ON LIFTS OF STRICTLY ERGODIC SUBSHIFTS BY PERMUTATIVE SLIDING BLOCK CODES

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ABSTRACT. Permutative sliding block codes – in the sense of Hedlund – give rise to finite-to-one extensions of subshifts. We provide criteria under which minimality and unique ergodicity are preserved by this 'lifting procedure'. These findings are illustrated by means of some natural example families, which we use to obtain strictly ergodic finite-to-one extensions of substitution subshifts (including the case of Fibonacci, Tribonacci, silver mean and noble means substitutions). We further study the interplay between permutative sliding block codes and Toeplitz flows. In particular, we find examples which demonstrate that a permutative lift of a strictly ergodic subshift may be minimal, but not uniquely ergodic – a phenomenon which cannot occur for primitive substitution subshifts.
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1. INTRODUCTION

1.1. **Context and overview.** The study of extension structures has a long tradition in ergodic theory and topological dynamics. Early seminal results include the structure theorems by Furstenberg and Veech for minimal distal flows [Fur63, Vee70] and the Furstenberg-Zimmer Structure Theorem for ergodic measure-preserving transformations [Zim76, Fur77]. Further progress with a particular view towards number-theoretic applications has been made in [HK05, HKM10] (see also [HK18]). In the context of entropy theory, symbolic extensions of smooth dynamical systems have been studied in [DN05, DM09, Bur11]. Apart from these particular examples, there exists a wealth of further results on extension structures of dynamical systems in the literature – we restrict here to cite only to a biased selection [AG77, FG78, SY12, GGY18, GHS$^+$25, QY25].

More recently, different topological and measure-theoretic concepts of finite extension structures and their relations to dynamical properties have received considerable attention (e.g. [LTY15, DG16, Gla18, FGJO21, GRJY21, BHJ24, GRLT24, LXZ25]). At the same time, multivariate versions of classical notions like stability, sensitivity, equicontinuity or tameness have equally attracted significant interest [HLSY21, LY21, LYY22, LY23, LLTY24]. These are again closely related to respective notions of finite extension structures [SYZ08, HLSY21, BHJ24, GRLT24, Hau25, LXZ25, Liu26]. From a broader viewpoint, these developments contribute towards a structure theory of zero entropy systems in topological dynamics.

In this context, the construction and description of examples is of vital interest. The problem is that the main interest does not lie in arbitrary finite-to-one extension of a given system – otherwise the direct product with any finite system would do – but rather in those extensions that keep essential dynamical properties of the underlying system. In particular, much of the above-mentioned theory on finite extensions has been developed in the setting of minimal or even strictly ergodic systems, so that finite extensions which preserve these properties are of particular interest. However, the construction of minimal or strictly ergodic extensions is in general a non-trivial problem and equally has a long history (e.g. [Anz51, Fur61, GW79, Lem87, Lem88, HPS92, Bje05, CP08, Jäg09, DG16, DPS21, HJ25]).

In this article, we focus on a specific pathway that leads to a wealth of new examples of strictly ergodic finite extensions of symbolic systems. The principal idea already goes back to the work of Hedlund on sliding block codes [Hed69], but it has apparently never been fully exploited in this context so far. We consider the class of *permutative* sliding block codes, which has been described in some detail in [Hed69, Sections 6 and 17]. One of their structural properties is the fact that they are always constant-to-one endomorphisms of the whole shift space. This means that they can be used to lift any subshift to a finite-to-one extension by simply taking the preimage, which we will refer to as a *permutative lift* of the subshift. However, it is a priori not clear to what extent dynamical properties are preserved by this procedure. The main questions that we will focus on here are the following.

- If φ is a permutative sliding block code and Y is a minimal subshift, is the extension $\varphi^{-1}(Y)$ minimal as well?
- If φ is a permutative sliding block code and Y is a uniquely ergodic subshift, is the extension $\varphi^{-1}(Y)$ uniquely ergodic as well?

It turns out that there is no general answer to these questions. In fact, on the one hand it is easy to see that even a fixed point of the shift can have non-minimal, and therefore also non-uniquely ergodic, lifts under permutative sliding block codes (a simple example will be given below). On the other hand, the Thue-Morse subshift, viewed as an extension of the period doubling subshift, provides a paradigmatic example in which strict ergodicity is perserved during this lifting procedure. Moreover, the answers to the above questions can be independent – in Section 6, we provide an example of a strictly ergodic Toeplitz flow whose lift under a simple permutative sliding block code is minimal, but not uniquely ergodic.

Hence, our aim is to establish criteria – both in terms of the subshift Y and the sliding block code φ – which allow to ensure that the lift of a strictly minimal/uniquely ergodic subshift Y under φ is again minimal/uniquely ergodic. It turns out that the results we obtain hinge on the study of the behaviour of certain *return permutations* associated to a permutative sliding block code, which thus turn out to be a crucial concept in our context. We are convinced that these considerations will equally be instrumental in addressing a wealth of further question that naturally occur in this setting, in particular concerning further dynamical spectral properties and quantities or the study of broader classes of examples. The latter may also serve as a useful testing ground in the context of the above-mentioned recent developments in the structure theory of topologial dynamics.

1.2. **Presentation of the main results.** In order to give a more precise statement of the main concepts and results, we need to introduce some notation and terminology. Given a finite alphabet $\mathcal{A}_\kappa = \{0, \dots, \kappa-1\}$, $\kappa \in \mathbb{N}$, we call $\mathcal{A}_\kappa^{\mathbb{Z}}$ the *shift space* over the alphabet $\mathcal{A}_\kappa$ and define the *(left) shift map* $\sigma : \mathcal{A}_\kappa^{\mathbb{Z}} \to \mathcal{A}_\kappa^{\mathbb{Z}}$ by $\sigma(x)_n = x_{n+1}$ for all $n \in \mathbb{Z}$. By $\mathcal{A}_\kappa^* = \bigcup_{n\in\mathbb{N}_0} \mathcal{A}_\kappa^n$, we denote the set of all finite words with letters in $\mathcal{A}_\kappa$ and write $|w|$ for the length of $w \in \mathcal{A}_\kappa^*$. Given $y \in \mathcal{A}_\kappa^{\mathbb{Z}}$ and integers $k < n$, we let $y_{[k,n]} = y_k \dots y_n$. A *subshift* Y is a closed σ-invariant subset of $\mathcal{A}_\kappa^{\mathbb{Z}}$ and $\mathcal{W}(Y)$ denotes the set of all finite words that occur in sequences of Y.

A continuous map $\varphi : \mathcal{A}_\kappa^{\mathbb{Z}} \to \mathcal{A}_\kappa^{\mathbb{Z}}$ is called a *shift endomorphism* if it commutes with the shift map, that is, $\varphi \circ \sigma = \sigma \circ \varphi$. A *block code* of length $\ell \in \mathbb{N}$ is a map $h : \mathcal{A}_\kappa^\ell \to \mathcal{A}_\kappa$, and we denote by $\mathcal{F}(\kappa, \ell)$ the family of such mappings. Each block code h induces a shift endomorphism η_h, referred to as a *sliding block code*, by

$$\eta_h(x)_n \;=\; h\left(x_{[n,n+\ell-1]}\right)$$

for all $n \in \mathbb{Z}$. Similarly, for each $m \in \mathbb{N}$, h defines a map $h_m : \mathcal{A}_\kappa^{m+\ell-1} \to \mathcal{A}_\kappa^m$ by $h_m(v)_n = h\left(v_{[n,n+\ell-1]}\right)$, $n = 0, \dots, m-1$. The seminal Curtis-Hedlund-Lyndon Theorem states that, up to post-composition with a shift iterate, every shift endomorphism is given by a sliding block code [Hed69].

Following [Hed69, Section 6], we call a block code $h \in \mathcal{F}(\kappa, \ell)$ (and likewise its associated sliding block code η_h) *permutative* if for all $u \in \mathcal{A}_\kappa^{\ell-1}$ the mappings

$$a \mapsto h(ua) \quad \text{and} \quad a \mapsto h(au)$$

are permutations of the alphabet $\mathcal{A}_\kappa$. In this context, we will call elements of $\mathcal{A}_\kappa^{\ell-1}$ *prefixes* or *suffixes*, depending on the situation. An important observation is the following: given a permutative block code $h \in \mathcal{F}(\kappa, \ell)$, a prefix $u \in \mathcal{A}_\kappa^{\ell-1}$ and an arbitrary sequence $y \in \mathcal{A}_\kappa^{\mathbb{Z}}$, the permutative structure of h allows to construct – via a straightforward induction – a unique sequence $\xi^u(y)$ with prefix u and image y under η_h. More precisely, we have

Proposition 1.1 ([Hed69]). *Let $h \in \mathcal{F}(\kappa,\ell)$ be permutative. Then for every $u \in \mathcal{A}_\kappa^{\ell-1}$ there exists a continuous mapping*

$$\xi^u : \mathcal{A}_\kappa^{\mathbb{Z}} \to \mathcal{A}_\kappa^{\mathbb{Z}} \quad , \quad y \mapsto \xi^u(y)$$

such that $\xi^u(y)_{[0,\ell-2]} = u$ and $\eta_h(\xi^u(y)) = y$ holds for all $y \in \mathcal{A}_\kappa^{\mathbb{Z}}$. Further, we have

$$\eta_h^{-1}(y) \;=\; \left\{\xi^u(y) \mid u \in \mathcal{A}_\kappa^{\ell-1}\right\} \;, \tag{1.1}$$

and consequently $\#\eta_h^{-1}(y) = \kappa^{\ell-1}$ for all $y \in \mathcal{A}_\kappa^{\mathbb{Z}}$.

Moreover, if $u \neq u' \in \mathcal{A}_\kappa^{\ell-1}$, the sequences $\xi^u(y)$ and $\xi^{u'}(y)$ are $(\ell-1)$-separated, meaning that they do not coincide on any interval of $\ell-1$ consecutive integers.

