CONSTRUCTION OF SMOOTH ISOMORPHIC AND FINITE-TO-ONE EXTENSIONS OF IRRATIONAL ROTATIONS WHICH ARE NOT ALMOST AUTOMORPHIC

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ABSTRACT. Due to a result by Glasner and Downarowicz, it is known that a minimal system is mean equicontinuous if and only if it is an isomorphic extension of its maximal equicontinuous factor. The majority of known examples of this type are almost automorphic, that is, the factor map to the maximal equicontinuous factor is almost one-to-one. The only cases of isomorphic extensions which are not almost automorphic are again due to Glasner and Downarowicz, who in the same article provide a construction of such systems in a rather general topological setting.

Here, we use the Anosov-Katok method in order to provide an alternative route to such examples and to show that these may be realised as smooth skew product diffeomorphisms of the two-torus with an irrational rotation on the base. Moreover – and more importantly – a modification of the construction allows to ensure that lifts of these diffeomorphism to finite covering spaces provide novel examples of finite-to-one topomorphic extensions of irrational rotations. These are still strictly ergodic and share the same dynamical eigenvalues as the original system, but show an additional singular continuous component of the dynamical spectrum.

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1. INTRODUCTION

The celebrated Halmos-von Neumann Theorem provides a classification, up to isomorphism, of ergodic measure-preserving dynamical systems with discrete dynamical spectrum. Moreover, any such system can be realised as a rotation on some compact abelian group [vN32, HvN42]. From the measure-theoretic viewpoint, this provides a rather complete picture for the class of dynamical systems with discrete spectrum. However, topological realisations of such systems can still show a surprising variety of different behaviours. One particular subclass that has recently attracted considerable attention are mean equicontinuous systems [LTY15, GRM15, LYY21, HLY11, GRLZ19]. In the minimal case, Downarowicz and Glasner showed that these are exactly those topological dynamical systems which are measure-theoretically isomorphic to their maximal equicontinuous factor (MEF) via the respective continuous factor map [DG16]. Such systems are called isomorphic extensions (of the MEF). Equivalently, these systems are characterised by discrete spectrum with continuous eigenfunctions. A generalisation of these results to the non-minimal case and more general group actions is provided in [FGL22]. Subsequent work has concentrated on characterising different types of mean equicontinuous systems in terms of invertibility properties of the factor map to the MEF. For instance, the fact that almost all points of a strictly ergodic system are injectivity points of the factor map, which implies mean equicontinuity, is equivalent to the stronger property of diam mean equicontinuity [GR17, GRJY21].

Examples of mean equicontinuous systems in the literature are abundant. In particular, these include the classes of regular Toeplitz flows [JK69, Wil84, MP79, Dow05] and regular model sets arising from Meyer's cut and project method [Mey72, Sch00, Sch00, Moo00, BLM07]. In both cases, the factor map is almost surely injective, so that the dynamics are diam mean equicontinuous. Examples of mean equicontinuous systems whose factor maps are not almost surely injective are given by certain irregular Toeplitz flows (e.g. [Wil84]) and irregular models sets [FGJO21]. In these cases, the systems are almost automorphic, meaning that the factor maps are almost one-to-one, i.e. the set of injectivity points is residual.

Mean equicontinuous systems for which the factor map to the MEF has no singular fibres are much more difficult to find. In fact, to the best of our knowledge, the only non-trivial examples[1] were so far given by Glasner and Downarowicz in [DG16], who showed that homeomorphisms with these properties are generic in certain spaces of extensions of minimal group rotations. One aim of this note is to provide an alternative construction of such examples based on the well-known Anosov-Katok method [AK70, FK04]. As a byproduct, we also obtain the smoothness of the resulting diffeomorphisms.

Theorem 1.1. *There exist C^∞-diffeomorphisms φ of the two-torus with the following properties.*

- *(a) φ is a skew product over some irrational rotation $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1, x \mapsto x + \alpha \mod 1$.*
- *(b) φ is totally strictly ergodic[2] and mean equicontinuous, with the rotation R_α as its maximal equicontinuous factor and the projection to the first coordinate as the factor map.*
- *(c) The unique φ-invariant measure μ is the projection of the Lebesgue measure λ on $\mathbb{T}^1$ onto some measurable graph, that is, it is of the form $\mu = (\mathrm{Id}_{\mathbb{T}^1} \times \gamma)_*\lambda$, where $\gamma : \mathbb{T}^1 \to \mathbb{T}^1$ is measurable.*

Note that since the projection to the first coordinate is the factor map to the MEF, all fibres are circles. In particular, there exist no injectivity points. A genericity statement similar to that in [DG16] can also be obtained (see Remark 4.1(a)), but we will not focus on this issue.

The price we have to pay for the smoothness of the examples is that of a more restricted setting. While the construction in [DG16] allows to choose an arbitrary strictly ergodic systems as factor, our examples are always extensions of irrational rotations with Liouvillean rotation number. However, on the positive side, the Anosov-Katok construction allows to exert additional control over the lifts of the resulting torus diffeomorphisms to finite covering spaces, and also of all iterates. We can thus ensure that all these mappings are uniquely ergodic. This entails that the finite lifts do not have additional dynamical eigenvalues, so that their discrete spectrum coincides with that of the original system and there has to be a continuous part of the dynamical spectrum. A classical result of Katok and Stepin on cyclic approximations [KS67], combined with further modifications of the construction, allows to ensure that this new part of the spectrum is singular continuous. Altogether, we obtain the following.

Theorem 1.2. *For any $m \in \mathbb{N}$ with $m \geqslant 2$, there exist C^∞-diffeomorphisms φ of the two-torus with the following properties.*

- *(a) φ is a skew product over some irrational rotation $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1, x \mapsto x + \alpha \mod 1$.*
- *(b) φ is totally strictly ergodic and a measure-theoretic m to 1 extension of the irrational rotation R_α, which is the maximal equicontinuous factor of the system.*
- *(c) The unique φ-invariant measure μ is the projection of the Lebesgue measure λ on $\mathbb{T}^1$ onto some m-valued measurable graph, that is, it is of the form $\mu = \sum_{j=1}^m (\mathrm{Id}_{\mathbb{T}^1} \times \gamma_j)_*\lambda$, where $\gamma_j : \mathbb{T}^1 \to \mathbb{T}^1$ are measurable functions for $j = 1, \ldots, m$ and $\gamma_i(x) \neq \gamma_j(x)$ λ-almost surely for all $i \neq j$.[3]*

[1] A 'trivial' example would be a homeomorphism of the circle with a unique fixed point. In this case, the MEF is just a single point.

[2] All iterates of φ are strictly ergodic.

[3] We call such systems *m:1-topomorphic extensions (of the MEF)* in analogy to the notion of *topoisomorphic extensions*, for which the topological factor map to the MEF is measure-theoretically one-to-one and hence measure-theoretic isomorphism.

(d) The dynamical spectrum of φ is given by the (discrete) dynamical spectrum of R_α and a singular continuous component.

In the case $m = 2$, these examples are measure-theoretically similar to (generalised) Thue-Morse subshifts [Kea68], and also to strictly ergodic irregular Toeplitz flows constructed by Iwanik and Lacroix in [IL94]. In both cases, the systems are also measure-theoretically finite-to-one extensions of their MEF, exhibit the same discrete spectrum as the MEF and equally show an additional singular continuous part of the spectrum. However, the topological structure of these examples is quite different, since Toeplitz flows always have a residual set of injectivity points for the projection to the MEF, whereas almost all fibres over the MEF of the generalised Thue-Morse subshift contain exactly two points.

Note that as measure theoretically finite-to-one extensions of equicontinuous systems, the examples provided by Theorem 1.1 and 1.2 have zero entropy. An interesting question, which we have to leave open here, is to understand if their complexity can be adequately described by some notion of slow entropy on a suitable scale or other topological invariants (see [KT97, Pet16, FGJ16]).

Structure of the article: In Section 2, we provide all the required preliminaries on topological dynamics, spectral theory, mean equicontinuity and the Anosov Katok method. In Section 3, we show that mean equicontinuity is a G_δ-property. This observation has been made already in [DG16], but in order to simplify the Anosov Katok construction carried out in Section 4 we provide a different G_δ-characterisation of mean equicontinuity here. Finally, in Section 5, we discuss how to modify the construction in order to ensure that all iterates of all finite lifts will still be strictly ergodic, no new dynamical eigenvalues occur and the additional spectral component is singular continuous.

2. Preliminaries

2.1. Topological and measure-preserving dynamics. We refer to standard textbooks such as [Aus88, BS02, Wal82, KH97] for the following basic facts on topological dynamics and ergodic theory. Throughout this article, a *topological dynamical system* (*tds*) is a pair (X,φ), where X is a compact metric space and φ is a homeomorphism of X. We say φ (or (X,φ)) is *minimal* if there exists no non-empty φ-invariant compact subset of X. Equivalently, φ is minimal if for all $x \in X$ the φ-orbit $\mathcal{O}_\varphi(x) = \{\varphi^n(x) \mid n \in \mathbb{Z}\}$ of x is dense in X. The tds (X,φ) is called *equicontinuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x,y) < \delta$ implies $d_X(\varphi^n(x),\varphi^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$. In this case, there is an equivalent metric on X such that φ becomes an isometry. When (X,φ) is both equicontinuous and minimal, then X can be given the structure of a compact abelian group with group operation $\oplus$ such that φ is just the rotation by some element from X, that is, there exists $\alpha \in X$ such that $\varphi(x) = x \oplus \alpha$ for all $x \in X$. We write $x \ominus y$ for $x \oplus (\ominus y)$ in this situation, where $\ominus y$ is the inverse of y. Since minimal rotations on compact abelian groups are always uniquely ergodic, the same holds for minimal equicontinuous systems.

Another tds (Y,ψ) is called a *factor* of (X,φ) with *factor map* $\pi : X \to Y$ if π is continuous and onto and satisfies $\pi \circ \varphi = \psi \circ \pi$. If in addition π is a homeomorphism, we say (X,φ) and (Y,ψ) are *conjugate*. Note that both minimality and equicontinuity are inherited by factors. Since factor maps are in general not unique, we will sometimes also refer to the triple (Y,ψ,π) as a factor in order to specify the factor map. We call such a triple a *maximal equicontinuous factor (MEF)* of (X,φ) if (Y,ψ) is equicontinuous and for any other equicontinuous factor (Z,ρ,p) there exists a unique factor map q between (Y,ψ) and (Z,ρ) such that $p = q \circ \pi$. The existence of a MEF is ensured by the following statement, which also addresses the question of uniqueness.

