On the onset of diffusion in the kicked Harper model

Tobias Jäger Andres Koropecki Fabio Armando Tal

Abstract

We study a standard two-parameter family of area-preserving torus diffeomorphisms, known in theoretical physics as the *kicked Harper model*, by a combination of topological arguments and KAM-theory. We concentrate on the structure of the parameter sets where the rotation set has empty and non-empty interior, respectively, and describe their qualitative properties and scaling behaviour both for small and large parameters. This confirms numerical observations about the onset of diffusion in the physics literature. As a byproduct, we obtain the continuity of the rotation set within the class of Hamiltonian torus homeomorphisms.

1 Introduction

The *kicked Harper map* is a parameter family of torus diffeomorphisms that is given by

$$f_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2 \ , \quad (x,y) \mapsto (x + \alpha \sin(2\pi(y + \beta \sin(2\pi x))), y + \beta \sin(2\pi x)) \ , \tag{1.1}$$

with parameters $\alpha, \beta \in \mathbb{R}$. It is the composition of a vertical and a horizontal skew shift: if we let

$$v_\beta(x,y) = (x, y + \beta \sin(2\pi x)) \ , \tag{1.2}$$
$$h_\alpha(x,y) = (x + \alpha \sin(2\pi y), y) \ , \tag{1.3}$$

then

$$f_{\alpha,\beta} = h_\alpha \circ v_\beta \ . \tag{1.4}$$

This also allows to see the maps $f_{\alpha,\beta}$ are all *Hamiltonian torus diffeomorphisms*, by which we mean that they are homotopic to the identity, preserve area and have zero

T. Jäger: Friedrich Schiller University Jena, Institute of Mathematics; e-mail: tobias.jaeger@uni-jena.de

A. Koropecki: Universidade Federal Fluminense, Instituto de Matemática e Estatística; e-mail: ak@id.uff.br

F.A.Tal: Universidade de São Paulo, Instituto de Matemática e Estatística, e-mail: fabiotal@ime.usp.br

Mathematics Subject Classification (2010): Primary 37E30, 37E45; Secondary 70H08

Lebesgue rotation number. As (1.1) presents one of the simplest ways to produce explicit examples of Hamiltonian torus diffeomorphism, one can see it as a standard family that may serve as a reference for the study of their dynamics and rotational behaviour. Moreover, this model has been associated to a variety of problems in theoretical physics, including the motion of magnetic field lines, wave-particle interactions, dynamics of particle accelerators or laser-plasma coupling, and both its classical and quantum dynamics have been studied with computational methods by a variety of authors (see, for example [HH84, LKFA90, Leb98, SA97, SA98, Shi02, Zas05, Zas07] and references therein). In the context of KAM-theory (1.1) provides a natural example of area-preserving diffeomorphisms that do not satisfy the classicial twist condition. This fact gives rise to a number of phenomena that have been studied, again in theoretical and computational physics, under the names of *meandering KAM-tori*, *separatrix reconnection* or the appearence of *twin chains* (e.g. [HH84, Leb98, SA97, Shi02]). The existence and breakup of a *shearless KAM-torus*[3] has been related to the *onset of diffusion and global chaos* in the kicked Harper map [Shi02].

The purpose of this article is to study this model from the viewpoint of rotation theory, which provides a rigorous framework for the description of (some of) the above-mentioned phenomena. The main topological invariant in this theory is the *rotation set* of a torus homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, homotopic to the identity and with lift $F : \mathbb{R}^2 \to \mathbb{R}^2$, which has been introduced by Misiurewicz and Ziemian in [MZ89] as

$$\rho(F) \;=\; \left\{ \rho \in \mathbb{R}^2 \;\middle|\; \exists n_i \nearrow \infty,\; z_i \in \mathbb{R}^2 :\; \rho = \lim_{i\to\infty} \left(F_i^n(z_i) - z_i\right)/n_i \right\} \;. \quad (1.5)$$

As a basis for our further investigations, we first show the continuity of the map $(\alpha, \beta) \mapsto \rho(F_{\alpha,\beta})$. This follows from a general result on the continuous dependence of rotation set for Hamiltonian homeomorphisms (Theorem 3.1 below). The onset of diffusion and global chaos in (1.1) then corresponds to the appearence of rotation sets with non-empty interior, and we aim at a better understanding of this transition by studying the two complementary parameter regions with empty and non-empty interior of the rotation set, as shown in Figure 1. The analysis of these sets is simplified by the fact that the maps $F_{\alpha,\beta}$ have a number of symmetries, which directly translate into symmetries of the rotation sets. In particular, the latter are invariant under the reflexions along the horizontal and vertical axis (see Section 2). Combined with the convexity of the rotation set, this implies the following

Proposition 1.1. *For any $\alpha, \beta \in \mathbb{R}$, the rotation set $\rho(F_{\alpha,\beta})$ is either*

(i) reduced to $\{(0,0)\}$;

(ii) a non-degenerate segment contained in the horizontal or vertical axis, with midpoint in the origin;

(iii) a set with non-empty interior.

We denote by $F_{\alpha,\beta}$ the canonical lift of $f_{\alpha,\beta}$ to $\mathbb{R}^2$ and let

$$\begin{aligned} \mathcal{E} &= \left\{(\alpha,\beta) \in \mathbb{R}^2 \mid \operatorname{int}(\rho(F_{\alpha,\beta})) = \emptyset\right\} \;, \\ \mathcal{N} &= \left\{(\alpha,\beta) \in \mathbb{R}^2 \mid \operatorname{int}(\rho(F_{\alpha,\beta})) \neq \emptyset\right\} \;, \end{aligned} \quad (1.6)$$

[3] A KAM-torus that has an extremal rotation number, so that no twist condition (*shear*) can be satisfied in any of its neighbourhoods.

so that $\mathcal{N} = \mathbb{R}^2 \setminus \mathcal{E}$. The following result provides a theoretical basis for the numerical approximations of these sets in Figure 1 (explained below) and thereby also provides a justification for similar computations in [Shi02]. By $|I|$, we denote the length of an interval $I \subseteq \mathbb{R}$. By $\pi_i : \mathbb{R}^2 \to \mathbb{R}$ we denote the canonical projection to the i-th coordinate, and we use the same notation for other product spaces like $\mathbb{T}^2$ or $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$.

Proposition 1.2. *For $i = 1, 2$, we have $|\pi_i(\rho(F_{\alpha,\beta}))| > 0$ if and only there exists some $z \in \mathbb{R}^2$ and $n \in \mathbb{N}$ such that $|\pi_i\left(F^n_{\alpha,\beta}(z) - z\right)| \geq 1$.*

Due to the symmetries mentioned above, this immediately entails

Corollary 1.3. *We have $\mathrm{int}(\rho(F)) \neq \emptyset$ if and only if there exist $z_1, z_2 \in \mathbb{R}$ such that $|\pi_i\left(F^n_{\alpha,\beta}(z_i) - z_i\right)| \geq 1$ for $i = 1, 2$.*

Since this allows us to detect rotation sets with nonempty interior by considering the absolute displacement of orbits instead of asymptotic averages, this provides a simple numerical procedure to approximate the sets $\mathcal{N}$ and $\mathcal{E}$. In order to obtain Figure 1, for each parameter (pixel), a test point is chosen and iterated 4 million times. If the maximal observed displacement is greater than 1 in both directions, the pixel is painted white. The process is repeated for a large number of test points. The white region can thus be seen as a (lower) approximation of $\mathcal{N}$. In the red region, which corresponds to $\mathcal{E}$, the color scheme corresponds to the maximum of the observed displacements between 0 and 1.

The fact that the rotation set has empty interior in a neighborhood of the coordinate axes (removing the origin) is easily explained by the KAM phenomenon: if one parameter is fixed (and nonzero) and the other is small enough, the dynamics in a neighborhood of one of the axes is a small perturbation of an integrable twist map, so that KAM tori persist and force the boundedness of orbits in the transverse direction. Hence, for any $\alpha \neq 0$ there exists $\beta_0 > 0$ such that $\{(\alpha, \beta) : |\beta| \leq \beta_0\} \subset \mathcal{E}$. A quantitative refinement of this statement will be given in Theorem 1.8 below. In contrast, when both parameters are large, one can guarantee the creation of *rotational horseshoes*, leading to a rotation set with nonempty interior. This entails the following

Proposition 1.4. *If $|\alpha| \geq 1/2$ and $|\beta| \geq 1/2$ then $(\alpha, \beta) \in \mathcal{N}$.*

The transition between the diffusive and the non-diffusive regime is a subtle problem that is still poorly understood, even in the classical Chirikov-Taylor standard family [Chi79, CS08]. A non-trivial qualitative feature of the sets $\mathcal{E}$ and $\mathcal{N}$ that can be observed in Figure 1 is the fact that a thin cusp of $\mathcal{N}$ seems to extend along the diagonal towards the origin. This is confirmed by the following results.

Theorem 1.5. *We have $\rho(F_{\alpha,\beta}) = \{(0, 0)\}$ if and only if $(\alpha, \beta) = (0, 0)$.*

Due to further symmetries of the rotation set on the diagonal explained in Section 2, this directly implies

Corollary 1.6. *We have $\mathrm{int}(\rho(F_{\alpha,\alpha})) = \emptyset$ if and only if $\alpha = 0$.*

Hence, the diagonal is indeed contained in $\mathcal{N}$. Further, the following statement confirms the cusp-like shape of $\mathcal{N}$ near the origin.

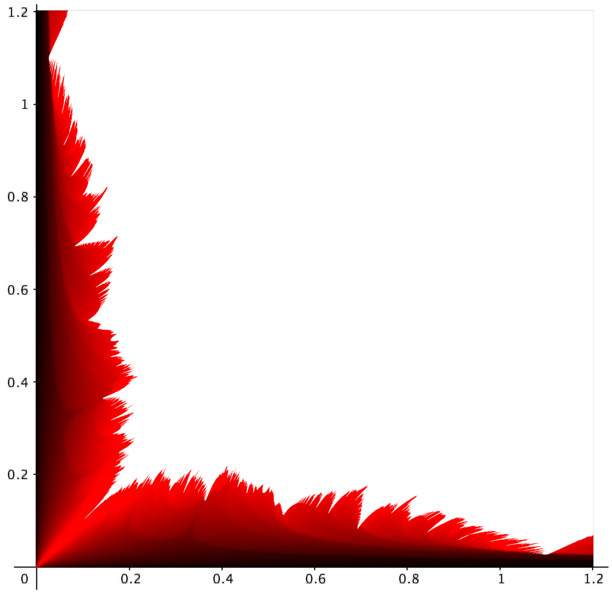

Figure 1: The parameter region $\mathcal{E}$ on which the rotation set of $f_{\alpha,\beta}$ has empty interior is shown in red, whereas the white region corresponds to parameters with non-empty interior rotation set. The red colour scheme indicates the amount of vertical movement (below the diagonal) or horizontal movement (above the diagonal). Dark red corresponds to very small displacements, whereas light red indicates displacments close to the critical threshold of one.

