

Monotonicity of maximal equicontinuous factors and an application to toral flows

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November 15, 2017

Abstract

We show that for group actions on locally connected spaces the maximal equicontinuous factor map is always monotone, that is, the preimages of single points are connected. As an application, we obtain that if the maximal continuous factor of a homeomorphism of the two-torus is minimal, then it is either (i) an irrational translation of the two-torus, (ii) an irrational rotation on the circle or (iii) the identity on a singleton.

1 Introduction

The notion of maximal equicontinuous factor (MEF) is a key concept in topological dynamics and plays a significant role for the understanding, description and classification of many important system classes, including Toeplitz flows and other almost automorphic minimal subshifts in symbolic dynamics [Aus88, Dow05] or model sets and their associated Delone dynamical systems in aperiodic order (e.g. Schlottmann [Sch00, BG13]). It is also closely related to the notion of (continuous) dynamical eigenfunctions, since the latter can be viewed as factor maps to a circle rotation and the MEF comprises all information about such equicontinuous factors.

However, despite its significance, little attention has been given to additional structural properties that the MEF and its associated factor map may inherit from the considered dynamical system or the ambient space. One such property is monotonicity, in the sense that the preimages of singletons under the factor map are connected subsets of the phase space. This often plays an important role when it comes to using information about the factor to obtain insight about the extension. A paradigmatic example in this context is Poincaré's classification of orientation-preserving circle homeomorphisms [Poi85], which ensures that if the rotation number of such a map is irrational, then it has an irrational circle rotation as a factor. In this setting, the factor map is known to be monotone, and this is crucial for deriving further information such as the unique ergodicity or the uniqueness of the minimal set of the considered circle homeomorphism (e.g. [KH97]).

Considerable efforts have been made over the last decades to extend Poincaré's approach to higher dimensions, and in particular to the two-dimensional torus [MZ89, Fra88, Fra89, KT14, BdCH16]. In particular, several conditions in terms of rotation vectors and their convergence properties have been established that ensure the existence of irrational circle rotations or irrational torus translations as factors [Jäg09, JT16, JP15, JP15]. However, in all these situations it has not been known whether these irrational factors coincide with the MEF. Likewise, the monotonicity has only been known for factor maps to irrational circle rotations, where it can be shown *'by hand'* (see [JP15]), but not in the case where the factor is an irrational translation of the torus.

It turns out that monotonicity of the maximal equicontinuous factor map can be established in considerable generality.

Theorem A. *Suppose that T is a topological group acting on a locally connected compact Hausdorff space X . Then the factor map to the MEF of (X, T) is monotone.*

The proof is based on a careful analysis the regionally proximal relation, which can be used to define the maximal equicontinuous structure relation in a constructive. The classes of the former are shown to be connected, and this carries over to the equivalence classes of the latter.

As an application, we then turn to toral flows and obtain a classification of the possible minimal equicontinuous factors.

Theorem B. *Suppose that f is a homeomorphism of the two-torus. If the MEF of (\mathbb{T}^2, f) is minimal, then it must be one of the following three:*

- (i) *an irrational translation on the two-torus;*
- (ii) *an irrational rotation on the circle;*
- (iii) *the identity on a singleton.*

Acknowledgement. *TJ has been supported by a Heisenberg professorship of the German Research Council (DFG grant OE 538/6-1).*

2 Monotonicity of the maximal equicontinuous factor

2.1 Notions and preliminaries

A *flow* is a triple (X, T, f) where X is a topological space, T is a topological group and f is a continuous map of $X \times T$ to X , such that $f(\cdot, e)$ is the identity on X (e is the identity element of T) and such that $f(f(\cdot, s), t) = f(\cdot, st)$ for $s, t \in T$. Each $t \in T$ defines a homeomorphism $f^t: X \rightarrow X$ by $f^t := f(\cdot, t)$. To simplify notation we usually write (X, T) instead of (X, T, f) and denote the action of $t \in T$ on X by f^t . A flow (X, T) is called *discrete* if $T = \mathbb{Z}$.

Let X, Y be sets and $R \subseteq X \times Y$. For $x \in X$ and $y \in Y$ define

$$[x]R := \{y' \in Y; (x, y') \in R\} \text{ and } R[y] := \{x' \in X; (x', y) \in R\}.$$

For $M \subseteq X$ and $N \subseteq Y$ define

$$[M]R := \bigcup_{x \in M} [x]R \text{ and } R[N] := R(N) := \bigcup_{y \in N} R[y].$$

For $S \subseteq Y \times Z$ define

$$RS := S \circ R := \{(x, z) \in X \times Z; \exists y \in Y: (x, y) \in R \text{ and } (y, z) \in S\}.$$

If $X = Y$ define $R^{-1} := \{(y, x); (x, y) \in R\}$.

Note that $(RS)Q = R(SQ)$, $([x]S)Q = [x](SQ)$ and $(RS)[y] = R(S[y])$ for sets W, X, Y, Z , elements $x \in X$, $y \in Y$ and relations $R \subseteq W \times X$, $S \subseteq X \times Y$ and $Q \subseteq Y \times Z$. Thus we usually omit the brackets and write shortly RSQ , $[x]SQ$, and $RS[y]$ respectively.

During this paper we will identify a function $f: X \rightarrow Y$ with its graph $\{(f(x), x); x \in X\}$. Note that this graph is considered to be a subset of $Y \times X$. Under this identification we obtain the usual notation

$$f(M) = \bigcup_{x \in M} \{y \in Y; (y, x) \in f\} = \{f(x); x \in M\}$$

for all $M \subseteq X$ and

$$f^{-1}(N) = [N]f = \bigcup_{y \in N} \{x \in X; (y, x) \in f\} = \{x \in X; f(x) \in N\}$$

for all $N \subseteq Y$.