Theorem 2.1 ([Aus88, Theorem 9.1, p. 125])**.** *Every topological dynamical system (X,φ) has a MEF (Y,ψ,π). If $(\hat{Y},\hat{\psi},\hat{\pi})$ is another MEF, then there is a unique conjugacy $h : (Y,\psi) \to$*

$(\hat{Y}, \hat{\psi})$ *such that* $\hat{\pi} = h \circ \pi$. *In particular, the systems* (Y, ψ) *and* $(\hat{Y}, \hat{\psi})$ *are conjugate in this case.*

Remark 2.2. (a) Despite the lack of uniqueness, we will often refer to a MEF (Y, ψ) of (X, φ) as *'the MEF'*, in particular in situations where we are only interested in conjugacy-invariant properties.

(b) Note also that once we have fixed an equicontinuous tds (Y, ψ) as the MEF of a minimal system (X, φ), the corresponding factor map π is unique modulo post-composition with a rotation on Y (where we refer to the above-mentioned group structure of minimal equicontinuous systems). The reason is the fact that in this case, given two different factor maps $\pi_1, \pi_2 : X \to Y$, the Y-valued function $\pi_1 \ominus \pi_2$ is continuous and φ-invariant, and therefore constant by minimality.

A *measure-preserving dynamical system (mpds)* is a quadruple $(X, \mathcal{A}, \mu, \varphi)$ consisting of a probability space $(X, \mathcal{A}, \mu)$ and a measurable transformation $\varphi : X \to X$ that preserves the measure μ, that is, $\varphi_*\mu = \mu$, where $\varphi_*\mu(A) = \mu(\varphi^{-1}(A))$. An mpds is *ergodic* if every φ-invariant set $A \in \mathcal{A}$ has measure 0 or 1. This is equivalent to the validity of the assertion of the Birkhoff Ergodic Theorem: for any $f \in L^1(\mu)$, there holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ \varphi^i(x) = \int_X f \, d\mu \tag{1}$$

for μ-almost every $x \in X$. Given two mpds $(X, \mathcal{A}, \mu, \varphi)$ and $(Y, \mathcal{B}, \nu, \psi)$, we call a measurable map $h : X \to Y$ a *measure-theoretic isomorphism* if there exist sets $A \in \mathcal{A}$, $B \in \mathcal{B}$ such that $\mu(A) = \nu(B) = 1$, $h : A \to B$ is a bi-measurable bijection, $h_*\mu = \nu$ and $\varphi \circ h = \psi \circ h$ on A.

The mpds we consider will mostly be topological, that is, X will be a compact metric space, $\mathcal{A} = \mathcal{B}(X)$ the Borel σ-algebra on X, μ a Borel measure and φ a homeomorphism. In particular, this means that φ is a bi-measurable bijection. For any tds (X, φ), the existence of at least one φ-invariant probability measure is ensured by the Krylov-Bogolyubov Theorem. If there exists exactly one invariant measure – which is necessarily ergodic in this case – we call a tds *uniquely ergodic*. In this case, the Uniform Ergodic Theorem states that the convergence of the ergodic averages in (1) is uniform for any continuous function f on X. Actually, the same holds if φ admits multiple invariant measures, but the integral of the function f is the same with respect to all of them. This a more or less direct consequence of the Krylov-Bogolyubov procedure and can be extended, to families of continuous functions that are compact in the uniform topology, in the following way.

Theorem 2.3 (Simultaneous Uniform Ergodic Theorem)**.** *Suppose that* (X, φ) *is uniquely ergodic with invariant measure* μ. *For any compact family* $\mathcal{F} \subseteq \mathcal{C}(X, [0,1])$ *of continuous functions and* $x \in X$*, the simultaneous ergodic averages*

$$A_n : X \times \mathcal{F} \longrightarrow \mathbb{R} \quad , \quad (x, f) \longmapsto \frac{1}{n} \sum_{i=0}^{n-1} f\left(\varphi^i(\cdot)\right)$$

converge uniformly to the function $(x, f) \mapsto \int f \, \mathrm{d}\mu$.

We omit the proof, which is a straightforward adaptation of the standard argument for the Uniform Ergodic Theorem (see e.g. [Wal82, Theorem 6.19]).

2.2. SPECTRAL THEORY OF DYNAMICAL SYSTEMS. Given an mpds $(X, \mathcal{A}, \mu, \varphi)$, the associated *Koopman operator* is given by

$$U_\varphi : L^2_\mu(X) \to L^2_\mu(X) \quad , \quad f \mapsto f \circ \varphi \, .$$

Since μ is φ-invariant, U_φ is a unitary operator, so that $\sigma(U_\varphi) \subseteq \mathbb{S}^1$. It is well-known that spectral properties of U_φ are closely related to dynamical properties of the system. For instance, ergodicity of φ is equivalent to the simplicity of 1 as an eigenvalue, and weak mixing of the system is equivalent the absence of any further eigenvalues.

For any continuous function $f:\sigma(U_\varphi)\to\mathbb{C}$, the continuous functional calculus yields the existence of a bounded linear operator $f(U_\varphi)$ on $L^2_\mu(X)$ such that the mapping

$$\mathcal{C}(\sigma(U_\varphi),\mathbb{C})\to\mathcal{C}^*(U_\varphi)=\{f(U_\varphi)\mid f\in\mathcal{C}(\sigma(U_\varphi),\mathbb{C})\}\quad,\quad f\mapsto f(U_\varphi)$$

is an isomorphism of $\mathcal{C}^*$-algebras. Given $g\in L^2_\mu(X)$, this further allows to define a bounded linear functional

$$\ell_g:\mathcal{C}(\sigma(U_\varphi),\mathbb{C})\to\mathbb{C}\quad,\quad f\mapsto\langle f(U_\varphi)g,g\rangle$$

and thus, by virtue of the Riesz Representation Theorem, a Borel measure μ_g on $\sigma(U_\varphi)\subseteq\mathbb{S}^1$ such that

$$\langle f(U_\varphi)g,g\rangle\;=\;\int_{\sigma(U_\varphi)} f\,d\mu_g\;.$$

The measure μ_g is called the *spectral measure associated to* g. Moreover, there exists an orthogonal decomposition

$$L^2_\mu(X)\;=\;L^2_\mu(X)_{\mathrm{pp}}\oplus L^2_\mu(X)_{\mathrm{sc}}\oplus L^2_\mu(X)_{\mathrm{ac}}$$

of $L^2_\mu(X)$, where

$$\begin{aligned}L^2_\mu(X)_{\mathrm{pp}}&=\{g\in L^2_\mu(X)\mid\mu_g\text{ is pure point}\}\;,\\ L^2_\mu(X)_{\mathrm{sc}}&=\{g\in L^2_\mu(X)\mid\mu_g\text{ is singular continuous }\}\;,\\ L^2_\mu(X)_{\mathrm{ac}}&=\{g\in L^2_\mu(X)\mid\mu_g\text{ is absolutely continuous}\}\;.\end{aligned}$$

In our setting the reference measure with respect to which the singularity and absoluteness (of continuity) is defined is the Lebesgue measure on the circle.

The spectra $\sigma_{\mathrm{pp}}(U_\varphi)$, $\sigma_{\mathrm{sc}}(U_\varphi)$ and $\sigma_{\mathrm{ac}}(U_\varphi)$ obtained from the restriction of U_φ to these subspaces are called the *discrete/pure point*, *singular continuous* and *absolutely continuous part*, or *component*, of the dynamical spectrum of U_φ. Note that the different spectral parts need not be disjoint.

In the case of purely discrete spectrum, it turns out that a system is uniquely characterised, up to isomorphism, by the group of its dynamical eigenvalues.

Theorem 2.4 (Halmos–von Neumann, [vN32, HvN42]). *An ergodic mpds $(X,\mathcal{A},\mu,\varphi)$ has purely discrete spectrum if and only if it is measure-theoretically isomorphic to a minimal rotation of a compact abelian group equipped with its Haar measure. Moreover, two mpds with purely discrete spectrum are are isomorphic if and only if they have the same group of eigenvalues.*

In order to prove the existence of a singular continuous spectral component, we will use a classical result from the theory of approximations by periodic transformations presented in [KS67]. An mpds $(X,\mathcal{A},\mu,\varphi)$ admits *cyclic approximation by periodic transformations (capt) with speed $s:\mathbb{N}\to\mathbb{R}^+_0$* if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of bijective bi-measurable transformations on X and a sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of finite measurable partitions of X, $\mathcal{P}_n=\{P_{n,1},\dots,P_{n,K_n}\}$, such that for all $n\in\mathbb{N}$

(P1) φ_n cyclically permutes the elements of $\mathcal{P}_n$;
(P2) for each $A\in\mathcal{A}$, there exist $A_n\in\sigma(\mathcal{P}_n)$ such that $\mu(A\Delta A_n)\xrightarrow{n\to\infty}0$;
(P3) $\sum_{i=1}^{K_n}\mu(\varphi(P_{n,i})\Delta\varphi_n(P_{n,i}))\leq s(K_n)$.

Here, $\sigma(\mathcal{P}_n)$ denotes the σ-algebra generated by $\mathcal{P}_n$.

Theorem 2.5 ([KS67, Corollary 3.1]). *If an mpds $(X,\mathcal{A},\mu,\varphi)$ admits cyclic approximation by periodic transformations (capt) with speed $s:\mathbb{N}\to\mathbb{R}^+_0$ and $\lim_{n\to\infty}ns(n)=0$, then U_φ has no absolutely continuous spectral component, that is, $\sigma_{\mathrm{ac}}(U_\varphi)=\varnothing$.*

2.3. MEAN EQUICONTINUITY. A tds (X, φ) is called mean equicontinuous if for all $\varepsilon > 0$ there is $\delta > 0$ such that $d_X(x, y) < \delta$ implies

$$d_{\mathsf{B}}(x,y) \ = \ \overline{\lim_{n\to\infty}} \ \frac{1}{n}\sum_{i=0}^{n-1} d(\varphi^i(x),\varphi^i(y)) \ < \ \varepsilon \tag{2}$$

It turns out that in the minimal case, mean equicontinuity implies unique ergodicity and is moreover equivalent to a certain invertibility property of the factor map onto the MEF.

Theorem 2.6 ([DG16, Thm. 2.1]). *Suppose (X, φ) is minimal and (Y, ψ, π) is a MEF. Denote by ν the unique invariant measure of (Y, ψ). Then the following are equivalent.*

(1) (X, φ) is mean equicontinuous.
(2) (X, ψ) is uniquely ergodic with unique invariant measure μ and π is a measure-theoretic isomorphism between the mpds $(X, \mathcal{B}(X), \mu, \varphi)$ and $(Y, \mathcal{B}(Y), \nu, \psi)$.