Theorem 1.7. *Suppose that* $\lambda \in [0,1)$. *Then*

$$\alpha_0(\lambda) \;=\; \inf\{\alpha > 0 \mid (\alpha, \lambda\alpha) \in \mathcal{N}\} \;\; (= \inf\{\alpha > 0 \mid (\lambda\alpha, \alpha) \in \mathcal{N}\}) \;>\; 0\,.$$

(Note that the alternative definition of α_0 *in brackets is justified by the symmetries of the rotation set discussed in Section 2.) Moreover,* α_0 *is uniformly bounded away from zero on any compact subinterval of* $[0,1)$.

Turning away from the vicinity of the origin, we then focus on large parameters near the coordinate axes. Here, Figure 2 reveals both a periodic structure combined with a decay in height of the region $\mathcal{E}$ as the parameter α tends to infinity. Both the periodicity and the scaling behaviour are explained in [Shi02] on a heuristic level, by rescaling the maps $F_{\alpha,\beta}$ in a suitable neighbourhood of the critical line $\mathbb{R} \times \{1/4\}$ and relating them to a quadratic approximation, given by the *standard non-twist map* (see [SA97, SA98]). As the argument relies on some a priori assumptions that are hard to verify, it is difficult to convert it into a rigorous proof. However, we can at least use these ideas to obtain analytic estimates for the scaling behaviour. Given $\alpha \in \mathbb{R}$, we let

$$\beta^-(\alpha) \;=\; \inf\{\beta > 0 \mid (\alpha,\beta) \in \mathcal{N}\} \quad \text{and} \quad \beta^+(\alpha) \;=\; \sup\{\beta > 0 \mid (\alpha,\beta) \in \mathcal{E}\}\,. \tag{1.7}$$

Then we have

Theorem 1.8. *There exists constants* $0 < c < C$ *such that*

$$\frac{c}{\sqrt{\alpha}} \;\le\; \beta^-(\alpha) \;\le\; \beta^+(\alpha) \;\le\; \frac{C}{\sqrt{\alpha}} \tag{1.8}$$

for all $\alpha \geq 1$.

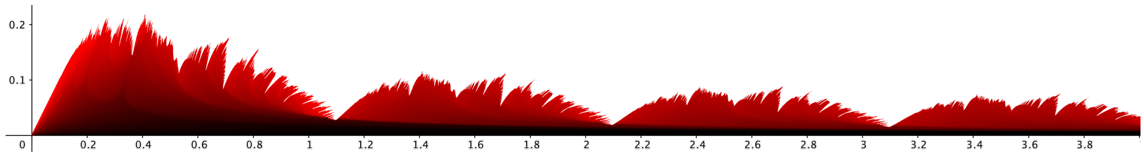

Figure 2: The picture shows the part of the parameter set $\mathcal{E}$ that lies below the diagonal, with α between 0 and 4. It reveals a seemingly periodic structure, combined with a decay in the height of the region when α becomes large.

The paper is organised as follows. In Section 2, we collect a number of basic facts about the kicked Harper map, including a description of its symmetries, the local analysis of canonical fixed points and a proof of Proposition 1.4. Section 2.2 deals with the diffusion threshold provided by Proposition 1.2. The continuous dependence of the rotation set of Hamiltonian torus diffeomorphisms is proved in Section 3. Section 4 provides the proofs of Theorems 1.5 and 1.7 and their corollaries, describing the cusp of $\mathcal{N}$ along the diagonal. The proof of Theorem 1.8 about the scaling behaviour for large parameters is then given in Section 5. We conclude with Section 6, presenting some additional remarks on the family as well as a few interesting questions for further work.

Acknowledgments. Most of this research was carried out while AK and FT spent a year at the Friedrich Schiller University of Jena as visiting professors. The stay of AK was supported by a Mercator Fellowship of the German Research Council (DFG-grant OE 538/9-1). The stay of FT was made possible by a Willhelm Friedrich Bessel Award of the Alexander von Humboldt Foundation. TJ acknowlegdes support by a Heisenberg grant of the German Research Council (DFG-grant OE 538/4-1).

2 Basic observations on the kicked Harper map

The aim of this section is to collect a number of basic observations about the symmetries and the fixed and periodic points of the maps $f_{\alpha,\beta}$ given by (1.1) and their lifts $F_{\alpha,\beta}$. In order to simplify notation, we let

$$s(x) \;=\; \sin(2\pi x)\;,$$

so that

$$F_{\alpha,\beta}(x,y) \;=\; (x+\alpha s(y+\beta s(x)), y+\beta s(x))\;.$$

We further let

$$H_\alpha(x,y) \;=\; (x+\alpha s(y),y) \quad \text{and} \quad V_\beta(x,y) \;=\; (x,y+\beta s(x))\;.$$

Then we have $F_{\alpha,\beta} = H_\alpha \circ V_\beta$, which allows to see that $F_{\alpha,\beta}$ is area-preserving and also extends to a biholomorphic diffeomorphism of $\mathbb{C}^2$.

2.1 Symmetries

The maps $f_{\alpha,\beta}$ and their lifts $F_{\alpha,\beta}$ have a number of natural symmetries, which directly translate to symmetries of their rotation sets $\rho(F_{\alpha,\beta})$ and the map $(\alpha,\beta) \mapsto (\rho(F_{\alpha,\beta}))$. In order to describe these, we make use of the following basic facts concerning the transformation of rotation sets.

By $\mathrm{Hom}(X)$ we denote the space of homeomorphisms of a topological space X. Further we let $\pi : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the canonical projection. Given a lift H of an element of $\mathrm{Hom}(\mathbb{T}^2)$, denote by $[H]$ the unique element of $\mathrm{GL}(2,\mathbb{Z})$ such that $z \mapsto H(z) - [H](z)$ is $\mathbb{Z}^2$-periodic. Suppose that F, G are lifts of elements of $\mathrm{Hom}_0(\mathbb{T}^2)$, where $\mathrm{Hom}_0(\mathbb{T}^2) = \{f \in \mathrm{Hom}(\mathbb{T}^2) \mid f$ is homotopic to the identity$\}$. Then (compare [Kwa92, Lemma 1]),

$$F \circ H = H \circ G \implies \rho(F) \; = \; [H]\rho(G) \; . \tag{2.1}$$

To properly state the symmetries, consider the following linear involutions:

$$S_1 \colon (x,y) \mapsto (-x,y), \quad S_2 \colon (x,y) \mapsto (x,-y), \quad S \colon (x,y) \mapsto (-x,-y), \quad D \colon (x,y) \mapsto (y,x)$$

We will also use the general fact that $\rho(F^{-1}) = S\rho(F)$. Futher, note that

$$H_\alpha \circ D = D \circ V_\alpha \quad \text{and} \quad V_\beta \circ D = D \circ H_\beta. \tag{2.2}$$

which, noting that $H_\alpha^{-1} = H_{-\alpha}$ and $V_\beta^{-1} = V_{-\beta}$, implies

$$F_{\alpha,\beta} \circ D = D \circ F^{-1}_{-\beta,-\alpha} \tag{2.3}$$

Moreover,

$$H_\alpha \circ S_i = S_i \circ H_{-\alpha} \quad \text{and} \quad V_\beta \circ S_i = S_i \circ V_{-\beta} \quad \text{for } i \in \{1,2\}, \tag{2.4}$$

which implies

$$F_{\alpha,\beta} \circ S_i = S_i \circ F_{-\alpha,-\beta} \quad \text{for } i \in \{1,2\}, \tag{2.5}$$

and, since $S = S_1 \circ S_2$, we also have

$$F_{\alpha,\beta} \circ S = S \circ F_{\alpha,\beta}. \tag{2.6}$$

Reversibility. For any $G \in \{H_\alpha \circ S_1,\ H_\alpha \circ S_2,\ S_1 \circ V_\beta,\ S_2 \circ V_\beta\}$, one has

$$F_{\alpha,\beta} \circ G = G \circ F^{-1}_{\alpha,\beta}, \tag{2.7}$$

and $G^2 = \mathrm{id}$, as one can directly verify using (2.4). This property of a map being conjugated to its inverse by means of an involution is often referred to as *reversibility* of the dynamics.

Reflection symmetries. First note that from (2.6) and (2.1) we see that

$$\rho(F_{\alpha,\beta}) = S\rho(F_{\alpha,\beta}) = \rho(F^{-1}_{\alpha,\beta}). \tag{2.8}$$

On the other hand, since $H_{-\alpha} = H_\alpha^{-1}$ and $V_{-\beta} = V_\beta^{-1}$, one easily verifies that

$$H_\alpha \circ F_{-\alpha,-\beta} = F^{-1}_{\alpha,\beta} \circ H_\alpha.$$

In particular, since $[H_\alpha] = \mathrm{id}$, the above equation and (2.1) imply

$$\rho(F_{-\alpha,-\beta}) = \rho(F_{\alpha,\beta}^{-1}) \overset{(2.8)}{=} \rho(F_{\alpha,\beta}). \tag{2.9}$$

From (2.5) and (2.1) we se that $\rho(F_{\alpha,\beta}) = S_i\rho(F_{-\alpha,-\beta})$ for $i \in \{1,2\}$, so the above implies

$$\rho(F_{\alpha,\beta}) = S_i\rho(F_{\alpha,\beta}) \quad \text{for } i \in \{1,2\}. \tag{2.10}$$

In other words, the rotation set is symmetric with respect to the two coordinate axes.

Remark 2.1. Together with the convexity of the rotation set and the fact that $\rho(F_{\alpha,\beta})$ always contains the origin (since the origin is a fixed point), the symmetry with respect to the two coordinate axes implies Proposition 1.1.

We also have, using (2.3),

$$\rho(F_{\alpha,\beta}) \overset{(2.1)}{=} D\rho(F_{-\beta,-\alpha}^{-1}) \overset{(2.8)}{=} D\rho(F_{-\beta,-\alpha}) \overset{(2.9)}{=} D\rho(F_{\beta,\alpha}). \tag{2.11}$$

Rotation symmetry. Consider the rotation $R : (x, y) \mapsto (-y, x)$ by $\pi/2$. Noting that $R = S_1D$, we see that

$$\rho(F_{\alpha,\beta}) \overset{(2.10)}{=} S_1\rho(F_{\alpha,\beta}) \overset{(2.11)}{=} S_1D\rho(F_{\beta,\alpha}) = R\rho(F_{\beta,\alpha}). \tag{2.12}$$

In particular, $\rho(F_{\alpha,\alpha}) = R\rho(F_{\alpha,\alpha})$ so that for parameters in the diagonal $\alpha = \beta$ the rotation set is invariant under rotations by angle $\pi/2$.

Remark 2.2. The above symmetries, in particular (2.11) and (2.12), imply that the set $\mathcal{N}$ is symmetric with respect to the diagonal and to rotations by $\pi/2$ around the origin, which allows us to restrict our attention to parameters $0 \leq \beta \leq \alpha$ below the diagonal in order to analyze the structure of the parameter sets $\mathcal{N}$ and $\mathcal{E}$.

Remark 2.3. The previous analysis relies only on the fact that s is an odd function. Therefore, it also applies if one replaces s by any 1-periodic odd continuous function.