For a metric space (X, d) denote $B_\varepsilon := \{(x, y) \in X^2; d(x, y) < \varepsilon\}$ and $\overline{B}_\varepsilon = \{(x, y) \in X^2; d(x, y) \leq \varepsilon\}$ for $\varepsilon > 0$. With the above notation we have that $B_\varepsilon[x]$ is the open ball around x with radius ε and $B_\varepsilon(M) = \bigcup\{B_\varepsilon[m]; m \in M\}$ for $M \subseteq X$.

Let X be a compact Hausdorff space. Then the collection \mathcal{U}_X of all neighbourhoods of the diagonal $\Delta \subseteq X \times X$ is a uniformity which generates the topology of X . Furthermore \mathcal{U}_X is the only uniformity that is compatible with the topological structure of X .

If S is an equivalence relation on a set X , denote by X/S the corresponding partition by equivalence classes. If S is an equivalence relation on a topological space X , then the *quotient space* X/S is equipped with the *quotient topology*, i.e. the finest topology on X/S such that the *factor map* $\pi_S: X \rightarrow \mathcal{P}: x \mapsto S[x]$ is continuous. The following lemma can be found in [Dav07, Proposition 1].

Lemma 2.1. *Let X be a topological space and S an equivalence relation on X such that all elements of X/S are closed. The following statements are equivalent.*

- (i) *For each $M \in X/S$ and each open $U \subseteq X$ with $M \subseteq U$ there exists an open set $V \subseteq X$ with $M \subseteq V$ and such that whenever $N \in X/S$ satisfies $N \cap V \neq \emptyset$ we have $N \subseteq U$.*
- (ii) *For every open subset $U \subseteq X$ the set $U^* := \bigcup\{M \in X/S; M \subseteq U\}$ is open.*
- (iii) *The projection mapping π_S is a closed mapping, i.e. π_S maps closed sets to closed sets.*

An equivalence relation S is called *upper semicontinuous*, if it satisfies one of the properties in Lemma 2.1 and if every $M \in X/S$ is a compact subset of X . A binary relation R on a topological space X is called *closed*, if it is closed as a subset of $X \times X$. The next Proposition shows that on compact Hausdorff spaces the notions of closed equivalence relations and upper semi-continuous equivalence relations coincide. We include a proof for the convenience of the reader.

Proposition 2.2. *Let X be a compact Hausdorff space and S an equivalence relation on X . The following statements are equivalent.*

- (i) *S is an upper semicontinuous.*
- (ii) *S is closed.*
- (iii) *X/S is a compact Hausdorff space.*

Proof. The equivalence of (ii) and (iii) is shown in [Rot88, Theorem 8.2.]. To obtain that (i) implies (iii) let S be upper semi-continuous. Let $M, N \in X/S$. Since X is Hausdorff and M resp. N are compact in X , these sets are also closed in X . Since compact Hausdorff spaces are normal¹ there are disjoint open sets U and V in X such that $M \subseteq U$ and $N \subseteq V$. Let $\tilde{U} := \{M' \in X/S; M' \subseteq U\}$ and define \tilde{V} analogously. By property (ii) of Lemma 2.1 the sets $\pi_S^{-1}(\tilde{U}) = U^*$ and $\pi_S^{-1}(\tilde{V}) = V^*$ are open in X . Thus \tilde{U} and \tilde{V} are open in X/S . Since \tilde{U} and \tilde{V} are disjoint neighbourhoods of M resp. N we have proven X/S to be Hausdorff.

To show the implication from (iii) to (i) assume X/S to be a compact Hausdorff space, i.e. S to be closed, and note that by Lemma 2.1 it is sufficient to show that every $M \in X/S$ is compact and that π_S is a closed map. Every $M \in X/S$ is of the form $S[x] = \pi_1(S \cap (X \times \{x\}))$ for some $x \in M$ and therefore closed. Here π_1 denotes the projection to the first coordinate. Since X is compact we have shown X/S to consist of compact sets. Further π_S is a closed mapping as a continuous function from a compact Hausdorff space to a Hausdorff space. \square

If X and Y are topological spaces, we call $R \subseteq X \times Y$ *monotone*, if $[x]R$ is particular we call a function $f: X \rightarrow Y$ to be monotone if $f^{-1}(\{y\}) = [y]f$ is connected for every $y \in Y$. Furthermore an equivalence relation S on a topological space X is monotone if and only if π_S is monotone, since $\pi_S^{-1}(\{x\}) = S[x] = [x]S$ for every $x \in X$.

If S is an equivalence relation on a topological space X , then the collection of all $M \subseteq X$ such that M is a connected component of some $N \in X/S$ is a partition of X into connected sets. We call the corresponding equivalence relation S_{mon} the *monotone refinement relation*

¹A topological space X is called *normal*, if every two disjoint closed sets of X have disjoint open neighborhoods.

of S . In [Dav07, I.4.] it is shown, that the monotone refinement relation S_{mon} of an upper semicontinuous equivalence relation S on a Hausdorff space X is upper semicontinuous. Proposition 2.2 therefore implies the following.

Proposition 2.3. *If X is a compact Hausdorff space and S is a closed equivalence relation on X , then the monotone refinement relation of S is also closed.*

If (X, T) is a flow, a binary relation R on X is called T -invariant, if for all $x \in X$ and all $t \in T$ we have $f^t([x]R) \subseteq [f^t(x)]R$. Note that this is the case if and only if $f^t([x]R) = [f^t(x)]R$ for all $x \in X$ and all $t \in T$.