Given a uniquely ergodic tds (X, φ) and a factor (Y, ψ, π) (which is automatically uniquely ergodic as well), we say (X, φ) is an isomorphic extension of (Y, ψ) if π is a measure-theoretic isomorphism between the two mpds $(X, \mathcal{B}(X), \mu, \varphi)$ and $(Y, \mathcal{B}(Y), \nu, \psi)$, where μ and ν are the unique invariant measures for φ and ψ, respectively. Hence, condition (2) above can be rephrased by saying that (X, φ) is an isomorphic extension of (Y, ψ).

Note that the above statement implies, in particular, that the dynamical spectrum of mean equicontinuous systems coincides with that of the MEF and is therefore purely discrete.

The mapping $d_{\mathsf{B}} : X \times X \to \mathbb{R}_0^+$ defined in (2) is always a pseudo-metric on X. It is called the *Besicovitch pseudo-metric.* For mean equicontinuous systems, it provides a way to directly define a MEF of the system.

Proposition 2.7 ([DG16]). *Suppose (X, φ) is mean equicontinuous. Define an equivalence relation on X by*

$$x \sim y \quad \Leftrightarrow \quad d_{\mathsf{B}}(x,y) = 0 \ .$$

Then the quotient system $(X/\sim, \varphi/\sim)$ together with the canonical projection as a factor map is a MEF.

The proof of this fact in [DG16] is implicit – it is contained in the proof of [DG16, Theorem 2.1].

2.4. THE ANOSOV-KATOK METHOD. The Anosov-Katok method is arguably one of the best-known and most widely used constructions in smooth dynamics and allows to obtain a broad scope of examples with particular combinations of dynamical properties. Although many readers will already be familiar with the general method, we provide a brief introduction in order to fix notation and comment on some specific issues that will be relevant in our context. The construction of mean equicontinuous diffeomorphism of the two-torus will then be carried out in Section 4, while the modification required to obtain the finite-to-one extensions in Theorem 1.2 will be discussed in Section 5.

We restrict to the case of tori $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and denote by $\mathrm{Homeo}\left(\mathbb{T}^d\right)$ the space of homeomorphisms of the d-dimensional torus, by $\mathcal{C}^k\left(\mathbb{T}^d\right)$ the space of k-times differentiable torus endomorphisms (including the cases $k = \infty$ and $k = \omega$, were the later stands for 'real-analytic') and let

$$\mathrm{Diffeo}^k\left(\mathbb{T}^d\right) \ = \ \left\{\varphi \in \mathrm{Homeo}(\mathbb{T}^d) \mid \varphi, \varphi^{-1} \in \mathcal{C}^k\left(\mathbb{T}^d\right)\right\} \ .$$

We will identify $\mathrm{Diffeo}^0\left(\mathbb{T}^d\right)$ and $\mathrm{Homeo}\left(\mathbb{T}^d\right)$. Further, we denote the supremum metric on $\mathcal{C}^0\left(\mathbb{T}^d\right)$ by $d_{\sup}$ and let

$$d_k(\varphi,\psi) \ = \ \max_{i=0}^{k} \max\left\{d_{\sup}\left(\varphi^{(i)},\psi^{(i)}\right), d_{\sup}\left(\left(\varphi^{-1}\right)^{(i)},\left(\psi^{-1}\right)^{(i)}\right)\right\} \ .$$

be the standard metric on the space of torus diffeomorphisms. By $B^k_\varepsilon(\psi)$, we denote the ε-ball around ψ in $\mathrm{Diffeo}^k(\mathbb{T}^d)$.

Our aim is to recursively construct a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of torus diffeomorphisms according to the following scheme.

- Each φ_n will be of the form $H_n \circ R_{\rho_n} \circ H_n^{-1}$, where $R_\rho : \mathbb{T}^d \to \mathbb{T}^d,\ x \mapsto x + \rho$ denotes the rotation with rotation vector $\rho \in \mathbb{T}^d$ and $H_n \in \mathrm{Diffeo}^\infty\left(\mathbb{T}^d\right)$.
- The H_n will be of the form $H_n = h_1 \circ \ldots \circ h_n$, where each $h_n \in \mathrm{Diffeo}^\infty\left(\mathbb{T}^d\right)$ commutes with the rotation $R_{\rho_{n-1}}$, that is, $h_n \circ R_{\rho_{n-1}} = R_{\rho_{n-1}} \circ h_n$. Note that consequently we have that $H_n \circ R_{\rho_{n-1}} \circ H_n^{-1} = \varphi_{n-1}$. Hence, at this stage, we have introduced the new conjugating map h_n, but the system has not changed yet.
- When going from step $n-1$ to n in the construction, we choose h_n first and only pick the new rotation vector ρ_n afterwards. Therefore, the continuity of the mapping $\rho \mapsto H_n \circ R_\rho \circ H_n^{-1}$ (with respect to the metric d_k for any $k \in \mathbb{N}$) allows to control the difference between φ_{n-1} and φ_n in the respective metric.
- As a consequence, we can ensure that the resulting sequence $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{Diffeo}^k\left(\mathbb{T}^d\right)$ for any $k \in \mathbb{N}$, simply by recursively choosing ρ_n sufficiently close to ρ_{n-1} with respect to d_n in the n-th step of the construction. This ensures that the φ_n converge to some limit $\varphi \in \mathrm{Diffeo}^\infty\left(\mathbb{T}^d\right)$.

So far, the above items explain how to ensure the convergence of the constructed sequence $(\varphi_n)_{n\in\mathbb{N}}$, but they do not yet specify how to obtain any particular dynamical properties. It turns out, however, that the method is tailor-made to realise any G_δ-properties in the space $\mathrm{Homeo}(\mathbb{T}^d)$. Suppose we want to ensure that our limit diffeomorphism φ belongs to a set of torus homeomorphism $A \subseteq \mathrm{Homeo}\left(\mathbb{T}^d\right)$ which is G_δ, that is, it is of the form $A = \bigcap_{n\in\mathbb{N}} U_n$ with $U_n \subseteq \mathrm{Homeo}\left(\mathbb{T}^d\right)$ open. As discussed below, both minimal and uniquely ergodic torus homeomorphisms can be characterised in this way. We then proceed as follows.

i) By choosing h_n and ρ_n accordingly, we ensure that $\varphi_n \in U_n$. How this is done exactly depends on the property that defines A. This is actually the crucial step in the construction, and we will provide details further below.
ii) Since U_n is open, there exists $\eta_n > 0$ such that $\overline{B^0_{\eta_n}(\varphi_n)} \subseteq U_n$. By ensuring that the distance between φ_j and φ_{j+1} is small and decays sufficiently fast for all $j \geq n$, this yields $\varphi = \lim_{j\to\infty} \varphi_j \in \overline{B^0_{\eta_n}(\varphi_n)} \subseteq U_n$ as well. Since this works for all $n \in \mathbb{N}$, we obtain $\varphi \in A$.
iii) In order to ensure $\varphi_n \in U_n$, it will often be convenient not to go from φ_{n-1} to φ_n directly, but to pass through some intermediate map $\hat{\varphi}_{n-1}$ instead. This is, for example, useful to ensure that the limit system is minimal and/or uniquely ergodic. For instance, we may first choose a totally irrational rotation number $\hat{\rho}_{n-1}$ and define $\hat{\varphi}_{n-1} = H_n \circ R_{\hat{\rho}_{n-1}} \circ H_n^{-1}$. Then $\hat{\varphi}_{n-1}$ is minimal and uniquely ergodic, since this is true for the irrational rotation $R_{\hat{\rho}_{n-1}}$. If U_n is defined in such a way that it contains all minimal/uniquely ergodic torus homeomorphisms, then $\hat{\varphi}_{n-1} \in U_n$ is automatic. If φ_n is then chosen sufficiently close to $\hat{\varphi}_{n-1}$ (by choosing ρ_n close to $\hat{\rho}_{n-1}$), we obtain $\varphi_n \in U_n$ as required. Further, if both $\hat{\rho}_{n-1}$ and ρ_n are close enough to ρ_{n-1}, then φ_n will also be close to φ_{n-1} in $\mathrm{Diffeo}^k(\mathbb{T}^d)$.
iv) In Section 4, we will actually use a further modification of the above scheme and pass through an additional third map $\tilde{\varphi}_{n-1} = H_n \circ R_{\tilde{\rho}_{n-1}} \circ H_n^{-1}$, where $\tilde{\rho}_{n-1}$ is irrational, but not totally irrational (so the entries of the rotation vector are rationally related).

The following works in high generality for any compact metric space X. Using a very similar argument as made in ii) we obtain

Proposition 2.8. *Let $A = \bigcap_{n\in\mathbb{N}} U_n \subseteq \mathrm{Homeo}(X) =: Z$ where the U_n are open w.r.t. d_0. There exists a sequence $\left(\eta_n^A\right)_{n\in\mathbb{N}}$ of functions $\eta_n^A : Z^n \longrightarrow \mathbb{R}^+$ such that if a sequence $(\varphi_n)_{n\in\mathbb{N}} \in Z^{\mathbb{N}}$ satisfies*

$$d_0(\varphi_n, \varphi_{n+1}) < \eta_n^A(\varphi_1, \ldots, \varphi_n) \quad \textit{and} \quad \varphi_n \in U_n \tag{3}$$

for all $n \in \mathbb{N}$, then it converges with limit $\varphi = \lim_{n\to\infty} \varphi_n \in A$.

Sketch of proof. If we set $\eta_n^A < 2^{-n}$ any sequence satisfying (3) will be Cauchy and converge to some $\varphi \in Z$ as Z is complete. The idea is now to choose η^A small enough such that (3) implies $\varphi_n \in U_k$ for $n \geqslant k$. This can be done by fixing some $\delta_k > 0$ such that $\overline{B_{\delta_k}(\varphi_k)} \subseteq U_k$ and then make sure that (3) implies $d_0(\varphi_n, \varphi_k) < \delta_k$, i.e. $\varphi_n \in B_{\delta_k}(\varphi_k)$. One way to do that and to encode all the previously imposed conditions into η_n^A is to recursively ensure that

$$\eta_n^A(\varphi_1, \dots, \varphi_n) < \min\left(\frac{\delta_n}{2}, \min_{k=1}^{n-1}\left(\eta_k^A(\varphi_1, \dots, \varphi_k) - d_0(\varphi_k, \varphi_n)\right)\right). \tag{4}$$

Now we argue that if $(\varphi_n)_{n\in\mathbb{N}}$ satisfies (3) then (4) inductively implies

$$\eta_k^A(\varphi_1, \dots, \varphi_k) > d_0(\varphi_k, \varphi_n) \tag{5}$$

for any $n > k$ and thus both $\eta_n^A > 0$ and $\varphi_n \in B_{\delta_k}(\varphi_k)$ (as $\eta_k^A(\varphi_1, \dots, \varphi_k) < \delta_k$). For $n = k+1$ (5) holds trivially. Now for the induction step $n \to n+1$ we have that

$$d_0(\varphi_{n+1}, \varphi_n) < \eta_n^A(\varphi_1, \dots, \varphi_n) < \eta_k^A(\varphi_1, \dots, \varphi_k) - d_0(\varphi_k, \varphi_n).$$

The triangle inequality now implies (5).