The mappings ξ^u will be called *(continuous) inverse branches* of η_h.

We should point out that some of these assertions are not explicitely stated in [Hed69], while others are obtained as consequences of more general results on surjective sliding block codes. We therefore provide a short direct proof in Section 3.

The inductive construction of the sequences $\xi^u(y)$ also works on a finite level, so that for each finite word $w \in \mathcal{A}_\kappa^*$ and each prefix $u \in \mathcal{A}_\kappa^{\ell-1}$ there exists a unique word $\xi^u(w) \in \mathcal{A}_\kappa^{|w|+\ell-1}$ such that $\xi^u(w)_{[0,\ell-2]} = u$ and $h_{|w|}(\xi^u(w)) = w$. This allows to associate, to any given permutative block code h, the following family of permutations on the set of prefixes. Given a finite word $w \in \mathcal{A}^*$, we define

$$\Phi_w^{(h)} : \mathcal{A}_\kappa^{\ell-1} \to \mathcal{A}_\kappa^{\ell-1} \quad , \quad u \mapsto \xi^u(w)_{[|w|,|w|+\ell-2]}$$

and call this the *prefix permutation induced by* w. In other words, $\Phi_w^{(h)}(u)$ is defined as the suffix of the unique lift $\xi^u(w)$ of the word w with prefix u.

Proposition 1.2. *Suppose that $h \in \mathcal{F}(\kappa,\ell)$ is permutative. Then for every $w \in \mathcal{A}_\kappa^*$, the mapping $\Phi_w^{(h)}$ is a permutation of $\mathcal{A}_\kappa^{\ell-1}$. Moreover, given $w, v \in \mathcal{A}_\kappa^*$, we have*

$$\Phi_{wv}^{(h)} \;=\; \Phi_v^{(h)} \circ \Phi_w^{(h)} \;.$$

The proof is given in Section 3. Altogether, the above concepts lead to a useful structural result, namely a skew-product representation for permutative lifts of subshifts.

Theorem 1.3. *Suppose that $h \in \mathcal{F}(\kappa,\ell)$ is a permutative block code, $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a subshift and $X = \eta_h^{-1}(Y)$. Let*

$$\gamma : X \to Y \times \mathcal{A}_\kappa^{\ell-1} \quad , \quad x \mapsto \left(\eta_h(x), x_{[0,\ell-2]}\right) \tag{1.2}$$

and

$$T : Y \times \mathcal{A}_\kappa^{\ell-1} \to Y \times \mathcal{A}_\kappa^{\ell-1} \quad , \quad (y,u) \mapsto (\sigma(y), \Phi_{y_0}(u)) \;. \tag{1.3}$$

Then γ and T are homeomorphisms and γ is a conjugacy between (X,σ) and $(Y \times \mathcal{A}_\kappa^{\ell-1}, T)$.

A particularly significant are prefix permutations induced by return words: we call

$$\mathcal{R}_h^Y(w) \;=\; \{\Phi_{wv} \mid wvw \in \mathcal{W}(Y)\}$$

the set of *return permutations* for the word w.

Remark 1.4. *We note that $\mathcal{R}_h^Y(w)$ could also be replaced by the possibly larger set $\tilde{\mathcal{R}}_h^Y(y) = \{\Phi_v \mid vw \in \mathcal{W}(Y)$ and vw starts with $w\}$. All the results below still hold in this case and the proofs remain valid with only minor technical modifications.*

Further, we say a subset P of the permutation group $\mathrm{Per}\left(\mathcal{A}_\kappa^{\ell-1}\right)$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$ if this holds for the subgroup $\langle P\rangle$ generated by P. These notions now allow to state equivalent criteria for the minimality of permutative lifts.

Theorem 1.5. *Suppose that $h \in \mathcal{F}(\kappa,\ell)$ is a permutative block code and $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a minimal subshift. Let $X = \eta_h^{-1}(Y)$. Then the following are equivalent:*

(i) (X,σ) *is minimal;*
(ii) *for every word $w \in \mathcal{W}(Y)$, the return permutation set $\mathcal{R}(w)$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$;*

(iii) there exists a sequence of words $w^{(n)} \in \mathcal{W}(Y)$ such that $|w^{(n)}| \xrightarrow{n\to\infty} \infty$ and for all $n \in \mathbb{N}$ the return permutation set $\mathcal{R}(w^{(n)})$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$;
(iv) there exists a sequence of words $w^{(n)} \in \mathcal{W}(Y)$ such that $w^{(n+1)}$ starts with $w^{(n)}$, $|w^{(n)}| \xrightarrow{n\to\infty} \infty$ and for all $n \in \mathbb{N}$ the family $\bigcup_{k\geqslant n} \mathcal{R}(w^{(k)})$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$.

The proof is given in Section 4.1. As we will see below, the last formulation in (iv) is tailor-made for the application to substitution subshifts.

In order to ensure the unique ergodicity of permutative lifts, there are actually two possibilities. First, we establish a general criterion, similar in spririt to Theorem 1.5, based on the concept of return permutations. This will be used in the study of (permutative lifts of) Toeplitz flows in Section 6. Since its formulation requires some additional notation and technical preparations, we refer to Section 5 for the precise statement. For the special case of primitive substitution subshifts, it turns out that one can alternatively show that whenever a permutative lift is minimal, it is linearly recurrent (and hence uniquely ergodic [Dur00][Bru22]). While a general proof of this fact will be given in [GJK], we provide a short direct argument for the case of the noble means substitutions in Section 5.1.

In order to illustrate our findings and to demonstrate that the above minimality criteria can be applied efficiently, we consider a natural two-parameter family of permutative block codes, given by

$$h_{\kappa,\ell} : \mathcal{A}_\kappa^\ell \to \mathcal{A}_\kappa \quad , \quad x_0 \dots x_{\ell-1} \mapsto x_{\ell-1} - x_0 \bmod \kappa \ ,$$

with integer parameters $\kappa, \ell \geqslant 2$. For simplicity, we write $\eta_{\kappa,\ell}$ instead of $\eta_{h_{\kappa,\ell}}$. We then apply these to obtain permutative lifts of some classical examples of substitution subshifts. Recall that the noble means substitution with parameter $p \in \mathbb{N}$ on the alphabet $\mathcal{A}_2 = \{0,1\}$ is given by the substitution rule $0 \mapsto 0^p1,\ 1 \mapsto 0$. It includes the Fibonacci substitution for $p = 1$ and is conjugate to the silver mean substitution for $p = 2$ [BG13, Section 4.4].

Theorem 1.6. *Let $X_{\mathrm{nm},p}$ denote the subshift generated by the noble means substitution with parameter $p \in \mathbb{N}$. Then $\eta_{\kappa,\ell}^{-1}(X_{\mathrm{Fib}})$ is strictly ergodic for all $\kappa, \ell \geqslant 2$.*

The statement remains valid when the noble means substitution is replaced by the the Tribonacci substitution (where $\kappa \geqslant 3$ is required for Tribonacci). The details are given in Sections 4.3 (for minimality) and 5.1 (for unique ergodicity). A very interesting case is the period doubling substitution, induced by the substitution rule $\Gamma_{\mathrm{pd}} : \mathcal{A}_2 \mapsto \mathcal{A}_2^*$, $1 \mapsto 10$, $0 \mapsto 11$. In this case, the minimality of the lift depends on the values of κ and ℓ.

Theorem 1.7. *Let X_{pd} denote the period doubling subshift. Then the lift $X = \eta_{\kappa,\ell}^{-1}(X_{\mathrm{pd}})$ is strictly ergodic if $\ell \geqslant 2$ is even and if $\ell = 3$ and κ is odd. In all other cases, X is non-minimal.*

We note that this statement covers a very classical example, which is the factor relation between the period doubling subshift X_{pd} and the Thue-Morse subshift X_{TM}. In fact, the latter coincides with the lift of X_{pd} under the sliding block code $\eta_{2,2}$. Moreover, lifts of Toeplitz subshifts via $\eta_{2,2}$ have been studied in the context of $\mathbb{Z}_2$-extensions and their spectral theory (e.g. [Lem87, Lem88, FKL88]). We also note that $\eta_{2,2}^2 = \eta_{2,3}$. Hence, the lift of the Thue-Morse subshift under $\eta_{2,2}$ conincides with $\eta_{2,3}^{-1}(X_{\mathrm{pd}})$ and is therefore not minimal by Theorem 1.6. An easier example of a subshift with a non-minimal lift under $\eta_{2,2}$ is given by the fixed point $0^{\mathbb{Z}}$, with preimages $0^{\mathbb{Z}}$ and $1^{\mathbb{Z}}$.

The prefix permutations for $h_{2,2}$ are particularly easy to determine: we have

$$\Phi_w^{h_{2,2}} : \mathcal{A}_2 \to \mathcal{A}_2 \quad , \quad a \mapsto a + \sum_{j=0}^{|w|-1} w_j \mod 2 \ .$$

This entails the following simplified minimality criterion for the special case of $h_{2,2}$.

Corollary 1.8. *Let $Y \subseteq \mathcal{A}_2^{\mathbb{Z}}$ be a minimal subshift. Then $X = \eta_{2,2}^{-1}(Y)$ is minimal if and only if there exists sequences of words $w^{(n)}, v^{(n)} \in \mathcal{W}(Y)$ such that $|w^{(n)}| \xrightarrow{n\to\infty} \infty$, $w^{(n)}v^{(n)}w^{(n)} \in \mathcal{W}(Y)$ and $\sum_{j=0}^{|w^{(n)}v^{(n)}|-1} (w^{(n)}v^{(n)})_j$ is odd for all $n \in \mathbb{N}$.*

A more detailed discussion is given in Section 4.2. Finally, we study the interplay between permutative sliding block codes and Toeplitz subshifts in Section 6. The main results can be summarised as follows.