If $\pi: X \rightarrow Y$ is a factor map between the flows (X, T) and (Y, T) . Then $\{\pi^{-1}(\{y\}); y \in Y\}$ is a partition of X and the corresponding equivalence relation S is closed and T -invariant. Furthermore X/S and Y are isomorphic. If S is a closed and T -invariant equivalence relation, then $(X/S, T)$ is a factor of (X, T) . We will therefore write factors of a flow (X, T) as $(X/S, T)$, where S is a closed and T -invariant equivalence relation.

2.2 Monotonicity of maximal equicontinuous factors

The *regionally proximal relation* is defined as $S_{rp} = \bigcap_{\alpha \in \mathcal{U}_X} \overline{\alpha T}$, where $\alpha T := \bigcup_{t \in T} f^t \alpha f^{(t^{-1})}$ (see [Aus88]).

Theorem 2.4 ([Aus88, Chapter 9, Theorems 1 and 3]). *If (X, T) is a flow on a compact Hausdorff space, there is a smallest closed T -invariant equivalence relation S_{eq} such that the quotient flow $(X/S_{eq}, T)$ is equicontinuous. Furthermore S_{eq} is the smallest closed and T -invariant equivalence relation which contains the regionally proximal relation S_{rp} .*

We call S_{eq} the *equicontinuous structure relation* of the flow (X, T) . Furthermore the factor flow $(X/S_{eq}, T)$ is called the *maximal equicontinuous factor* of (X, T) .

Remark 2.5. For a large class of minimal flows (which include those for which the acting group is abelian, as well as point distal flows) the regionally proximal relation S_{rp} is an equivalence relation. Since it is always closed invariant, it coincides with the equicontinuous structure relation S_{eq} in these cases [Aus88, p.141].

In this section will show that S_{rp} and S_{eq} are monotone, whenever (X, T) is a flow on a compact and locally connected Hausdorff space. Hence, the respective factor map $\pi_{S_{eq}}$ is monotone as well. As S_{rp} is reflexive the following Proposition shows that it is sufficient to prove that S_{rp} is monotone.

Proposition 2.6. *Let (X, T) be a flow on a compact Hausdorff space and S a reflexive and monotone binary relation on X . Let \hat{S} be the smallest closed T -invariant equivalence relation that contains S . Then \hat{S} is monotone.*

Proof. We will show that \hat{S} equals its monotone refinement relation \hat{S}_{mon} . Clearly $\hat{S}_{mon} \subseteq \hat{S}$. Since \hat{S} is the intersection over all closed and T -invariant equivalence relations R such that $S \subseteq R$ it is left to show that \hat{S}_{mon} is closed, T -invariant and $S \subseteq \hat{S}_{mon}$.

Since \hat{S} is a closed equivalence relation, Proposition 2.3 implies that \hat{S}_{mon} is closed. In order to show \hat{S}_{mon} to be T -invariant, let $x \in X$ and $t \in T$. Since $f^t: \hat{S}[x] \rightarrow f^t(\hat{S}[x])$ is a homeomorphism and \hat{S} is T -invariant we know that $f^t(\hat{S}_{mon}[x])$ is a connected component of $f^t(\hat{S}[x]) = \hat{S}[f^t(x)]$. Note that $\hat{S}_{mon}[f^t(x)]$ is the connected component of $\hat{S}[f^t(x)]$ that contains $f^t(x)$. Thus $f^t(\hat{S}_{mon}[x]) = \hat{S}_{mon}[f^t(x)]$ and we have shown \hat{S}_{mon} to be T -invariant.

To show $S \subseteq \hat{S}_{mon}$ let $x \in X$. The reflexivity of S implies $x \in S[x] \subseteq \hat{S}[x]$. Since $\hat{S}_{mon}[x]$ is the connected component of $\hat{S}[x]$ that contains x and $S[x]$ is a connected subset of $\hat{S}[x]$ that contains x , it follows that $S[x] \subseteq \hat{S}_{mon}[x]$. \square

We now split up the proof of the fact that the regionally proximal relation S_{rp} of a flow (X, T) on a locally connected and compact Hausdorff space is monotone into several lemmas.

Lemma 2.7. *Let X be a topological space and $R \subseteq X \times X$ reflexive and monotone.*

(i) *If M is a nonempty and connected subset of X , then $[M]R$ is connected.*

(ii) If $f: X \rightarrow X$ is a homeomorphism, then fRf^{-1} is monotone.

(iii) If $(R_\iota)_{\iota \in I}$ is a family of reflexive and monotone binary relations on X , then $\bigcup_{\iota \in I} R_\iota$ is monotone.

Proof. To show (i) let $x \in M$. The set $[x]R \cup M$ is a union of connected sets with nonempty intersection. Furthermore for each $x \in M$ we have $\emptyset \neq M \subseteq [x]R \cup M$. Thus $[M]R = \bigcup_{x \in M} [x]R = \bigcup_{x \in M} ([x]R \cup M)$ is connected as an union of connected sets with nonempty intersection.

To show (ii) let $x \in X$ and note that $[x]f = f^{-1}(\{x\})$ is connected as the continuous image of a connected set. Thus (i) implies $[x](fR) = [[x]f]R$ to be connected. As f is continuous we obtain the image $[x](fRf^{-1}) = ([x](fR))f^{-1} = f([x](fR))$ to be connected.

