MULTIVARIATE MEAN EQUICONTINUITY FOR FINITE-TO-ONE TOPOMORPHIC EXTENSIONS

J. BREITENBÜCHER, L. HAUPT, AND T. JÄGER

Abstract. In this note, we generalise the concept of topo-isomorphic extensions and define finite topomorphic extensions as topological dynamical systems whose factor map to the maximal equicontinuous factor is measure-theoretically at most m-to-one for some $m \in \mathbb{N}$. We further define multivariate versions of mean equicontinuity, complementing the notion of multivariate mean sensitivity introduced by Li, Ye and Yu, and then show that any m-to-one topomorphic extension is mean $(m+1)$-equicontinuous. This falls in line with the well-known result, due to Downarowicz and Glasner, that strictly ergodic systems are isomorphic extensions if and only if they are mean equicontinuous. While in the multivariate case we can only conjecture that the converse direction also holds, the result provides an indication that multivariate equicontinuity properties are strongly related to finite extension structures. For minimal systems, an Auslander-Yorke type dichotomy between multivariate mean equicontinuity and sensitivity is shown as well.

1. Introduction

A topological system (tds) (X, φ), given by a compact metric space X and a homeomorphism $\varphi : X \to X$, is called *mean equicontinuous* if the *Besicovitch pseudo-metric* d_B given by

$$d_B(x,y) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(\varphi^i(x), \varphi^i(y)\right)$$

is continuous with respect to the original metric d on X. This notion was introduced by Li, Tu and Ye in [LTY15], who also proved that any minimal mean equicontinuous tds is uniquely ergodic. Moreover, Downarowicz and Glasner showed that this property is closely related to the extension structure of (X, φ). We denote by (Y, ψ) the *maximal equicontinuous factor* (MEF) of (X, φ) and by $\pi : X \to Y$ the corresponding factor map. Then in the minimal case (X, φ) is mean equicontinuous if and only if it is uniquely ergodic and π is a measure-theoretic isomorphism between the two systems (X, φ) and (Y, ψ) equipped with their respective unique invariant probability measures [DG16]. In this situation, we say (X, φ) is a *topo-isomorphic extension* of (Y, ψ). This seminal result prompted further research in different directions. It was generalised in [FGL22] to the non-minimal case and more general group actions. In [GRJY21], various subclasses of topo-isomorphic extensions, defined in terms of additional invertibility properties of the factor map π, were characterised by intrinsic dynamical properties of the system. At the same time, multivariate versions of mean sensitivity – the counterpart to mean equicontinuity – were introduced by Li, Ye and Yu [LYY22] and further studied by various authors [LY21, LY23].

Broadly speaking, the aim of this note is to establish a link between the multivariate version of mean equicontinuity – complementary to the notion of multivariate mean sensitivity in [LYY22] – and the extension structure of the system, similar to the result on topo-isomorphic extensions in [DG16]. We also refer to [SYZ08, HLSY21] for related results concerning the interplay between dynamical properties finite-to-one extension structures.

In order to be more precise, we need to introduce some notation. Given $m \in \mathbb{N}, m \geq 2$ and $x_1, \ldots, x_m \in X$, we let

$$\mathrm{D}_m(x_1, \ldots, x_m) = \min_{1\leq i<j\leq m} d(x_i, x_j) \quad \text{and} \quad \mathrm{D}_m^{\max}(x_1, \ldots, x_m) = \max_{1\leq i<j\leq m} d(x_i, x_j) .$$

We define the *Besicovitch m-distance* of $x_1, \ldots, x_m$ as

$$\overline{\mathrm{D}}_m^{\varphi}(x_1, \ldots, x_m) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{D}_m \left(\varphi^i(x_1), \ldots, \varphi^i(x_m)\right)$$

and say (X, φ) is *mean m-equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathrm{D}_m^{\max}(x_1, \ldots, x_m) < \delta$ implies $\overline{\mathrm{D}}_m^{\varphi}(x_1, \ldots, x_m) < \varepsilon$.

The corresponding extension structure is defined as follows. First, given a map $\gamma : Y \to X^m$ that satisfies $\pi \circ \gamma_i(y) = y$ for all $y \in Y$ and $i \in \{1, \ldots, n\}$, we denote the corresponding point set by $\Gamma = \{\gamma_i(y) \mid y \in Y,\ i = 1, \ldots, m\}$. Then, we call (X, φ) an *m:1 topomorphic extension (of its MEF)* if m is the least integer such that there exists a measurable map $\gamma : Y \to X^n$ which satisfies $\mu(\Gamma) = 1$ for any φ-invariant measure μ on X. Equivalently, one could require that any φ-invariant measure is supported on at most m points in every fibre $\pi^{-1}(y),\ y \in Y$ (see Section 3 for details and further discussion). We note that mean 2-equicontinuity is just mean equicontinuity and a 1:1 topomorphic extension is just a topo-isomorphic extension in the sense of [LTY15, DG16, FGL22]. Hence, for minimal systems mean 2-equicontinuity and a 1:1 topomorphic extension structure are equivalent by [DG16], as mentioned above.

Here, our main result reads as follows.

Theorem 1.1. *Let $m \in \mathbb{N}$ and suppose that (X, φ) is a minimal m:1 topomorphic extension of its MEF (Y, ψ). Then (X, φ) is mean $(m + 1)$-equicontinuous.*

We actually believe that, as in the case $m = 1$, the converse holds as well. However, we do not pursue this problem here and only discuss briefly in Section 5 why a proof of this fact – if it is true – needs to be more intricate in the multivariate case.

Conjecture 1.2. *Let $m \in \mathbb{N}$ and suppose that (X, φ) is minimal. Then (X, φ) is mean $(m + 1)$-equicontinuous, but not mean m-equicontinuous, if and only if it is an m:1 topomorphic extension of its MEF.*

Further, a classical result due to Auslander and Yorke in [AY80] states that a minimal tds is either equicontinuous or sensitive (has sensitive dependence on initial conditions). As shown by Li, Tu and Ye in [LTY15], an analogue holds for the mean versions of these notions as well. Following [LYY22], we say (X, φ) is *mean m-sensitive* if there exists $\varepsilon > 0$ such that for every open set $U \subseteq X$ there exist points $x_1, \ldots, x_m \in B_\delta(x)$ with $\overline{\mathrm{D}}_m^{\varphi}(x_1, x_2, \ldots, x_m) \geq \varepsilon$.