Theorem 1.9. *(a) For any permutative block code h, there exists a regular Toeplitz subshift Y such that $\eta_h^{-1}(Y)$ is strictly ergodic.*

(b) For any permutative block code h, there exists a regular Toeplitz subshift Y such that $\eta_h^{-1}(Y)$ is not minimal.

(c) For any Toeplitz subshift Y, there exists a permutative block code h such that $\eta_h^{-1}(Y)$ is not minimal.

(d) There exists a regular Toeplitz subshift Y and a permutative block code h such that $\eta_h^{-1}(Y)$ is minimal, but not uniquely ergodic.

These assertions are restated and proven below as (a) Theorem 6.14, (b) Proposition 6.12, (c) Theorem 6.11 and (d) Theorem 6.13. In all cases, the block code and the Toeplitz subshift share the same alphabet. One question which we leave open here is the following:

Question 1.10. *Given a minimal/strictly ergodic (Toeplitz) subshift $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$, does there exist a permutative sliding block code $h \in \mathcal{F}(\kappa, \ell)$ of length $\ell \geqslant 2$ such that $X = \eta_h^{-1}(Y)$ is minimal/strictly ergodic?*

The article is organised as follows: in Section 2, we provide the required background knowledge on symbolic and topological dynamics and sliding block codes. In Section 3, we consider inverse branches and induced prefix permutations of permutative sliding block codes and provide a proof of the skew product representation in Theorem 1.3. In Section 4, we prove the equivalence of the minimality criteria in Theorem 1.5 and apply these to the lifts of substitution systems under the parameter family $\eta_{\kappa,\ell}$ introduced above. The question of unique ergodicity is treated in Section 5. Finally, in Section 6 we take a look at Toeplitz flows and their lifting properties in order to prove the assertions stated in Theorem 1.9.

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2. Preliminaries

2.1. **Shift spaces.** As before, given $\kappa \in \mathbb{N}$ we let $\mathcal{A}_\kappa = \{0, \ldots, \kappa - 1\}$ and $\mathcal{A}_\kappa^* = \bigcup_{n \in \mathbb{N}_0} \mathcal{A}_\kappa^n$, where $\mathcal{A}_\kappa^0 = \{e\}$, where e is the empty word. We further define $|w|$ as the length of the word $w \in \mathcal{A}_\kappa^*$, so that $w = w_0 \ldots w_{|w|-1}$. Given two words w, v with $|w| \leqslant |v|$, we write $w \lhd v$ if there exists $k \in \{0, \ldots, |v| - |w|\}$ such that $w_j = v_{k+j}$ for all $j = 0, \ldots, |w| - 1$ and say w *is contained in* v in this case. The same notation and terminology is used when v is an infinite (one- or two-sided) sequence. The concatenation of two words $w, v \in \mathcal{A}_\kappa^*$ is written as wv, and the same notation is used if v is an infinite one-sided sequence. The concatenation of p copies of a finite word w will be denoted by w^p. Given $k < m \in \mathbb{Z}$ and $x \in \mathcal{A}_\kappa^{\mathbb{Z}}$, we write $x_{[k,m]} = x_k \ldots x_m$. Analogous notation will be used for finite words in $\mathcal{A}_\kappa^*$ or one-sided sequences in $\mathcal{A}_\kappa^{\mathbb{N}}$. Given any subset $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$, we let

$$\mathcal{W}(Y) = \{w \in \mathcal{A}_\kappa^* \mid \exists y \in Y : w \lhd y\} .$$

and $\mathcal{W}_n(Y) = \mathcal{W}(Y) \cap \mathcal{A}_\kappa^n$ for $n \in \mathbb{N}$. If $Y = \{\xi\}$, we write $\mathcal{W}(\xi)$ instead of $\mathcal{W}(\{\xi\})$.

We equip the shift space $\mathcal{A}_\kappa^{\mathbb{Z}}$ with the metric $d(x, y) = e^{-\min\{|n| \mid n \in \mathbb{Z},\, x_n \neq y_n\}}$, which makes it a compact and totally disconnected metric space. Given a finite word $w \in \mathcal{A}_\kappa^{\{-n \ldots n\}}$, we define the corresponding *cylinder set of level* n as

$$[w]_n = \left\{x \in \mathcal{A}_\kappa^{\mathbb{Z}} \mid x_k = w_k \text{ for all } k = -n, \ldots, n\right\} .$$

Similarly, given $x \in \mathcal{A}_\kappa^{\mathbb{Z}}$, we let $[x]_n = [x_{-n} \dots x_n]_n$. Note that thus $[x]_n = B_{e^{-n}}(x)$, where as usual $B_\delta(x)$ denotes the open ball of radius δ around x. More generally, we also consider finite sets $M \subseteq \mathbb{Z}$ of positions and 'words' $w \in \mathcal{A}^M$, and then write

$$[w]_M \;=\; \left\{x \in \mathcal{A}^{\mathbb{Z}} \mid x_k = w_k \text{ for all } k \in M\right\} \ .$$

Note that thus $[w]_n = [w]_{\{-n,\dots,n\}}$ (and in particular $[w]_n \neq [w]_{\{n\}}$ if $n \neq 0$). Finally, given $w \in \mathcal{A}_\kappa^n$, we will also use the notation

$$[w]_n^+ \;=\; [w]_{\{0,\dots,n-1\}} \;=\; \left\{x \in \mathcal{A}_\kappa^{\mathbb{Z}} \mid x_k = w_k \text{ for all } k = 0,\dots,n-1\right\} \ .$$

By $\sigma : \mathcal{A}_\kappa^{\mathbb{Z}} \to \mathcal{A}_\kappa^{\mathbb{Z}},\ (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$, we denote the (left) shift map on $\mathcal{A}_\kappa^{\mathbb{Z}}$. For any collection of words $W \subseteq \mathcal{A}_\kappa^*$, we let

$$X_W \;=\; \left\{x \in \mathcal{A}_\kappa^{\mathbb{Z}} \mid \mathcal{W}(x) \subseteq W\right\} \tag{2.1}$$

and also write X_ξ instead of $X_{W(\xi)}$. This obviously defines a (possibly empty) subshift in $\mathcal{A}_\kappa^{\mathbb{Z}}$.

2.2. **Shift endomorphisms as factor maps between subshifts.** A continuous mapping $\eta : \mathcal{A}_\kappa^{\mathbb{Z}} \to \mathcal{A}_\kappa^{\mathbb{Z}}$ that satisfies $\sigma \circ \eta = \eta \circ \sigma$ is called a *shift endomorphism*. For any $\ell \in \mathbb{N}$, we denote the set of *block codes* of length ℓ by

$$\mathcal{F}(\kappa,\ell) \;=\; \left\{h : \mathcal{A}_\kappa^\ell \to \mathcal{A}_\kappa\right\} \ .$$

Given $m \geqslant 1$, any $h \in \mathcal{F}(\kappa,\ell)$ induces mappings $h_m : \mathcal{A}_\kappa^{m+\ell-1} \to \mathcal{A}_\kappa^m$ and a map $\eta_h : \mathcal{A}_\kappa^{\mathbb{Z}} \to \mathcal{A}_\kappa^{\mathbb{Z}}$, which are given by

$$h_m(w)_j \;=\; h\left(w_{[j,j+\ell-1]}\right) \quad \text{for } j = 0,\dots,m-1\ , \tag{2.2}$$

$$\eta_h(x)_j \;=\; h\left(x_{[j,j+\ell-1]}\right) \quad \text{for } j \in \mathbb{Z}\ , \tag{2.3}$$

where $w \in \mathcal{A}_\kappa^{m+\ell-1}$ and $x \in \mathcal{A}_\kappa^{\mathbb{Z}}$, respectively. The map η_h will be referred to as the the *sliding block code* induced by h.

A closed σ-invariant subset $X \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is called a *subshift*. Given $\ell \in \mathbb{N}$, we denote the collection of all maps $h : \mathcal{W}_\ell(X) \to \mathcal{A}$ by $\mathcal{F}_X(\kappa,\ell)$. Then, similar to above, any $h \in \mathcal{F}_X(\kappa,\ell)$ induces mappings

$$h_m : \mathcal{W}_{m+\ell-1}(X) \to \mathcal{A}_\kappa^m \quad \text{and} \quad \eta_h^X : X \to \mathcal{A}_\kappa^{\mathbb{Z}}\ ,$$

which are again defined by (2.2) and (2.3), respectively. If we let $Y = \eta_h^X(X)$, then η_h^X is a factor map from (X,σ) to (Y,σ). Recall here that $\eta : X \to Y$ is a *factor map* if η is continuous and surjective and $\sigma \circ \eta = \eta \circ \sigma$. Note also that any block code $h \in \mathcal{F}_X(\kappa,\ell)$ can be extended to a block code in $\mathcal{F}(\kappa,\ell)$, since the images of any words in $\mathcal{A}_\kappa^\ell \backslash \mathcal{W}_\ell(X)$ can be chosen freely in such an extension.

The following proposition is a version of the classical Curtis-Hedlund-Lyndon Theorem [Hed69] restricted to subshifts (and the original statement is recovered when $X = Y = \mathcal{A}^{\mathbb{Z}}$). The proof does not differ from [Hed69] in any essential way, but it is included for the convenience of the reader.

Theorem 2.1 (Curtis-Hedlund-Lyndon). *Suppose that $X, Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ are subshifts and $\eta : X \to Y$ is a factor map. Then there exist $k \in \mathbb{Z}$, $\ell \in \mathbb{N}$ and $h \in \mathcal{F}_X(\kappa,\ell)$ such that $\eta = \sigma^{-k} \circ \eta_h^X$.*

Note that if φ is a factor map between the subshifts (X,σ) and (Y,σ), then so is $\eta = \sigma^k \circ \varphi$. Hence, we will always assume without loss of generality that $k = 0$.