For $x \in X$ and $\iota \in I$ we obtain $x \in [x]R_\iota$ and that $[x]R_\iota$ is connected. Thus $[x](\bigcup_{\iota \in I} R_\iota) = \bigcup_{\iota \in I} ([x]R_\iota)$ is connected as a union of connected sets with non empty intersection. This proves (iii). \square

Lemma 2.8. *Let X be a locally connected and compact Hausdorff space and \mathcal{U}_X the unique uniformity on X . Then every $\alpha \in \mathcal{U}_X$ contains a monotone $\beta \in \mathcal{U}_X$.*

Proof. Let $\alpha \in \mathcal{U}_X$. Note that \mathcal{U}_X is the set of all neighbourhoods of the diagonal $\Delta \subseteq X \times X$. Thus for $x \in X$ there is an open neighbourhood U_x of x with $U_x \times U_x \subseteq \alpha$. Since X is locally connected there is a connected neighbourhood V_x of x such that $V_x \subseteq U_x$. Let $\beta := \bigcup_{x \in X} V_x \times V_x$. Clearly $\beta \subseteq \alpha$ and β is a neighbourhood of Δ , i.e. $\beta \in \mathcal{U}_X$. To show that β is monotone let $x \in X$ and note that

$$[x]\beta = \{z \in X; \exists y \in X : x, z \in V_y\} = \bigcup \{V_y; y \in X \text{ and } x \in V_y\}$$

is a union of connected subsets of X whose intersection is nonempty since it contains x . \square

The following lemma is a straightforward generalisation of the Cantor intersection theorem for countable intersections.

Lemma 2.9. *Let X be a compact Hausdorff space and $(X_\alpha)_{\alpha \in A}$ be a decreasing net of nonempty and closed subsets of X such that for each $\alpha \in A$ there is $\beta \in A$ with $\beta \geq \alpha$ and such that X_β is connected. Then $\bigcap_{\alpha \in A} X_\alpha$ is nonempty and connected.*

Applying the previous Lemma, we obtain the following results about the closure and the intersection of monotone equivalence relations.

Lemma 2.10. *Let X be a compact Hausdorff space.*

(i) *If X is locally connected and R is a monotone and reflexive relation on X , then the closure \overline{R} of R in $X \times X$ is also monotone.*

(ii) *If $(R_\alpha)_{\alpha \in A}$ is a net of closed and reflexive binary relations on X such that for all $\alpha \in A$ there is $\beta \in A$ with $\alpha \leq \beta$ and such that R_β is monotone, then $\bigcap_{\alpha \in A} R_\alpha$ is monotone.*

Proof. Note that $\overline{R} = \bigcap_{(\alpha, \beta) \in \mathcal{U}_X^2} \alpha R \beta$. For $x \in X$ we obtain

$$[x]\overline{R} = \bigcap_{(\alpha, \beta) \in \mathcal{U}_X^2} ([x]\alpha R \beta) = \bigcap_{\alpha \in \mathcal{U}_X} \overline{[x]\alpha R}.$$

Set $X_\alpha = \overline{[x]\alpha R}$ and order \mathcal{U}_X by reversed set inclusion to obtain a decreasing net $(X_\alpha)_{\alpha \in \mathcal{U}_X}$ of nonempty and closed subsets of X . For $\alpha \in \mathcal{U}_X$ there is a monotone $\gamma \in \mathcal{U}_X$ such that $\gamma \subseteq \alpha$ by Lemma 2.8. Thus $[x]\gamma$ is connected and we find $X_\gamma = \overline{[x]\gamma R} = \overline{[[x]\gamma]R}$ to be connected by Lemma 2.7(i). Lemma 2.9 therefore applies to $(X_\alpha)_{\alpha \in \mathcal{U}_X}$ and we obtain that $[x]\overline{R} = \bigcap_{\alpha \in \mathcal{U}_X} X_\alpha$ is connected. This shows (i).

To see (ii) let $x \in X$ and note that $[x](\bigcap_{\alpha \in A} R_\alpha) = \bigcap_{\alpha \in A} ([x]R_\alpha)$. Set $X_\alpha := [x]R_\alpha$ for $\alpha \in A$. This defines a decreasing net $(X_\alpha)_{\alpha \in A}$ of non empty and closed subsets of X . For $\alpha \in A$ there is $\beta \in A$ with $\alpha \leq \beta$ and such that R_β is monotone. Thus $X_\beta = [x]R_\beta$ is connected and Lemma 2.9 yields $[x](\bigcap_{\alpha \in A} R_\alpha) = \bigcap_{\alpha \in A} X_\alpha$ to be connected. \square

Proposition 2.11. *Let (X, T) be a flow on a compact and locally connected Hausdorff space. Then the regionally proximal relation S_{rp} is monotone.*

Proof. Note that $S_{rp} = \bigcap_{\alpha \in \mathcal{U}_X} \overline{\alpha T}$. Denote $R_\alpha = \overline{\alpha T} = \overline{\bigcup_{t \in T} f^t \alpha f^{(t^{-1})}}$ for $\alpha \in \mathcal{U}_X$ and order \mathcal{U}_X by reversed inclusion to obtain a decreasing net $(R_\alpha)_{\alpha \in \mathcal{U}_X}$ of closed and reflexive binary relations on X . For $\alpha \in \mathcal{U}_X$ there is a monotone $\beta \in \mathcal{U}_X$ such that $\beta \subseteq \alpha$. For $t \in T$ we know f^t to be a homeomorphism. Lemma 2.7(ii) therefore implies $f^t \beta f^{(t^{-1})}$ to be monotone. Note that $f^t \beta f^{(t^{-1})}$ is reflexive. Lemma 2.7(iii) and Lemma 2.10(i) therefore yield $R_\beta = \overline{\bigcup_{t \in T} f^t \beta f^{(t^{-1})}}$ to be monotone. We have shown that for $\alpha \in \mathcal{U}_X$ there is $\beta \in \mathcal{U}_X$ such that $\beta \subseteq \alpha$ and such that R_β is monotone. Thus Lemma 2.10(ii) can be applied and shows S_{rp} to be monotone. \square

Combining Proposition 2.6 and Proposition 2.11 we obtain the following Theorem.