In total $\varphi_m \in B_\eta(\varphi_n) \subseteq \overline{B_\eta(\varphi_n)}$ for all $m > n$. So $\varphi \in \overline{B_\eta(\varphi_n)} \subseteq U_n$ for all $n \in \mathbb{N}$ and thus $\varphi \in \bigcap_{n\in\mathbb{N}} U_n = A$. □

2.5. G_δ-CHARACTERISATION OF STRICT ERGODICITY. It is well-known that the set $\mathrm{Homeo}^{\mathrm{se}}(X)$ of strictly ergodic homeomorphisms of X is G_δ in $\mathrm{Homeo}(X)$. For the convenience of the reader, we include a short proof of this folklore result.

We first show that minimality is G_δ. Let $\varepsilon > 0$. A set $A \subseteq X$ is called *ε-dense* if $B_\varepsilon(A) = X$. Observe that the mapping ξ is minimal if and only if for any $\varepsilon > 0$ there is $M \in \mathbb{N}$ such that $\{x, \varphi(x), \dots, \varphi^M(x)\}$ is ε-dense for any $x \in X$. Furthermore, the k-th iterate of a homeomorphism depends continuously on that homeomorphism. Therefore

$$U_{M,\varepsilon}^{\min} = \left\{\psi \in \mathrm{Homeo}(X,X) \,\middle|\, \forall x \in X : \{x, \psi(x), \dots, \psi^M(x)\} \text{ is } \varepsilon\text{-dense}\right\}$$

is open in the supremum norm. If $\mathrm{Homeo}^{\min}(X)$ denotes the set of all minimal homeomorphisms of X, the above yields

$$\mathrm{Homeo}^{\min}(X) = \bigcap_{\varepsilon\in\mathbb{Q}^+} \bigcup_{M\in\mathbb{N}} U_{M,\varepsilon}^{\min}.$$

This means in particular that $\mathrm{Homeo}^{\min}(X)$ is G_δ.

Now, we turn to unique ergodicity. Fix a dense set $\{s_n \mid n \in \mathbb{N}\} \subset \mathcal{C}(X,\mathbb{R})$. Given $n \in \mathbb{N}$ and a continuous map $\xi : X \to X$, we can assign to any $g \in \mathcal{C}(X,\mathbb{R})$ its n-step ergodic average

$$A_n^\xi g = \frac{1}{n}\sum_{i=0}^{n-1} g \circ \xi^i.$$

It is well known (e.g. [EW10, Thm. 4.10, p. 105]) that ξ is uniquely ergodic if and only if for every $k \in \mathbb{N}$ the sequence of functions $(A_n^\xi s_k)_{n\in\mathbb{N}}$ converges pointwise to a constant. For $g \in \mathcal{C}(X,\mathbb{R})$ we denote by $V(g)$ its variation over X, that is

$$V(g) := \sup_{x\in X} g(x) - \inf_{y\in X} g(y).$$

Given any ψ-invariant probability measure μ, we have $\int A_n^\psi g \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$. This implies that if $V(A_n^\psi g) \xrightarrow{n\to\infty} 0$, then $A_n^\psi g \xrightarrow{n\to\infty} \int g \,\mathrm{d}\mu$.

As $A_k^\xi g$ depends continuously on ξ and $V(f)$ depends continously on f, we see that

$$U_{n,K,\beta}^{\mathrm{ue}} := \left\{\psi \in \mathrm{Homeo}(X,X) \,\middle|\, V(A_K^\psi s_n) < \beta\right\}$$

is open in the supremum metric. In particular, the set of those transformations for which the ergodic average of s_n eventually stabilises at a variation below β, given by

$$U_{n,\beta}^{\mathrm{ue}} = \bigcup_{K\in\mathbb{N}} \bigcap_{k>K} U_{n,k,\beta}^{\mathrm{ue}}$$

is G_δ. We notice that $\psi \in \bigcap_{k\in\mathbb{N}} \bigcap_{\beta\in\mathbb{Q}^+} U^{\mathrm{ue}}_{k,\beta}$ if and only if $A^\psi_n s_n$ converges to $\int s_n \,\mathrm{d}\mu$ for any $n \in \mathbb{N}$ if and only if ψ is uniquely ergodic. This yields that the set $\mathrm{Homeo}^{\mathrm{ue}}(X)$ of uniquely ergodic homeomorphisms of X is G_δ.

Altogether, this shows that $\mathrm{Homeo}^{\mathrm{se}}(X) = \mathrm{Homeo}^{\min}(X) \cap \mathrm{Homeo}^{\mathrm{ue}}(X)$ is a G_δ-set as claimed.

3. G_δ-CHARACTERISATION OF MEAN EQUICONTINUITY FOR SKEW PRODUCTS

As discussed in the previous section, in order to construct mean equicontinuous systems via the Anosov Katok method, it is instrumental to have an explicit G_δ-characterisation of mean equicontinuity available. In principle, such a characterisation is already contained in [DG16]. However, the latter uses the fact that mean equicontinuity is equivalent to the existence of a unique self-joining on the product space $X \times X$ over the MEF as a common factor. Since we want to avoid working in the product space, as this would rather complicate the construction in the next section, we provide an alternative characterisation here. As in [DG16], we make use of the fact that we are in a skew product setting and the factor map is given *a priori* (by the projection to the first coordinate). We formulate the statement in abstract terms, as it might be useful in other situations as well.

Proposition 3.1. *Let (X,φ) be a tds and (Y,ψ,π) an equicontinuous factor. Then (Y,ψ,π) is a MEF of (X,φ) and (X,φ) is an isomorphic extension of (Y,ψ) if and only if for all $\varepsilon > 0$ there exists some $K \in \mathbb{N}$ such that, for all $x, y \in X$, we have*

$$\pi(x) = \pi(y) \quad \Longrightarrow \quad \frac{1}{K}\sum_{i=0}^{K} d_X(\varphi^i(x), \varphi^i(y)) < \varepsilon \,. \tag{6}$$

Remark 3.2. (a) Denote by $\mathrm{Homeo}^{\mathrm{eq}}(Y)$ the space of equicontinuous homeomorphisms of Y. Consider the space

$$\mathcal{E}(\pi) \;=\; \{\varphi \in \mathrm{Homeo}(X) \mid \exists \psi \in \mathrm{Homeo}^{\mathrm{eq}}(Y) : \pi\circ\varphi = \psi\circ\pi\}$$

with the subspace

$$\mathcal{E}^{\mathrm{iso}}(\pi) \;=\; \left\{\varphi \in \mathcal{E}(\pi) \,\middle|\, \begin{array}{c}\exists\psi \in \mathrm{Homeo}(Y) : (Y,\psi,\pi) \text{ is the MEF of } (X,\varphi)\\ \text{and } (X,\varphi) \text{ is its isomorphic extension}\end{array}\right\} .$$

Let

$$U_n^{\mathrm{iso}}(\pi) \;=\; \{\varphi \in \mathcal{E}(\pi) \mid \exists K \in \mathbb{N} : (6) \text{ holds with } \varepsilon = 1/n\} \,.$$

Then, by the above statement, we have $\mathcal{E}^{\mathrm{iso}}(\pi) = \bigcap_{n\in\mathbb{N}} U_n^{\mathrm{iso}}(\pi)$. As the sets $U_n^{\mathrm{iso}}(\pi)$ are open, this implies that $\mathcal{E}^{\mathrm{iso}}(\pi)$ is a G_δ-set in $\mathcal{E}(\pi)$.

Note here that property (6) only depends on the factor map π, but not on the map ψ acting on the factor space.

(b) A similar characterisation could be given with a fixed factor system (Y,ψ) on the base. However, in the context of the Anosov Katok construction, where the base systems of the approximating diffeomorphisms will be circle rotations with varying rotation numbers, the independence of ψ in the above characterisation is crucial.

(c) According to Proposition 2.8, there are mappings

$$\eta_n^{\mathrm{me}} : \mathrm{Homeo}(\mathbb{T}^d)^n \longrightarrow [0, 2^{-n}]$$

such that if a sequence $(\tilde\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{E}(\pi)$ satisfies $\tilde\varphi_n \in U_n^{\mathrm{iso}}(\pi)$ and

$$d_0\left(\tilde\varphi_n, \tilde\varphi_{n+1}\right) < \eta_n^{\mathrm{me}}\left(\tilde\varphi_1, \dots, \tilde\varphi_n\right)$$

for all $n \in \mathbb{N}$, then its limit $\varphi = \lim_{n\to\infty} \tilde\varphi_n$ exists and belongs to $\mathcal{E}^{\mathrm{iso}}(\pi)$.

Proof of Proposition 3.1. First, assume that (Y,ψ,π) is a MEF and (X,φ) is an isomorphic extension of (Y,ψ). Then, by Theorem 2.7, $\pi(x)=\pi(y)$ implies $d_{\mathrm{B}}(x,y)=0$. Let

$$\mathcal{J}_\pi(X) \;=\; \{(x,y)\in X\times X \mid \pi(x)=\pi(y)\}\ .$$

Then $d_{\mathrm{B}}(x,y)$ is the ergodic average of the function d_X for the action of $\varphi\times\varphi$ on $X\times X$. Since these ergodic averages are identically zero on the compact invariant set $\mathcal{J}_\pi(X)$, we have that $\int_{\mathcal{J}_\pi(X)} d_X(x,y)\ d\gamma(x,y) = 0$ for all $\varphi\times\varphi$-invariant measures γ on $\mathcal{J}_\pi(X)$ (in fact, by [DG16, Proposition 2.5], there is only one such measure when (X,φ) is mean equicontinuous). The Uniform Ergodic Theorem therefore implies that the functions

$$a_K(x,y) \;=\; \frac{1}{K}\sum_{i=0}^{K} d_X\left(\varphi^i(x),\varphi^i(y)\right)$$

uniformly converge to zero as $K\to\infty$. Hence, we have $a_K<\varepsilon$ for sufficiently large $K\in\mathbb{N}$, which is just an equivalent reformulation of (6).