Theorem 1.3. *A minimal tds (X, φ) is mean m-equicontinuous if and only if it is not mean m-sensitive.*

It is noteworthy that the finite topomorphic extension structure defined above is shared by a broad scope of classical examples. Probably the best-know case of 2:1 topomorphic extensions (over the dyadic odometer) are subshifts induced by the Thue-Morse substitution and their generalisations [Kea68]. As Keane already remarked in [Kea68], similar substitutions on alphabets with m symbols should lead to m:1 topomorphic extensions in an analogous way. Moreover, all constant length substitution induce subshifts that have a finite topomorphic extension structure [Kam72, Dek78]. Further examples include certain irregular Toeplitz flows constructed by Williams [Wil84] and by Iwanik and Lacroix [IL94], and similar examples have been be obtained in the class of irregular model sets [FGJO21]. Smooth examples of minimal skew-products on the two-torus with the same extension structure have recently been constructed in [HJ].

The paper is organised as follows. In Section 2, we provide the necessary background and preliminaries. In Section 3, we discuss basic properties and alternative definitions of finite-to-one topomorphic extensions. The Auslander-Yorke type dichotomy (Theorem 1.3) is shown in Section 4, alongside with pointwise characterisations of multivariate mean equicontinuity and sensitivity. The proof of Theorem 1.1 is then given in Section 5, where we also include a brief discussion of Conjecture 1.2.

2. Notation and preliminaries

We assume that the reader is familiar with standard notions of topological dynamics and ergodic theory, as provided, for instance, in [Wal82, Pet83, KH97, EW10] A *measure-preserving dynamical system (mpds)* is a quadruple $(X, \mathcal{B}, \mu, \varphi)$ consisting of a probability space $(X, \mathcal{A}, \mu)$ and a bi-measurable bijective transformation $\varphi : X \to X$ which preserves the measure μ. In our context, X will in most cases be a compact metric space, $\mathcal{B} = \mathcal{B}(X)$ the Borel σ-algebra generated by the topology on X and φ a homeomorphism of X. Since any compact metric space is Polish, $(X, \mathcal{B}(X))$ is a standard Borel space in this situation.[1]

Any measure that is mentioned in the following will implicitly be understood to be a probability measure, unless explicitly stated otherwise. If μ is a measure on a Borel space $(X, \mathcal{B}(X))$, we denote by

$$\mathrm{supp}(\mu) \;=\; \{x \in X \mid \mu(U) > 0 \text{ for all open neighbourhoods } U \text{ of } x\}$$

the topological support of μ.

Suppose now that $(X, \mathcal{B}, \mu, \varphi)$ and $(Y, \mathcal{A}, \nu, \psi)$ are two mpds. Then a measurable map $\pi : X \to Y$ is called a *(measure-theoretic) factor map* and $(Y, \mathcal{A}, \nu, \psi)$ a *(measure-theoretic) factor* of $(X, \mathcal{B}, \mu, \varphi)$ if $\pi \circ \varphi = \psi \circ \pi$ holds μ-almost surely. If additionally there exist subsets $X_0 \subseteq X$, $Y_0 \subseteq Y$ of full measure such that $\pi : X_0 \to Y_0$ is a bi-measurable bijection, we call π an *isomorphism* of mpds and say the two systems are *(measure-theoretically) isomorphic.*

Theorem 2.1 (Rokhlin's skew product theorem, [Gla03, Theorem 3.18])**.** *Suppose that $(X, \mathcal{B})$ and $(Y, \mathcal{A})$ are standard Borel spaces, $(X, \mathcal{B}, \mu, \varphi)$ is an ergodic mpds and $(Y, \mathcal{A}, \mu, \psi)$ is a factor mpds with factor map $\pi : X \to Y$. Then there exists a standard Lebesgue space[2] $(Z, \mathcal{C}, \lambda)$ and an $\mathcal{A} \otimes \mathcal{C}$-bi-measurable bijection $\rho : Y \times Z \to Y \times Z$ preserving $\nu \otimes \lambda$ such that the two systems $(X, \mathcal{B}, \mu, \varphi)$ and $(Y \times Z, \mathcal{A} \otimes \mathcal{C}, \nu \otimes \lambda, \rho)$ are isomorphic.*

Further, the transformation ρ can be chosen such that it has skew product form

$$\rho : Y \times Z \to Y \times Z \quad , \quad (y, z) \mapsto (\psi(y), \rho_y(z))$$

where $\rho_y : Z \to Z$ preserves the measure λ for all $y \in Y$. Moreover, the isomorphism $\iota : X \to Y \times Z$ can be chosen such that it satisfies $p_Y \circ \iota = \pi$, where $p_Y : Y \times Z \to Y$ denotes the canonical projection to Y.

Whenever we invoke Rokhlin's Theorem in the following, we always assume that the transformation ρ and the isomorphism ι satisfy the additional assertions stated above.

We will need to use a direct implication of this statement for the structure of ergodic measures of (measure-theoretically) finite-to-one extensions. Recall that if X, Y are Polish spaces, μ is a Borel probability measure on X and $\nu = \pi_* \mu$, then there exists a mapping

$$Y \times \mathcal{B}(X) \;\to\; [0,1] \quad , \quad (y, A) \mapsto \mu_y(A)$$

such that

- for every $y \in Y$, the mapping $A \mapsto \mu_y(A)$ is a Borel probability measure on X;
- for every $A \in \mathcal{B}(X)$, the function $y \to \mu_y(A)$ is integrable and

$$\mu(A) \;=\; \int_Y \mu_y(A)\; d\nu(y)\ .$$

Such a mapping, which we will also denote as $(\mu_y)_{y \in Y}$, is called a *conditional probability distribution* of μ over π. It is unique modulo modifications on a subset of Y of measure zero and we have $\mu_y(\pi^{-1} y) = 1$ for ν-almost every $y \in Y$. We refer to the measures μ_y as *fibre measures.*

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[1]Recall that a *standard Borel space* is a measurable space of the form $(X, \mathcal{B}(X))$, where X is a Polish space.