Theorem 2.12. *The maximal equicontinuous factor $(X/S_{eq}, T)$ of a flow (X, T) on a compact and locally connected Hausdorff space has a monotone factor map, i.e. the equicontinuous structure relation S_{eq} is monotone.*

3 On monotone factors of flows on the torus

Our main application of Theorem 2.12 is the following.

Theorem 3.1. *If the maximal equicontinuous factor of a homeomorphism of \mathbb{T}^2 is minimal, then it is either conjugate to a flow on \mathbb{T}^2 , conjugate to an irrational circle rotation, or conjugate to a flow on a single point.*

3.1 Notions and preliminaries

A flow (X, T) is called *effective*, if for all $t \in T$ not being the identity element the mapping f^t is not the identity on X . A flow (X, T) is said to be *strongly effective*, if for all $t \in T$ not being the identity element the mapping f^t is fix point free. Note that every strongly effective flow is effective and that the reverse is valid for minimal flows, as shown in [Aus88, Chapter 1].

If (X, T) is a flow, then the set F of all $t \in T$ such that f^t is the identity on X is a closed and normal subgroup. Thus $G := T/F$ defines an action on X by $f^{[t]}(x) := f^t(x)$ for $[t] \in G$ and $x \in X$. The resulting flow (X, G) is effective and G is called the *effective subgroup*.

If (X, T) is a flow and (Y, T) is a factor with the factor map $\pi: X \rightarrow Y$, then (Y, T) is said to be *relatively effective* with respect to (X, T) , if for all $t \in T$ we have that f^t is the identity on X , whenever f^t is the identity on Y . Note that (Y, T) is relatively effective with respect to (X, T) if and only if the effective subgroups of (X, T) and (Y, T) coincide, i.e. $t \in T$ acts as the identity on X , whenever t acts as the identity on Y . Thus if (Y, T) is relatively effective with respect to (X, T) these flows can be considered to be effective simultaneously, by going over to the effective subgroup of T if necessary. Note furthermore, that every effective factor of a flow is relatively effective.

Lemma 3.2. *If $(X/S, T)$ is a strongly effective factor of a flow (X, T) , then (X, T) is strongly effective.*

Proof. If $t \in T$ and x is a fix point of f^t in X , then $S[x]$ is a fix point of f^t in Y . \square

During this section denote by $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the two-torus and by $p: \mathbb{R}^2 \rightarrow \mathbb{T}^2$ the canonical projection. A subset $M \subseteq \mathbb{T}^2$ is called *bounded*, if every connected component of $p^{-1}(M)$ is bounded. Furthermore a subset $M \subseteq \mathbb{T}^2$ is called *unbounded*, if M is not bounded. An open subset $U \subseteq \mathbb{T}^2$ is called *doubly-essential*, if U contains two homotopically non trivial curves which are not homotopic to each other. An open subset $U \subseteq \mathbb{T}^2$ is called *inessential*, if all curves in U are homotopically trivial. Furthermore an open subset $U \subseteq \mathbb{T}^2$ is called *essential*, if U is not doubly essential and not inessential. Note that an open subset $U \subseteq \mathbb{T}^2$

is essential, if and only if U contains a homotopically non trivial curve γ and every other curve is either homotopically trivial or homotopic to γ . A closed subset $M \subseteq \mathbb{T}^2$ is called *inessential*, if $\mathbb{T}^2 \setminus M$ is doubly essential; *essential*, if $\mathbb{T}^2 \setminus M$ is essential; and *doubly essential*, if $\mathbb{T}^2 \setminus M$ is inessential. It is well-known that if M is a continuum², then M is bounded if and only if M is inessential.

A subset $M \subseteq \mathbb{T}^2$ is called a *topological disc*, if it is homeomorphic to the open disc D° , and a *topological annulus*, if it is homeomorphic to the open annulus $\mathbb{R} \times S^1$. Furthermore a continuum $M \subseteq \mathbb{T}^2$ is called an *annular continuum*, if its complement $\mathbb{T}^2 \setminus M$ is a topological annulus. It is a well known fact that a continuum M is an annular continuum if and only if M is essential and non-separating³ and that every open and simply connected subset $U \subseteq \mathbb{T}^2$ is a topological disc (as a consequence of the Riemann Mapping Theorem). The following version of the classical Moore Theorem can be found in [JKP13, Section 3]. See also [Dav07].

Theorem 3.3. (Moore) *If S an upper semi-continuous and monotone equivalence relation on the torus \mathbb{T}^2 such that each $M \in X/S$ is non separating and inessential, then \mathbb{T}^2/S is homeomorphic to \mathbb{T}^2 .*

If we assume S to be an upper semi-continuous and monotone equivalence relation, we obtain X/S to be a compact and connected metric space. If each $M \in \mathbb{T}^2/S$ is non separating and essential, i.e. an annular continuum, then the connected components of $\mathbb{T}^2 \setminus (M \cup N)$ are two topological annuli, whenever $M, N \in \mathbb{T}^2/S$ with $M \neq N$. Thus $(\mathbb{T}^2/S) \setminus \{M, N\}$ has two connected components. In [HoYo61, Theorem 2-28] it is shown that a compact and connected metric space X is homeomorphic to S^1 if and only if for all points $x, y \in X$ with $x \neq y$ the complement $X \setminus \{x, y\}$ has exactly two connected components. This proves the following proposition.