Conversely, suppose that for all $\varepsilon>0$ there exists $K\in\mathbb{N}$ such that (6) holds. We assume without loss of generality that ψ is an isometry. Denote by Δ_Y and Δ_X the diagonals in the respective product spaces $Y\times Y$ and $X\times X$. Note that thus $\mathcal{J}_\pi(X)=(\pi\times\pi)^{-1}(\Delta_Y)$.

Now, fix $\varepsilon>0$ and choose $K\in\mathbb{N}$ according to (6). This means that the function a_K is strictly smaller than ε on $\mathcal{J}_\pi(X)$. By compactness of $\mathcal{J}_\pi(X)$ and continuity of a_K, there exists $\delta_1>0$ such that $a_K<\varepsilon$ on $B_{\delta_1}(J_\pi(X))$. Due to the continuity of $\pi\times\pi$, there exists $\eta>0$ such that $A=(\pi\times\pi)^{-1}\left(\overline{B_\eta(\Delta_Y)}\right)\subseteq B_{\delta_1}(\mathcal{J}_\pi(X))$. Further, as π is uniformly continuous, there exists $\delta>0$ such that $\pi\times\pi(B_\delta(\mathcal{J}_\pi(X)))\subseteq B_\eta(\Delta_Y)$.

As ψ is an isometry, the set A is $\varphi\times\varphi$-invariant. Since d_{B} is equal to the ergodic average of a_K for the action of $(\varphi\times\varphi)^K$ and $a_K<\varepsilon$ on A, this yields $d_{\mathrm{B}}<\varepsilon$ on A. However, by the above choices, $d_X(x,y)<\delta$ implies $d_Y(\pi(x),\pi(y))<\eta$ and therefore $(x,y)\in A$. Hence, we obtain that $d_X(x,y)<\delta$ implies $d_{\mathrm{B}}(x,y)<\varepsilon$. As $\varepsilon>0$ was arbitrary, this means that (X,φ) is mean equicontinuous.

Therefore Theorem 2.6 implies that (X,φ) is an isomorphic extension of its MEF, which we denote by $(\hat{Y},\hat{\psi},\hat{\pi})$. By definition, we know that (Y,ψ) is a factor of the MEF, so the fibres of $\hat{\pi}$ are contained in the fibres of π. However, since the Besicovitch distance d_{B} between two points is zero on each fibre of π, and as this property characterises the fibres of the MEF due to Proposition 2.7 (note here that the fibres do not depend on the particular choice of the MEF), the fibres of π are also contained in the fibres of $\hat{\pi}$. Thus, π and $\hat{\pi}$ have the same fibres, which implies that (Y,ψ,π) is also a MEF and (X,φ) is an isomorphic extension of (Y,ψ). □

4. Mean equicontinuous skew products on the torus: Proof of Theorem 1.1

We are going to construct a mean equicontinuous diffeomorphisms of the two-torus which have skew product form

$$\varphi:\mathbb{T}^2\to\mathbb{T}^2\quad,\quad (x,y)\mapsto(x+\alpha,\varphi_x(y)) \tag{7}$$

and are such that the underlying irrational rotation $R_\alpha:\mathbb{T}^1\to\mathbb{T}^1,\ x\mapsto x+\alpha$ is the MEF and the factor map is given by the projection $\pi:\mathbb{T}^2\to\mathbb{T}^1,\ (x,y)\mapsto x$ to the first coordinate. In order to do so, we employ the Anosov Katok method as described in Section 2.4 and recursively define sequences of skew product diffeomorphism $(\varphi_n)_{n\in\mathbb{N}}$, $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$ and $(\hat{\varphi}_n)_{n\in\mathbb{N}}$ whose common limit φ will satisfy the assertions of Theorem 1.1. The general scheme of our inductive construction will be as follows.

- The mappings $\varphi_n,\tilde{\varphi}_n$ and $\hat{\varphi}_n$ will be of the form

$$\varphi_n = H_n\circ R_{\rho_n}\circ H_n^{-1}\ ,\quad \tilde{\varphi}_n = H_{n+1}\circ R_{\tilde{\rho}_n}\circ H_{n+1}^{-1}\quad\text{and}\quad \hat{\varphi}_n = H_{n+1}\circ R_{\hat{\rho}_n}\circ H_{n+1}^{-1}\ ,$$

where ρ_n is rational, $\tilde{\rho}_n = \alpha\rho_n$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\hat{\rho}_n$ is totally irrational. Further, for technical reasons, we require that

$$\rho_n = \left(\frac{p_n}{q_n}, \frac{p'_n}{q_n}\right) \quad \text{with } p_n, p'_n, q_n \in \mathbb{N} \text{ relatively prime .} \tag{8}$$

- The conjugating diffeomorphisms H_n will be of the form $H_n = h_1 \circ \ldots \circ h_n$, where h_{n+1} always commutes with the rotation R_{ρ_n}. Moreover, all h_n have skew product structure $h_n : (x, y) \mapsto (x, h_{n,x}(y))$, with fibre maps $h_{n,x} : \mathbb{T}^1 \to \mathbb{T}^1$.
- We choose the functions η_n^{se} and η_n^{me} and the sets $U_n^{\mathrm{iso}}(\pi)$ according to Proposition 2.8 and Remark 3.2(c).
- The approximating torus diffeomorphisms $\varphi_n, \tilde{\varphi}_n, \hat{\varphi}_n$ will be chosen such that for each $n \geq 2$ they satisfy

$$d_n(\varphi_n, \tilde{\varphi}_n) \leq \frac{1}{3}\min\left\{\eta_{n-1}^{\mathrm{se}}(\hat{\varphi}_1, \ldots, \hat{\varphi}_{n-1}), \eta_{n-1}^{\mathrm{me}}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1})\right\} \tag{9}$$

$$d_n(\tilde{\varphi}_n, \hat{\varphi}_n) \leq \frac{1}{3}\min\left\{\eta_{n-1}^{\mathrm{se}}(\hat{\varphi}_1, \ldots, \hat{\varphi}_{n-1}), \eta_{n}^{\mathrm{me}}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n})\right\} \tag{10}$$

$$d_n(\hat{\varphi}_n, \varphi_{n+1}) \leq \frac{1}{3}\min\left\{\eta_{n}^{\mathrm{se}}(\hat{\varphi}_1, \ldots, \hat{\varphi}_{n}), \eta_{n}^{\mathrm{me}}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n})\right\} \tag{11}$$

$$\tilde{\varphi}_n \in U_n^{\mathrm{iso}}(\pi) . \tag{12}$$

Note that these conditions together imply that

$$d_n(\tilde{\varphi}_n, \tilde{\varphi}_{n+1}) \leq \eta_n^{\mathrm{me}}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n) \tag{13}$$

$$d_n(\hat{\varphi}_n, \hat{\varphi}_{n+1}) \leq \eta_n^{\mathrm{se}}(\hat{\varphi}_1, \ldots, \hat{\varphi}_n) \tag{14}$$

for all $n \in \mathbb{N}$, which together with (12) means that the conditions of Proposition 2.8 and Remark 3.2(c) are met, where Proposition 2.8 is applied to the sequence $(\hat{\varphi}_n)_{n\in\mathbb{N}}$ and Remark 3.2(c) is applied to $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$. Note here that the diffeomorphisms $\hat{\varphi}_n$ are all strictly ergodic, since they are conjugate to a totally irrational torus rotation. Consequently, the common limit φ of all the sequences, whose existence is also guaranteed by (9)–(11) (note that $\eta_n^{\mathrm{me}}, \eta_n^{\mathrm{se}} \leq 2^{-n}$), is both strictly ergodic and mean equicontinuous, with π as the factor map to the MEF. The MEF is then given by $(\mathbb{T}^1, R_\alpha, \pi)$, where $\alpha = \lim_{n\to\infty} \pi(\rho_n)$.
- Note that conditions (9)–(11) can always be ensured by choosing the rotation vectors $\rho_n, \tilde{\rho}_n, \hat{\rho}_n$ and ρ_{n+1} sufficiently close to each other. The reason is the fact that we have $\varphi_n = H_{n+1} \circ R_{\rho_n} \circ H_{n+1}^{-1}$ due to the commutativity between h_{n+1} and R_{ρ_n} combined with the continuity of $\rho \mapsto H_{n+1} \circ R_\rho \circ H_{n+1}^{-1}$. Therefore, the only issue that remains to be addressed is to ensure that the intermediate maps $\tilde{\varphi}_n$ are indeed contained in $U_n^{\mathrm{iso}}(\pi)$.

In order to start the induction, we let $H_1 = h_1 = \mathrm{Id}_{\mathbb{T}^2}$ and choose φ_1 to be an arbitrary rational rotation on $\mathbb{T}^2$ whose rotation vector satisfies (8). Note that the inductive assumptions (9)–(11) are all still empty at this point.

Now, suppose that $\varphi_1, \ldots, \varphi_N$, $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{N-1}$ and $\hat{\varphi}_1, \ldots, \hat{\varphi}_{N-1}$ have been constructed such that (9)–(12) hold for all $n = 1, \ldots, N-1$. We have $\rho_n = \left(\frac{p_n}{q_n}, \frac{p'_n}{q_n}\right)$, where $p_n, p'_n, q_n \in \mathbb{N}$ are relatively prime. The aim is to choose h_{n+1} and $\tilde{\rho}_n$ in such a way that $\tilde{\varphi}_n \in U_n^{\mathrm{iso}}(\pi)$, that is, $\tilde{\varphi}_n$ satisfies (6) with $\varepsilon = 1/n$.

To that end, we note that orbits of R_{ρ_n} move along closed curves of the form

$$L(\rho_n, t) = \{(xp_n/q_n, t + xp'_n/q_n) \mid x \in [0, q_n)\} ,$$

which are parametrised by the functions

$$\ell_{\rho_n,t} : \mathbb{T}^1 \to L(t, \rho_n) \quad , \quad x \mapsto (xp_n, t + xp'_n) .$$

We now dwell on this insight a bit further in order to see how we need to choose h_{n+1}. First, observe that the mapping $\ell_{\rho_n,t}$ conjugates the one-dimensional rotation r_{1/q_n} on $\mathbb{T}^1$

and the restriction of R_{ρ_n} to $L(t,\rho_n)$, that is, $R_{\rho_n} \circ \ell_{\rho_n,t} = \ell_{\rho_n,t} \circ r_{1/q_n}$. Consequently, the orbits of $\varphi_n = H_{n+1} \circ R_{\rho_n} \circ H_{n+1}^{-1}$ move along the curves $H_{n+1}(L(\rho_n,t))$, and $H_{n+1} \circ \ell_{\rho_n,t}$ provides a conjugacy between the action of φ_n on these curves and the rational rotation r_{1/q_n}. Moreover, if we change the rotation vector ρ_n to $\tilde{\rho}_n = \alpha\rho_n$, where α is an irrational real number, then the orbits of $\tilde{\varphi}_n$ still move along the same curves, but now the action of $\tilde{\varphi}_n$ on these curves is conjugate to the irrational rotation r_{α/q_n} on $\mathbb{T}^1$ (again with conjugacy $H_{n+1} \circ \ell_{\rho_n,t}$).