Translation symmetries. From the fact that $s(x + 1/2) = -s(x)$, we obtain some additional symmetries. Consider the translations

$$T_1 \colon (x, y) \mapsto (x + 1/2, y), \quad T_2 \colon (x, y) \mapsto (x, y + 1/2)$$

Then it is easily checked that

$$F_{\alpha,\beta} \circ T_1 = T_1 \circ F_{\alpha,-\beta} \quad \text{and} \quad F_{\alpha,\beta} \circ T_2 = T_2 \circ F_{-\alpha,\beta}. \tag{2.13}$$

Consequently (2.1) implies

$$\rho(F_{\alpha,\beta}) \;=\; \rho(F_{\alpha,-\beta}) \;=\; \rho(F_{-\alpha,\beta}) \;=\; \rho(F_{-\alpha,-\beta})\,. \tag{2.14}$$

Note that the last equality can be derived from the first one, applied to $F_{-\alpha,\beta}$, or similarly from the fact that if we let $T = T_1 \circ T_2$ then we have

$$F_{\alpha,\beta} \circ T \;=\; T \circ F_{-\alpha,-\beta} \tag{2.15}$$

by (2.13). We also note that these facts only depend on the fact that $s(x + 1/2) = -s(x)$ and will remain true if s is replaced by any other function with this property.

2.2 A threshold for diffusion

As we saw in Proposition 1.1, if $\rho(F_{\alpha,\beta})$ has empty interior it is contained in one of the coordinate axes. It is known that when the rotation set is a nondegenerate interval, the displacement in the direction perpendicular to the interval is uniformly bounded (see [Dav16, GKT14]). In our case, the reversibility of the dynamics allows us to obtain a direct proof and an explicit bound.

Proposition 2.4. *If $z \in \mathbb{R} \times \{0\}$ and $F^n_{\alpha,\beta}(z) \in \mathbb{R} \times \{k/2\}$ for some $k, n \in \mathbb{Z}$, then $F^{2n}_{\alpha,\beta}(z) = z + (0,k)$. In particular, $(0, \frac{k}{2n}) \in \rho(F_{\alpha,\beta})$. An analogous property holds in the horizontal direction.*

Proof. Suppose $z = (t,0)$ and $F^n_{\alpha,\beta}(z) = (t', k/2)$. Letting $G = H_\alpha \circ S_2$, one has $G(z) = z$, so by the reversibility equation (2.7) we deduce

$$G(F^n_{\alpha,\beta}(z)) = F^{-n}_{\alpha,\beta}(G(z)) = F^{-n}_{\alpha,\beta}(z),$$

and noting that $s(-k/2) = 0$ we see that

$$G(F^n_{\alpha,\beta}(z)) = H_\alpha(S_2(t', k/2)) = (t', -k/2) = F^n_{\alpha,\beta}(z) - (0,k).$$

Therefore $F^{-n}_{\alpha,\beta}(z) = F^n_{\alpha,\beta}(z) - (0,k)$, which yields $z = F^{2n}_{\alpha,\beta}(z) - (0,k)$, and the claim follows. The analogous claim for the horizontal direction is proven similarly using $G = S_1 \circ V_\beta$. □

Corollary 2.5. *If $\pi_i(\rho(F_{\alpha,\beta})) = \{0\}$, then $|\pi_i(F^n_{\alpha,\beta}(z) - z)| < 1$ for all $z \in \mathbb{R}^2$ and $n \in \mathbb{Z}$.*

Proof. We consider the case $i = 2$, the case $i = 1$ is analogous. Assuming that there exist $z_0 \in \mathbb{R}^2$ and $n \in \mathbb{Z}$ such that $|\pi_2(F^n_{\alpha,\beta}(z_0) - z_0)| \geq 1$, we need to prove that $\pi_2(\rho(F_{\alpha,\beta})) \neq \{0\}$.

Since $z \mapsto \pi_2(F^n_{\alpha,\beta}(z) - z)$ attains the value 0, by the intermediate value theorem we may choose z_0 such that $|\pi_2(F^n_{\alpha,\beta}(z_0) - z_0)| = 1$. Moreover, since $F_{\alpha,\beta}(-z) = -F_{\alpha,\beta}(z)$ we may assume $\pi_2(F^n_{\alpha,\beta}(z_0) - z_0) = 1$ (replacing z_0 by $-z_0$ if necessary). In addition, we may assume that $z_0 \in \mathbb{R} \times (-1, 0]$ since z_0 can be replaced by any of its integer translates.

Consider first the case where $z_0 \in \mathbb{R} \times (-1/2, 0]$, so that $F^n_{\alpha,\beta}(z_0) \in \mathbb{R} \times (1/2, 1]$. Then the image by $F^n_{\alpha,\beta}$ of the half-plane $H_0 = \{(x,y) : y \leq 0\}$ is bounded from above and intersects $\{(x,y) : y > 1/2\}$, which implies that its boundary $\partial F^n_{\alpha,\beta}(H_0) = F^n_{\alpha,\beta}(\mathbb{R} \times \{0\})$ also intersects $\{(x,y) : y > 1/2\}$. Since the line $\mathbb{R} \times \{0\}$ contains fixed points of $F_{\alpha,\beta}$, we see that $F^n_{\alpha,\beta}(\mathbb{R} \times \{0\})$ also intersects $\{(x,y) : y < 1/2\}$, and therefore it intersects the line $\mathbb{R} \times \{1/2\}$. Thus the previous proposition implies that $\pi_2(\rho(F_{\alpha,\beta})) \neq \{0\}$.

In the case that $z_0 \in \mathbb{R} \times (-1, -1/2]$, an analogous argument shows that $F^n_{\alpha,\beta}(\mathbb{R} \times \{-1/2\})$ intersects $\mathbb{R} \times \{0\}$, and the previous proposition again implies $\pi_2(\rho(F_{\alpha,\beta})) \neq \{0\}$, completing the proof. □

2.3 Local analysis for irrotational fixed points

For parameters $\alpha, \beta \neq 0$, the map $F_{\alpha,\beta}$ has, up to integer translations, exactly four fixed points: $(0,0), (0,1/2), (1/2,0)$ and $(1/2,1/2)$. In order to analyze their stability, note that the Jacobian of $F_{\alpha,\beta}$ is given by

$$DF_{\alpha,\beta}(x,y) \;=\; \begin{pmatrix} 4\pi^2\alpha\beta\cos(2\pi y'))\cos(2\pi x)+1 & 2\pi\alpha\cos(2\pi y')) \\ 2\pi\beta\cos(2\pi x) & 1 \end{pmatrix} \qquad (2.16)$$

where $y' = y + b\sin(2\pi x)$. At the origin, this simplifies to

$$DF_{\alpha,\beta}(0,0) \;=\; \begin{pmatrix} 4\pi^2\alpha\beta+1 & 2\pi\alpha \\ 2\pi\beta & 1 \end{pmatrix}. \qquad (2.17)$$

If $\alpha\beta \neq 0$, the eigenvalues are

$$\lambda_1^{(0,0)} \;=\; 2\pi^2\alpha\beta - 2\pi(\alpha\beta(\pi^2\alpha\beta+1))^{1/2}+1 \;<1\,, \qquad (2.18)$$
$$\lambda_2^{(0,0)} \;=\; 2\pi^2\alpha\beta + 2\pi(\alpha\beta(\pi^2\alpha\beta+1))^{1/2}+1 \;>1\,, \qquad (2.19)$$

with eigenvectors

$$v_1^{(0,0)} \;=\; \begin{pmatrix} -((\alpha\beta(\pi^2\alpha\beta+1))^{1/2}-\pi\alpha\beta)/\beta \\ 1 \end{pmatrix} \quad \text{and} \qquad (2.20)$$
$$v_2^{(0,0)} \;=\; \begin{pmatrix} ((\alpha\beta(\pi^2\alpha\beta+1))^{1/2}+\pi\alpha\beta)/\beta \\ 1 \end{pmatrix} \qquad (2.21)$$

In $(1/2,1/2)$ the Jacobian is

$$DF_{\alpha,\beta}(0,0) \;=\; \begin{pmatrix} 4\pi^2\alpha\beta+1 & -2\pi\alpha \\ -2\pi\beta & 1 \end{pmatrix}. \qquad (2.22)$$

It has the same eigenvalues, $\lambda_1^{(\frac{1}{2},\frac{1}{2})} = \lambda_1^{(0,0)}$ and $\lambda_2^{(\frac{1}{2},\frac{1}{2})} = \lambda_2^{(0,0)}$, but with eigenvectors

$$v_1^{(\frac{1}{2},\frac{1}{2})} \;=\; \begin{pmatrix} ((\alpha\beta(\pi^2\alpha\beta+1))^{1/2}-\pi\alpha\beta)/\beta \\ 1 \end{pmatrix} \quad \text{and} \qquad (2.23)$$
$$v_2^{(\frac{1}{2},\frac{1}{2})} \;=\; \begin{pmatrix} -((\alpha\beta(\pi^2\alpha\beta+1))^{1/2}+\pi\alpha\beta)/\beta \\ 1 \end{pmatrix} \qquad (2.24)$$

Thus $(0,0)$ and $(1/2,1/2)$ are hyperboic fixed points. The remaining two fixed points are $(0,1/2)$ and $(1/2,0)$. The Jacobian at those points is

$$\begin{pmatrix} 1-4\pi^2\alpha\beta & \pm 2\pi\alpha \\ \mp 2\pi\beta & 1 \end{pmatrix}. \qquad (2.25)$$

Note that its trace is $2-4\pi^2\alpha\beta$ which can only be equal to 2 if either α or β is 0. In particular the fixed points are elementary, meaning that 1 is not an eigenvalue of the Jacobian.

2.4 Rotational periodic orbits and a priori lower bounds on the rotation set

Suppose that $\alpha, \beta \geq n$ for some $n \in \mathbb{N}$. Choose $x, y \in [0,1]$ with $s(x) = n/\beta$ and $s(y) = n/\alpha$. Then it is easily checked that

$$F_{\alpha,\beta}(\pm x, \pm y) \ = \ (x \pm n, y \pm n)$$

so that $\{(n,n), (-n,n), (n,-n), (-n,-n)\} \subseteq \rho(F_{\alpha,\beta})$. Hence, by convexity of the rotation set $[-n,n]^2 \subseteq \rho(F_{\alpha,\beta})$. In the particular case $\alpha = \beta = n$, we even obtain the equality $\rho(F_{n,n}) = [-n,n]^2$, as the maximal displacement in the x- and y-direction is exactly n.

