtained by averaging over the van Hove sequence given by one-dimensional balls $(B_M^1(0))_{M \in \mathbb{N}}$ and N -dimensional balls $(B_M(0))_{M \in \mathbb{N}}$, respectively. Fix some $\xi \in \mathbb{T}^{N+1}$ and assume without loss of generality that there is $t \in \mathbb{R}$ with $\xi = [0, t]_{\mathcal{L}}$. Let $\varepsilon > 0$ be smaller than $r := \frac{1}{2} \cdot \min\{\inf_{p \neq q \in \lambda(W+t)} \|p - q\|, \inf_{p \neq q \in \lambda_1(W)} |p - q|\}$. Given $M \in \mathbb{N}$, let $S_1(\varepsilon, M)$ be (ε, M) -spanning for $\Omega(\lambda_1(W))$ with minimal cardinality $P_1(\varepsilon, M) := \sharp S_1(\varepsilon, M)$. Our goal is to construct a set $S^\xi(\varepsilon, M)$ which is (ε, M) -spanning for $\beta^{-1}(\xi)$ and satisfies

$$\sharp S^\xi(\varphi, \varepsilon, M) \leq P_1(\varepsilon, M)^{\kappa \text{Leb}(B_{M+1/\varepsilon}^{N-1}(0))}. \quad (5.2)$$

To that end, recall that two Delone sets $\Lambda, \Gamma \in \beta^{-1}(\xi)$ satisfy $\max_{s \in B_M(0)} d(\Lambda - s, \Gamma - s) < \varepsilon$ if for all $s \in B_M(0)$ we have $\Lambda \cap B_{1/\varepsilon}(s) = \Gamma \cap B_{1/\varepsilon}(s)$ (see Remark 2.11). Since $\lambda(W+t)$ can be covered by pseudolines, this is the case if for all such s and each $p \in B_{1/\varepsilon}(s) \cap \lambda(W+t)$ we have $G_{W+t}(\mathbf{m}_p) \cap \Lambda \cap B_{1/\varepsilon}(s) = G_{W+t}(\mathbf{m}_p) \cap \Gamma \cap B_{1/\varepsilon}(s)$. Note that this is equivalent to

$$\pi_1(G_{W+t}(\mathbf{m}_p) \cap \Lambda \cap B_{1/\varepsilon}(s)) = \pi_1(G_{W+t}(\mathbf{m}_p) \cap \Gamma \cap B_{1/\varepsilon}(s)), \quad (5.3)$$

since $\pi_1|_{G_{W+t}(\mathbf{m}_p)}$ is injective.

Now, given $\Gamma \in \beta^{-1}(\xi)$ with $\Gamma = \lim_{j \rightarrow \infty} \lambda(W + t_j)$ (see (2.2)), observe that

$$\begin{aligned} \pi_1(G_{W+t}(\mathbf{m}) \cap \Gamma) &= \pi_1(G_{W+t}(\mathbf{m}) \cap \lim_{j \rightarrow \infty} \lambda(W + t_j)) \\ &= \lim_{j \rightarrow \infty} \pi_1(G_{W+t_j}(\mathbf{m}) \cap \lambda(W + t_j)) \end{aligned} \quad (5.4)$$

which is an element of $\Omega(\lambda_1(W))$ due to (5.1). Hence, by definition of $S_1(\varepsilon, M)$, there is $\Delta \in S_1(\varepsilon, M)$ with $\max_{s \in B_M^1(0)} d(\pi_1(G_{W+t}(\mathbf{m}) \cap \Gamma) - s, \Delta - s) < \varepsilon$. In particular, we have

$$\pi_1(G_{W+t}(\mathbf{m}) \cap \Gamma \cap B_{M+1/\varepsilon}(0)) \subseteq \Delta + \delta \quad (5.5)$$

for some $\delta \in \mathbb{R}$ with $|\delta| < \varepsilon$. Since $\varepsilon < r$, we have that for fixed \mathbf{m} and Δ there is at most one such δ for which (5.5) is satisfied for some $\Gamma \in \lambda(W+t)$. If (5.5) holds, we say Γ realises the local configuration of Δ along $G_{W+t}(\mathbf{m})$. We define an equivalence relation \sim on $\beta^{-1}(\xi)$ by putting $\Gamma \sim \Lambda$ if Γ and Λ realise the same local configuration along $G_{W+t}(\mathbf{m}_p)$ ($p \in B_{1/\varepsilon}(s) \cap \lambda(W+t)$). The above shows: $\max_{s \in B_M(0)} d(\Lambda - s, \Gamma - s) < \varepsilon$ if $\Lambda \sim \Gamma$.