Given two points $z,z' \in \mathbb{T}^2$ with $\pi(z) = \pi(z')$ we may choose $x, t_z, t_{z'} \in \mathbb{T}^1$ such that $z = H_{n+1}\left(\ell_{\rho_n,t_z}(x)\right)$ and $z' = H_{n+1}\left(\ell_{\rho_n,t_{z'}}(x)\right)$. For the average distance of the iterates of these two points along their orbits, we obtain

$$\begin{aligned} \frac{1}{n}\sum_{i=0}^{n-1} d\left(\tilde{\varphi}_n^i(z), \tilde{\varphi}_n^i(z')\right) &= \frac{1}{n}\sum_{i=0}^{n-1} d\left(\tilde{\varphi}_n^i\left(\ell_{\rho_n,t_z}(x)\right), \tilde{\varphi}_n^i\left(\ell_{\rho_n,t_{z'}}(x)\right)\right) \\ &= \frac{1}{n}\sum_{i=0}^{n-1} F_{n,t_z,t_z'} \circ r^i_{\alpha/q_n}(x) \ , \end{aligned} \tag{15}$$

where

$$F_{n,t,t'} : \mathbb{T}^1 \to \mathbb{R}^+ \quad , \quad x \mapsto d\left(H_{n+1} \circ \ell_{\rho_n,t}(x), H_{n+1} \circ \ell_{\rho_n,t'}(x)\right) \ .$$

By unique ergodicity of the irrational rotation r_{α/q_n}, the averages in (15) converge uniformly to $\int_{\mathbb{T}^1} F_{n,t,t'}(x)\, dx$. As the family $\{F_{n,t,t'} \mid t,t' \in \mathbb{T}^1\}$ is compact, the Simultaneous Uniform Ergodic Theorem 2.3 implies that this convergence is even uniform in the parameters $t,t' \in \mathbb{T}^1$. This means that for any $\kappa > 0$ there exists $K \in \mathbb{N}$ such that

$$\left| \frac{1}{K}\sum_{i=0}^{K-1} d\left(\tilde{\varphi}_n^i(z), \tilde{\varphi}_n^i(z')\right) - \int_{\mathbb{T}^1} F_{n,t_z,t_{z'}}(x)\, ds \right| \ < \ \kappa$$

holds for all $z,z' \in \mathbb{T}^1$ with $\pi(z) = \pi(z')$. Hence, in order to ensure the validity of (12), it suffices to let $\kappa = 1/2n$ and to choose h_{n+1} in such a way that

$$\int_{\mathbb{T}^1} F_{n,t,t'}(x)\, dx \ \leq \ \frac{1}{2n} \tag{16}$$

holds for all $t,t' \in \mathbb{T}^1$.

Now, constructing such a mapping h_{n+1} is not difficult, albeit somewhat technical. We first define h_{n+1} on the vertical strip $S = I \times \mathbb{T}^1$, where $I = [0, 1/q_n]$. We fix $\delta > 0$ and choose some circle diffeomorphism g, homotopic to the identity, such that $g\left(\mathbb{T}^1 \setminus B_\delta(1/2)\right) \subseteq B_\delta(0)$. For instance, g could be the projective action of a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ with sufficiently large $\lambda > 0$. Then we choose a smooth homotopy $G : [0,1] \times \mathbb{T}^1 \to \mathbb{T}^1$, $(x,y) \mapsto G_x(y)$ between $G_0 = \mathrm{Id}_{\mathbb{T}^1}$ and $G_1 = g$ such that $G_x = \mathrm{Id}_{\mathbb{T}^1}$ for all x in some neighbourhood of zero. We assume $\delta \in [0, 1/4q_n]$, let $\hat{I} = [\delta, 1/q_n - \delta]$ and choose a smooth mapping $T : S \to S$ such that $T(x,y) = (x, T_x(y))$ where $T_x(y) = y + \frac{x}{1/q_n - 2\delta}$ for all $(x,y) \in \hat{I}$ and $T_x = \mathrm{Id}_{\mathbb{T}^1}$ for all x in a neighbourhood of zero. Thus, the image of a horizontal line segment $\hat{I} \times \{y\}$ under T 'wraps' around the torus exactly once in the vertical direction.

Using these auxiliary mappings, we let

$$h_{n+1} : S \to S \quad , \quad (x,y) \mapsto \begin{cases} G_{x/\delta} \circ T(x,y) & 0 \leq x \leq \delta \\ g \circ T(x,y) & \delta < x < 1/q_n - \delta \\ G_{(1-x)/\delta} \circ T(x,y) & 1/q_n - \delta \leq x \leq 1/q_n \end{cases} \ . \tag{17}$$

Then h_{n+1} is smooth on S and coincides with the identity on a neighbourhood of ∂S.

Now, we first focus on that segment of a curve $L(\rho_n, t)$ passing through S, which is parametrised by the mapping $p_n^{-1} I \to \mathbb{T}^1,\ x \mapsto \ell_{\rho_n,t}(x)$. Note that there are p_n such pieces, which all differ by an additive constant that is a multiple of $1/p_n$. On $p_n^{-1}\hat{I}$, this function has constant slope $\frac{1}{1/q_n-2c}+\frac{p_n}{q_n}$. As a consequence, it passes through the set $\hat{I}\times B_\delta(1/2)$ at most twice over the interval $\hat{S}$ and we obtain that

$$\mathrm{Leb}_{\mathbb{T}^1}\left(\left\{x\in p_n^{-1}I \mid T_x\circ\ell_{\rho_n,t}(x)\in B_\delta(1/2)\right\}\right) \;\le\; 2\delta+\frac{\delta}{p_nq_n}\ . \tag{18}$$

Since g maps the complement of the interval $B_\delta(1/2)$ into the interval $B_\delta(0)$ and (18) holds for all $t\in\mathbb{T}^1$, the images of respective segments of two different curves $L(q_n,t), L(q_n,t')$ under T will be 2δ-close to each other most of the time. More precisely, we obtain

$$\mathrm{Leb}_{\mathbb{T}^1}\left(\left\{x\in p_n^{-1}I \mid |h_{n+1,s}\circ\ell_{\rho_n,t}(x)-h_{n+t,s}\circ\ell_{\rho_n,t'}(x)|\ge 2\delta\right\}\right) \;\le\; 4\delta+\frac{2\delta}{p_nq_n}\ . \tag{19}$$

So far, we have only defined h_{n+1} on S and only considered the restriction of the curves $\ell_{\rho_n,t}$ to the interval $p_n^{-1}I$, which only parametrises the $1/p_nq_n$-th part of the whole curves $L(\rho_n,t)$. However, if we extend the definition of h_{n+1} by commutativity to all of $\mathbb{T}^2$, i.e. by setting $h_{n+1|R^k_{\rho_n}(S)}=R^k_{\rho_n}\circ h_{n+1|S}\circ R^{-k}_{\rho_n}$, $k=1,\dots,q_n-1$, then the behaviour of all p_nq_n segments of pairs of curves $h_{n+1}(L(q_n,t))$ and $h_{n+1}(L(q_n,t'))$ will be the same – we are simply looking at a rotated version of the same situation. Therefore, we obtain the estimate

$$\mathrm{Leb}_{\mathbb{T}^1}\left(\left\{x\in\mathbb{T}^1 \mid |h_{n+1,x}\circ\ell_{\rho_n,t}(x)-h_{n+1,x}\circ\ell_{\rho_n,t'}(x)|\ge 2\delta\right\}\right) \;\le\; 6\delta p_nq_n\ . \tag{20}$$

Since H_n is uniformly continuous, we may choose δ in such a way that $d(x,y)<2\delta$ implies $d(H_n(x),H_n(y))<\frac{1}{4n}$. Then (20) implies

$$\mathrm{Leb}_{\mathbb{T}^1}\left(\left\{s\in\mathbb{T}^1 \mid |H_{n+1,s}\circ\ell_{\rho_n,t}(s)-H_{n+t,s}\circ\ell_{\rho_n,t'}(s)|\ge 1/4n\right\}\right) \;\le\; 6\delta p_nq_n\ . \tag{21}$$

When δ is sufficiently small (say $\delta<1/24np_nq_n$), this finally yields (16).

In order to complete the induction step, we now choose $\tilde{\rho}_n=\alpha\rho_n$, where $\alpha\in\mathbb{R}\smallsetminus\mathbb{Q}$ is sufficiently close to 1 such that (9) holds. After that, we can take $\hat{\rho}_n$ to be any totally irrational rotation vector, close enough to $\tilde{\rho}_n$ to ensure (10), and finally choose a new rational rotation vector ρ_{n+1} that is close enough to $\hat{\rho}_n$ to ensure (11) and satisfies (8) holds. This completes the inductive construction and therefore the proof of Theorem 1.1.

Remark 4.1. (a) The above construction starts with an arbitrary rational rotation R_{ρ_1} and allows to ensure that the resulting limit diffeomorphism φ is arbitrarily close to R_{ρ_1}. Together with the G_δ-property of mean equicontinuity and strict ergodicity, this implies that the set of skew product diffeomorphisms which satisfy the assertions of Theorem 1.1 form a residual subset of the space

$$\overline{\mathrm{Cob}}(\mathbb{T}^2,\pi) \;=\; \left\{H\circ R_\rho\circ H^{-1}\mid \rho\in\mathbb{T}^2,\ H\in\mathrm{Diffeo}^k(\mathbb{T}^2),\ \pi\circ H=\pi\right\}\ ,$$

where $k\in\mathbb{N}_0\cup\{\infty\}$ is arbitrary. The analogous observation has been made in [DG16].

(b) All the torus maps in the above construction, and hence also the resulting diffeomorphisms φ, may be chosen as the projective actions of quasiperiodic $\mathrm{SL}(2,\mathbb{R})$-cocycles (compare [HP06]).