Similarly, if $\alpha, \beta \geq 1/2$, we can choose $x, y \in [0,1]$ such that $s(x) = 1/2\beta$ and $s(y) = 1/2\alpha$. Then we obtain that $F_{\alpha,\beta}(x,0) = F_{\alpha,\beta}(x,1/2)$ and further $F_{\alpha,\beta}(x,1/2) = F_{\alpha,\beta}(x,1)$ (note that $s(0) = s(1/2) = 0$). Consequently $(0,1/2) \in \rho(F_{\alpha,\beta})$, and in the same way we obtain $(0,-1/2), (1/2,0), (-1/2,0) \in \rho(F_{\alpha,\beta})$. Hence, the rotation set contains the square $\mathcal{Q} = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1/2\}$ in this case, and in particular has non-empty interior. This proves Proposition 1.4. As a consequence, this means that we can restrict to parameters in the $1/2$-neighbourhood of the coordinate axes when analyzing the sets $\mathcal{N}$ and $\mathcal{E}$. We note also that, when $\alpha = \beta = 1/2$, then one can verify that $\langle F^2_{1/2,1/2}(x,y) - (x,y); (\sqrt{2}/2, \sqrt{2}/2)\rangle \leq 1$. This, together with the symmetries of the rotation set, imply that in this case the rotation set is exactly $\mathcal{Q}$.

3 Continuity of the rotation set for Hamiltonian lifts of torus homeomorphisms

In this section we prove a general result which in particular implies the continuous dependence of the rotation set of $F_{\alpha,\beta}$ on the parameters (α, β). Recall that a homeomorphism f of the two-torus is said to be *Hamiltonian* if it is homotopic to the identity, preserves the Lebesgue measure λ on $\mathbb{T}^2$ and has a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ with zero *Lebesgue rotation vector*, that is,

$$\rho_\lambda(F) = \int_{[0,1]^2} F(z) - z \, d\lambda(z) \ = (0,0) \, . \tag{3.1}$$

The lift F is called a *Hamiltonian lift* in this case. We denote by $\mathrm{Hom}_0(\mathbb{T}^2)$ the space of torus homeomorphism homotopic to the identity and let

$$\mathrm{Hom}_0^{\mathrm{ap}}(\mathbb{T}^2) \ = \ \left\{ f \in \mathrm{Hom}_0\left(\mathbb{T}^2\right) \mid f \text{ is area-preserving} \right\} \, .$$

The respective spaces of lifts will be denoted by $\widehat{\mathrm{Hom}_0}(\mathbb{T}^2)$ and $\widehat{\mathrm{Hom}_0^{\mathrm{ap}}}(\mathbb{T}^2)$, respectively. Further, we denote by $\mathrm{Ham}_0(\mathbb{T}^2)$ the space of Hamiltonian torus homeomorphisms and by $\widehat{\mathrm{Ham}_0}(\mathbb{T}^2)$ the space of their lifts that satisfy (3.1). Note that we have

$$\mathrm{Ham}_0(\mathbb{T}^2) \ \subseteq \ \mathrm{Hom}_0^{\mathrm{ap}}(\mathbb{T}^2) \ \subseteq \ \mathrm{Hom}_0(\mathbb{T}^2) \, ,$$

and the same inclusions hold for the respective lift spaces. The statement on the continuity of the rotation set then reads as follows.

Theorem 3.1. *The mapping*

$$F \mapsto \rho(F)$$

is continuous on $\widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ *(with respect to the* $\mathcal{C}^0$*-topology on* $\widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ *and the Hausdorff metric on the space of compact subsets of* $\mathbb{R}^2$*).*

In order to prove Theorem 3.1, let us first explain how existing results can be used to reduce the problem to the case when $F \in \widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ has a rotation set of the form

$$\rho(F) = \{tu \mid a \leq t \leq b\}, \quad \text{where } u \in \mathbb{Z}^2 \text{ and } a < 0 < b\,. \tag{3.2}$$

When $\rho(F)$ is a singleton, continuity follows from the upper semi-continuity of the rotation set [MZ89]. Likewise, when $\rho(F)$ has non-empty interior continuity of ρ in F follows from [MZ91, Theorem B]. Thus, we can assume that $\rho(F)$ is a line segment, which contains the origin due to the Hamiltonian condition $\rho_\lambda(F) = 0$. Moreover, [CT15, Theorem 70] implies that this segment is exactly of the form (3.2).

Hence, suppose from now on that $F \in \widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ satisfies (3.2) and fix $\varepsilon > 0$. Let v, v' be two rational vectors (i.e., in $\mathbb{Q}^2$,) in $\rho(F)$ which are ε-close to the endpoints. If we can show that $v, v' \in \rho(G)$ for any sufficiently small perturbation g of F in $\widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$, then by the convexity of the rotation set the whole segment between v and v' will be contained in $\rho(G)$. This yields the lower semicontinuity of ρ in F. Together with the upper semicontinuity from [MZ89] this gives the continuity of ρ in F. Therefore, it suffices to prove the following statement, which can be applied to two pairs of rational points w, v and w', v' chosen close enough to the endpoints of $\rho(F)$.

Proposition 3.2. *Suppose that rotation set of* $F \in \widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ *is a line segment containing* $w \in \rho(F) \cap \mathbb{Q}^2 \setminus \{0\}$*. Then, for each* $v \in \mathbb{Q}^2$ *lying in the interval* $I_w = \{tw \mid t \in (0,1)\}$*, there exists a neighborhood* $\mathcal{U}$ *of* F *in* $\widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ *such that every* $G \in \mathcal{U}$ *satisfies* $v \in \rho(G)$*.*

Proof. Since $\rho(F^n) = n\rho(F)$, we can replace F by an adequate power to assume that v and w have integer coordinates. Note that since n depends on (the denominators of) v and w, so does the neighbourhood $\mathcal{U}$ chosen below.

Hence, we assume $v, w \in \mathbb{Z}^2$. Since F is a lift of an area-preserving homeomorphism of $\mathbb{T}^2$ and its rotation set is a segment containing w, the main result of [Fra95] implies that there exists z_0 such that $F(z_0) = z_0 + w$. Fix $0 < \epsilon < 1/4$ such that $F(B_\epsilon(z_0))$ has diameter smaller than $1/4$. Let $\mathcal{U}$ be a neighbourhood of F in $\widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ with the property that every element $G \in \mathcal{U}$ is such that $G(z_0) \in B_\epsilon(z_0 + w)$ and $G(B_\epsilon(z_0))$ has diameter smaller than $1/4$. We claim that $v \in \rho(G)$ for any such G. To show this, we consider an area-preserving homeomorphism h defined on $B_\epsilon(z_0 + w)$ which is the identity in the boundary of the disk and such that $h(G(z_0)) = z_0 + w$. We extend h to a homeomorphism $H \in \widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ by $H(z) = h(z-\nu)+\nu$ if $z \in B_\epsilon(z_0+w)+\nu$ for some $\nu \in \mathbb{Z}^2$, and $H(z) = z$ otherwise. One easily verifies that $H \in \widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$, and $G' := HG$ satisfies $G'(z_0) = z_0 + w$.

Thus $w \in \rho(G')$, and since $G' \in \widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$, we also have $0 \in \rho(G')$. By convexity this implies $I_w \subset \rho(G')$. It then follows from [Fra95, Prop. 2.1] that there exists $z_1 \in \mathbb{R}^2$ such that $G'(z_1) = z_1 + v$. If $z_1 \in B_\epsilon(z_0)$, then

$$\|G'(z_0) - G'(z_1)\| = \|(z_0 + w) - (z_1 - v)\| \geq \|w - v\| - \|z_0 - z_1\| \geq 1 - \epsilon > 3/4\,,$$

which contradicts the fact that $G'(B_\epsilon(z_0)) = G(B_\epsilon(z_0))$ has diameter at most $1/4$. Thus $z_0 \notin B_\epsilon(z_0)$, and the same argument applied to its integer translations show that $z_1 \notin \bigcup_{u \in \mathbb{Z}^2} B_\epsilon(z_0) + u$. Since G' coincides with G outside of this set, we conclude that $G(z_1) = G'(z_1) = z_1 + v$. In particular $v \in \rho(G)$ as claimed. □

Remark 3.3. We note that the above argument can also be modified in order to give an alternative and elementary proof of [MZ91, Theorem B]. In order to show the persistence of a rotation vector $v \in \mathrm{int}(\rho(F)) \cap Q^2$, it suffices to choose three rational vectors $w_1, w_2, w_3 \in \rho(F)$ such that v is contained in the interior of their convex hull and to repeat the above construction simultaneously for three fixed points z_1, z_2, z_3 realising w_1, w_2, w_3, respectively.

4 The cusp along the diagonal

In this section, we concentrate on parameters on or close to the diagonal, with the aim to verify (in a qualitative way) the cusp form of the set $\mathcal{N}$ in this region. First, we note from (2.12) that the rotation set of $\rho(F_{\alpha,\alpha})$ is invariant under the rotation by angle $\pi/2$, and therefore it cannot be a line segment of positive length. Hence, Corollary 1.6, which states that the rotation set always has non-empty interior on the diagonal (excluding the origin) is an immediate consequence of Theorem 1.5.

4.1 Absence of irrotational dynamics for $(\alpha, \beta) \neq 0$: Proof of Theorem 1.5

A torus homeomorhism $f \in \mathrm{Hom}_0(\mathbb{T}^2)$ is called *irrotational* if it has a lift F that satisfies $\rho(F) = \{(0,0)\}$. In this case, we also say the lift F is irrotational. The aim of this section is to show that for the kicked Harper map this case can only occur when $\alpha = \beta = 0$, which is the statement of Theorem 1.5.

When $\alpha = 0$ and $\beta \neq 0$ or vice versa, then it is obvious that $\rho(F_{\alpha,\beta})$ is a non-degenerate segment. Hence it remains to consider the case $\alpha\beta \neq 0$. For such parameters, as discussed in Section 2, the fixed points of $F_{\alpha,\beta}$ are all elementary. As $F_{\alpha,\beta}$ extends to a biholomorphic mapping of $\mathbb{C}^2$, we know due to Ushiki's Theorem [HK02, p. 289] that $F_{\alpha,\beta}$ does not admit any saddle connections between hyperbolic fixed points. Therefore, Theorem 1.5 is a consequence of the following more general result on irrotational Hamiltonian torus homeomorphisms.

Theorem 4.1. *Suppose that f is a Hamiltonian torus diffeomorphism with a lift $F \in \widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ such that $\rho(F) = \{(0,0)\}$. Then F either has a non-elementary fixed point, or it admits a saddle connection between hyperbolic fixed points.*

Proof. Assume that every fixed point of F is elementary. Since $\rho(F) = \{(0,0)\}$, every fixed point of f is lifted to a fixed point of F, and since elementary fixed points are isolated, f has finitely many fixed points. This implies that the fixed point set of f is inessential (contained in some topological open disk in $\mathbb{T}^2$), and one may apply

[CT15, Corollary I] which implies that F has uniformly bounded displacements, that is, we have

$$\sup_{z\in\mathbb{R}^2, n\in\mathbb{Z}} \|F^n(z) - z\| \ < \ \infty \, . \tag{4.1}$$

Let H_0 be the half-plane $\mathbb{R} \times (-\infty, 0)$, and consider $H_1 = \bigcup_{n\in\mathbb{Z}} F^n(H)$, which is F-invariant and Γ_1-invariant where $\Gamma_1(x,y) = (x+1,y)$. Let H be the union of H_1 with all bounded connected components of the complement of H_1. Then H is still F- and Γ_1-invariant, bounded from above, and moreover it is a simply connected open set. Let U_0 denote the projection of H to the annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R} \simeq \mathbb{R}^2/\langle\Gamma_1\rangle$. The map F induces a homeomorphism $\tilde{F}\colon \mathbb{A} \to \mathbb{A}$ which leaves U_0 invariant and commutes with the map $\tilde{\Gamma}_2\colon \mathbb{A} \to \mathbb{A}$ induced by the corresponding translation $\Gamma_2 : (x,y) \mapsto (x, y+1)$ of $\mathbb{R}^2$.