Proof. As φ is continuous, so is the map $\varphi_0 : X \to \mathcal{A}_\kappa,\ x \mapsto \varphi(x)_0$. Since φ_0 is finite-valued, it must be locally constant. Due to compactness, this means that X can be written as a finite union of cylinder sets $C_1,\dots,C_n$ such that φ_0 is constant on each C_i. If k is the maximal level of these cylinder sets, then $\varphi_0(x) = \varphi(x)_0$ only depends on $x_{-k},\dots,x_k$. Hence, there exists a mapping $h : \mathcal{A}_\kappa^{2k+1} \to \mathcal{A}_\kappa$ such that $\varphi(x)_0 = h\left(x_{-k}\dots x_k\right) = h\left(\sigma^{-k}(x)_{[0,2k+1]}\right)$. The fact that φ commutes with the shift σ now yields $\varphi(x)_n = h\left(\sigma^{-k}(x)_{[n,n+2k+1]}\right)$, which means that $\varphi = \eta_h \circ \sigma^{-k} = \sigma^{-k} \circ \eta_h$ as claimed. □

A subshift Y is called *linearly recurrent* or *linearly repetitive* if there exists some constant $L \in \mathbb{N}$ such that $w \lhd v$ holds for all $w, v \in \mathcal{W}(Y)$ whenever $|v| \geqslant L|w|$.

2.3. **Topological dynamics.** Throughout this article, a *topological dynamical system (tds)* will be a pair (X, T), where X is a compact metric space and $T : X \to X$ a homeomorphism. The *orbit* of a point $x \in X$ under T is given by $\mathcal{O}_T(x) = \{T^n(x) \mid n \in \mathbb{Z}\}$. Similarly, the *forward* and *backward* orbits of $x \in X$ are defined as $\mathcal{O}_T^+(x) = \{T^n(x) \mid n \geqslant 0\}$ and $\mathcal{O}_T^-(x) = \{T^n(x) \mid n \leqslant 0\}$, respectively. The tds (X, T) is called *minimal* if all orbits are dense, that is, $\overline{\mathcal{O}_T(x)} = X$ for all $x \in X$. In this case, all forward and backward orbits are dense as well. In general, a point $x \in X$ is called *almost periodic* (with respect to T) if $\left(\overline{\mathcal{O}_T(x)}, T\right)$ is minimal. Note that when X is a subshift, then almost periodicity of a sequence $x \in X$ is equivalent to the fact that every subword $w \lhd x$ occurs infinitely often in x, with uniformly bounded gaps between successive occurrences.

A tds (X, T) is called *uniquely ergodic* if there exists a unique T-invariant probability measure μ on X. This is equivalent to the fact that

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} f \circ T^j(x) \;=\; \int_X f \, d\mu \tag{2.4}$$

holds for all $x \in X$. In this case, the convergence in (2.4) is uniform in $x \in X$. A tds (X, T) is called *strictly ergodic* if it is both minimal and uniquely ergodic. We note that this holds for all linearly recurrent subshifts.

3. Permutative block codes: inverse branches and skew product form

As mentioned, a block code $h \in \mathcal{F}(\kappa, \ell)$ is called *permutative* if the two mappings

$$\rho_u^{(h)} \;:\; \mathcal{A}_\kappa \to \mathcal{A}_\kappa \quad , \quad a \mapsto h(ua) \tag{3.1}$$

$$\lambda_u^{(h)} \;:\; \mathcal{A}_\kappa \to \mathcal{A}_\kappa \quad , \quad a \mapsto h(au) \tag{3.2}$$

are both permutations of the alphabet $\mathcal{A}_\kappa$ for all $u \in \mathcal{A}_\kappa^{\ell-1}$. when no ambiguity can arise, we will drop the reference to h and simply write $\rho_u,\ \lambda_u$. The block code h is called *right permutative* if the ρ_u are permutations for all $u \in \mathcal{A}_\kappa^{\ell-1}$, and *left permutative* if this holds for the λ_u. [Hed69, Section 17] shows that $h \in \mathcal{F}(\kappa, \ell)$ is permutative if and only if $\#\eta_h^{-1}(y) = \kappa^{\ell-1}$. Non-permutative sliding block codes can be constant-to-one as well, but in this case the fibre cardinality must be strictly smaller. An example is given in [Hed69, Section 6].

Now, suppose that h is permutative, $w \in \mathcal{A}_\kappa^m$ for some $m \geqslant 1$, $a \in \mathcal{A}_\kappa$ and $v \in h_m^{-1}(w)$. If $v_{[m,m+\ell-2]} = u$ then $v\rho_u^{-1}(a)$ is the unique word of length $m + \ell$ that is mapped to wa by h_{m+1}. Likewise, if $v_{[0,\ell-2]} = u$, then $\lambda_u^{-1}(a)v$ is the unique word of length $m + \ell$ which is mapped to aw. Using this fact, a straightforward induction shows that for every finite word $w \in \mathcal{A}_\kappa^*$ and $u \in \mathcal{A}_\kappa^{\ell-1}$ there exist unique words $\xi^u(w), \zeta^u(w) \in \mathcal{A}_\kappa^{|w|+\ell-1}$ such that

$$h_{|w|}(\xi^u(w)) \;=\; h_{|w|}(\zeta^u(w)) \;=\; w$$

and $\xi^u(w)$ starts with u, whereas $\zeta^u(w)$ ends with u. Similarly, for any sequence $y \in \mathcal{A}_\kappa^{\mathbb{Z}}$ and a prefix $u \in \mathcal{A}_\kappa^{\ell-1}$, there exists a unique sequence $\xi^u(y) \in \mathcal{A}_\kappa^{\mathbb{Z}}$ such that

$$\xi^u(y)_{[0,\ell-2]} \;=\; u \quad \text{ and } \quad \eta_h\left(\xi^u(y)\right) \;=\; y \;. \tag{3.3}$$

Given a permutative block code $h \in \mathcal{F}(\kappa, \ell)$ and $w \in \mathcal{A}_\kappa^*$, this allows to define the *prefix permutation induced by* w as

$$\Phi_w^{(h)} : \mathcal{A}_\kappa^{\ell-1} \to \mathcal{A}_\kappa^{\ell-1} \;,\; u \mapsto \xi^u(w)_{[|w|,|w|+\ell-2]} \;. \tag{3.4}$$

In the same way, we define the *suffix permutation induced by* w as

$$\Psi_w^{(h)} : \mathcal{A}_\kappa^{\ell-1} \to \mathcal{A}_\kappa^{\ell-1} \quad , \quad u \mapsto \zeta^u(w)_{[0,\ell-2]} \;. \tag{3.5}$$

As a direct consequence of the definitions, we obtain that $\Phi_w(\Psi_w(u)) = u$ and $\Psi_w(\Phi_w(u)) = u$ for all $u \in \mathcal{A}_\kappa^{\ell-1}$. Hence, these maps are inverse to each other and must therefore be permutations of $\mathcal{A}_\kappa^{\ell-1}$. We now prove Propositions 1.1 and 1.2 stated in the introduction.

Proof of Proposition 1.2. The fact that the mappings Φ_w are permutations already follows from the discussion above. It remains to show that

$$\Phi_{wv} \ = \ \Phi_v \circ \Phi_w \tag{3.6}$$

holds for all $w, v \in \mathcal{A}_\kappa^*$. To that end, note that for any $u \in \mathcal{A}_\kappa^{\ell-1}$, we have that

$$\xi^u(wv) \ = \ \xi^u(w)_{[0,|w|-1]}\xi^{\Phi_w(u)}(v) \ . \tag{3.7}$$

Since $\xi^u(w)_{[0,|w|-1]}\xi^{\Phi_w(u)}(v)$ ends with $\Phi_v(\Phi_w(u))$, this proves (3.6). □

As mentioned, the assertions of Proposition 1.1 follow from more general results in [Hed69], but we include a short direct argument for the permutative case.

Proof of Proposition 1.1. The existence of the mappings ξ^u already follows from the discussion above. Continuity is a consequence of the fact that $\xi^u(y)_{[-n,n+\ell-1]} = \xi^u(y_{[-n,n]})$ only depends on $y_{-n}, \ldots, y_n$. Further, (1.1) follows directly from the existence and uniqueness of the sequences $\xi^u(y)$ – each element $x \in \eta_h^{-1}(y)$ coincides with $\xi^u(y)$ for $u = x_{[0,\ell-2]}$.

In order to see that different elements of the same fibre are $(\ell-1)$-separated, note that similar to above, for each $n \in \mathbb{N}$ there exists a unique sequence $\xi^{u,n}(y) \in \mathcal{A}_\kappa^{\mathbb{Z}}$ such that $\eta_h(\xi^{u,n}(y)) = y$ and $\xi^{u,n}(y)_{[n,n+\ell-2]} = u$ (for $n = 0$, we have $\xi^{u,0}(y) = \xi^u(y)$). Now, if $x, x' \in \eta_h^{-1}(y)$ agree on some interval $[n, n+\ell-2]$ with $n \in \mathbb{Z}$, say $x_{[n,n+\ell-2]} = x'_{[n,n+\ell-2]} = u$, this yields $x = \xi^{u,n}(y) = x'$. Hence, different elements of the same fibre can never coincide on any interval of length $\ell - 1$. □

Remark 3.1. *(a) In* [Hed69] *the question of the separation of fibre elements is considered in the more general setting of surjective sliding block codes, where proofs are more involved. The fact that different fibre elements of a permutative sliding block codes are always $(\ell-1)$-separated follows from* [Hed69, Theorem 11.5]*, in combination with the constant cardinality of the fibres.*

(b) We note that with the notation of the previous proof, we have

$$\xi^{u,n}(y) \ = \ \begin{cases} \xi^{\Psi_{y_{[0,n-1]}}(u)}(y) & \text{if } n \geqslant 0 \\ \xi^{\Phi_{y_{[n,-1]}}(u)}(y) & \text{if } n < 0 \end{cases}$$

We can now establish the skew product representation stated in Theorem 1.3.

