Strange non-chaotic attractors and non-smooth saddle-node bifurcations

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A quasiperiodically forced (qpf) interval or circle map is a continuous map \( T : \mathbb{T}^1 \times X \to \mathbb{T}^1 \times X \) (with \( X \) either a compact interval \([a,b]\) or the circle \( \mathbb{T}^1 \), respectively) which has the following skew-product structure:

\[
T(\theta, x) = (\theta + \omega, T_\theta(x))
\]

where \( \omega \in \mathbb{T}^1 \) is irrational and \( T_\theta(x) := \pi_2 \circ T(\theta, x) \) are called the fiber maps. An invariant graph is a function \( \varphi : \mathbb{T}^1 \to X \) that satisfies \( T_\theta(\varphi(x)) = \varphi(\theta + \omega) \) \( \forall \theta \in \mathbb{T}^1 \).

One phenomenon which has evoked considerable interest in qpf systems in theoretical physics is the existence of non-continuous invariant graphs with negative Lyapunov exponents,\(^1\) which were called ‘strange non-chaotic attractors’ (SNA) due to their combination of non-chaotic dynamics with a fractal geometry (see e.g. [1]–[5]). However, despite the vast amount of numerical results, the only examples where the existence of these objects was shown rigorously so far are so-called pinched skew-products - a simple type of model systems introduced in [1] (see also [6]) - and quasiperiodic SL(2\( \mathbb{R} \))-cocycles, with the particular case of quasiperiodic Schrödinger cocycles. For the later, Herman described in [7], amongst other results, how at the top of the spectrum of the associated almost-Mathieu operator SNA’s are created by the collision of a stable and an unstable invariant curve. Unfortunately, these arguments depend crucially on the linear structure of the cocycles and cannot be extended to other types of qpf systems, where numerical studies indicate that such ‘non-smooth saddle-node bifurcations’ are quite common as well (e.g. [5]).

In [8] the parameter family

\[
(\theta, x) \mapsto (\theta + \omega, \arctan(\alpha x) - \beta \cdot (1 - \sin(\pi \theta)))
\]

is studied,\(^2\) and a purely dynamical proof for the existence of SNA created in non-smooth saddle-node bifurcations is derived. The results can be summarized as follows:

**Theorem 1** Suppose \( \omega \) is diophantine and \( \alpha \) is sufficiently large (depending on \( \omega \)). Then there exists a unique bifurcation parameter \( \beta_0 \), such that for \( \beta < \beta_0 \) the system (2) has exactly three invariant graphs, all of which are continuous, and for \( \beta > \beta_0 \) there exists only one (continuous) invariant graph. At \( \beta = \beta_0 \) there exist three invariant graphs \( \varphi^- < \psi \leq \varphi^+ \), of which only \( \varphi^- \) is continuous. \( \lambda(\varphi^+) \) is negative, such that \( \varphi^+ \) is an SNA, whereas \( \lambda(\psi) > 0 \). Further, the topological closures of the point sets corresponding to \( \psi \) and \( \varphi^+ \) are equal.

In the following, we shall briefly describe the dynamical mechanism responsible for such non-smooth bifurcations, and which determines the strategy for the proof of the above

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\(^1\)Given that the fiber maps are all differentiable, the Lyapunov exponent of an invariant graph is defined as \( \lambda(\varphi) := \int_0^1 \log T_\theta'(\varphi(\theta)) \, d\theta \).

\(^2\)In fact, the same results hold under more general conditions. In particular the functions \( F(x) = \arctan \alpha x \) and \( g(\theta) = (1 - |\sin(\pi \theta)|) \) in (2) could be replaced by \( C^1 \)-distortions.
result in [8]. In order to do so, we will concentrate on the behaviour of the upper invariant graph $\varphi^+$ as the parameter $\beta$ is varied and approaches $\beta_0$ from below (see Figure 1): The characteristic pattern which precedes the collision in Figure 1 seems to be quite general and can be observed similarly in many other parameter families, including quasiperiodic Schrödinger cocycles. (See [8] for more examples.) Figure 2 gives an heuristic explanation for this behaviour.