We note that $\tilde{F}$ preserves some finite non-atomic measure μ of full support. This can be seen by noting that the sets $A_k = \Gamma_2^k(\Gamma_2(U_0) \setminus U_0)$ are bounded, invariant, and $\tilde{F}$ preserves the measure μ_k given by the Lebesgue measure restricted to the interior of A_k. Note that the boundary of each A_k is a closed nowhere dense set, which implies that $\bigcup_{k\in\mathbb{Z}} \operatorname{int}(A_k)$ is dense. Since each μ_k is finite (and $\mu_k(A_k)$ does not depend on k), letting $\mu = \sum_{k\in\mathbb{Z}} 2^{-|k|}\mu_k$ we obtain a finite $\tilde{F}$-invariant measure of full support.

Let $\mathbb{S}^2 = \mathbb{A} \cup \{+\infty, -\infty\}$ be the usual compactification of $\mathbb{A}$ by topological ends (where $+\infty$ is the end on which U does not accumulate), and $U = U_0 \cup \{-\infty\}$, which is an open topological disk. Extending $\tilde{F}$ (by fixing $\pm\infty$) we have an orientation-preserving homeomorphism of $\mathbb{S}^2$ which leaves invariant the open topological disk U. The fact that the original map is irrotational implies that the prime ends rotation number of $\tilde{F}$ in U is 0 (this follows, for instance, from [FLC03, Props. 5.3-5.4], or more directly from [Mat10]). Moreover, in a neighborhood of ∂U, the fixed points of $\tilde{F}$ are elementary (because, as elements of $\mathbb{A}$, they are projections of fixed points of F). Since $\tilde{F}$ also preserves a finite measure of full support in $\mathbb{S}^2$, it follows from Theorem 1.4 of [KN18] that ∂U either contains a degenerate fixed point or consists entirely of hyperbolic fixed points and saddle connections. □

4.2 Pinching at the origin: Proof of Theorem 1.7

We already know from Corollary 1.6 that the set $\mathcal{N}$ of parameters where the rotation set has nonempty interior includes $\{(\alpha,\alpha) : \alpha \neq 0\}$, and therefore a neighborhood of the latter set (since $\mathcal{N}$ is open). On the other hand, Figure 1 suggests that the set of parameters with nonempty interior (in the first quadrant) has a cusp shape at the origin; that is, every line through the origin other than the diagonal contains an interval around the origin where the rotation set has empty interior. This is confirmed by Theorem 1.7. We slightly reformulate the latter and prove the following statement. Note that due to the symmetries described in Section 2, it suffices to consider parameters below the diagonal.

Theorem 4.2. *For every $0 \leq \lambda < 1$ there exists $\alpha_0 = \alpha_0(\lambda) > 0$ such that for every $\alpha \in [0, \alpha_0(\lambda)]$ the rotation set of $F_{\alpha,\lambda\alpha}$ has empty interior. Moreover, $\alpha_0 : [0,1) \to (0,+\infty)$ can be chosen continuous.*

Remark 4.3. What we actually show is that if α is chosen smaller than $\alpha_0(\lambda)$, then there exist horizontal invariant closed curves (KAM-curves) for $F_{\alpha,\lambda\alpha}$. This implies that the rotation set is contained in the horizontal axis.

Proof of Theorem 4.2. Consider the vector field

$$W^{\lambda,\alpha}(x,y) \;=\; (s(y+\alpha\lambda s(x)), \lambda s(x)) \;,$$

and denote the corresponding flow by $\Phi^{\lambda,\alpha}$. If we perform Euler's method for the numerical integration of $W^{\lambda,\alpha}$, then the map we obtain for time step α is exactly

$$F_{\alpha,\lambda\alpha}(x,y) \;=\; (x,y) + \alpha W^{\lambda,\alpha}(x,y) \;.$$

Let $n_\alpha = \lfloor 1/\alpha \rfloor$. Although we have a dependence between the vector field $W^{\lambda,\alpha}$ and the time step α, standard estimates on the convergence of the Euler method (as, for instance, provided by Theorem 7.3 in the appendix) imply that the C^∞-distance between $F^{n_\alpha}_{\alpha,\lambda\alpha}$ and the time-one map $\Phi^{\lambda,\alpha}_1$ converges to zero as $\alpha \to 0$. (Note here that there exist uniform bounds for all derivatives of the vector fields $W^{\lambda,\alpha}$ with $\alpha \in [0,1]$.) As at the same time $\Phi^{\lambda,\alpha}_1$ clearly converges to $\Phi^{\lambda,0}_1$, this means that for α sufficiently small the map $F^{n_\alpha}_{\alpha,\lambda\alpha}$ is C^∞-close to $\Phi^{\lambda,0}_1$.

However, the flow $\Phi^\lambda = \Phi^{\lambda,0}$ with $\lambda \in [0,1]$ is easy to analyse. It is a conservative flow, which lifts a flow of $\mathbb{T}^2$ with two hyperbolic singularities at $(0,0)$ and $(1/2,1/2)$ and two elliptic ones at $(0,1/2)$ and $(1/2,0)$. When $\lambda = 1$, the hyperbolic singularities have saddle connections as shown in Figure 4.2(a). When $\lambda < 1$, one may easily verify that these connections are replaced by homoclinic connections as in Figure 4.2(b). The region complementary to the elliptic islands on $\mathbb{T}^2$ consists of two essential horizontal annuli A_1 and A_2, and the dynamics on the A_i is integrable, that is, all its orbits are essential (horizontal) circles. Moreover, by the smoothness of the flow, and since each point is these annuli is periodic, the function assigning to each point its period is also smooth and constant on each invariant circle. Note that the set of invariant circles has a natural topology where it is homeomorphic to an open interval of $\mathbb{R}$. Finally, since the boundary of these annuli contain singularities, the function assigning to each circle the period of its point cannot be constant and thus must be strictly monotone in some sub-interval. Therefore one may find a smaller annulus $A_0 \subset A_1$ foliated by invariant circles such that Φ^λ_1 is an integrable twist map on A_0 By the KAM theorem [Her83], any map sufficiently C^∞-close to Φ^λ_1 will have horizontal invariant circles. In particular, if α is small enough, $F^{n_\alpha}_{\alpha,\lambda\alpha}$ has some horizontal invariant circle C, and therefore so has $F_{\alpha,\lambda\alpha}$. Hence, $\rho(F_{\alpha,\lambda\alpha})$ must be contained in a horizontal segment. This proves that for α small enough, $\rho(F_{\alpha,\lambda\alpha})$ has empty interior.

Finally, we note that due to the stability of the KAM-circles, we may choose α_0 such that it is uniformly bounded away from 0 on any compact subinterval of $[0,1)$. Reducing α_0 further if necessary, we can therefore choose it continuously as a function $[0,1) \to (0,+\infty)$. □

5 Large parameters: Proof of Theorem 1.8

Recall that

$$\beta^-(\alpha) \;=\; \inf\{\beta > 0 \mid \mathrm{int}(\rho(F_{\alpha,\beta})) \neq \emptyset\} \;, \tag{5.1}$$

$$\beta^+(\alpha) \;=\; \sup\{\beta > 0 \mid \mathrm{int}(\rho(F_{\alpha,\beta})) = \emptyset\} \;, \tag{5.2}$$

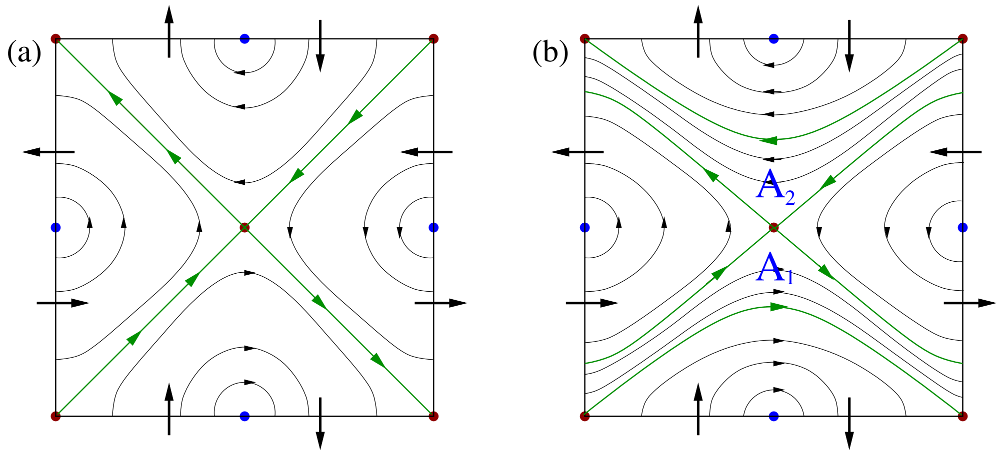

Figure 3: Schematic picture of the (projections of the) vector fields $W^{\lambda,0}$ and the corresponding flows Φ^λ on the torus: (a) the case $\lambda = 1$; (b) the case $\lambda < 1$, with the two invariant annuli A_1 and A_2, bounded by homoclinic saddle-connections (in green).

and Theorem 1.8 asserts that both these quantities are of order $1/\sqrt{\alpha}$ for large α, in the sense that there exists constants $0 < c < C$ such that

$$\frac{c}{\sqrt{\alpha}} \;\leq\; \beta^-(\alpha) \;\leq\; \beta^+(\alpha) \;\leq\; \frac{C}{\sqrt{\alpha}} \,. \tag{5.3}$$

We will treat the lower and upper estimate in two separate lemmas.

Lemma 5.1. *There exists a constant $C > 0$ such that $\beta^+(\alpha) \leq \frac{C}{\sqrt{\alpha}}$ for all $\alpha \geq 1/2$.*

We note that whenever $\alpha \geq 1/2$, we have that $[-1/2, 1/2] \times \{0\} \subseteq \pi_1(\rho(F_{\alpha,\beta}))$. Hence, in order to find an upper bound on $\beta^+(\alpha)$ it suffices to show that the rotation set is not contained in the horizontal axis for any β larger than the desired bound.