[2]A *standard Lebesgue space* $(Z, \mathcal{C}, \lambda)$ is a standard Borel space $(Z, \mathcal{C})$ equipped with a probability measure λ such that $\mathcal{C}$ is complete with respect to λ (that is, any subset of a null set in $\mathcal{B}$ is again contained in $\mathcal{C}$).

Now, let $(X,\mathcal{B},\mu,\varphi)$ be an extension of $(Y,\mathcal{A},\nu,\psi)$ with corresponding factor map π and suppose that μ is ergodic. Further, assume that the support $\mathrm{supp}(\mu_y)$ has finite cardinality ν-almost surely. Then, since μ_y is the push-forward of λ under the mapping $z \mapsto \iota^{-1}(y,z)$, the measure λ in Theorem 2.1 is supported on a finite set as well. We can therefore assume that the space Z itself is finite, that is, $Z = \{z_1,\dots,z_k\}$ for some $k \in \mathbb{N}$. If we let $\gamma_i(y) = \iota^{-1}(y,z_i)$, $i = 1,\dots,k$, then the situation we arrive at is the following: We have a measurable multivalued function

$$\gamma : Y \to X^k \quad , \quad y \mapsto \gamma(y) = (\gamma_1(y),\dots,\gamma_k(y))$$

such that

- $\pi(\gamma_i(y)) = y$ holds for ν-almost every $y \in Y$ and all $i = 1,\dots,k$;
- $\gamma_i(y) \neq \gamma_j(y)$ holds for ν-almost every $y \in Y$ and all $1 \leq i < j \leq k$.

and the measure μ_γ is of the form

$$\mu_\gamma \;=\; \frac{1}{k}\sum_{i=1}^{k} \gamma_{i*}\nu \ . \tag{1}$$

We will call a measure of the form (1) a *graph measure of multiplicity* k. The term is motivated by the situation where (X,φ) is a skew product system, that is, $X = Y \times \Xi$ and $\varphi(y,\xi) = (\psi(y),\varphi_y(\xi))$ for all $(y,\xi) \in X$, and π is just the canonical projection from $X = Y \times \Xi$ to Y. In this case $\gamma_i(y) = (y,\hat{\gamma}_i(y))$ for some measurable function $\hat{\gamma}_i : Y \to \Xi$ and the measure μ_γ is supported on the union of the graphs of $\hat{\gamma}_1,\dots,\hat{\gamma}_k$.

Altogether, we obtain the following.

Lemma 2.2. *Suppose that the mpds $(X,\mathcal{B},\mu,\varphi)$ is an extension of $(Y,\mathcal{A},\nu,\psi)$. Further, assume that μ is ergodic and $\sharp\,\mathrm{supp}(\mu_y) < \infty$ ν-almost surely. Then μ is a graph measure of finite multiplicity $k \in \mathbb{N}$. In particular, we have $\sharp\,\mathrm{supp}(\mu_y) = k$ for ν-almost every $y \in Y$.*

Finally, given a tds (X,φ), a φ-invariant measure μ and $n \in \mathbb{N}$, an n-fold (self-)joining of μ is a $\varphi^{\times n}$-invariant measure $\hat{\mu}$ on X^n that satisfies $\pi_{i*}\hat{\mu} = \mu$ for $i = 0,\dots,n$. Here

$$\varphi^{\times n} : X^n \to X^n \quad , \quad (x_1,\dots,x_n) \mapsto (\varphi(x_1),\dots,\varphi(x_n)) \ .$$

If (Y,ψ) is factor of (X,φ) with factor map $\pi : X \to Y$ and $\hat{\mu}$ is supported on the set

$$X^n_\pi \;=\; \{x \in X^n \mid \pi(x_1) = \ldots = \pi(x_n)\} \ ,$$

then $\hat{\mu}$ is called an *n-fold joining over the common factor* (Y,ψ). There is always at least one such joining: if μ disintegrates as $(\mu_y)_{y\in Y}$, then it can be obtained by integration of the fibre measures $\hat{\mu}_y = \bigotimes_{i=1}^n \mu_y$ with respect to ν. We refer to [Gla03, dlR23] for further background on joinings.

3. Finite topomorphic extensions

The aim of this section is to provide two seemingly weaker, but alternative definitions of finite topomorphic extensions. First, instead of requiring the existence of an m-valued mapping $\gamma : Y \to X^m$, $m \in \mathbb{N}$, whose corresponding point set supports all φ-invariant measures, it suffices to require that all these measures are supported on some measurable set that intersects the fibres $\pi^{-1}(y)$ in at most m points. The *'graph structure'* of this set then comes for free. Secondly, it also suffices to require that any φ-invariant measure is supported on at most m points in every fibre. In both cases, the proof of the equivalence to the original definition hinges on an application of Rokhlin's Skew Product Theorem.

Proposition 3.1. *Suppose (X,φ) is a tds with uniquely ergodic MEF (Y,ψ) and corresponding factor map $\pi : X \to Y$. Denote by ν the unique ψ-invariant measure. Then the following are equivalent for all $m \in \mathbb{N}$.*

(i) (X,φ) is an m:1 topomorphic extension of (Y,ψ).

(ii) m is the least integer such that there exists a measurable set $M \subseteq X$ with $\mu(M) = 1$ for all φ-invariant Borel probability measures on X and $\sharp\pi^{-1}\{y\} \cap M = m$ for ν-almost every $y \in Y$.

(iii) For every φ-invariant measure μ on X, the fibre measures μ_y are ν-almost surely supported on m points.

In fact, the assumption of unique ergodicity of (Y, ψ) is not strictly necessary. However, since this is always satisfied for minimal systems and these are our main focus, we use it here both for the sake of simplicity and because some additional information is available in this case (see Addendum 3.2).