5. Total strict ergodicity for lifts and non-existence of additional eigenvalues

Given any $l,m\in\mathbb{N}$, $\mathbb{T}^{(l,m)}=\mathbb{R}/l\mathbb{Z}\times\mathbb{R}/m\mathbb{Z}$ is a canonical finite covering space of the torus $\mathbb{T}^2$. For any torus homeomorphism ψ homotopic to the identity, there exist lifts $L^\psi_{(l,m;s)}:\mathbb{T}^{(l,m)}\to\mathbb{T}^{(l,m)}$ with $s\in A(l,m)=(\mathbb{Z}/l\mathbb{Z})\times(\mathbb{Z}/m\mathbb{Z})$, which are uniquely determined by the requirement that $L^\psi_{(l,m;s)}(0)\in[s_1,s_1+1)\times[s_2,s_2+1)$. Note that two different lifts $L^\psi_{(l,m;s)}$ and $L^\psi_{(l,m;s)}$ are conjugate by the integer translation $(x,y)\mapsto(x+(s_1'-s_1),y+(s_2'-s_s))$ on $\mathbb{T}^{(l,m)}$ and thus share the same dynamical properties.

We denote the iterates of these lifts by $L^{\psi,j}_{(l,m;s)}=\left(L^\psi_{(l,m;s)}\right)^j$. Note that $L^{\psi,j}_{(l,m;s)}$ may differ from $L^{\psi^j}_{(l,m;s)}$ by an integer translation. For any rotation R_ρ on $\mathbb{T}^2$, its lift $L^{R_\rho}_{(l,m;s)}$

is conjugate to the torus rotation $R_{((\rho_1+s_1)/l,(\rho_2+s_2)/m)}$, where a conjugacy $h_{(l,m)}$ is simply given by rescaling, that is, $h_{(l,m)}(x,y) = (x/l, y/m)$. Obviously, this does not affect the arithmetical properties of the rotation vector (rational, irrational or totally irrational). We can also rescale the lifts $L^{\psi}_{(l,m;s)}$ to obtain homeomorphisms $\ell^{\psi}_{(l,m;s)} = h_{(l,m)} \circ L^{\psi}_{(l,m;s)} \circ h^{-1}_{(l,m)}$ of the standard torus $\mathbb{T}^2$. Further, given torus homeomorphisms φ, ψ homotopic to the identity, we have

$$d_k(\varphi,\psi) = \min_{s\in A(l,m)} d_k\left(L^{\varphi}_{(l,m;0)}, L^{\psi}_{(l,m;s)}\right) \geq \min_{s\in A(l,m)} d_k\left(\ell^{\varphi}_{(l,m;0)}, \ell^{\psi}_{(l,m;s)}\right) \tag{22}$$

5.1. Total strict ergodicity of the lifts. In order to ensure that all iterates of all lifts of the diffeomorphism φ from Theorem 1.1 are strictly ergodic as well, we may now modify the construction in Section 4 by replacing conditions (9)–(11) with the following stronger assumptions: we recursively choose sequences $s_n^{(l,m)}, \tilde{s}_n^{(l,m)}, \hat{s}_n^{(l,m)}$ such that

$$\begin{aligned} d_n\left(L^{\varphi_n}_{\left(l,m,s_n^{(l,m)}\right)}, L^{\tilde{\varphi}_n}_{\left(l,m,\tilde{s}_n^{(l,m)}\right)}\right) &= d_n\left(\varphi_n,\tilde{\varphi}_n\right) ,\\ d_n\left(L^{\tilde{\varphi}_n}_{\left(l,m,\tilde{s}_n^{(l,m)}\right)}, L^{\hat{\varphi}_n}_{\left(l,m,\hat{s}_n^{(l,m)}\right)}\right) &= d_n(\tilde{\varphi}_n,\hat{\varphi}_n) ,\\ d_n\left(L^{\hat{\varphi}_n}_{\left(l,m,\hat{s}_n^{(l,m)}\right)}, L^{\varphi_{n+1}}_{\left(l,m,s_{n+1}^{(l,m)}\right)}\right) &= d_n(\hat{\varphi}_n,\varphi_{n+1}) \end{aligned}$$

hold for all $n \in \mathbb{N}$. Then, in the n-th step of the induction, we exert control over the speed of convergence not only for the original maps $\varphi_n, \hat{\varphi}_n, \tilde{\varphi}_n$, but also for all iterates of all lifts up to level n. To that end, we require that for all $l,m,j = 1,\ldots,n$ we have

$$d_n\left(\ell^{\varphi_n,j}_{\left(l,m;s_n^{(l,m)}\right)}, \ell^{\tilde{\varphi}_n,j}_{\left(l,m;\tilde{s}_n^{(l,m)}\right)}\right) \leq \frac{1}{3}\min\left\{\eta^{\mathrm{se}}_{n-1}\left(\ell^{\hat{\varphi}_1,j}_{\left(l,m;\hat{s}_1^{(l,m)}\right)},\ldots,\ell^{\hat{\varphi}_{n-1},j}_{\left(l,m;\hat{s}_{n-1}^{(l,m)}\right)}\right), \eta^{\mathrm{me}}_{n-1}\left(\ell^{\tilde{\varphi}_1,j}_{\left(l,m;\tilde{s}_1^{(l,m)}\right)},\ldots,\ell^{\tilde{\varphi}_{n-1},j}_{\left(l,m;\tilde{s}_{n-1}^{(l,m)}\right)}\right)\right\} \tag{23}$$

$$d_n\left(\ell^{\tilde{\varphi}_n,j}_{\left(l,m;\tilde{s}_n^{(l,m)}\right)}, \ell^{\hat{\varphi}_n,j}_{\left(l,m;\hat{s}_n^{(l,m)}\right)}\right) \leq \frac{1}{3}\min\left\{\eta^{\mathrm{se}}_{n-1}\left(\ell^{\hat{\varphi}_1,j}_{\left(l,m;\hat{s}_1^{(l,m)}\right)},\ldots,\ell^{\hat{\varphi}_{n-1},l}_{\left(l,m;\hat{s}_{n-1}^{(l,m)}\right)}\right), \eta^{\mathrm{me}}_{n}\left(\ell^{\tilde{\varphi}_1,l}_{\left(l,m;\tilde{s}_1^{(l,m)}\right)},\ldots,\ell^{\tilde{\varphi}_{n},j}_{\left(l,m;\tilde{s}_{n}^{(l,m)}\right)}\right)\right\} \tag{24}$$

$$d_n\left(\ell^{\hat{\varphi}_n,j}_{\left(l,m;\hat{s}_n^{(l,m)}\right)}, \ell^{\varphi_{n+1},j}_{\left(l,m;s_{n+1}^{(l,m)}\right)}\right) \leq \frac{1}{3}\min\left\{\eta^{\mathrm{se}}_{n}\left(\ell^{\hat{\varphi}_1,j}_{\left(l,m;\hat{s}_1^{(l,m)}\right)},\ldots,\ell^{\hat{\varphi}_{n},j}_{\left(l,m;\hat{s}_{n}^{(l,m)}\right)}\right), \eta^{\mathrm{me}}_{n}\left(\ell^{\tilde{\varphi}_1,j}_{\left(l,m;\tilde{s}_1^{(l,m)}\right)},\ldots,\ell^{\tilde{\varphi}_{n},j}_{\left(l,m;\tilde{s}_{n}^{(l,m)}\right)}\right)\right\} \tag{25}$$

With the same reasoning as in Section 4, we now obtain that for any $(l,m) \in \mathbb{N}^2$ and $j \in \mathbb{N}$ the sequence $\ell^{\hat{\varphi}_n,j}_{\left(l,m,\hat{s}_n^{(l,m)}\right)}$ converges to a strictly ergodic diffeomorphism, which is a rescaled lift $\ell^{\varphi,j}_{(l,m,s)}$ of an iterate of $\varphi = \lim_{n\to\infty}\hat{\varphi}_n$.

Let $\psi = \ell^{\varphi}_{(1,m;0)}$. The invariant measure of φ is of the form $\mu = (\mathrm{Id}_{\mathbb{T}^1} \times \gamma)_*\mathrm{Leb}_{\mathbb{T}^1}$, where $\gamma : \mathbb{T}^1 \to \mathbb{T}^1$ is the measurable function whose graph supports μ. Consequently, if we let

$$\gamma_j : \mathbb{T}^1 \to \mathbb{T}^1 \quad , \quad x \mapsto \frac{\gamma(x)+j-1}{m} \;, \quad j = 1,\ldots,m, \tag{26}$$

then

$$\mu^{\psi} = \frac{1}{m}\sum_{j=1}^{m}(\mathrm{Id}_{\mathbb{T}^1} \times \gamma_j)_*\mathrm{Leb}_{\mathbb{T}^1} \tag{27}$$

defines an invariant measure for the rescaled lift ψ. By unique ergodicity, it is the only ψ-invariant measure. This proves assertions (a)–(c) of Theorem 1.2.

5.2. NON-EXISTENCE OF ADDITIONAL EIGENVALUES. We consider a torus diffeomorphism φ that satisfies the assertions (a)–(c) of Theorem 1.2. Fix $m \in \mathbb{N}$ and let $\psi = \ell^{\varphi}_{(1,m;0)}$ as above. Recall that both mappings are skew products over the irrational rotation $r_\alpha : x \mapsto x + \alpha$. Our aim is to show that ψ has the same discrete dynamical spectrum as φ, that is, there exist no additional dynamical eigenvalues for ψ.

Suppose that $\gamma : \mathbb{T}^1 \to \mathbb{T}^1$ is the measurable function whose graph supports the unique φ-invariant measure μ, that is, $\mu = (\mathrm{Id}_{\mathbb{T}^1} \times \gamma)_* \mathrm{Leb}_{\mathbb{T}^1}$. Then, as discussed in the previous section, the unique ψ-invariant measure μ^ψ is given by (27). Now, suppose for a contradiction that $f \in L^2_{\mu^\psi}(\mathbb{T}^2)$ is an eigenfunction of U_ψ with a new eigenvalue λ that is not contained in the group of eigenvalues $M(\alpha) = \{\exp(2\pi i k\alpha) \mid k \in \mathbb{Z}\}$ of U_φ. Then f cannot be constant in the fibres (that is, independent of the second coordinate y), since in this case $x \mapsto f(x,0)$ would define an eigenfunction of r_α with eigenvalue λ, contradicting the fact that the eigenvalue group of r_α is $M(\alpha)$ as well. Further, the function

$$g : \mathbb{T}^1 \to \mathbb{T}^1 \quad , \quad x \mapsto \prod_{j=1}^{m} f(x, \gamma_j(x))$$

is an eigenfunction of r_α with eigenvalue λ^m, since we have

$$\psi(\{(x,\gamma_1(x)),\ldots,(x,\gamma_m(x))\}) \;=\; \{(x+\alpha,\gamma_1(x+\alpha)),\ldots,(x+\alpha,\gamma_m(x+\alpha))\}$$

and therefore

$$g(x+\alpha) \;=\; \prod_{j=1}^{m} f \circ \psi(x,\gamma_j(x)) \;=\; \lambda^m \prod_{j=1}^{m} f(x,\gamma_j(x)) \;=\; \lambda^m g(x)$$

$\mathrm{Leb}_{\mathbb{T}^1}$-almost surely on $\mathbb{T}^1$. Hence, g is an eigenfunction of r_α. This implies $\lambda^m = \exp(2\pi i k\alpha)$ for some $k \in \mathbb{Z}$, so that λ must be of the form

$$\lambda \;=\; \exp\left(2\pi i \left(\frac{k\alpha + p}{m}\right)\right)$$

for some $(k,p) \in (\mathbb{Z} \times \{0,\ldots,m-1\}) \setminus (m\mathbb{Z} \times \{0\})$.