In order to do so, we will use a geometric argument that essentially relies on the fact that the horizontal shift H_α induces a strong shear in most parts of the phase space. As the proof does not use any specific properties of the kicked Harper model and the construction may also be useful in other situations, we work in a slighly more general setting. Consider homeomorphisms of $\mathbb{T}^2$ of the following type: Let $\varphi, \psi, : \mathbb{R} \to \mathbb{R}$ be two continuous and 1-periodic functions. Let $H_\varphi, V_\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$H_\varphi(x,y) \;=\; (x + \varphi(y), y) \,, \tag{5.4}$$
$$V_\psi(x,y) \;=\; (x, y + \psi(x)) \,, \tag{5.5}$$

and define $F_{\varphi,\psi} = H_\varphi \circ V_\psi$ (note that, with this notation, the Harper map $F_{\alpha,\beta}$ should be denoted $F_{\alpha s,\beta s}$). Given a 1-periodic continuous function $\gamma : \mathbb{R} \to \mathbb{R}$, let

$$\mathrm{Var}_\gamma(\delta) = \min_{t\in\mathbb{R}} \left(\max_{x\in[t,t+\delta]} \gamma(x) - \min_{x\in[t,t+\delta]} \gamma(x) \right)$$

be the minimal variation that the function γ has on an interval of length δ.

Proposition 5.2. *Let φ, ψ be such that $\min_{x\in\mathbb{R}} \psi(x) \leq 0 < \max_{x\in\mathbb{R}} \psi(x) = \beta$, and such that there exists $\delta \leq \beta/2$ such that $\mathrm{Var}_\varphi(\delta) \geq 2$. Then $\pi_2\left(\rho(F_{\varphi,\psi})\right) \geq \beta - \delta$.*

Proof. Let α_0 be the line segment joining $(0,0)$ to $(1,0)$. We will show by induction that, for every $n \geq 1$, there exists a curve $\alpha_n \subset F_{\varphi,\psi}(\alpha_{n-1})$ such that $\max \pi_1(\alpha_n) - \min \pi_1(\alpha_n) = 1$ and $\alpha_n \subset \mathbb{R} \times [n(\beta-\delta), n(\beta-\delta)+\delta]$. The last property clearly implies the proposition, as it shows that there are points in α_0 whose vertical displacement after n iterations is $\geq n(\beta - \delta)$.

Given $n \geq 1$, suppose that α_{n-1} satisfies the inductive assumption (which is trivial for α_0) and let $a_{n-1} = \min_{z\in\alpha_n} \pi_1(z)$. There exists some $x_0, x_1 \in [a_{n-1}, a_{n-1}+1]$ such that $\psi(x_0) = 0$ and $\psi(x_1) = \beta$. Note that, by the induction hypothesis, there exists y_0, y_1 such that both (x_0, y_0) and (x_1, y_1) belong to α_{n-1}, and

$$\alpha_{n-1} \subseteq [a_{n-1}, a_{n-1}+1] \times [(n-1)(\beta-\delta), (n-1)(\beta-\delta)+\delta] .$$

Note further that $V_\psi(x_0, y_0) = (x_0, y_0)$ and $\delta \leq \beta/2$, so

$$\pi_2(V_\psi(x_0, y_0)) \leq (n-1)(\beta-\delta) + \delta \leq n(\beta-\delta) ,$$

and $V_\psi(x_1, y_1) = (x_1, y_1 + \beta)$, so

$$\pi_2(V_\psi(x_1, y_1)) \geq (n-1)(\beta-\delta) + \beta = n(\beta-\delta) + \delta .$$

Moreover, $V_\psi(\alpha_{n-1})$ is still contained in the strip $[a_{n-1}, a_{n-1}+1] \times \mathbb{R}$. One deduces that there exists a sub-arc γ of $V_\psi(\alpha_{n-1})$ contained in $[a_{n-1}, a_{n-1}+1] \times [n(\beta-\delta), n(\beta-\delta)+\delta]$ such that γ intersects both the upper and lower boundaries of this rectangle.

Now, as $\mathrm{Var}_\varphi(\delta) \geq 2$, we may find y_0', y_1' in $[n(\beta-\delta), n(\beta-\delta)+\delta]$ such that $\varphi(y_1') - \varphi(y_0') \geq 2$. Let x_0', x_1' be such that both (x_0', y_0') and (x_1', y_1') belong γ. Note that

$$\pi_1\left(H_\varphi((x_0', y_0'))\right) \leq a_{n-1} + 1 + \varphi(y_0') \leq a_{n-1} + \varphi(y_1') - 1,$$

and

$$\pi_1\left(H_\varphi((x_1', y_1'))\right) \geq a_{n-1} + \varphi(y_1').$$

Moreover, $H_\varphi(\gamma)$ is contained in the strip $\mathbb{R} \times [n(\beta-\delta), n(\beta-\delta)+\delta]$. Choosing $a_n = a_{n-1} + \varphi(y_1') - 1$ one deduces the existence of a subarc α_n of $H_\varphi(\gamma) \subset F_{\varphi,\psi}(\alpha_{n-1})$ such that

$$\alpha_n \subseteq [a_n, a_n+1] \times [n(\beta-\delta), n(\beta-\delta)+\delta]$$

and α_n intersects both the left and right boundaries of this rectangle, proving the induction assumption for n and thus the proposition. □

Proof of Lemma 5.1. Let $s(x) = \sin(2\pi x)$ as before. Choose $C_1 > 0$ such that $\mathrm{Var}_s(\delta) > C_1\delta^2$ for all $\delta > 0$. Note that such a constant exists, since s is quadratic at its critical points. We have that $\mathrm{Var}_{\alpha s}(\delta) = \alpha \mathrm{Var}_s(\delta)$. Therefore, if $\alpha \geq \frac{8}{C_1\beta^2}$, we get that $\mathrm{Var}_{\alpha s}(\beta/2) > 2$. Taking $C = \frac{8}{C_1}$, we get by Proposition 5.2 that if $\alpha \geq C/\beta^2$, then $\rho(F_{\alpha,\beta}) = \rho(F_{\alpha s,\beta s})$ is not contained in $\mathbb{R} \times \{0\}$. Hence, $(\alpha, \beta) \in \mathcal{N}$ in this case, thus proving that $\beta^+(\alpha) \leq C/\sqrt{\alpha}$ for all $\alpha \geq 1/2$. □

Lemma 5.3. *There exists a constant $c > 0$ such that for any $\alpha \geq 1$ we have that $\beta^-(\alpha) \geq c/\sqrt{\alpha}$.*

Proof. It will be convenient to consider the maps $G_{\alpha,\beta} = V_\beta \circ H_\alpha$ instead of $F_{\alpha,\beta}$. Note that since $G_{\alpha,\beta} = V_\beta \circ F_{\alpha,\beta} \circ V_\beta^{-1}$, we have $\rho(F_{\alpha,\beta}) = \rho(G_{\alpha,\beta})$ and may therefore replace $F_{\alpha,\beta}$ by $G_{\alpha,\beta}$ in the definition of $\beta^-(\alpha)$ in 1.8. Further for any $\alpha_0 > 0$ the restriction of $F_{\alpha_0,0}$ to $\mathbb{R}\times[-1/8, 1/8]$ is a lift of the completely integrable twist map $f_{\alpha_0,0}$ on the annulus A obtained by projecting the corresponding strip $\mathbb{R} \times [-1/8, 1/8]$. By the KAM theorem, $f_{\alpha_0,0}$ has stable KAM-circles, and thus $f_{\alpha,\beta}$ has a horizontal KAM-circle whenever (α, β) is close enough to $(\alpha_0, 0)$. By a compactness argument, this guarantees that for each $M > 0$ there is a constant c_M such that $\beta^-(\alpha) > c_M > 0$ whenever $\alpha \in [1, M]$. As a consequence, it will be sufficient to prove the estimate of the lemma for large enough values of α.

Let $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ and $\kappa = 4\pi^2 = |s''(1/4)|$ and consider the parameter family of annular diffeomorphisms $S_{\alpha,\beta}\colon \mathbb{A} \to \mathbb{A}$ lifted by

$$\tilde{S}_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}^2 \quad , \quad (x, y) \ \mapsto \ V_\beta(x + \alpha - \kappa y^2, y) \ .$$

We note that this family is sometimes referred to as the *standard non-twist map* (e.g. [SA98, SA98]). For each $\varepsilon > 0$ and $\alpha_0 \in \mathbb{R}$, the restriction of the map $S_{\alpha_0,0}$ to $\mathbb{T}^1 \times [\varepsilon, 1]$ is a completely integrable twist map and therefore has stable horizontal KAM-circles. Moreover, we have $S_{\alpha_0+1,\beta} = S_{\alpha_0,\beta}$, so that α_0 can be viewed as an element of $\mathbb{T}^1$. Hence, by compactness we obtain that there exist constants $b, \varepsilon_0 > 0$ and $k_0 \in \mathbb{N}$ such that any smooth injective map $G : \mathbb{A} \to \mathbb{A}$ whose restriction to $\mathcal{A} = \mathbb{T}^1 \times [0, 1]$ is ε_0-close to $S_{\alpha_0,0|\mathcal{A}}$ in the $\mathcal{C}^{k_0}$-metric for some $\alpha_0 \in \mathbb{R}$ has horizontal KAM-circles.

Now, given $\alpha, \beta \in \mathbb{R}$, consider the rescaling

$$\tilde{G}_{\alpha,\beta} \ = \ \Phi_\alpha \circ G_{\alpha,\beta} \circ \Phi_\alpha^{-1}$$

of $G_{\alpha,\beta}$, where $\Phi_\alpha(x, y) = (x, \sqrt{\kappa\alpha}(y - 1/4))$. Note that $\Phi_\alpha \circ V_\beta = V_{\sqrt{\kappa\alpha}\beta} \circ \Phi_\alpha$, and therefore

$$\tilde{G}_{\alpha,\beta} \ = \ V_{\sqrt{\kappa\alpha}\beta} \circ \Phi_\alpha \circ H_\alpha \circ \Phi_\alpha^{-1} \ = V_{\sqrt{\kappa\alpha}\beta} \circ \tilde{G}_{\alpha,0} \ .$$

Let $\widehat{G}_{\alpha,\beta} : \mathbb{A} \to \mathbb{A}$ be the homeomorphism naturally induced by $\tilde{G}_{\alpha,\beta}$ on $\mathbb{A}$. Then it can be checked that, due to the above rescaling, the maps $\widehat{G}_{\alpha_0+n,0|\mathcal{A}}$ converge to $S_{\alpha_0,0|\mathcal{A}}$ as $n \to \infty$ in the $\mathcal{C}^k$-topology for any $k \in \mathbb{N}$. (Note here that $\widehat{G}_{\alpha_0+n,0} \neq \widehat{G}_{\alpha_0,0}$ for $n \in \mathbb{N} \setminus \{0\}$, since the rescaling that is carried out before projecting to $\mathcal{A}$ is different for the two maps.) Moreover, the convergence is uniform in $\alpha_0 \in [0, 1]$. Hence, there exists a constant $M > 0$ such that such that for any $\alpha > M$ the map $\widehat{G}_{\alpha,0|\mathcal{A}}$ is $\varepsilon_0/2$-close to $S_{\alpha,0|\mathcal{A}}$ in the $\mathcal{C}^{k_0}$-topology.