Proof. Let the least integers for which the statements in (i), (ii) and (iii) hold be denoted by m, m' and m'', respectively. A priori, we also allow values $+\infty$. However, we show $m \geq m' \geq m'' \geq m$, which proves the asserted equivalence.

$m \geq m'$: suppose $\gamma : Y \to X^m$ is such that the corresponding point set Γ supports all φ-invariant probability measures. Then it suffices to set $M = \Gamma$.

$m' \geq m''$: Let $M_y = M \cap \pi^{-1}\{y\}$. Then $\mu(M) = 1$ implies $\mu_y(M_y) = 1$ for ν-almost every $y \in Y$. Hence, μ_y is supported on m' points ν-almost surely.

$m'' \geq m$: We first show that there exist at most m different ergodic φ-invariant measures on X. Assume otherwise and let $\mu^1, \ldots, \mu^{m+1}$ denote different φ-invariant ergodic measures. For each of these measures, the fibre measures μ^i_y are supported on at most m'' points ν-almost surely. This means that if we apply Rokhlin's Skew Product Theorem to μ^i, we obtain that $(X, \mathcal{B}(X), \mu^i, \varphi)$ is isomorphic via an isomorphism ι_i to a skew product system $(Y \times Z_i, \mathcal{B}(Y) \otimes \mathcal{C}_i, \nu \otimes \lambda_i, \rho_i)$ where λ_i is the equidistribution (by ergodicity) on a finite set $\hat{Z}_i = \{z^i_1, \ldots, z^i_{k_i}\}$ with $k_i \leq m''$. The measure μ^i_y is then supported on the set $\hat{X}_i(y) = \iota_i^{-1}\left(\{y\} \times \hat{Z}_i\right)$. By ergodicity, the sets $X_1(y), \ldots, X_{m''+1}(y)$ are ν-almost surely disjoint. However, this means that the fibre measures μ_y of the measure $\mu = \frac{1}{m''+1}\sum_{i=1}^{m''+1} \mu^i$ are supported on the sets $X(y) = \bigcup_{i=1}^{m''+1} X_i(y)$ of cardinality $\sum_{i=1}^{m''+1} k_i > m''$, contradicting the assumption.

Hence, if $\mu^1, \ldots, \mu^\ell$ are all the different ergodic φ-invariant measures on X, then $\ell \leq m''$. More precisely, the above argument yields that if the fibre measures of μ^i are supported on k_i points in ν-almost every fibre, then $\tilde{m} = \sum_{i=1}^{\ell} k_i \leq m''$. Given $j \in \{1, \ldots, \tilde{m}\}$, we can write j in a unique way as $j = \sum_{i=1}^{s-1} k_i + t$, where $s \in \{1, \ldots, \ell\}$ and $t \in \{1, \ldots, k_s\}$. If we then define

$$\gamma_j(y) \ = \ \iota_s^{-1}(z^s_t)$$

we obtain a measurable mapping $\gamma : Y \to X^{\tilde{m}}$ that satisfies $\mu^i(\Gamma) = 1$ for all $i = 1, \ldots, \ell$, and hence $\mu(\Gamma) = 1$ for all φ-invariant measures by ergodic decomposition. □

In combination with Lemma 2.2, the preceding proof of Proposition 3.1 actually yields some additional information.

Addendum 3.2. *Suppose we are in the situation of Proposition 3.1. Then there exist at most m φ-invariant measures $\mu_1, \ldots, \mu_\ell$, where $\ell \leq m$, all of which are graph measures. Further, if k_i denotes the multiplicity of the graph measure μ_i, then $\sum_{i=1}^{\ell} k_i = m$.*

4. An Auslander-Yorke type dichotomy

Before we turn to a closer inspection of multivariate mean equicontinuity, we first want to collect some basic facts concerning the mappings $\mathrm{D}^{\max}_m$, D_m and $\overline{\mathrm{D}}^{\varphi}_m$ introduced above. We will refer to these as *multidistances*.

4.1. THE MULTIDISTANCES D_m AND $\mathrm{D}_m^{\max}$. Given $m \in \mathbb{N}$ and $\mathbf{x} \in X^m$, we define $(\mathbf{x}\,|_i\, z)$ by $(\mathbf{x}\,|_i\, z)_j = x_j$ if $j \neq i$ and $(\mathbf{x}\,|_i\, z)_i = z$, i.e. $(\mathbf{x}\,|_i\, z) = (x_1, \ldots, x_{i-1}, z, x_{i+1} \ldots x_m)$. We say that a map $\mathrm{D} : X^m \to \mathbb{R}_0^+$ satisfies the *polygon inequality* if and only if for any $\mathbf{x} \in X^m$ and $z \in X$ we have

$$\mathrm{D}(\mathbf{x}) \leqslant \sum_{i=1}^{m} \mathrm{D}\left(\mathbf{x}\,|_i\, z\right) \ .$$

If additionally D is *symmetric*, that is, for any tuple $(x_1, \ldots, x_m) \in X^{m+1}$ and any permutation $\sigma \in \mathrm{Sym}(m)$ we have

$$\mathrm{D}(x_1, \ldots, x_m) = \mathrm{D}(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \ ,$$

and *positive semi-definite*, meaning $\mathrm{D}(x, \ldots, x) = 0$ for all $x \in X$, we call D an *m-multidistance*.

Remark 4.1. A notable example of a multidistance, not needed in this work, is the measure of the simplex spanned by m points in euclidean space $\mathbb{R}^{m-1}$.

Lemma 4.2. *D_m and $\mathrm{D}_m^{\max}$ are m-multidistances.*

Proof. Symmetry and positive semi-definiteness are clear. We will show the polygon inequality. For $m = 1$ we have the original metric and the original triangle inequality so let $m \geqslant 2$. Let $\mathbf{x} = (x_1, \ldots, x_m) \in X^m$ and $z \in X$ be arbitrary.