We first assume that $k = 0$. In this case, f is an eigenfunction of ψ^m with eigenvalue $\lambda^m = 1$. As f is non-constant, this contradicts the ergodicity of ψ^m.

Secondly, assume that $p = 0$. In this case, we consider the rescaled lift $\tilde{\psi} = \ell^{\psi}_{(1,m,0)}$ of ψ, which is now a skew product over the rotation $r_{\alpha/m}$. The eigenfunction f transforms to an eigenfunction

$$\tilde{f}(x,y) \;=\; f(mx,y)$$

of $U_{\tilde{\psi}}$, which still has the same eigenvalue $\lambda = \exp(2\pi i k\alpha/m)$. However, this is now an eigenvalue of the underlying rotation $r_{\alpha/m}$, which corresponds to the eigenfunction $g(x,y) = \exp(2\pi i k x/m)$ of $U_{\tilde{\psi}}$. As g is constant in y for ν-almost every x, but $\tilde{f}$ is not, the two eigenfunctions cannot coincide. Since they have the same eigenvalue, this contradicts the unique ergodicity of $\tilde{\psi}$ (note that $\tilde{\psi}$ is still a lift of the original map φ and is therefore uniquely ergodic).

Finally, we consider the case where $k \neq 0 \neq p$. In this case, f is an eigenfunction of ψ^m, with eigenvalue $\exp(2\pi i k\alpha)$. However, ψ^m is a rescaled lift of φ^m, which has the same properties as φ (it satisfies the assertions of Theorem 1.1), but has underlying rotation number $m\alpha$. This means that we are in exactly the same situation as in the case $p = 0$ above, and again arrive at a contradiction.

Altogether, this shows that ψ has exactly the same dynamical eigenvalues as φ. However, the two systems cannot be isomorphic, as measure-theoretic factor maps into group rotations are uniquely determined up to post-composition with a rotation and the canonical factor map from ψ to φ is m:1. Due to the Halmos-von Neumann Theorem, ψ and φ cannot have the same (purely discrete) dynamical spectrum. This means that the spectrum of U_ψ must have a continuous component.

5.3. SINGULARITY OF THE CONTINUOUS SPECTRAL COMPONENT. In order to complete the proof of Theorem 1.2, our aim now is to show that the Anosov-Katok construction of φ can be modified such that the map ψ defined in the last section has a singular continuous spectral component. To that end, we need to show that φ and all its lifts admit cyclic approximation by periodic transformations with speed $o(1/n)$, in the sense of Theorem 2.5. The main problem here lies in the fact that – unlike in Anosov-Katok construction in an area-preserving setting – the unique invariant measure μ of the transformation φ is not known *a priori*. Therefore, it is necessary to control both the size of the symmetric differences between the images of partition elements under φ_n and the eventual limit φ and also the limit measure of these sets at the same time. Recall that, in the end, we need to show that there exist suitable partitions $\mathcal{P}_n$ that satisfy conditions (P1)–(P3) from Section 2.2. This will exclude the existence of an absolutely continuous spectral component and thus complete the proof.

We adopt the notation from the main construction in Section 4. In particular, q_n is the denominator of the rotation vector ρ_n of the n-th approximating diffeomorphism φ_n. Note that ρ_n was chosen only after the n-th conjugating diffeomorphism H_n was defined. Hence, we can require that

$$d(x,y) < 2/q_n \ \Rightarrow \ d(H_n(x), H_n(y)) < 1/n \ . \tag{28}$$

We define the partition $\mathcal{P}_n$ as $\mathcal{P}_n = \{P_{n,i,j} \mid i,j = 0,\dots,q_n-1\}$, where

$$P_{n,i,j} = H_n\left([i/q_n,(i+1)/q_n) \times [j/q_n,(j+1)/q_n)\right) \ .$$

Note that $\varphi_n = H_n \circ R_{\rho_n} \circ H_n^{-1}$ cyclically permutes the elements of $\mathcal{P}_n$ due to the fact that $\rho_n = (p_n/q_n, p_n'/q_n)$ with p_n, p_n', q_n relatively prime (8). Moreover, due to (28), the maximal diameter of an element of $\mathcal{P}_n$ is at most $1/n$, which implies (P2) due to the regularity of the measure μ. Hence, both (P1) and (P2) are satisfied.

It remains to show (P3) with sufficiently fast speed of convergence. We choose an arbitrary function $s : \mathbb{N} \to \mathbb{R}^+$ which satisfies $\lim_{n\to\infty} ns(n) = 0$, so that Theorem 2.5 will be applicable. As $K_n = \sharp\,\mathcal{P}_n = q_n^2$, we have to ensure that

$$\sum_{i,j=0}^{q_n-1} \mu\left(\varphi_n\left(P_{n,i,j}\right) \triangle \varphi\left(P_{n,i,j}\right)\right) \ \le \ s(q_n^2) \ . \tag{29}$$

In order to do so, we need to introduce further inductive assumptions into the construction carried out in Section 4 that we already modified by (23)–(25) above.

Suppose that $n \in \mathbb{N}$ and H_{n+1} has already been chosen, but not the rotation vectors $\tilde{\rho}_n$, $\hat{\rho}_n$ and ρ_{n+1} (which then define $\tilde{\varphi}_n$, $\hat{\varphi}_n$ and φ_{n+1}). Let $\mu_n = (H_{n+1})_* \operatorname{Leb}_{\mathbb{T}^2}$ and note that independent of the choice $\hat{\rho}_n$ (assuming total irrationality), this is the unique invariant measure of $\hat{\varphi}_n = H_{n+1} \circ R_{\rho_n} \circ H_{n+1}^{-1}$. For $i,j = 1,\dots,q_n$, we choose continuous functions $f_{n,i,j} : \mathbb{T}^2 \to [0,1]$ such that

$$\partial\left(\varphi_n\left(P_{n,i,j}\right)\right) \ \subseteq \ \operatorname{int}\left(f_{n,i,j}^{-1}(1)\right)$$

and

$$\int_{\mathbb{T}^2} f_{n,i,j}\, d\mu_n \ < \ s(q_n^2)/q_n^2 \ .$$

For the latter condition, note that since φ_n simply permutes the elements of $\mathcal{P}_n$, the set $\partial\left(\varphi_n\left(P_{n,i,j}\right)\right)$ is simply the boundary of another partition element, and therefore a smooth curve that has measure zero with respect to μ_n (which has smooth density with respect to Lebesgue, since it is the image of the Lebesgue measure under the smooth diffeomorphism H_n).

If $\tilde{\rho}_n$ and subsequently $\hat{\rho}_n$ are chosen sufficiently close to ρ_n, so that $\tilde{\varphi}_n$ and $\hat{\varphi}_n$ are close to φ_n, then we have

$$\hat{\varphi}_n\left(P_{n,i,j}\right) \Delta \varphi_n\left(P_{n,i,j}\right) \ \subseteq \ \operatorname{int}\left(f_{n,i,j}^{-1}(1)\right) \tag{30}$$

By unique ergodicity, there exists $M_n \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{T}^2} \frac{1}{M_n} \sum_{l=1}^{M_n} f_{n,i,j} \circ \hat{\varphi}_n^l(x) \ < \ s(q_n^2)/q_n^2 \ . \tag{31}$$

Since both (30) and (31) are open conditions, we may now choose $\delta_n > 0$ such that, for all $\psi \in \overline{B_{\delta_n}(\hat{\varphi}_n)}$, the following conditions hold.

$$\psi\left(P_{n,i,j}\right)\Delta\varphi_n\left(P_{n,i,j}\right) \quad \subseteq \quad \operatorname{int}\left(f_{n,i,j}^{-1}(1)\right) \tag{32}$$

$$\sup_{x\in\mathbb{T}^2} \frac{1}{M_n} \sum_{l=1}^{M_n} f_{n,i,j} \circ \psi^l(x) \quad < \quad s(q_n^2)/q_n^2 \ . \tag{33}$$

We can now require, throughout the inductive construction in Section 4, that for all $n \in \mathbb{N}$ we have

$$d_0\left(\hat{\varphi}_n, \hat{\varphi}_m\right) \ \le \ \delta_m \qquad \text{for all } m = 1, \ldots, n-1 \ . \tag{34}$$

For this, when going from n to $n+1$, it suffices to ensure that

$$\max\left\{d_0\left(\hat{\varphi}_n, \varphi_{n+1}\right),\ d_0\left(\varphi_{n+1}, \tilde{\varphi}_{n+1}\right),\ d_0\left(\tilde{\varphi}_{n+1}, \hat{\varphi}_{n+1}\right)\right\} \ < \ \frac{1}{3}\min_{m=1}^{n} \delta_m - d_0\left(\hat{\varphi}_n, \hat{\varphi}_m\right) \ .$$

This, in turn, is simply achieved by a sufficiently small variation of the rotation vectors when choosing $\rho_{n+1}, \tilde{\rho}_{n+1}$ and $\hat{\rho}_{n+1}$. In particular, it does not contradict any other recursive assumptions that we have made elsewhere during the construction.

As a consequence, the resulting limit φ will still satisfy (32) and (33) (with ψ replaced by φ). However, if μ denotes the unique φ-invariant measure, then the above conditions imply that, for all $n \in \mathbb{N}$,

$$\mu\left(\varphi\left(P_{n,i,j}\right)\Delta\varphi_n\left(P_{n,i,j}\right)\right) \overset{(32)}{\le} \int_{\mathbb{T}^2} f_{n,i,j}\, d\mu \overset{(33)}{\le} s(q_n^2)/q_n^2 \ .$$

This proves (29), so that Theorem 2.5 yields the absence of singular continuous spectrum for U_φ. Hence, assertion (d) of Theorem 1.2 holds, which completes the proof.

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