Proof of Theorem 1.3. Suppose that $h \in \mathcal{F}(\kappa, \ell)$ is a permutative block code, $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a subshift and $X = \eta_h^{-1}(Y)$. Let

$$\gamma : X \to Y \times \mathcal{A}_\kappa^{\ell-1} \quad , \quad x \mapsto \left(\eta_h(x), x_{[0,\ell-2]}\right) \ , \tag{3.8}$$

$$T : Y \times \mathcal{A}_\kappa^{\ell-1} \to Y \times \mathcal{A}_\kappa^{\ell-1} \quad , \quad (y,u) \mapsto (\sigma(y), \Phi_{y_0}(u)) \ . \tag{3.9}$$

We need to show that both γ and T are homeomorphisms and γ is a conjugacy between the systems (X, σ) and $(Y \times \mathcal{A}_\kappa^{\ell-1}, T)$.

The continuity of both mappings is obvious from the definitions. Due to Proposition 1.1, the inverse of the map γ is explicitely given by

$$\gamma^{-1} : Y \times \mathcal{A}_\kappa^{\ell-1} \to X \quad , \quad (y,u) \mapsto \xi^u(y) \ .$$

The map T is bijective since both σ and the mappings Φ_{y_0} with $y_0 \in \mathcal{A}_\kappa$ are bijective. Hence, both γ and T are continuous bijections between compact Hausdorff spaces, and therefore homeomorphisms. The fact that γ is indeed a conjugacy follows from a direct computation:

$$\begin{aligned} \gamma(\sigma(x)) \ &= \ \left(\eta_h \circ \sigma(x), \sigma(x)_{[0,n-2]}\right) \ = \ \left(\sigma \circ \eta_h(x), \Phi_{\eta_h(x)_0}(x_{[0,n-2]})\right) \\ &= \ T\left(\eta_h(x), x_{[0,n-2]}\right) \ = \ T\left(\gamma(x)\right). \end{aligned}$$

□

For later use, we also include the following

Lemma 3.2. *Consider a permutative block code $h \in \mathcal{F}(\mathcal{A}, \ell)$. Then for any $w, v \in \mathcal{A}_\kappa^*$ and $u \in \mathcal{A}_\kappa^{\ell-1}$, the mapping $\mathcal{A}_\kappa^{\ell-1} \to \mathcal{A}_\kappa^{\ell-1}$, $u' \mapsto \Phi_{wu'v}(u)$ is a bijection.*

Note that we include the cases where w or v equals the empty word e.

Proof. Fix $w, v \in \mathcal{A}_\kappa^*$ and $u \in \mathcal{A}_\kappa^{\ell-1}$. Let $\bar{u} \in \mathcal{A}_\kappa^{\ell-1}$. Then the word $\xi^u(w)\zeta^{\bar{u}}(v)$, which starts with u and ends in $\bar{u}$, is mapped by $h_{|w|+|v|+\ell-1}$ to a word $wu'v$, where $u' \in \mathcal{A}_\kappa^{\ell-1}$ is given by $u' = h_{\ell-1}(\Phi_w(u)\Psi_v(\bar{u}))$. Hence, we have that

$$\xi^u(wu'v) \;=\; \xi^u(w)\zeta^{\bar{u}}(v)$$

ends with $\bar{u}$, so that $\Phi_{wu'v}(u) = \bar{u}$ by definition of the induced prefix permutation $\Phi_{wu'v}$. This proves the surjectivity of the mapping $u' \mapsto \Phi_{wu'v}(u)$, which must therefore be a permutation of $\mathcal{A}_\kappa^{\ell-1}$ as claimed. □

4. Minimality of lifts under permutative sliding block codes

4.1. **Minimality via return permutations.** The main goal of this section is to provide a proof of Theorem 1.5. However, we start with the observation that the lift of a minimal subshift can always be decomposed into a finite number of minimal components. This is a consequence of the following

Theorem 4.1 ([Hed69, Theorems 12.4])**.** *Let $h \in \mathcal{F}(\kappa, \ell)$ be a permutative block code. If $y \in \mathcal{A}_\kappa^{\mathbb{Z}}$ is almost periodic, then any $x \in \eta_h^{-1}(y)$ is almost periodic.*

Corollary 4.2. *Let $h \in \mathcal{F}(\kappa, \ell)$ be a permutative block code and $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ a minimal subshift. Then $X = \eta_h^{-1}(Y)$ is the disjoint union of at most $\kappa^{\ell-1}$ minimal sets.*

Proof. Due to Theorem 4.1, every point in X is almost periodic and therefore contained in some minimal set. Consequently, X is a disjoint union of minimal sets. Since Y is minimal, we must have $\eta_h(M) = Y$ for every shift-invariant closed subset $M \subseteq X$. Therefore every minimal set must intersect each fibre $\eta_h^{-1}(y)$, $y \in Y$. However, as the fibres contain only $\kappa^{\ell-1}$ points by Proposition 1.1, there can also be at most $\kappa^{\ell-1}$ minimal sets. □

Note that both these statments hold in the more general situaton that η_h is surjective (but h is not necessarily permutative), see [Hed69].We can now turn to the

Proof of Theorem 1.5. We note that (ii)⇒(iv) is obvious and show the implications (i)⇒(ii), (iv)⇒(iii) and (iii)⇒(i).

(i)⇒(ii). Let $w \in \mathcal{W}(Y)$ and $u, u' \in \mathcal{A}_\kappa^{\ell-1}$. Then both $\xi^u(w)$ and $\xi^{u'}(w)$ are contained in

$$\mathcal{W}(X) \;=\; \bigcup_{k=0}^{\ell-1} \mathcal{A}_\kappa^k \cup \bigcup_{u \in \mathcal{A}_\kappa^{\ell-1}} \xi^u(\mathcal{W}(Y)) \, .$$

Hence, by minimality, every $x = \xi^u(y) \in X$ contains both $\xi^u(w)$ and $\xi^{u'}(w)$ infinitely often as subwords. By shifting if necessary, we may assume $x_{[0,|w|+\ell-2]} = \xi^u(w)$, which implies $y_{[0,|w|-1]} = w$. Further, if $\xi^u(y)_{[k,k+|w|+\ell-2]} = \xi^{u'}(w)$, then we must also have $y_{[k,k+|w|-1]} = w$ and $\xi^u(y)_{[k+1,k+\ell-1]} = u'$. Consequently, we obtain $\Phi_{wv}(u) = u'$, where $v = y_{[|w|,k]}$. Since $u, u' \in \mathcal{A}_\kappa^{\ell-1}$ were arbitrary, this shows that $\mathcal{R}(w)$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$.

(iv)⇒(iii). Suppose that (iv) holds, fix $n \in \mathbb{N}$ and let $k \geqslant n$. Let $\Phi_{w^{(k)}v}$ be a return permutation of $w^{(k)}$, that is, $w^{(k)}vw^{(k)} \in \mathcal{W}(Y)$. Since $w^{(k)}$ starts with $w^{(n)}$, we have $w^{(k)} = w^{(n)}w'$ for some $w' \in \mathcal{A}_\kappa^*$. This means that $w^{(n)}w'vw^{(n)} \in \mathcal{W}(Y)$, so that $\Phi_{w^{(k)}v} = \Phi_{w^{(n)}w'v} \in \mathcal{R}(w^{(n)})$. As $k \geqslant n$ and $\Phi_{w^{(k)}v} \in \mathcal{R}(w^{(k)})$ were arbitrary, this shows that $\bigcup_{k \geqslant n} \mathcal{R}\left(w^{(k)}\right) \subseteq \mathcal{R}\left(w^{(n)}\right)$. Therefore $\mathcal{R}\left(w^{(n)}\right)$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$, as required in (iii).

(iii)⇒(i). Suppose for a contradiction that (iii) holds, but there exist a minimal strict subset $M \subsetneq X$. Let $M' = X \backslash M$ and note that M' is compact as a finite union of minimal sets by Corallary 4.2. Then, since M and M' must have a positive distance to each other, there exists $n \in \mathbb{N}$ such that

$$\mathcal{W}_{|w^{(n)}|+\ell-1}(M) \cap \mathcal{W}_{|w^{(n)}|+\ell-1}(M') \;=\; \varnothing \, .$$

In order to simplify notation, we let $w = w^{(n)}$ and $m = |w^{(n)} + \ell - 1|$ for the remainder of this proof. Now, let $u, u' \in \mathcal{A}_\kappa^{\ell-1}$ be such that $\xi^u(w) \in \mathcal{W}_m(M)$ and $\xi^{u'}(w) \in \mathcal{W}_m(M')$. Note that since M and M' intersect every fibre due to the minimality of (Y, σ), such u, u' must always exist. Using the fact that $\mathcal{R}(w)$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$, we can choose a finite sequence $u^{(0)} = u, u^{(1)}, \ldots, u^{(q)} = u'$ in $\mathcal{A}_\kappa^{\ell-1}$ and a corresponding sequence of return permutations $\Phi_{wv^{(i)}} \in \mathcal{R}(w)$, $i = 0, \ldots, q-1$, such that

$$\Phi_{wv^{(i)}}\left(u^{(i)}\right) \;=\; u^{(i+1)}$$

holds for $i = 0, \ldots, q-1$. We now show, by finite induction on i, that

$$\xi^{u^{(i)}}(w) \in \mathcal{W}_m(M) \tag{4.1}$$

for $i = 0, \ldots, q$, contradicting the assumption that $\xi^{u^{(q)}}(w) = \xi^{u'}(w) \in \mathcal{W}_m(M')$ and $\mathcal{W}_m(M) \cap \mathcal{W}_m(M') = \emptyset$.