Further, there exists $\delta > 0$ such that for any $\alpha \in \mathbb{R}$ and $\tilde{\beta} \in (0, \delta)$ the map $\tilde{G}_{\alpha,\tilde{\beta}/\sqrt{\kappa\alpha}} = V_{\tilde{\beta}} \circ \tilde{G}_{\alpha,0}$ is $\varepsilon_0/2$-close to $\tilde{G}_{\alpha,0}$ in the $\mathcal{C}^{k_0}$-topology. As a consequence, we obtain that for any $\beta \in (0, \delta/\sqrt{\kappa\alpha})$ the map $\widehat{G}_{\alpha,\beta|\mathcal{A}}$ is $\varepsilon_0/2$-close to $\widehat{G}_{\alpha,0|\mathcal{A}}$, and thus ε_0-close to $S_{\alpha,0|\mathcal{A}}$ in the $\mathcal{C}^{k_0}$-topology when $\alpha > M$. By the above, this means that $\widehat{G}_{\alpha,\beta}$ has invariant KAM-circles. However, as $\widehat{G}_{\alpha,\beta}$ is just a rescaling of the projection of $G_{\alpha,\beta}$ to $\mathbb{A}$, this means that the rotation set of $G_{\alpha,\beta}$ is confined to the

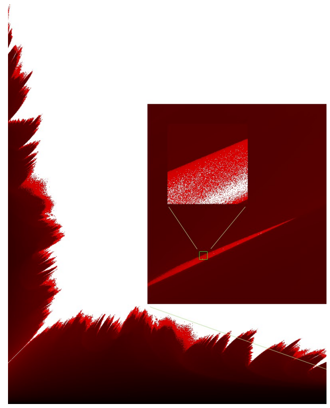

Figure 4: More detailed look at the sets $\mathcal{E}$ and $\mathcal{N}$ with zoomed regions where numerical estimates indicate the possibility of non-connectedness of the sets. The first zoomed square corresponds approximately to the parameter region $[0.915, 0.935] \times [0.065, 0.085]$. The second zoomed square corresponds approximately to the parameter region $[0.920, 0.921] \times [0.069, 0.070]$.

horizontal axis, that is, $\rho(G_{\alpha,\beta}) \subseteq \mathbb{R} \times \{0\}$. Letting $c = \min\{\delta/\sqrt{\kappa}, c_M\}$ (where c_M is the constant from the beginning of the proof) we conclude that $\beta^-(\alpha) \geq c/\sqrt{\alpha}$ for all $\alpha \geq 1$, as required. $\square$

6 Questions and final remarks

The kicked Harper map, by which we mean the whole parameter family $(f_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}}$, shows a rich variety of different dynamical behaviours and phenomena. We believe that its study as a paradigmatic example of smooth torus dynamics can be extremely fruitful and may lead to general insights about torus dynamics and rotation theory on surfaces that go well beyond the context of this particular example. With the results presented above, we have merely scratched at the surface of a multitude of intriguing open problems that can be investigated in this context. In the remainder of this section, we collect a few directions in which future research on this topic may be oriented.

6.1 Structure of the parameter regions

The aim for a better understanding of the structure of the parameter regions $\mathcal{E}$ and $\mathcal{N}$ leads to a number of further questions concerning their qualitative and quantitative properties. First of all, in analogy to the well-known problems in the study of Julia sets in complex dynamics, one may ask

- Are the sets $\mathcal{E}$ and $\mathcal{N}$ connected? Are they locally connected?

As Figure 4 shows, even connectedness should not be taken for granted.

Another problem that we leave open here is that of the seemingly periodic structure of the set $\mathcal{E}$ observed in Figure 2 (and described previously in [Shi02]). In mathematical terms, one may formulate it as follows. Let $M_2((x,y),t) = (x,ty)$

Conjecture 6.1. *The sequence* $A_n = M_2((\mathcal{E} \cap [n,n+1] \times [0,1]) - (n,0), \sqrt{n})$ *converges in Hausdorff distance to the set*

$$A = \{(\alpha,\beta) \in [0,1]^2 \mid S_{\alpha,\beta} \textit{ admits unbounded orbits}\},$$

where $S_{\alpha,\beta}$ *is the standard non-twist map introduced in the proof of Lemma 5.3.*

Various further questions may be asked about the tongue structure that appears in Figures 1 and 2. On a heuristic level, it seems plausible that the tongues of the region $\mathcal{N}$ that reach into the region $\mathcal{E}$ should somehow correspond to 'resonances' appearing in the dynamics that make it easier to break all KAM-circles, so that diffusion can take place. This should correspond to the appearence and disappearence of certain periodic orbits. However, the precise mechanisms are not at all clear to us. We refer to [HH84, Shi02, Leb98, LKFA90, Zas07] for more details and some phenomenological descriptions.

6.2 Monotonicity properties

Another aspect that is not well-understood and prompts a multitude of questions is that of the dependence of the rotation set on the parameters. Apart from the continuity derived in Section 3, little is known. Specifically, one may ask about monotonicity properties: when do $0 \leq \alpha \leq \tilde{\alpha}$ and $0 \leq \beta \leq \tilde{\beta}$ imply $\rho(F_{\alpha,\beta}) \subseteq \rho(F_{\tilde{\alpha},\tilde{\beta}})$. For instance, we have a natural upper bound $\rho(F_{\alpha,\beta}) \subseteq [-\alpha,\alpha] \times [-\beta,\beta]$ on the rotation set. However, while this upper bound grows monotonically with the parameters, the same is not true in general for the rotation set itself.

One way to see this is to consider parameters $\alpha = 0$ and $\beta \in (0,1)$. In this case, we have $\rho(F_{0,\beta}) = \{0\} \times [-\beta,\beta]$. However, for any parameter pair (α,β) an average vertical displacement of β is only possible if an orbit stays exactly on the vertical line $\{1/4\} \times \mathbb{R}$, or converges to it. This is not possible for $\alpha \in (0,1)$, so that $(0,\beta) \notin \rho(F_{\alpha,\beta})$ in this case, and therefore $\rho(F_{0,\beta}) \nsubseteq \rho(F_{\alpha,\beta})$.

In contrast to this, numerical simulations based on [PPGJ17] suggest that the rotation set behaves monotonically along the diagonal.

Conjecture 6.2. *If* $0 \leq \alpha \leq \tilde{\alpha}$*, then* $\rho(F_{\alpha,\alpha}) \subseteq \rho(F_{\tilde{\alpha},\tilde{\alpha}})$.

6.3 Mode-locking

A well-known and -studied phenomenon in the context of rotation theory is that of mode-locking, which refers to the stability rotation numbers, vectors or sets under perturbations of the system. In the context of torus dynamics, it was shown in [Pas14] that there exists an open and dense subset of $\mathrm{Hom}_0(\mathbb{T}^2)$ on which the rotation set is locally constant and a rational polygon, and in [GK17] the same statement was shown to hold when restricted to $\mathrm{Hom}_0^{\mathrm{ap}}(\mathbb{T}^2)$. However, it is not clear if the analogous statement is still true if one restricts to the parameter family $(f_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}}$, although the recent results in [LCAZ15], showing that for any analytic one parameter family

$G_t \in \mathrm{Hom}_0^{\mathrm{ap}}(\mathbb{T}^2)$ the rotation set cannot strictly increase over a whole interval $I \subseteq \mathbb{R}$ (that is, $\rho(G_s) \subset \mathrm{int}(\rho(G_t))$ cannot hold for all $s < t$ in I), point in that direction.

So, the following questions are open.

- Is it true that there exists an open and dense set $M \subseteq \mathbb{R}^2$ such that the mapping $(\alpha, \beta) \mapsto \rho(F_{\alpha,\beta})$ is locally constant on M?
- Is it true that whenever the mapping $(\alpha, \beta) \mapsto \rho(F_{\alpha,\beta})$ is locally constant, the rotation set is a rational polygon?
- Is there an open and dense set $A \subseteq \mathbb{R}$ such that the mapping $\alpha \mapsto \rho(F_{\alpha,\alpha})$ is locally constant and only has rational polygons as images on A.

In this context, we note that the numerical computation or approximation of rotation sets is an intricate problem in itself, such that it is difficult to obtain numerical evidence concerning the occurrence or density of mode-locking. We refer to [PPGJ17] for details on the numerical aspects. Using the algorithm developed there, it is possible to identify some specific mode-locked regions in the kicked Harper family, for instance around parameters $(\alpha, \beta) = (0.66, 0.66)$.

6.4 Shape of rotation sets

Another general open problem in torus dynamics is that of the possible shapes of rotation sets. Due to [MZ89], it is know that the rotation set of a torus homeomorphism is always convex, and Kwapisz showed that every rational polygon (a polygon with all vertices in $\mathbb{Q}^2$) are realised. Moreover, a few examples of non-polygonal rotation sets have been described [Kwa95, BdCH16], but all of these only have a countable number of extremal points. Hence, it is completely open if a set like the unit disk may appear as the rotation set of a torus homeomorphism.

- Which sets do appear as rotation sets in the family $(F_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}}$?

6.5 Phase space

Finally, questions that are typically studied in the context of the Arnol'd standard family (of area-preserving twist maps) may equally be asked for the kicked Harper model.

- Do elliptic islands exist for all/Lebesgue-almost all parameters $\alpha, \beta \neq 0$.
- Conversely, are there parameters for which the kicked Harper map is topologically transitive/ergodic with respect to Lebesgue?
- What is the Lebesgue measure of the complement of the union of all KAM-circles/elliptic islands?

6.6 Transverse foliation

A number of recent advances in surface dynamics have been based on the concept of transverse foliations (Brouwer-Le Calvez foliations) and a related forcing theory developed in [CT15]. As we have not made use of this theory, we refrain from going into more detail here. However, for readers that are familiar with the topic, we want to point out that the existence of a transverse foliation (which in general follows from

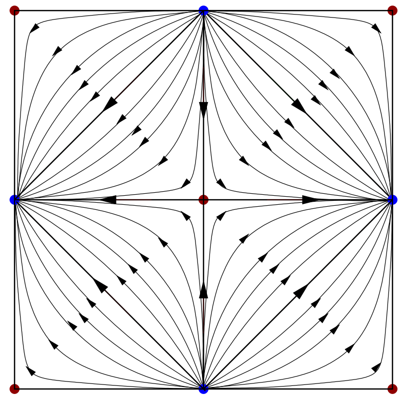

Figure 5: Transverse foliation for the kicked Harper map with $\alpha, \beta \neq 0$.

the work of Le Calvez in [LC05]) can be seen quite easily for the kicked Harper model. If one considers the homotopy $(h^t)_{t\in[0,1]}$ between the identity and $f_{\alpha,\beta}$ given by

$$h^t : \mathbb{R}^2 \to \mathbb{R}^2 \quad , \quad (x,y) \mapsto \begin{cases} (x, y + 2t\beta s(x)), & 0 \leq t \leq 1/2 \\ (x + (2t-1)\alpha s(y + \beta s(x)), y + \beta s(x)) \ , & 1/2 < t \leq 1 \end{cases}$$

then each path of this isotopy that connects a point (x, y) to its image under $f_{\alpha,\beta}$ consists of a vertical and a horizontal segment (possibly degenerate). Moreover, the orientation of these segments is only determined by the quadrant of $\mathbb{T}^2$ in which the segment starts. This allows to see that the oriented foliation shown in Figure 5 is positively transverse to the dynamics, that is, the paths of the homotopy the leaves of the foliation in a transverse way from left to right, for all parameters $\alpha, \beta \neq 0$ at the same time. The four common fixed points $(0, 0), (1/2, 0), (0, 1/2), (1, 1)$ of the kicked Harper maps are singularities of the foliation.