First we consider $\mathrm{D}_m^{\max}$. Let $a, b \in \{1, \ldots, m\}$ be maximizing indices, that is, $d(x_a, x_b) = D_m^{\max}(x_1, \ldots, x_m)$. Then we have

$$D_m^{\max}\left(\mathbf{x}\,|_i\, z\right) \geqslant D_m^{\max}(\mathbf{x})$$

for all $i \notin \{a, b\}$. So a fortiori we have

$$\sum_{i=0}^{m} D_m^{\max}\left(\mathbf{x}\,|_i\, z\right) \geqslant D_m^{\max}(\mathbf{x})\,.$$

Now we consider D_m. Pick minimizing indices $a, b \in \{1, \ldots, m\}$, that is, $d(x_a, x_b) = D_m(x_1, \ldots, x_m)$. For ease of notation, let $x_{m+1} = z$. Likewise, for any $i \in \{1, \ldots, m\}$, pick indices $a_i, b_i \in \{1, \ldots, m+1\} \setminus \{i\}$ such that $d(x_{a_i}, x_{b_i}) = D_m((\mathbf{x}\,|_i\, x_{m+1}))$. If for some $j \in \{1, \ldots, m\}$ neither $a_j = m+1$ nor $b_j = m+1$, we are done, as then

$$\sum_{i=1}^{m} D_m\left(\mathbf{x}\,|_i\, x_{m+1}\right) \geqslant D_m((\mathbf{x}\,|_j\, x_{m+1})) = d(x_{a_j}, x_{b_j}) \geqslant D_m(\mathbf{x}) \ .$$

So without loss of generality $b_i = m+1$ for any i. Now observe that $i \mapsto a_i \in \{1, \ldots, m\}$ can not be constant, as $a_i \neq i$ by construction. Pick $k, l \in \{1, \ldots, m\}$ such that $a_k \neq a_l$. Now the triangle inequality implies

$$\begin{aligned}\sum_{i=1}^{m} D_m\left(\mathbf{x}\,|_i\, x_{m+1}\right) &\geqslant D_m\left(\mathbf{x}\,|_k\, x_{m+1}\right) + D_m\left(\mathbf{x}\,|_l\, x_{m+1}\right) \\ &= d(x_{a_k}, x) + d(x_{a_l}, x) \\ &\geqslant d(x_{a_k}, x_{a_l}) \geqslant D_m(\mathbf{x})\,.\end{aligned}$$

□

The multidistance property of D_m directly carries over to $\overline{\mathrm{D}}_m^{\varphi}$ in the limit.

Corollary 4.3. *If (X, φ) is a tds, then $\overline{\mathrm{D}}_m^{\varphi}$ is an m-multidistance*

4.2. Pointwise multivariate mean equicontinuity and sensitivity. Recall that a tds (X,φ) is mean m-equicontinuous, with $m \geq 2$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathrm{D}_m^{\max}(\mathbf{x}) < \delta$ implies $\overline{\mathrm{D}}_m^{\varphi}(\mathbf{x}) < \varepsilon$ for all $\mathbf{x} \in X^m$. Conversely, (X,φ) is mean m-sensitive if for some $\varepsilon > 0$ and every open set $U \subseteq X$ there exists $\mathbf{x} \in U^m$ with $\overline{\mathrm{D}}_m^{\varphi}(\mathbf{x}) \geq \varepsilon$. The aim of this section is to provide pointwise characterisations of these properties. Note that neither minimality nor any other recurrence assumption is required here.

Given $\varepsilon > 0$, a point $x \in X$ is called an *mean ε-m-equicontinuity point* if and only if there is a $\delta > 0$ such that $\overline{\mathrm{D}}_m^{\varphi}(x, x_2, \dots, x_m) < \varepsilon$ for all $x_2, \dots, x_m \in B_\delta(x)$. It is called a *mean m-equicontinuity point* if it is a mean ε-m-equicontinuity point for all $\varepsilon > 0$. Conversely, we say point $x \in X$ is *mean m-sensitive (mean ε-m-sensitive)* if it is not a mean m-equicontinuity point (mean ε-m-equicontinuity point).

Proposition 4.4. *Let (X,φ) be a tds and $m \geq 2$.*

(a) (X,φ) is mean m-equicontinuous if and only if every $x \in X$ is a mean m-equicontinuity point.

(b) (X,φ) is mean m-sensitive if and only if there exists $\varepsilon > 0$ such that every $x \in X$ is mean ε-m-sensitive.

Note that for the case $m = 2$ this result is contained in [LTY15]. In the following proof, we essentially adapt the respective arguments to the multivariate case.

Proof. (a) If (X,φ) is mean m-equicontinuous, then it is obvious that all points are mean m-equicontinuity points. Conversely, assume that every $x \in X$ is a mean m-equicontinuity point and fix $\varepsilon > 0$. For every $x \in X$, choose $\delta(x) > 0$ such that $\overline{\mathrm{D}}_m^{\varphi}(x, x_2, \dots, x_m) < \frac{\varepsilon}{m}$ for all $x_2, \dots, x_m \in B_{\delta(x)}(x)$. Then $\mathcal{U} = \{B_{\delta(x)}(x) \mid x \in X\}$ is an open cover of X. Choose any $\delta > 0$ smaller than the Lebesgue covering number of $\mathcal{U}$. Then, if $\mathbf{x} = (x_1, \dots, x_m) \in X^m$ satisfies $\mathrm{D}_m^{\max}(x_1, \dots, x_m) < \delta$, we have that $\{x_1, \dots, x_m\} \subseteq B_\delta(x_1) \subseteq B_{\delta(\xi)}(\xi)$ for some $\xi \in X$. Consequently, the polygonal inequality yields

$$\overline{\mathrm{D}}_m^{\varphi}(\mathbf{x}) \;\leq\; \sum_{i=1}^{m} \overline{\mathrm{D}}_m^{\varphi}((\mathbf{x}\,|_i\,\xi)) \;<\; m \cdot \frac{\varepsilon}{m} \;=\; \varepsilon\ .$$

This shows the mean m-equicontinuity of (X,φ).