For $i = 0$, condition (4.1) holds by choice of $u = u^{(0)}$. For the induction step $i \to i+1$, choose $y \in \mathcal{A}_\kappa^{\mathbb{Z}}$ such that $y_{[0,|wv^{(i)}w|-1]} = wv^{(i)}w$. Then, since $\xi^{u^{(i)}}(w)$ is contained in $\mathcal{W}_m(M)$ (and therefore not in $\mathcal{W}_m(M')$), we must have $\xi^{u^{(i)}}(y) \in M$. However, we also have $\Phi_{wv^{(i)}}\left(u^{(i)}\right) = u^{(i+1)}$, which implies $\xi^{u^{(i)}}(y)_{[|wv^{(i)}|,|wv^{(i)}|+\ell-2]} = u^{(i+1)}$ and therefore

$$\xi^{u^{(i)}}(y)_{[|wv^{(i)}|,|wv^{(i)}|+|w|+\ell-2]} \;=\; \xi^{u^{(i+1)}}(w)\;.$$

Hence, the word $\xi^{u^{(i+1)}}(w)$ appears in the sequence $\xi^{u^{(i)}}(y) \in M$ and thus lies in $\mathcal{W}_m(M)$, as required. This completes the proof. $\square$

4.2. **An example family.** As specific examples, we consider the family of block codes $h_{\ell,\kappa} \in \mathcal{F}(\kappa, \ell)$, given by

$$h_{\ell,\kappa} : \mathcal{A}_\kappa^\ell \to \mathcal{A}_\kappa \quad , \quad w_0 \ldots w_{\ell-1} \mapsto w_{\ell-1} - w_0 \bmod \kappa\;, \tag{4.2}$$

with integer parameters $\kappa, \ell \geqslant 2$. Recall that we write $\eta_{\ell,\kappa}$ instead of $\eta_{h_{\ell,\kappa}}$. In the further discussion, it will be convient to identify the alphabet $\mathcal{A}_\kappa$ with the cyclic group $\mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$ of order κ. Likewise, we consider $\mathcal{A}_\kappa^n$ as the direct product group $\mathbb{Z}_\kappa^n$, equipped with the coordinate-wise addition as group structure. In this context, we will sometimes write $(w_0, \ldots, w_{n-1})$ instead of $w_0 \ldots w_{n-1}$ for words in $\mathcal{A}_\kappa^n$. Given $w, v \in \mathcal{A}_\kappa^*$ such that $v = \xi^u(w)$ for some $u \in \mathcal{A}_\kappa^{\ell-1}$, we see from (4.2) that $v_{n+\ell-1} = v_n + w_n \bmod \kappa$ and hence, by induction,

$$v_{n+k\ell-1} \;=\; v_n + \sum_{j=0}^{k-1} w_{n+j(\ell-1)} \bmod \kappa\;. \tag{4.3}$$

This allows for a straightforward computation of the induced prefix permutations. For $k = 0, \ldots, \ell-2$, we let

$$t^k(w) \;=\; \sum_{j=0}^{\lfloor(|w|-k-1)/(\ell-1)\rfloor} w_{k+j(\ell-1)} \bmod \kappa\;, \tag{4.4}$$

where $\lfloor\alpha\rfloor = \sup\{n \in \mathbb{Z} \mid n \leqslant \alpha\}$ for all $\alpha \in \mathbb{R}$. In other words, $t^k(w)$ is the sum of the entries of w over all positions in the set $(k + (\ell-1)\mathbb{Z}) \cap [0, |w|-1]$. Then, if $|w|$ is a multiple of $\ell - 1$ (which is the simpler case) we have

$$\Phi_w(u_0, \ldots, u_{\ell-2}) \;=\; (u_0, \ldots, u_{\ell-2}) + (t^0(w), \ldots, t^{\ell-2}(w))\;, \tag{4.5}$$

(with addition is carried out in $\mathcal{A}_\kappa^{\ell-1} \simeq \mathbb{Z}_\kappa^{\ell-1}$). In general, we have

$$\begin{aligned} \Phi_w(u_0, \ldots, u_{\ell-2}) \;&=\; (u_{|w| \bmod (\ell-1)}, \ldots, u_{\ell-2}, u_0, \ldots, u_{|w|-1 \bmod (\ell-1)}) \\ &\quad + (t^{|w| \bmod (\ell-1)}(w), \ldots, t^{\ell-2}(w), t^0(w), \ldots, t^{|w|-1 \bmod (\ell-1)}(w))\;. \end{aligned} \tag{4.6}$$

Remark 4.3. *For the application of Theorem 1.5, we need to ensure that the family of return permutations for a given word w acts transitively on $\mathcal{A}_\kappa^{\ell-1}$. This is certainly true if it contains the two permutations*

$$\pi : (u_0, u_1, \ldots, u_{\ell-2}) \mapsto (u_1, \ldots, u_{\ell-2}, u_0) \quad \textit{and}$$
$$\pi' : (u_0, u_1, \ldots, u_{\ell-2}) \mapsto (u_0, \ldots, u_{k-1}, u_k + 1, u_{k+1}, \ldots, u_{\ell-2})$$

for some $k \in \{0, \ldots, \ell-2\}$. When $h = h_{\kappa,\ell}$ for some $\kappa, \ell \geqslant 2$, so that the return permutations are given by (4.5) and (4.6), this is always the case if there exist $v, v' \in \mathcal{A}_\kappa^*$ such that

- $wvw, wv'w \in \mathcal{W}(Y)$,
- $|wv| \bmod \kappa = 1$;
- $(t^0(wv), t^1(wv), \ldots, t^{\ell-2}(wv)) = (0, 0, \ldots, 0)$;
- $|wv'| \bmod \kappa = 0$;
- $\exists! k \in \{0, \ldots, \ell-2\} : t^k(wv') = 1$ *and* $t^j(wv') = 0$ *for all* $j \neq k$.

In this case, we have $\Phi_{wv} = \pi$ and $\Phi_{wv'} = \pi'$. This will be instrumental for the application to substitution subshifts in Section 4.3 below.

In the special case $\ell = 2$, all induced prefix permutations are simply rotations on the cyclic group $A_\kappa \simeq \mathbb{Z}_\kappa$. More precisely, we have

$$\Phi_w(a) \;=\; a + \sum_{j=0}^{|w|-1} w_j \mod \kappa \; .$$

This means, in particular, that all induced prefix permutations commute with each other. Moreover, in this case the skew product map T in Theorem 1.3 is a group extension, since all fibre maps $\Phi_{y_0} : a \mapsto a + y_0 \bmod \kappa$ are rotations on the cyclic group $\mathcal{A}_\kappa$. If $w \in \mathcal{A}_\kappa^*$ contains a subword $\tilde{w} \lhd w$, say $w = w'\tilde{w}w''$, and $wvw \in \mathcal{W}(Y)$, then we also have $\tilde{w}(w''vw')\tilde{w} \in \mathcal{W}(Y)$ (since this is a subword of wvw) and (3.6) together with the commutativity yields

$$\Phi_{\tilde{w}(w''vw')} \;=\; \Phi_{w'} \circ \Phi_{\tilde{w}(w''vw')} \circ \Phi_{w'}^{-1} \;=\; \Phi_{wv} \; .$$

As a consequence, we obtain $\mathcal{R}(w) \subseteq \mathcal{R}(\tilde{w})$. Further, in the situation of Theorem 1.5(iii), the minimality of Y together with the fact that $|w^{(n)}| \xrightarrow{n\to\infty} \infty$ imply that for each $n \in \mathbb{N}$ there exists $N \geqslant n$ such that $w^{(n)} \lhd w^{(k)}$ holds for all $k \geqslant N$. This leads to the following simplified criterion for the minimality of lifts under the sliding block codes $\eta_{2,\kappa}$, $\kappa \in \mathbb{N}$.

Proposition 4.4. *Let $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ be a minimal subshift, $\kappa \geqslant 2$ and $X = \eta_{2,\kappa}^{-1}(Y)$. Then X is minimal if and only if there exists a sequence of words $w^{(n)}, v^{(n)} \in \mathcal{W}(Y)$ such that*

(i) $w^{(n)}v^{(n)}w^{(n)} \in \mathcal{W}(Y)$ *for all* $n \in \mathbb{N}$;
(ii) $|w^{(n)}| \xrightarrow{n\to\infty} \infty$;
(iii) *the sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}_\kappa$ given by*

$$t_n \;=\; t^0\left(w^{(n)}v^{(n)}\right) \;=\; \sum_{k=0}^{|w^{(n)}v^{(n)}|-1} \left(w^{(n)}v^{(n)}\right)_k \mod \kappa \tag{4.7}$$

has the property that $\{t_k \mid k \geqslant N\}$ generates $\mathbb{Z}_\kappa$ for all $N \in \mathbb{N}$.

Proof. Suppose that (i)–(iii) hold and fix $n \in \mathbb{N}$. By minimality, we can choose $N \geqslant n$ such that $w^{(n)} \lhd w^{(k)}$ for all $k \geqslant N$. Then, by the above, we have $\mathcal{R}(w^{(k)}) \subseteq \mathcal{R}(w^{(n)})$ for all $k \geqslant N$. Therefore $\mathcal{R}(w^{(N)})$ contains all the permutations

$$\Phi_{v^{(k)}w^{(k)}} : \mathcal{A}_\kappa \to \mathcal{A}_\kappa \quad , \quad a \mapsto a + t_k \bmod \kappa$$

with $k \geqslant N$. Since $\{t_k \mid k \geqslant N\}$ generates $\mathbb{Z}_k$, this implies that $\mathcal{R}(w^{(n)})$ acts transitively on $\mathcal{A}_\kappa$. Hence, Theorem 1.5(iii) yields the minimality of X.