Finally, we set $S^\xi(\varphi, \varepsilon, M)$ to be a set which contains one representative for each equivalence class of \sim . Recall that the number of pseudolines that intersect $B_{M+1/\varepsilon}(0)$ is bounded by $\kappa \cdot \text{Leb}(B_{M+1/\varepsilon}^{N-1}(0))$ (see Lemma 5.14). Since there are at most $P_1(\varepsilon, M)$ possible configurations realised along each $G_{W+t}(\mathbf{m}_p) \cap B_{M+1/\varepsilon}(0)$, we obtain (5.2). Thus,

$$\begin{aligned} h_{\text{top}}^\xi(\varphi) &\leq \lim_{\varepsilon \rightarrow 0} \limsup_{M \rightarrow \infty} \frac{\kappa \text{Leb}(B_{M+1/\varepsilon}^{N-1}(0))}{\text{Leb}(B_M(0))} \log P_1(\varepsilon, M) \\ &= \lim_{\varepsilon \rightarrow 0} \limsup_{M \rightarrow \infty} \frac{2\kappa}{\sqrt{\pi}} \left(\frac{1}{M\varepsilon} + 1 \right)^{N-1} \frac{\log P_1(\varepsilon, M)}{\text{Leb}(B_M^1(0))} \\ &\leq \frac{2\kappa}{\sqrt{\pi}} \lim_{\varepsilon \rightarrow 0} \limsup_{M \rightarrow \infty} \frac{\log P_1(\varepsilon, M)}{\text{Leb}(B_M^1(0))} = 0 \end{aligned}$$

which finishes the proof. \square

6. INVARIANT MEASURES, DYNAMICAL SPECTRUM AND DIFFRACTION

In this part, we discuss the spectral properties of the model sets constructed in the previous sections. Suppose G is an abelian group. Recall that given a topological dynamical system (X, G) which preserves a measure μ , we say $f \in L_2(X, \mu)$ is an *eigenfunction* of (X, G) (equipped with μ) if there exists $\lambda \in \hat{G}$ such that $g.f = \lambda(g) \cdot f$ ($g \in G$), where $g.f(x) = f(gx)$ ($x \in X$). Here, \hat{G} denotes the dual of G . We say (X, G) has *pure point spectrum* if there exists an orthonormal basis of $L_2(X, \mu)$ which consists of eigenfunctions.

Let us recall some basic facts from the spectral theory of minimal equicontinuous topological dynamical systems (\mathbb{T}, G) . Due to Theorem 2.1, we may assume without loss of

generality that \mathbb{T} is a compact abelian group and $g\xi = \xi + \omega(g)$ for all $\xi \in \mathbb{T}$ and $g \in G$, where $\omega: G \rightarrow \mathbb{T}$ denotes a group homomorphism with dense image in \mathbb{T} . Note that for all $\lambda \in \hat{\mathbb{T}}$, $g \in G$, and $\xi \in \mathbb{T}$ we have $\lambda(\xi + \omega(g)) = \lambda(\omega(g)) \cdot \lambda(\xi)$. Moreover, observe that $\lambda(\omega(\cdot))$ is a character on G . Hence, every element of $\hat{\mathbb{T}}$ is an eigenfunction of (\mathbb{T}, G) (equipped with the unique invariant measure $\Theta_{\mathbb{T}}$). Recall that by the Peter Weyl Theorem, the characters of a compact group \mathbb{T} form an orthonormal basis of $L_2(\mathbb{T}) = L_2(\mathbb{T}, \Theta_{\mathbb{T}})$. This shows the following well-known fact: every minimal equicontinuous system has pure point spectrum with continuous eigenfunctions.

In the following, we consider a Delone dynamical system $(\Omega(\lambda(W)), G)$ corresponding to a CPS (G, H, \mathcal{L}) with proper window W . As before, let $\beta: \Omega(\lambda(W)) \rightarrow \mathbb{T}$ be the associated torus parametrisation, φ the translation action on $(\Omega(\lambda(W)))$ (see Section 2.3) and ω the G -action on \mathbb{T} (see Section 2.4).

Definition 6.1. A measurable map $\gamma: \mathbb{T} \rightarrow \Omega(\lambda(W))$ is referred to as an invariant graph (for $(\Omega(\lambda(W)))$) if

$$\forall s, u \in G, t \in H: \quad \beta \circ \gamma([s, t]_{\mathcal{L}}) = [s, t]_{\mathcal{L}} \quad \text{and} \quad \varphi(u, \gamma([s, t]_{\mathcal{L}})) = \gamma(\omega(u, [s, t]_{\mathcal{L}})). \quad (6.1)$$

Given an invariant graph γ , we define the associated *graph measure* by setting

$$\mu_{\gamma}(A) = \Theta_{\mathbb{T}}(\gamma^{-1}(A))$$

for all measurable $A \subseteq \Omega(\lambda(W))$. Observe that, since $\Theta_{\mathbb{T}}$ is ergodic, μ_{γ} is an ergodic measure of $(\Omega(\lambda(W)), G)$. Define

$$U_{\gamma}: L_2(\Omega(\lambda(W)), \mu_{\gamma}) \rightarrow L_2(\mathbb{T}), \quad f \mapsto f \circ \gamma.$$