Proposition 3.4. *If S an upper semi-continuous and monotone equivalence relation on the Torus \mathbb{T}^2 such that each $M \in X/S$ is non separating and essential, then \mathbb{T}^2/S is homeomorphic to S^1 .*

The following version of Brouwers fix point theorem will be useful.

Lemma 3.5. *If $U \subseteq \mathbb{T}^2$ is a topological disc and $f: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ a homeomorphism such that $f(\bar{U}) \subseteq U$, then f has a fixed point.*

3.2 Monotone factors of toral flows

We will now show that if the quotient flow under a monotone and upper semi-continuous equivalence relation is minimal, then either all equivalence classes are bounded, or all equivalence classes are unbounded.

Lemma 3.6. *Let S be a monotone upper semi-continuous equivalence relation on the torus \mathbb{T}^2 . Then $B := \{M \subseteq \mathbb{T}^2/S; M \text{ bounded}\}$ is open in \mathbb{T}^2/S .*

Proof. Let $\pi: \mathbb{R}^2 \rightarrow \mathbb{T}^2$ be the canonical lift of the torus. For $M \in B$ denote by C_M the set of all connected components of $\pi^{-1}(M)$. Note that all sets in C_M are bounded and closed, hence compact in \mathbb{R}^2 as M is bounded in \mathbb{T}^2 and S is upper semi-continuous. For $K \in C_M$ we have $C_M = \{K + v; v \in \mathbb{Z}^2\}$, since M is connected in \mathbb{T}^2 . Thus there is $\varepsilon > 0$ such that $B_\varepsilon(K)$ and $B_\varepsilon(L)$ are disjoint for distinct $K, L \in C_M$.

Let $U := \pi(B_\varepsilon(\pi^{-1}(M)))$ and $\mathcal{V} = \{N \in \mathbb{T}^2/S; N \subseteq U\}$. Note that $\pi^{-1}(U) = B_\varepsilon(\pi^{-1}(M))$ which implies that U is open. As S is upper semi-continuous, we obtain $\bigcup_{V \in \mathcal{V}} V$ to be open in \mathbb{T}^2 , i.e. V to be open in \mathbb{T}^2/S by Lemma 2.1(ii). Note that $M \in V$. To show that $V \subseteq B$ let $N \in V$. Then $\bigcup_{L \in C_N} L = \pi^{-1}(N) \subseteq \pi^{-1}(U) = B_\varepsilon(\pi^{-1}(M)) = \bigcup_{K \in C_M} B_\varepsilon(K)$. As each $L \in C_N$ is connected there is a unique $K \in C_M$ such that $L \subseteq B_\varepsilon(K)$, which implies L to be bounded in \mathbb{R}^2 . This proves N to be bounded in \mathbb{T}^2 . \square

²A *continuum* is a compact and connected topological space.

³A subset M of a topological space X is called *non-separating*, if $X \setminus M$ has at most one connected component.

Lemma 3.7. *Let S be a monotone upper semi-continuous T -invariant equivalence relation on the flow (\mathbb{T}^2, T) . If $(\mathbb{T}^2/S, T)$ is minimal and if there exists an inessential set $M \in \mathbb{T}^2/S$, then all sets in \mathbb{T}^2/S are inessential.*

Proof. Note that a set is inessential if and only if it is bounded. As boundedness of subsets of the torus is invariant under homeomorphisms from the torus to the torus we know $U := \{M \subseteq \mathbb{T}^2/S; M \text{ unbounded}\}$ to be T -invariant. Furthermore by Lemma 3.6 we know U to be closed in \mathbb{T}^2/S . Thus the minimality shows that either $U = \emptyset$ or $U = \mathbb{T}^2/S$. If we assume the existence of an inessential set in \mathbb{T}^2/S , then $U \neq \mathbb{T}^2/S$, hence $U = \emptyset$. Thus all elements of \mathbb{T}^2/S are inessential. \square

Theorem 3.8. *Let (\mathbb{T}^2, T) be a strongly effective flow. Every monotone minimal factor $(\mathbb{T}^2/S, T)$ of (\mathbb{T}^2, T) is either conjugate to a flow on \mathbb{T}^2 , conjugate to a flow on the circle S^1 or conjugate to a flow on a single point.*

Proof. First assume that there exists a doubly essential set $M \in \mathbb{T}^2/S$. As no other element of \mathbb{T}^2/S can be doubly essential we obtain M to be a fix point of f^t for each $t \in T$. As $(\mathbb{T}^2/S, T)$ is assumed to be minimal, this implies that $M = \mathbb{T}^2$, i.e. that $\mathbb{T}^2/S = \{\mathbb{T}^2\}$ is the flow on a single point.