Conversely, suppose that X is minimal. Choose some $y \in Y$ and let $w^{(n)} = y_{[0,n]}$. Then by minimality $\xi^1\left(w^{(n)}\right)$ occurs in $\xi^0(y)$, say $\xi^1\left(w^{(n)}\right) = \xi^0(y)_{[k,k+n]}$, where we can assume without loss of generality that $k > n$. If we let $v^{(n)} = y_{[n+1,k-1]}$, so that $w^{(n)}v^{(n)} = y_{[0,k-1]}$, then we obtain that $y_{[0,k+n]} = w^{(n)}v^{(n)}w^{(n)}$ and

$$\Phi_{w^{(n)}v^{(n)}}(0) \;=\; \Phi_{y_{[0,k-1]}}(0) \;=\; 1 \; ,$$

which further implies $t_n = t^0\left(w^{(n)}v^{(n)}\right) = 1$. Hence, properties (i)–(iii) are satisfied. □

Note that in the special case $\kappa = 2$, Corollary 1.8 follows as a direct consequence.

4.3. **Applications to substitution systems.** The aim of this section is to illustrate the above minimality criteria by the application to some well-known substitution systems. More precisely, we will have a look at the the noble means substitution with parameter $p \in \mathbb{N}$ (nm, p) (which includes the Fibonacci and the silver mean substitution), the Tribonacci substitution (Tri) and the period doubling substitution (pd). It will become clear from the proofs below that there is considerable potential for further generalisations. However, our intention here is not to perform a systematic study of the largest-possible classes of substitutions to which our methods can be applied – this will be subject of subsequent work [GJK]. Instead, the results presented below should rather be understood as a 'proof of principle'.

In order to avoid redundancies, we fix some uniform notation for all examples. First, the respective substitution rule will be denoted by Γ_α, where $\alpha \in \{(\mathrm{nm}, p), \mathrm{Tri}, \mathrm{pd}\}$. The n-th substitution word is defined as

$$\tau^{(n)} \;=\; \Gamma_\alpha^n(0)\ , \tag{4.8}$$

with the exception of period doubling, where we let $\tau^{(n)} = \Gamma_{\mathrm{pd}}^n(1)$ (see Section 4.3.3 for an explanation). Note that we keep the dependence of the substitution words on α implicit. Each of these substitutions induces a subshift X_α, which is defined via (2.1) as $X_\alpha = X_{W(\alpha)}$, where $W(\alpha) = \left\{ w \in \mathcal{A}_\kappa^* \mid \exists n \in \mathbb{N} : w \lhd \tau^{(n)} \right\}$. By

$$r_n \;=\; |\tau^{(n)}| \mod (\ell - 1)\ , \tag{4.9}$$

we denote the length of the n-th substitution word $\bmod\, \ell - 1$. Together with the coefficients

$$t_n^k \;=\; t^k\left(\tau^{(n)}\right) \;=\; \sum_{j=0}^{\lfloor (|w^{(n)}|-k-1)/(\ell-1) \rfloor} \tau^{(n)}_{k+j(\ell-1)} \mod \kappa\ , \tag{4.10}$$

for $n \in \mathbb{N}$ and $k = 0, \ldots, \ell - 2$, these parameters determine the prefix permutations $\Phi_{\tau^{(n)}}$, as described by (4.5) and (4.6) above. When $\ell = 2$, we also write

$$t_n \;=\; \sum_{j=0}^{|\tau^{(n)}|-1} \tau_j^{(n)} \tag{4.11}$$

instead of t_n^0. We refer to [BG13, Bru22] for a general background on substitution systems, as well as for further specific information about the examples discussed below.

4.3.1. *Noble means substitutions.* The *noble means substitution* with parameter $p \in \mathbb{N}$ is given by the substitution rule

$$\Gamma_{\mathrm{nm},p} : \mathcal{A}_2 \to \mathcal{A}_2^2 \quad , \quad \begin{Bmatrix} 0 \mapsto & 0^p 1 \\ 1 \mapsto & 0 \end{Bmatrix}\ . \tag{4.12}$$

Note that for $p = 1$, we obtain the Fibonacci substitution $0 \mapsto 01,\ 1 \mapsto 0$. Moreover, $\Gamma_{\mathrm{nm},2}$ is conjugate to the *silver mean substitution* $\Gamma_{\mathrm{silver}} : \mathcal{A}_2 \to \mathcal{A}_2^2,\ 0 \mapsto 010,\ 1 \mapsto 0$ by the concatenation with 0 from the left, respectively the right, in the sense that $\Gamma_{\mathrm{nm},2}^n(0)0 = 0\Gamma_{\mathrm{silver}}^n(0)$. Therefore, if X_{silver} denotes the subshift induced by the silver means substitution, then $X_{\mathrm{nm},2} = X_{\mathrm{silver}}$ (see [BG13, Section 4.4] for details). Consequently, we do not have to distinguish between these examples and only treat the general case of the noble means substitutions. The substitution words satisfy the recursive equation

$$\tau^{(n+1)} \;=\; \left(\tau^{(n)}\right)^p \tau^{(n-1)}\ . \tag{4.13}$$

Theorem 4.5. *For all parameters $p, \kappa, \ell \geqslant 2$, the subshift $X = \eta_{\kappa,\ell}^{-1}(X_{\mathrm{nm},p})$ is minimal.*

Proof. In order to apply Theorem 1.5(iv), we let $w^{(n)} = \tau^{(n-1)}$ and $v^{(n)} = \left(\tau^{(n-1)}\right)^{p-2}\tau^{(n-2)}$, so that we obtain $w^{(n)}v^{(n)} = \tau^{(n)}$. Note that in addition we have $w^{(n)}v^{(n)}w^{(n)} = \tau^{(n)}\tau^{(n-1)} \in \mathcal{W}(X_{\mathrm{nm},p})$, which means that $\Phi_{\tau^{(n)}} \in \mathcal{R}\left(w^{(n)}\right)$.

For the length parameters r_n, (4.13) yields $r_{n+1} = p \cdot r_n + r_{n-1} \bmod \kappa$. Consequently, we have

$$\begin{pmatrix} r_n \\ r_{n+1} \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix} \cdot \begin{pmatrix} r_{n-1} \\ r_n \end{pmatrix}\ .$$

As the matrix $\begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix}$ acting on the finite space $\mathbb{Z}_{\ell-1}^2$ is invertible, the sequence $(r_n)_{n\in\mathbb{N}}$ must be periodic. For the parameters t_n^k defined by (4.10), we obtain the recursive formulas

$$t_{n+1}^k \;=\; \left(\sum_{j=0}^{p-1} t_n^{k-jr_n \bmod (\ell-1)}\right) + t_{n-1}^{k-pr_n \bmod (\ell-1)} \mod \kappa \;. \tag{4.14}$$

If we define vectors $V_n = \left(t_{n-1}^0, \ldots, t_n^{\ell-2}, t_n^0, \ldots, t_n^{\ell-2}\right)^t$, then (4.14) translates to

$$V_{n+1} \;=\; A_{r_n} \cdot V_n \;,$$

where the matrices A_{r_n} acting on $\mathbb{Z}_\kappa^{2(\ell-1)}$ are again invertible. The latter comes from the fact that the parameters t_{n-1}^k, $k = 0, \ldots, \ell-2$, only appear once in the equations (4.14), and therefore V_n can be recovered from the data contained in V_{n+1}. Alternatively, one may also compute the determinants of the A_{r_n} over the ring $\mathbb{Z}_\kappa$, which are non-vanishing.

As the matrices A_{r_n} form a periodic sequence acting on a finite space, the vectors V_n must be periodic. This further yields the periodicity of the induced prefix permutations $\Phi_{\tau^{(n)}}$ via (4.6). Therefore, it suffices to show that the family of permutations $\{\Phi_{\tau^{(n)}} \mid n \in \mathbb{N}\}$ acts transitively on $\mathcal{A}^{\ell-1}$. However, this is already true for the two permutations

$$\begin{aligned} \Phi_{\tau^{(0)}} = \Phi_0 \quad &: \qquad u_0 \ldots u_{\ell-2} \;\mapsto\; u_1 \ldots u_{\ell-2} u_0 \;, \\ \Phi_{\tau^{(1)}} = \Phi_{0^{p-1}1} \quad &: \qquad u_0 \ldots u_{\ell-2} \;\mapsto\; u_{p \bmod (\ell-1)} \ldots u_{\ell-2} u_0 \ldots (u_{p-1 \bmod \ell-1} + 1) \;. \end{aligned}$$

Consequently, Theorem 1.5 implies the minimality of $X = \eta^{-1}(X_{\mathrm{nm},p})$ as required. □

4.3.2. *Tribonacci substitution.* The *Tribonacci subsitution* on three symbols is given by

$$\Gamma_{\mathrm{Tri}} : \mathcal{A}_3 \to \mathcal{A}_3^2 \quad , \quad \begin{Bmatrix} 0 \mapsto & 01 \\ 1 \mapsto & 02 \\ 2 \mapsto & 0 \end{Bmatrix} \;. \tag{4.15}$$

For the substitution words, we have the recursive formula

$$\tau^{(n+1)} \;=\; \tau^{(n)} \tau^{(n-1)} \tau^{(n-2)} \;. \tag{4.16}$$

The minimality of the lifts of the *Tribonacci subshift* X_{Trib} follows by the same general pattern as in the previous example. The decisive observation is that the substitution word with the lowest index appears only once in the recursive rule (4.16), which ensures the invertibility of the 'transfer matrices'. Again, we provide some details in the proof below.

Theorem 4.6. *For all integers $\kappa \geqslant 3$ and $\ell \geqslant 2$, the subshifts $X = \eta_{\kappa,\ell}^{-1}(X_{\mathrm{Tri}})$ are minimal.*

Proof. We let $w^{(n)} = \tau^{(n-1)}$ and $v^{(n)} = \tau^{(n-2)}\tau^{(n-3)}$, so that we have $w^{(n)}v^{(n)} = \tau^{(n)}$ and $w^{(n)}v^{(n)}w^{(n)} = \tau^{(n)}\tau^{(n-1)} \in \mathcal{W}(X_{\mathrm{Trib}})$. The latter yields $\Phi_{\tau^{(n)}} \in \mathcal{R}\left(w^{(n)}\right)$.