Appendix

Theorem 7.3 (C^r-convergence of Euler's method)**.** *For each $r \geq 1$ and $M > 0$ there exists $C_r = C_r(M) > 0$ such that the following property holds. Let $z \mapsto V(z)$ be a C^{r+1} vector field $(r \geq 1)$ in $\mathbb{R}^n$, and $z_0 \in \mathbb{R}^n$ a point such that the corresponding flow $\phi^t(z_0)$ is defined for all $t \in [0, 1]$, and assume that the C^{r+1} norm of $V(z)$ is at most M for all z in the ϵ-neighborhood U_ϵ of $\{\phi^t(z_0) : t \in [0, 1]\}$. Then the function $G_\delta(z) = z + \delta V(z)$ satisfies*

$$\|D^r G_\delta^n(z_0) - D^r \phi^{n\delta}(z_0)\| \leq C_r \delta$$

for all $0 < \delta < \min\{1, \epsilon\}/C_r$ and $n \leq \lfloor 1/\delta \rfloor$.

Sketch of the proof. Without loss of generality we assume $\epsilon < 1$. Denote by M_r the C^r norm of V in U_ϵ. Iterating G_δ produces an Euler approximation of the solutions of $z' = V(z)$, and we have the following well-known estimate for the error in Euler's method:

$$\|\phi^{n\delta}(z_0) - G_\delta^n(z_0)\| \leq \delta M_0(e^{M_1\delta(n+1)} - 1),$$

which holds for all n such that the right hand side is smaller than ϵ. In particular, if $C_0 = M_0(e^{2M_1} - 1)$ then

$$\|\phi^{n\delta}(z_0) - G_\delta^n(z_0)\| \le C_0\delta$$

holds whenever $\delta < \epsilon/C_0$ and $n \le n_\delta$. Thus the claim holds for $r = 0$.

To get a similar estimate for the derivatives, we will use the previous observations in a new vector field. To avoid cumbersome notation with higher order derivatives, we omit details about the spaces to which each object belongs; this should be clear from context.

We will use the following notation: $D_*^k V(z) = (DV(z), D^2V(z), \dots, D^kV(z))$. Let Γ_k be a C^∞ map such that if $f, g\colon \mathbb{R}^n \to \mathbb{R}^n$ are two C^k maps and $h = f \circ g$,

$$\Gamma_k(D_*^k f(g(z)), D_*^k g(z)) = D^k h(z).$$

An explicit formula for Γ_k can be given (for instance Faa di Bruno's formula).

Let $u = (z, u_1, \dots, u_r)$, $W_0(u) = V(z)$,

$$W_k(u) = \Gamma_k(D_*^k V(z), u_1, \dots, u_k)$$

and

$$W(u) = (W_0(z), W_1(z, u_1), \dots, W_r(z, u_1, \dots, u_r)).$$

Then it is easy to verify that the solution to

$$u' = W(u) \tag{7.1}$$

with initial condition $u(0) = v(z) := (z, I, 0, \dots, 0)$ is

$$\phi_r^t(v(z)) = (\phi^t(z), D\phi^t(z), \dots, D^r(\phi^t(z)). \tag{7.2}$$

Let $G_{r,\delta}(u) = u + \delta W(u)$ be the Euler approximation of the flow given by the vector field W. If U_ϵ^r denotes the ϵ-neighborhood of $\{\phi_r^t(v(z_0)) : t \in [0,1]\}$, we then know from the case $r = 0$ that there exists $C_r > 0$ such that whenever $\delta < \epsilon/C_r$ and $n \le n_\delta$,

$$\left\|\phi_r^{n\delta}(v(z_0)) - G_{r,\delta}^n(v(z_0))\right\| \le C_r\delta. \tag{7.3}$$

where C_r depends only on the C^1 norm of W in U_ϵ^r.

For $t \in [0,1]$ and $1 \le k \le r$, there is a uniform bound $\left\|D^k\phi_r^t(v(z_0))\right\| \le K$ depending only on M_r. This can be seen noting (for instance from Faa di Bruno's formula) that

$$\Gamma_k(D_*^k V(z), (u_1, \dots, u_k)) = \Lambda_k(D_*^k V(z), u_1, \dots, u_{k-1}) + DV(z)u_k,$$

where Λ_k is another (explicit) function, and applying Gronwall's inequality for each coordinate u_k in (7.1) inductively. We leave these details to the reader.

This implies that any $u \in U_\epsilon^r$ satisfies $\|u_i\| \le K + \epsilon \le K + 1$ for $1 \le i \le r$. Using this fact and the explicit form of W we see that the C^1 norm of W in U_ϵ^r is bounded by a constant depending only on M_{r+1}. In particular the constant C_r above depends only on M_{r+1}.

Since the r-th coordinate of $\phi_r^{n\delta}(v(z_0))$ is $D^r\phi^{n\delta}(z_0)$, in view of (7.3) and (7.2), to complete the proof it suffices to show that

$$G^n_{r,\delta}(v(z)) = (G^n_\delta(z), DG^n_\delta(z), \dots, D^r G^n_\delta(z)). \tag{7.4}$$

This clearly holds when $n = 0$ due to the defintion of $v(z_0)$; and for $n \geq 0$

$$G^{n+1}_{r,\delta}(v(z)) = G_{r,\delta}(G^n_{r,\delta}(v(z))) = G^n_{r,\delta}(v(z)) + \delta W(G^n_{r,\delta}(v(z))).$$

So assuming by induction that the claim holds for n, looking at the k-th coordinates we get

$$G^{n+1}_{r,\delta}(v(z))_k = D^k G^n_\delta(z) + \delta\Gamma_k(G^n_\delta(z), DG^n_\delta(z), \dots D^k G^n_\delta(z)).$$

Using the definition of Γ_k, this is equal to

$$D^k G^n_\delta(z) + \delta D^k V(G^n_\delta(z)) = D^k(G_\delta(G^n_\delta(z))) = D^k G^{n+1}_\delta(z),$$

which proves the induction step. This completes the proof.

□

References

[BdCH16] P. Boyland, A. de Carvalho, and T. Hall. New rotation sets in a family of torus homeomorphisms. *Invent. math.*, 204(3):895–937, 2016.

[Chi79] B.V. Chirikov. A universal instability of many-dimensional oscillator systems. *Phys. Rep.*, 52(5):264–379, 1979.

[CS08] B. Chirikov and D. Shepelyansky. Chirikov standard map. *Scholarpedia*, 3(3):3550, 2008.

[CT15] Patrice Le Calvez and Fábio Armando Tal. Forcing theory for transverse trajectories of surface homeomorphisms. *Inventiones mathematicae*, 212:619–729, 2015.

[Dav16] P. Davalos. On annular maps of the torus and sublinear diffusion. *Inst. Math. Jussieu*, pages 1–66, 2016.

[FLC03] J. Franks and P. Le Calvez. Regions of instability for non-twist maps. *Ergodic Theory Dynam. Systems*, 23(1):111–141, 2003.

[Fra95] J. Franks. The rotation set and periodic points for torus homeomorphisms. In N. Aoki et al., editor, *Dynamical systems and chaos. Vol. 1: Mathematics, engineering and economics. Singapore, World Scientific.*, pages 41–48, 1995.

[GK17] P.-A. Guihéneuf and A. Koropecki. Stability of the rotation set of area-preserving toral homeomorphisms. *Nonlinearity*, 30(3):1089–1096, 2017.

[GKT14] N. Guelman, A. Koropecki, and F. Tal. A characterization of annularity for area-preserving toral homeomorphisms. *Math. Z.*, 276(3-4):673–689, 2014.

[Her83] M. Herman. Sur les courbes invariantes par les difféomorphismes de l'anneau. *Astérisque*, 103–104, 1983.

[HH84] J.E. Howard and S.M. Hohs. Stochasticity and reconnection in hamiltonian systems. *Phys. Rev. A*, 29(1):418, 1984.

[HK02] B. Hasselblatt and A. Katok. *Handbook of dynamical systems*. Elsevier, 2002.

[KN18] Andres Koropecki and Meysam Nassiri. Boundary dynamics for surface homeomorphisms. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 1995–2015. World Sci. Publ., Hackensack, NJ, 2018.

[Kwa92] J. Kwapisz. Every convex polygon with rational vertices is a rotation set. *Ergodic Theory Dynam. Systems*, 12:333–339, 1992.

[Kwa95] J. Kwapisz. A toral diffeomorphism with a nonpolygonal rotation set. *Nonlinearity*, 8:461–476, 1995.

[LC05] P. Le Calvez. Une version feuilleté équivariante du théorème de translation de Brouwer. *Publ. Math. Inst. Hautes Études Sci.*, 102:1–98, 2005.

[LCAZ15] P. Le Calvez and S. Addas-Zanata. Rational mode locking for homeomorphisms of the 2-torus. *Preprint* `arXiv:1508.02597`, 2015.

[Leb98] P. Leboeuf. Normal and anomalous diffusion in a deterministic area-preserving map. *Physica D*, 116(1):8–20, 1998.

[LKFA90] P. Leboeuf, J. Kurchan, M. Feingold, and D.P. Arovas. Phase-space localization: topological aspects of quantum chaos. *Phys. Rev. Lett.*, 65(25):3076, 1990.

[Mat10] S. Matsumoto. Rotation sets of invariant separating continua of annular homeomorphisms. *Preprint* `arXiv:1011.3176`, 2010.

[MZ89] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *J. Lond. Math. Soc.*, 2(3):490–506, 1989.

[MZ91] M. Misiurewicz and K. Ziemian. Rotation sets and ergodic measures for torus homeomorphisms. *Fundam. Math.*, 137(1):45–52, 1991.

[Pas14] A. Passeggi. Rational polygons as rotation sets of generic torus homeomorphisms of the two torus. *J. Lond. Math. Soc.*, pages 235–254, 2014.

[PPGJ17] K. Polotzek, K. Padberg-Gehle, and T. Jäger. Set-valued numerical computation of rotation sets. *Preprint,* `http://users.minet.uni-jena.de/~tjaeger/preprints/RotationSets.pdf`, 2017.

[SA97] S. Shinohara and Y. Aizawa. The breakup condition of shearless kam curves in the quadratic map. *Progress of Theoretical Physics*, 97(3):379–385, 1997.

[SA98] S. Shinohara and Y. Aizawa. Indicators of reconnection processes and transition to global chaos in nontwist maps. *Progress of Theoretical Physics*, 100(2):219–233, 1998.

[Shi02] S. Shinohara. The threshold for global diffusion in the kicked harper map. *Phys. Lett. A*, 298(5):330–334, 2002.

[Zas05] G.M. Zaslavsky. *Hamiltonian chaos and fractional dynamics*. Oxford University Press on Demand, 2005.

[Zas07] G.M. Zaslavsky. *The physics of chaos in Hamiltonian systems*. World Scientific, 2007.