As a second case, we assume the existence of an inessential set in \mathbb{T}^2/S . As shown in Lemma 3.7, we obtain all sets in \mathbb{T}^2/S to be inessential. Let $M \in \mathbb{T}^2/S$. As M is inessential we know one connected component of $\mathbb{T}^2 \setminus M$ to be doubly essential and all other connected components to be inessential. Let U be an inessential connected component of $\mathbb{T}^2 \setminus M$. As U is inessential and open and M is connected, U is simply connected and thus a topological disc. Let $U^* := \{N \in \mathbb{T}^2/S; N \subseteq U\}$. As S is monotone, we obtain $U = \bigcup_{N \in U^*} N$, hence $U^* \neq \emptyset$. Since S is upper semi-continuous we have that U^* is open in \mathbb{T}^2/S . Thus the minimality of $(\mathbb{T}^2/S, T)$ implies the existence of $t \in T$ such that $f^t(M) \in U^*$, i.e. $f^t(M) \subseteq U$. Since $f^t(U)$ is inessential, it is mapped to an inessential connected component of $f^t(M)$, hence $f^t(U) \subseteq U$. As $f^t(\bar{U}) \subseteq f^t(M) \cup f^t(U) \subseteq U$, Lemma 3.5 yields the existence of a fix point of f^t , which contradicts the assumption on (\mathbb{T}^2, T) to be strongly effective. Thus there are no inessential connected components of $\mathbb{T}^2 \setminus M$ and M is non separating. We can now apply Moores Theorem (Theorem 3.3) to obtain a homeomorphism between \mathbb{T}^2/S and \mathbb{T}^2 .

For the remaining case assume that all sets in \mathbb{T}^2/S are essential. Let $M \in \mathbb{T}^2/S$. As each connected component of $\mathbb{T}^2 \setminus M$ is a union of essential sets, the complement of M cannot contain any inessential connected components. By connectedness of M , $\mathbb{T}^2 \setminus M$ cannot contain more than one essential connected component either, so M is non separating. Thus Proposition 3.4 yields \mathbb{T}^2/S to be homeomorphic to S^1 . \square

Remark 3.9. Note that the argument in the proof of Theorem 3.8 shows that one can obtain the homeomorphism class of \mathbb{T}^2/S by looking at some $M \in \mathbb{T}^2/S$. If M is inessential, then \mathbb{T}^2/S is homeomorphic to \mathbb{T}^2 and all sets in \mathbb{T}^2/S are inessential. If M is essential, then we obtain \mathbb{T}^2/S to be homeomorphic to S^1 and all sets in \mathbb{T}^2/S are essential. If M is doubly essential, then \mathbb{T}^2/S is homeomorphic to a point and $M = \mathbb{T}^2$.

If we assume the flow instead of the factor to be minimal we obtain the following.

Corollary 3.10. *Every monotone factor $(\mathbb{T}^2/S, T)$ of a minimal flow (\mathbb{T}^2, T) is either conjugate to a flow on \mathbb{T}^2 , conjugate to a flow on the circle S^1 or conjugate to a flow on a single point.*

Proof. Let G be the effective subgroup of T with respect to the flow (\mathbb{T}^2, T) . Note that (\mathbb{T}^2, G) is an effective and minimal flow, hence strongly effective. Since $(\mathbb{T}^2/S, G)$ is a factor of the minimal flow (\mathbb{T}^2, G) we obtain $(\mathbb{T}^2/S, G)$ to be minimal. As the factor map π_S is monotone, the statement follows from Theorem 3.8. \square

Combining Theorem 2.12 with Corollary 3.10 yields the following.

Corollary 3.11. *Every maximal equicontinuous factor of a minimal flow on the torus \mathbb{T}^2 is either conjugate to an equicontinuous flow on \mathbb{T}^2 , to an equicontinuous flow on the circle, or the flow on a single point.*

As another corollary of Theorem 3.8 we obtain the following.

Corollary 3.12. *Every monotone minimal relatively effective factor $(X/S, T)$ of a flow (\mathbb{T}^2, T) is either conjugate to a flow on \mathbb{T}^2 , conjugate to a flow on the circle or conjugate to a flow on a single point.*

Proof. Let G be the effective subgroup of T with respect to $(\mathbb{T}^2/S, T)$ and note that the induced flows (\mathbb{T}^2, G) and $(\mathbb{T}^2/S, G)$ are effective and $(\mathbb{T}^2/S, G)$ is a monotone minimal factor of (\mathbb{T}^2, G) . As $(\mathbb{T}^2/S, G)$ is minimal and effective, it is also strongly effective. Thus (\mathbb{T}^2, G) is strongly effective by Lemma 3.2, and Theorem 3.8 implies \mathbb{T}^2/S to be homeomorphic either to \mathbb{T}^2 , to S^1 , or to a single point. \square

If the flow is a \mathbb{Z} -action, then every minimal factor is effective as shown in the next lemma.

Lemma 3.13. *Every non trivial⁴ monotone minimal factor of a discrete flow $(\mathbb{T}^2, \mathbb{Z})$ is effective.*

Proof. If $(\mathbb{T}^2/S, \mathbb{Z})$ is a non trivial monotone minimal factor of $(\mathbb{T}^2, \mathbb{Z})$ that is not effective, then there is $n \in \mathbb{N}$ such that f^n is the identity in \mathbb{T}^2/S , hence the orbit of any $M \in X/S$ has only finitely many elements. As \mathbb{T}^2/S is assumed to be minimal this implies \mathbb{T}^2/S to have finitely many elements. But each of these elements is closed as a subset of the connected set \mathbb{T}^2 and we obtain \mathbb{T}^2/S to consist of exactly one element, a contradiction. \square

As the minimal discrete flows on the circle are the irrational circle rotations we obtain the following corollary from Lemma 3.13 and Corollary 3.12.