(b) Suppose that (X,φ) is mean m-sensitive. Then there exists some $\eta > 0$ such that for any non-empty open subset $U \subseteq X$ there is $\mathbf{x} \in U^m$ with $\overline{\mathrm{D}}_m^{\varphi}(\mathbf{x}) > \eta$. Let $x \in X$, $\delta > 0$ and $U = B_\delta(x)$. Choose $\mathbf{x} \in U^m$ with $\overline{\mathrm{D}}_m^{\varphi}(\mathbf{x}) > \eta$. Then by the multi-triangle inequality we have

$$\sum_{i=0} \overline{\mathrm{D}}_m^{\varphi}\left((\mathbf{x}\,|_i\,x)\right) \geqslant \overline{\mathrm{D}}_m^{\varphi}(\mathbf{x}) > \eta\,.$$

So $\overline{\mathrm{D}}_m^{\varphi}\left((\mathbf{x}\,|_i\,x)\right) > \frac{\eta}{m}$ for at least for one $i \in \{0, \dots, m\}$. This yields that x is a mean ε-m-sensitive point with $\varepsilon = \frac{\eta}{m}$. As $x \in X$ was arbitrary, this proves the first implication. The converse direction is again obvious. □

4.3. Auslander-Yorke type dichotomy. We again refer to [LTY15] for analogous results on the case $m = 2$. The key observation is the following.

Lemma 4.5. *Let (X,φ) be transitive with transitivity point x. Then either x is a mean m-equicontinuity point or the system is mean $(m+1)$-sensitive.*

Proof. Suppose that x is a transitive point, but not a mean m-equicontinuity point. Let $U \subseteq X$ be open. By definition, x is mean ε-m-sensitive for some $\varepsilon > 0$. However, this means that $\varphi^n(x)$ is mean ε-m-sensitive for any $n \in \mathbb{N}$. As x is a transitivity point, $\varphi^n(x) \in U$ for some $n \in \mathbb{N}$. Since U is open, we can choose $x_2, \dots, x_m \in U$ such that $\overline{\mathrm{D}}_m^{\varphi}(\varphi^n(x), x_2, \dots, x_m) \geq \varepsilon$. This proves mean m-sensitivity of (X,φ). □

For a minimal system (X,φ), this means that either the system is mean m-sensitive or all points are mean m-equicontinuity points. Due to Proposition 4.4(a), the latter implies that (X,φ) is mean m-equicontinuous. This proves Theorem 1.3, which we restate here as

Corollary 4.6 (Auslander-Yorke type Dichotomy)**.** *A minimal tds (X,φ) is mean m-equicontinuous if and only if it is not mean m-sensitive.*

4.4. Non-continuity of the Besicovitch multidistance. We want to close this section by pointing out an important difference between mean equicontinuity and multivariate mean equicontinuity ($m > 2$). As mentioned in the introduction, mean equicontinuity can be defined as the continuity of the Besicovitch pseudo-metric. This is different in the multivariate case, since continuity of the Besicovitch m-distance equally implies mean equicontinuity. Hence, if (X,φ) is mean m-equicontinuous for some $m > 2$, but not mean equicontinuous, the corresponding Besicovitch m-distance cannot be continuous. Examples of this type are given, for instance, by Thue-Morse subshifts ([Kea68], compare [LYY22, Theorem 4.6] and its proof) or by certain irregular Toeplitz flows [Wil84, IL94].

Lemma 4.7. *Assume that (X,φ) has an infinite MEF. If $\overline{\mathrm{D}}_m^\varphi$ is continuous, then (X,φ) is mean equicontinuous.*

Proof. Let $\pi : X \to Y$ be the factor map to the MEF (Y,ψ). Pick a ψ-invariant metric d_Y on Y and note that we can switch to equivalent metrics in order to ensure that $d(x,y) > d_Y(\pi(x),\pi(y))$ for any $x,y \in X$. Pick any $x_2 \in X$ and choose $x_3,\dots,x_m \in X$ such that $\pi(x_i) \neq \pi(x_j)$ for any $2 \le i < j \le m$. Define $c = \min_{2\leqslant i<j\leqslant m} d_Y(\pi(x_i),\pi(x_j)) > 0$. Now note that if $d_Y(\pi(x),\pi(x_2)) < \frac{c}{2}$, then $\min_{3\leqslant i\leqslant m} d_Y(\pi(x),\pi(x_j)) > \frac{c}{2}$. By invariance of d_Y this holds for any iterate.

Observe that $\overline{\mathrm{D}}_m^\varphi(x_2,x_2\dots,x_m) = 0$, so that continuity of $\overline{\mathrm{D}}_m^\varphi$ implies

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \mathrm{D}_m(\varphi^k(x),\varphi^k(x_2),\dots,\varphi^k(x_m)) = \overline{\mathrm{D}}_m^\varphi(x,x_2\dots,x_m) \xrightarrow{x\to x_2} 0 .$$

Expanding we obtain

$$\begin{aligned} &\mathrm{D}_m(\varphi^k(x),\varphi^k(x_2),\dots,\varphi^k(x_m)) \\ (2)\quad = \;& \min\Big\{d\big(\varphi^k(x),\varphi^k(x_2)\big), \min_{3\leqslant j\leqslant m} d\big(\varphi^k(x),\varphi^k(x_j)\big), \min_{2\leqslant i<j\leqslant m} d\big(\varphi^k(x_i),\varphi^k(x_j)\big)\Big\}. \end{aligned}$$

Note that the terms in (2) are uniformly bounded from below by $\frac{c}{2}$. So it is the term $d\big(\varphi^k(x),\varphi^k(x_j)\big)$ that must be responsible for $\overline{\mathrm{D}}_m^\varphi(x,x_2,\dots,x_m)$ getting arbitrarily small for x sufficiently close to x_2. This in turn implies that $d_{\mathsf{B}}(x,x_2)$ gets arbitrarily small for x sufficiently close to x_2. So x_2 is a mean equicontinuity point.