![Figure 1](image1.png)

Figure 1: The upper invariant graphs in the parameter family given by (2) with $\omega$ the golden mean and $\alpha = 10$. In (a) ($\beta = 1$) the invariant graph has a global minimum, which we refer to as the first ‘peak’. As $\beta$ is increased to 1.4, a second peak appears (b), which is steeper and grows faster than the first one. More peaks appear in (c) $\beta = 1.47$, (d) $\beta = 1.48$, (e) $\beta = 1.4808$ and (f) $\beta = 1.48095$. This pattern continues all up to the collision.

![Figure 2](image2.png)

Figure 2: For large $\alpha$, the map $F: x \mapsto \arctan(\alpha x)$ is strongly s-shaped. Thus, a small interval $I_e$ around the repelling fixed point $x = 0$ is highly expanded, whereas an interval $I_c$ above is strongly contracted (a). For the skew-product system, this gives rise to an expanding region $E$ around the 0-line and a contracting region $C$ above. Likewise, there is a contracting region below the 0-line, but as the forcing in (2) acts only downwards this is always a trapping region, and any interesting interaction takes place between $E$ and $C$. As long as the forcing parameter $\beta$ is not too large, the contracting region is still mapped inside itself, such that $\bigcap_{n \in \mathbb{N}} T^n(C)$ is the point set of an attracting and continuous invariant graph inside $T(C)$ (b). Thus, the this upper invariant graph will be close to the line $x_c - \beta(1 - |\sin(\pi \theta)|)$, where $x_c$ is the upper fixed point of $F$, and consequently it has a first peak corresponding to the maximum of the forcing function. As the forcing parameter is increased, the tip of this first peak will eventually enter the expanding region. At this point it will induce a second peak, which is steeper and thinner than the first one and moves faster by a factor corresponding to the expansion constant in $E$. As soon as this second peak reaches the expanding region as well, it induces a third one and so on . . . . Hence, it is not hard to imagine that this process gives rise to a non-continuous limit object as the bifucation parameter is approached.

When this basic idea is translated into a rigorous proof of Theorem 1, a crucial role is played by so-called ‘sink-source orbits’ — orbits that have a positive Lyapunov exponent both forwards and backwards in time. The existence of such a-typical orbits
is already known from quasiperiodic Schrödinger cocycles, where it is equivalent to Anderson localization of the associated operators. Now the same phenomenon is found in a purely dynamical setting. Sink-source orbits imply the existence of SNA (see Thm. 3.5 in [8]), such that the non-smoothness of the bifurcation in Theorem 1 is obtained as consequence of the following

Lemma 2 If \( \omega \) is diophantine and \( \alpha \) is sufficiently large (depending on \( \omega \)), then there exists a parameter \( \beta_0 \) such that (2) with \( \beta = \beta_0 \) has a sink-source orbit. (Of course, this \( \beta_0 \) is the same as in Theorem 1.

The described approach also allows to prove the existence of SNA in symmetric systems such as \((\theta, x) \mapsto (\theta + \omega, \arctan(\alpha x) + \beta(1 - 4d(\theta, 0)))\). Due to the inherent symmetry \( T_\theta(-x) = -T_\theta(x) \), any collision of invariant curves must involve three curves at the same time, such that in this situation there exist two SNA which embrace a non-continuous invariant graph with positive Lyapunov exponent. Again, the topological closures of all three objects coincide.

Finally, it should be mentioned that the results presented here should only be understood as a first step towards more general criteria which ensure the existence of strange non-chaotic attractors in a great variety of parameter families. Concerning this goal, great impetus should also be expected from recent work by Bjerklöv ([9] and [10]) who uses similar ideas in order to study quasiperiodic Schrödinger cocycles. As his approach is purely dynamical and does not really make use of the specific linear structure of cocycles, it should allow to treat more general cases in almost the same way.

References