Corollary 3.14. *Every monotone minimal factor of a discrete flow $(\mathbb{T}^2, \mathbb{Z})$ is either conjugate to a discrete flow on \mathbb{T}^2 , conjugate to an irrational circle rotation or conjugate to a discrete flow on a singleton.*

Remark 3.15. We note that an analogous result can be obtained for homeomorphisms of the closed annulus. In this case, if the maximal equicontinuous factor is minimal, then it can only be either an irrational rotation on the circle or the identity on a singleton.

Finally, we want to close with two consequences specifically concern the dynamics of torus homeomorphisms and directly relate to current developments in this topic (compare [Jäg09, JP15, KPS16, JT16]). Equivalent conditions for the existence of a semiconjugacy from an area-preserving torus homeomorphism to an irrational rotation of the two-torus have been established [Jäg09]. However, whether the factor map needs to be monotone in this situation has not been known so far. It now follows from Theorem 2.12 and Corollary 3.11.

Corollary 3.16. *Suppose that a homeomorphism f of the two-torus is homotopic to the identity and has an irrational translation ρ of the two-torus as a factor. Further, assume that the factor map is homotopic to the identity. Then ρ is the maximal equicontinuous factor and the factor map is monotone.*

Proof. As f has a two-dimensional irrational rotation ρ as a factor, Corollary 3.11 implies that the maximal equicontinuous factor of f is a two-dimensional irrational rotation ρ' as well. If both rotations are conjugate, so that ρ is the maximal equicontinuous factor (up to conjugacy), then monotonicity follows from Theorem 2.12.

Hence, suppose for a contradiction that ρ and ρ' are not conjugate. Suppose that ρ lifts to a translation $R_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $x \mapsto x + \alpha$ and ρ' lifts to a corresponding translation R_β . Let h be factor map to ρ and h' the maximal equicontinuous factor map.

The fact that the map h is homotopic to the identity implies that the rotation number of a suitable lift $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ equals α , that is, $\lim_{n \rightarrow \infty} (F^n(x) - x)/n = \alpha$ for all $x \in \mathbb{R}^2$

⁴A factor (Y, T) of a flow (X, T) is considered to be *trivial*, if Y consists of one point.

(e.g.). Suppose that the action of h' on the first homology group of \mathbb{T}^2 , which is isomorphic to \mathbb{Z}^2 , is given by the two-by-two integer matrix $A \in M^{2 \times 2}(\mathbb{Z})$. Then a standard argument as in [Kwa92, Jäg09] yields that $\beta = A\alpha$. However, if A is invertible, then this means that ρ and ρ' are conjugate by the induced linear torus automorphism T_A induced by A . If A is not invertible, then ρ' is a factor of ρ , with factor map T_A , but conversely ρ is not a factor of ρ' . This contradicts the fact that ρ' is the maximal equicontinuous factor. \square

If a torus homeomorphism f , homotopic to the identity, is semiconjugate to a one-dimensional irrational rotation $\rho : \mathbb{T}^1 \rightarrow \mathbb{T}^1$, then a construction in [BCJ15] shows that it is possible that all fibres $h^{-1}(\xi)$ of the factor map $h : \mathbb{T}^2 \rightarrow \mathbb{T}^1$ are hereditarily indecomposable continua (so-called pseudo-circles [Bin48]) and hence have a ‘wild’ topological structure. If the maximal equicontinuous factor is two-dimensional, then this is not possible anymore.

Recall that a continuum K is *decomposable* if it is the union of two non-empty subcontinua.

Corollary 3.17. *Suppose that the maximal equicontinuous factor of a torus homeomorphism f is (conjugate to) an irrational rotation ρ of the two-torus. Further, assume that h_1 is a factor map from f to a one-dimensional irrational rotation ρ_1 . Then the fibres of h_1 are finite unions of decomposable continua. Moreover, if the action $\mathbb{Z}^2 \rightarrow \mathbb{Z}$ of h on homology is of the form $v \mapsto \langle u, v \rangle$ for some irreducible vector $u \in \mathbb{Z}^2$, then the fibres of h are decomposable continua.*

Note that due to the characterisation in [JP15], the above continua are all *annular* (the complement of a topological annulus embedded in the torus).

Proof. Let h be the factor map from f to the maximal equicontinuous factor ρ . Then $h_1 = \tilde{h} \circ h$, where \tilde{h} is a factor map from ρ to ρ_1 . However, any factor map from a two-dimensional to a one-dimensional irrational rotation is of the form $x \mapsto \langle x, u \rangle$ for some integer vector $u \in \mathbb{Z}^2 \setminus \{0\}$. By performing a linear change of coordinates, we may assume that u is horizontal, that is, $u = (m, 0)$ for some $m \in \mathbb{N}$. This means that \tilde{h} is just the composition of the projection to the first coordinate and the covering map $\xi \mapsto m\xi \bmod 1$ on the circle.

If $m = 1$, then it follows from the construction in [JP15, Proof of Theorem 1] that the fibres of $\tilde{h} \circ h = \pi_1 \circ h$ are essential annular continua. However, we have that

$$(\pi_1 \circ h)^{-1}(\xi) = h^{-1}(\{\xi\} \times \mathbb{T}^1) = h^{-1}([0, 1/2]) \cup h^{-1}([1/2, 1])$$

and the two sets on the right are subcontinua of the fibre $(\pi_1 \circ h)^{-1}(\xi)$, since preimages of connected sets under monotone maps are connected.

If $m > 1$, then fibres of $h_1 = \tilde{h} \circ h$ are finite unions of fibres of $\pi_1 \circ h$. This completes the proof. \square

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