As $x_1 \in X$ was arbitrary, this shows that all points in X are mean equicontinuity points. Hence, by Proposition 4.4(a), (X,φ) is mean equicontinuous. □

5. Multivariate mean equicontinuity of finite topomorphic extensions

We again consider a tds (X,φ) with MEF (Y,ψ) and corresponding factor map $\pi : X \to Y$. Given $m \in \mathbb{N}$ and $\mathbf{x} \in X^m$, we let $\mathrm{D}_{Y,m}^{\max}(\mathbf{x}) = \max_{1\le i<j\le m} d_Y(\pi(x_i),\pi(x_j))$, where d_Y denotes the metric on Y. Then, we call (X,φ) *factor mean m-equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathrm{D}_{Y,m}^{\max}(\mathbf{x}) < \delta \quad \text{implies} \quad \overline{\mathrm{D}}_m^\varphi(\mathbf{x}) < \varepsilon .$$

Due to the continuity of the factor map π, it is obvious that factor mean m-equicontinuity implies mean m-equicontinuity. Therefore, Theorem 1.1 is a direct consequence of the following equivalence.

Theorem 5.1. *Let $m \in \mathbb{N}$. A minimal tds (X,φ) is an m:1 topomorphic extension of its MEF if and only if it is factor mean $(m+1)$-equicontinuous, but not factor mean k-equicontinuous for any $k \le m$.*

Proof. We first assume that (X,φ) is an m:1 topomorphic extension of its MEF, so that all φ-invariant measures are supported on a set $\Gamma = \{\gamma_i(y) \mid i = 1,\ldots,m,\ y \in Y\}$, where $\gamma : Y \to X^m$ is measurable. Let the ν be the unique invariant measure on (Y,ψ). By Addendum 3.2, there exist a finite number $\ell \leq m$ of ergodic measures on (X,φ), which we denote by $\mu_1,\ldots,\mu_k$. We assume without loss of generality that the metric d_Y on Y is ψ-invariant and that $d(x_1,x_2) \geqslant d_Y(\pi(x_1),\pi(x_2))$.

Let $\eta,\rho > 0$, where both parameters are arbitrary at the moment, but will be specified further later. By Lusin's Theorem, there is a compact set $K = K_\eta \subseteq Y$ with $\nu(K) > 1-\eta$ such that $\gamma|_K$ is continuous. Let

$$U \;=\; B_\rho\left(\bigcup_{i=1}^m \gamma_i(K)\right) .$$

Then, since all φ-invariant measures are supported on Γ and project to ν, we have that $\mu(U) > 1-\eta$ for any φ-invariant measure μ. As a consequence of the semi-uniform ergodic theorem [SS00], we obtain

$$\liminf_{n\to\infty} \frac{1}{N}\sum_{i=1}^n \mathbf{1}_U \circ \varphi^n(x) \geqslant 1-\eta$$

for any $x \in X$. Now, let $\mathbf{x} \in X^{m+1}$ and consider the set of simultaneous hitting times

$$T(\mathbf{x}) \;=\; \left\{t \in \mathbb{N} \,\middle|\, \varphi^t(x_j) \in U \text{ for all } j \in \{1,\ldots,m+1\}\right\} .$$

For any $\mathbf{x} = (x_0,\ldots,x_m) \in X^{m+1}$, this set has lower asymptotic density

$$a(\mathbf{x}) \;=\; \liminf_{N\to\infty} \frac{1}{N}\#\{T(\mathbf{x}) \cap \{1,\ldots,N\}\} \;\geqslant\; 1-(m+1)\eta .$$

Now, fix $\varepsilon > 0$. We need to find $\delta > 0$ such that $\mathrm{D}^{\max}_{Y,m+1}(\pi(\mathbf{x})) < \delta$ implies $\overline{\mathrm{D}}^{\varphi}_{m+1}(\mathbf{x}) < \varepsilon$. We have that

$$\begin{aligned}\overline{\mathrm{D}}^{\varphi}_{m+1}(\mathbf{x}) &\leqslant (1-a(\mathbf{x}))\cdot \mathsf{diam}(X) + a(\mathbf{x})\cdot\kappa(\mathbf{x})\\ &\leqslant (m+1)\cdot\eta\cdot\mathsf{diam}(X) + \kappa(\mathbf{x})\end{aligned}$$

holds for any $\mathbf{x} \in X^{m+1}$, where

$$\kappa(\mathbf{x}) \;=\; \sup_{n\in T(\mathbf{x})} \mathrm{D}_{m+1}(\varphi^n(\mathbf{x}))$$

As X is compact, $\mathsf{diam}(X) < \infty$. We can therefore choose η small enough such that

$$(m+1)\cdot\eta\cdot\mathsf{diam}(X) \;<\; \frac{\varepsilon}{2} .$$

It remains to find $\delta > 0$ such that $\mathrm{D}^{\max}_{Y,m+1}(\pi(\mathbf{x})) < \delta$ implies $\kappa(\mathbf{x}) \leq \frac{\varepsilon}{2}$. As the γ_j are continuous on K, there is $\alpha > 0$ such that

$$d_Y(y_1,y_2) < \alpha \Longrightarrow d(\gamma_j(y_1),\gamma_j(y_2)) < \frac{\varepsilon}{6}$$

holds for all $y_1,y_2 \in K$ and any $j \in \{1,\ldots,m\}$. We now fix $\rho < \min\left\{\frac{\alpha}{3},\frac{\varepsilon}{6}\right\}$ and let $\delta < \frac{\alpha}{3}$.

Suppose that $\mathrm{D}^{\max}_{Y,m+1}(\pi(\mathbf{x})) < \delta$ and let $n \in T(\mathbf{x})$ be arbitrary. For any $i \in \{1,\ldots,m+1\}$ we have $\varphi^n(x_i) \in U$. So there is $j_i \in \{1,\ldots,m\}$ and $y_i \in K$ such that $d(\varphi^n(x_i),\gamma_{j_i}(y_i)) < \rho$. Clearly $d_Y(y_i,\psi^n(\pi(x_i))) < \rho < \frac{\alpha}{3}$. Further $d_Y(\psi^n(\pi(x_i)),\psi^n(\pi(x_j))) < \delta < \frac{\alpha}{3}$. Invariance and triangle inequality together imply that $d_Y(y_i,y_j) < \alpha$. So for any $k \in \{1,\ldots,m\}$ we have $d(\gamma_k(y_i),\gamma_k(y_j)) < \frac{\varepsilon}{6}$.