Observe that

$$\begin{aligned} \langle U_{\gamma}f, U_{\gamma}g \rangle_{L_2(\mathbb{T})} &= \int_{\mathbb{T}} \overline{U_{\gamma}f} \cdot U_{\gamma}g \, d\Theta_{\mathbb{T}} = \int_{\mathbb{T}} (\overline{f} \cdot g) \circ \gamma \, d\Theta_{\mathbb{T}} = \int_{\Omega(\lambda(W))} \overline{f} \cdot g \, d\mu_{\gamma} \\ &= \langle f, g \rangle_{L_2(\Omega(\lambda(W)), \mu_{\gamma})}, \end{aligned}$$

for all $f, g \in L_2(\Omega(\lambda(W)), \mu_{\gamma})$. Furthermore, due to (6.1), we have $U_{\gamma}(g \circ \beta) = g$ for all $g \in L_2(\mathbb{T})$. Hence, U_{γ} is bijective. Finally, for each $u \in G$, (6.1) yields

$$u.(U_{\gamma}f)(\cdot) = f \circ \gamma(\omega(u, \cdot)) = f(\varphi(u, \gamma(\cdot))) = (u.f) \circ \gamma(\cdot) = U_{\gamma}(u.f)(\cdot).$$

Altogether, we have proven

Proposition 6.2. $(\Omega(\lambda(W)), G)$ equipped with a graph measure has pure point spectrum and all eigenfunctions are continuous.

Suppose for almost every $[s, t]_{\mathcal{L}} \in \mathbb{T}$, the fibre $\beta^{-1}([s, t]_{\mathcal{L}})$ contains a unique maximal element Γ_+ or a unique minimal element Γ_- with respect to set inclusion. Given the existence of such elements, we set

$$\gamma_{\pm}: \mathbb{T} \rightarrow \Omega(\lambda(W)), \quad [s, t]_{\mathcal{L}} \mapsto \Gamma_{\pm}([s, t]_{\mathcal{L}}).$$

Proposition 6.3. Suppose almost every fibre contains an element Γ_+ (Γ_-) as above. Then γ_+ (γ_-) is an invariant graph.

Proof. As the proofs for γ_+ and γ_- are similar, we omit the index \pm in the following. (6.1) is obvious. In order to see the measurability of γ , recall that $F: \mathbb{T} \rightarrow \mathcal{K}(\Omega(\lambda(W)))$, $[s, t]_{\mathcal{L}} \mapsto \beta^{-1}([s, t]_{\mathcal{L}})$, where $\mathcal{K}(\Omega(\lambda(W)))$ denotes the class of compact subsets of $\Omega(\lambda(W))$, is measurable (see the proof of Theorem 1.2). Lusin's Theorem hence yields the existence of compact sets $K_n \subseteq \mathbb{T}$ with $\Theta_{\mathbb{T}}(K_n) > 1 - 1/n$ on which F is continuous with respect to the Hausdorff topology on $\mathcal{K}(\Omega(\lambda(W)))$. Observe that hence, $\gamma|_{K_n}$ is continuous, too. Thus, γ is measurable with respect to the completion of the sigma algebra of the Borel sets of \mathbb{T} . \square

Remark 6.4. In the situation of the previous statement, the measure $\mu_{\gamma_{\pm}}$ is also referred to as *Mirsky measure* (see, for example, [39–41]).

Now, let W and V be as in Section 5. By means of Remark 4.4 and the above statement, we see that $(\Omega(\lambda(W)), \mathbb{R})$ allows for two invariant graphs γ_{\pm} (mapping each $\xi \in \mathbb{T}$ to the maximal [minimal] element of $\beta^{-1}(\xi)$). Moreover, taking into account that the torus parametrisation of $(\Omega(\lambda(W)), \mathbb{R})$ is almost everywhere 2-to-1, the proof of Lemma 4.7 also shows that the associated graph measures $\mu_{\gamma_{\pm}}$ of $(\Omega(\lambda(W)), \mathbb{R})$ are the only ergodic

measures of $(\Omega(\lambda(W)), \mathbb{R})$. Likewise, due to Lemma 4.5 and the unique ergodicity of $(\Omega(\lambda(V)), \mathbb{R})$, we have that $(\Omega(\lambda(V)), \mathbb{R})$ allows for a unique invariant graph γ (mapping each $\xi \in \mathbb{T}$ to the maximal element of $\beta^{-1}(\xi)$). We have thus proven

Theorem 6.5. $(\Omega(\lambda(W)), \mathbb{R})$ (equipped with any ergodic measure) as well as $(\Omega(\lambda(V)), \mathbb{R})$ (equipped with the unique invariant measure) have pure point dynamical spectrum with all eigenfunctions being continuous.

By an immediate application of [3, Theorem 3.2] and [3, Theorem 4.1] we hence get

Corollary 6.6. Every $\Gamma \in (\Omega(\lambda(V)), \mathbb{R})$ and μ -almost every $\Gamma \in (\Omega(\lambda(W)), \mathbb{R})$ (where μ is any invariant measure on $(\Omega(\lambda(W)), \mathbb{R})$) has pure point diffraction spectrum.

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