Note that we have a map $i \mapsto j_i$ that goes from $\{1,\ldots,m+1\}$ to $\{1,\ldots,m\}$. Therefore, the pigeon hole principle implies the existence of indices $a \neq b$ from $\{1,\ldots,m+1\}$ such that

$j_a = j_b$. We write $k = j_a$ for that common value and obtain

$$d(\varphi^n(x_a), \varphi^n(x_b)) \leqslant d(\varphi^n(x_a), \gamma_k(y_a)) + d(\gamma_k(y_a), \gamma_k(y_b)) + d(\gamma_k(y_b), \varphi^n(x_b))$$
$$< \rho + \frac{\varepsilon}{6} + \rho < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.$$

So for any $n \in T(\mathbf{x})$ there are indices $a \neq b$ such that $d(\varphi^n(x_a), \varphi^n(x_b)) < \frac{\varepsilon}{2}$ and therefore

$$\mathrm{D}_{m+1}(\varphi^n(\mathbf{x})) < \frac{\varepsilon}{2}.$$

This now implies $\kappa(\mathbf{x}) \leq \frac{\varepsilon}{2}$ as required and proves the mean $(m+1)$-equicontinuity of (X, φ).

For the converse direction, assume that (X, φ) is factor mean $(m+1)$-equicontinuous and suppose for a contradiction that it is not a k:1 topomorphic extension of its MEF (Y, ψ) for some $k \leq m$. Then, by Proposition 3.1(iii), there exists some φ-invariant measure μ such that $\sharp \operatorname{supp}(\mu) \geq m+1$ ν-almost surely. If we consider the $(m+1)$-fold joining $\hat{\mu}$ over the factor π given by $\hat{\mu}_y = \bigotimes_{i=1}^{m+1} \mu_y$, we have that

$$\hat{\mu}\left(X^{m+1} \setminus W_{m+1}\right) > 0,$$

where $W_{m+1} = \{\mathbf{x} \in X^{m+1} \mid \exists i \neq j : x_i = x_j\}$.

Due to the Ergodic Decomposition Theorem, there also exist an ergodic joining $\tilde{\mu}$ with this property. However, as $\mathrm{D}_{m+1}(\mathbf{x}) > 0$ on $X^{m+1} \setminus W_{m+1}$, we obtain $\int_{X^{m+1}} \mathrm{D}_{m+1}(\mathbf{x})\, d\tilde{\mu}(\mathbf{x}) > 0$ and hence $\overline{\mathrm{D}}^{\varphi}_{m+1}(\mathbf{x}) > 0$ $\tilde{\mu}$-almost surely. This shows the existence of points $(x_1, \ldots, x_{m+1}) \in X^{m+1}$ with the property that $\pi(x_1) = \ldots = \pi(x_{m+1})$ and $\overline{\mathrm{D}}^{\varphi}_{m+1}(x_1, \ldots, x_{m+1}) > 0$, contradicting the factor mean $(m+1)$-equicontinuity. □

Remark 5.2. The above proof follows the same overall strategy as the proof of the equivalence of mean equicontinuity and a topo-isomorphic extension structure in [DG16], with the necessary modifications for the multivariate case. However, the statement of Theorem 5.1 is weaker in the sense that we have to replace mean m-equicontinuity by factor mean m-equicontinuity. The reason for this is the following.

In [DG16], the direction from mean equicontinuity to the extension structure is proved by a contradiction argument, showing that any system that is not topo-isomorphic to its MEF is mean sensitive. The core part of the argument is to show the existence of a tuple $\mathbf{x} \in X^{m+1}$, located in some fibre $\pi^{-1}(y)$, such that $\overline{\mathrm{D}}^{\varphi}_m(\mathbf{x}) > 0$. This is done in a similar way in the proof above. What would be missing in order to show mean m-sensitivity (and not just lack of factor mean $(m+1)$-equicontinuity) is to prove that this tuple $\mathbf{x}$ can be found within an arbitrarily small ball. However, in the case $m = 1$ this comes for free, since it is known that $\pi(x_1) = \pi(x_2)$ implies $d_{\mathsf{B}}(x, y) = 0$ and therefore $\inf_{n \in \mathbb{Z}} d(\varphi^n(x_1), \varphi^n(x_2)) = 0$ in mean equicontinuous systems. Thus, one may simply replace x_1 and x_2 by suitable iterates in order to bring the two Besicovitch-separated points arbitrarily close.

When $m \geq 2$, however, the analogous statement is not true anymore. The fact that $\pi(x_1) = \ldots = \pi(x_{m+1})$ does not imply $\inf_{n \in \mathbb{N}} \mathrm{D}^{\max}_m(\varphi^n(\mathbf{x})) = 0$. In fact, a careful analysis of the examples in [HJ] reveals that the m:1 topomorphic extensions of irrational rotations constructed there allow no $(m+1)$-tuples in a single fibre that satisfy $\inf_{n \in \mathbb{N}} \mathrm{D}^{\max}_{m+1}(\varphi^n(\mathbf{x})) = 0$ and $\overline{\mathrm{D}}^{\varphi}_{m+1}(\mathbf{x})) = 0$ at the same time.

Altogether, this means that any argument allowing to prove Conjecture 1.2 needs to be substantially different and must take into account the behaviour of points across different fibres. We leave this problem open here.

Remark 5.3. Given Theorem 5, the validity of Conjecture 1.2 would imply that in the minimal case factor mean m-equicontinuity and mean m-equicontinuity are equivalent. We note that this is not true in the general (non-minimal) case. Simple counterexamples are given by Morse-Smale systems on the unit interval: suppose that $f : [0,1] \to [0,1]$ is a homeomophism with a finite number of fixed points $0 = x_0 < x_1 < \ldots < x_k = 1$ and $\lim_{n \to \infty} f^n(x) = x_i$ holds for all $x \in (x_{i-1}, x_i]$ and $i = 1, \ldots, k$. Then it is easy to check that the MEF of $([0,1], f)$ is trivial (a singleton $Y_0 = \{0\}$), the system is a k:1 topomorphic extension of Y_0, but it is always mean 3-equicontinuous.

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