The lengths r_n satisfy the recursive equation $r_{n+1} = r_n + r_{n-1} + r_{n-2} \bmod \kappa$, so that

$$\begin{pmatrix} r_{n-1} \\ r_n \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_{n-2} \\ r_{n-1} \\ r_n \end{pmatrix} .$$

Due to the bijectivity of the matrix, the sequence $(r_n)_{n\in\mathbb{N}}$ is periodic. For the parameters t_n^k given by (4.10), we have the recursive equations

$$t_{n+1}^k \;=\; t_n^k + t_{n-1}^{k-r_n \bmod (\ell-1)} + t_{n-2}^{k-r_n-r_{n-1} \bmod (\ell-1)} \mod \kappa \;. \tag{4.17}$$

Hence, the vectors $V_n = \left(t_{n-2}^0, \ldots, t_{n-2}^{\ell-2}, t_{n-1}^0, \ldots, t_n^{\ell-2}, t_n^0, \ldots, t_n^{\ell-2}\right)^t$ satisfy

$$V_{n+1} \;=\; A_{r_n, r_{n-1}} \cdot V_n \;,$$

with invertible matrices $A_{r_n,r_{n-1}}$ determined by (4.17). This yields the periodicity of $(V_n)_{n\in\mathbb{N}}$, and hence of $(\Phi_{\tau^{(n)}})_{n\in\mathbb{N}}$. Consequently, it suffices again to show that the family of permutations $\{\Phi_{\tau^{(n)}} \mid n \in \mathbb{N}\}$ acts transitively on $\mathcal{A}^{\ell-1}$. This is true since it contains the two permutations

$$\begin{aligned}\Phi_{\tau^{(0)}} = \Phi_0 \quad &: \quad u_0 \dots u_{\ell-2} \ \mapsto \ u_1 \dots u_{\ell-2} u_0 \ , \\ \Phi_{\tau^{(1)}} = \Phi_{01} \quad &: \quad u_0 \dots u_{\ell-2} \ \mapsto \ u_2 \dots u_{\ell-2} u_0 \dots (u_1 + 1) \ .\end{aligned}$$

As before, Theorem 1.5 now implies the minimality of $X = \eta_{\kappa,\ell}^{-1}(X_{\mathrm{Tri}})$. □

4.3.3. *Period doubling substitution.* The *period-doubling substitution* is given by

$$\Gamma_{\mathrm{pd}} : \mathcal{A}_2 \to \mathcal{A}_2^2 \quad , \quad \begin{Bmatrix} 0 \mapsto & 11 \\ 1 \mapsto & 10 \end{Bmatrix} . \tag{4.18}$$

We note that in comparision with the usual representation of the period doubling substitution, we interchange the roles of the symbols 0 and 1. The reason is the fact that in this way, the Thue-Morse subshift can be obtained as the lift of X_{pd} under $\eta_{2,2}$. Otherwise, we would have to replace $h_{2,2}$ by the block code $\tilde{h}_{2,2} : x_0 x_1 \mapsto x_1 - x_0 + 1 \bmod 2$ in order to obtain the same result. If we let

$$\tau^{(n)} \ = \ \Gamma_{\mathrm{pd}}^n(1) \quad \text{and} \quad \vartheta^{(n)} \ = \ \Gamma_{\mathrm{pd}}^n(0) \ , \tag{4.19}$$

then it is straightforward to check that

$$\tau^{(n+1)} \ = \ \tau^{(n)} \vartheta^{(n)}, \ \vartheta^{(n+1)} \ = \ \tau^{(n)} \tau^{(n)}. \tag{4.20}$$

Alternatively, we can also represent $\tau^{(n+1)}$ as

$$\tau^{(n+1)} \ = \ \tau^{(n)} \tau^{(n-1)} \tau^{(n-1)} \ . \tag{4.21}$$

Moreover, we have $|\tau^{(n)}| = |\vartheta^{(n)}| = 2^n$ for every $n \in \mathbb{N}$. We define a mapping

$$\breve{\ } : \mathcal{A}_\kappa^* \to \mathcal{A}_\kappa^* \quad , \quad w \mapsto \breve{w} = w_{[0,|w|-2]} \ , \tag{4.22}$$

thus reducing the length of each finite word by omitting the last symbol. Using the recursive relations (4.20), a straightforward induction shows that, for every $n \geqslant 1$,

$$\breve{\tau}^{(n)} \ = \ \breve{\vartheta}^{(n)}, \tag{4.23}$$

while

$$\tau_{2^n-1}^{(n)} \ = \ n + 1 \bmod 2 \ = \ 1 - \vartheta_{2^n-1}^{(n)} \ . \tag{4.24}$$

In order to prove Theorem 1.7, we distinguish four cases ((1) ℓ even; (2) $\ell = 3$, κ odd; (3) $\ell \geqslant 5$ odd; (4) $\ell = 3$, κ even) and treat each of them separately.

Proposition 4.7. *For any $\kappa \geqslant 2$ and any even $\ell \geqslant 2$, the lift $X = \eta_{\ell,\kappa}^{-1}(X_{\mathrm{pd}})$ of the period doubling subshift under $\eta_{\ell,\kappa}$ is minimal.*

Proof. Choose a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of integers with $2^{k_0} > \ell - 1$. We shall apply Theorem 1.5(iii) with $w^{(n)} = \breve{\tau}^{(k_n)}$.

For each $n \in \mathbb{N}$, choose $m_n > k_n$ such that $2^{m_n} > (\ell - 1) \cdot 2^{k_n}$. Since $\tau_{2^n}^{(n)}$ alternates between 0 and 1, we may assume that

$$\tau_{2^{m_n}-1}^{(m_n)} \ = \ 0 \quad \text{and} \quad \vartheta_{2^{m_n}-1}^{(m_n)} \ = \ 1 \tag{4.25}$$

by increasing m_n if necessary. Recall that by the recursive relations in (4.20), we have

$$\tau^{(m_n+2)} \ = \ \tau^{(m_n)} \vartheta^{(m_n)} \tau^{(m_n)} \tau^{(m_n)} \ .$$

We may further decompose $\tau^{(m_n+2)}$ into words of length 2^{k_n}

$$\begin{aligned}\tau^{(m_n+2)} \ = \ & \underbrace{v^{(0)} \dots v^{(2^{m_n-k_n}-1)}}_{\tau^{(m_n)}} \underbrace{v^{(2^{m_n-k_n})} \dots v^{(2\cdot 2^{m_n-k_n}-1)}}_{\vartheta^{(m_n)}} \\ & \underbrace{v^{(2\cdot 2^{m_n-k_n})} \dots v^{(3\cdot 2^{m_n-k_n}-1)}}_{\tau^{(m_n)}} \underbrace{v^{(3\cdot 2^{m_n-k_n})} \dots v^{(4\cdot 2^{m_n-k_n})-1}}_{\tau^{(m_n)}} \ ,\end{aligned} \tag{4.26}$$

where $v^{(i)} \in \{\tau^{(k_n)}, \vartheta^{(k_n)}\}$ for $i \in \{0, \ldots, 4 \cdot 2^{m_n - k_n} - 1\}$. Now fix $i \in 0, \ldots, \ell - 2$ and consider the subwords v and v' of $\tau^{(m_n+2)}$ of length $(\ell - 1) \cdot 2^{k_n}$ given by

$$\begin{aligned} v &= v^{(2^{m_n-k_n}-i-1)} \ldots v^{(2^{m_n-k_n}-i+\ell-3)} , \\ v' &= v^{(2 \cdot 2^{m_n-k_n}-i-1)} \ldots v^{(2 \cdot 2^{m_n-k_n}-i+\ell-3)} . \end{aligned}$$

Note that v and v' are exactly at distance $2^{m_n-k_n}$ of each other and $v^{(2^{m_n-k_n}-1)} \in \{\tau^{(k_n)}, \vartheta^{(k_n)}\}$. Further, since $\breve{\tau}^{(m_n)} = \breve{\vartheta}^{(m_n)}$, we have $v_j = v'_j$ if $j \neq (i+1) \cdot 2^{k_n} - 1$. Due to (4.25), we have m_n, $v_{(i+1) \cdot 2^{k_n}-1} = \tau^{(m_n)}_{2^{m_n}-1} = 0$ and $v'_{(i+1) \cdot 2^{k_n}-1} = \vartheta^{(m_n)}_{2^{m_n}-1} = 1$. Moreover, it follows from the decomposition (4.26) that $\Phi_v, \Phi_{v'} \in \mathcal{R}\left(\breve{\tau}^{(k_n)}\right)$. Since $\ell - 1$ divides $|v|$, formula (4.5) yields

$$\Phi_v(u_0, \ldots, u_{\ell-2}) = (u_0, \ldots, u_{\ell-2}) + (t^0(v), \ldots, t^{\ell-2}(v)) .$$

where the addition is taken coordinatewise modulo κ. Since $|v| = |v'|$, the same formula holds with v replaced by v'. As $\ell - 1$ is odd, we have $\gcd(2^{k_n}, \ell - 1) = 1$. Hence, the map

$$\alpha : \mathbb{Z}_{\ell-1} \to \mathbb{Z}_{\ell-1} \quad , \quad j \mapsto (j+1)2^{k_n} - 1 \bmod \ell - 1$$

is a bijection. From the observations above, it follows that $t^k(v) = t^k(v')$ whenever $k \neq \alpha(i)$, while $t^{\alpha(i)}(v') = t^{\alpha(i)} + 1 \bmod \kappa$. Therefore, for every $u \in \mathcal{A}_\kappa^{\ell-1}$ and $k \neq \alpha(i)$.

$$\left(\Phi_v^{\kappa-1} \circ \Phi_{v'}(u)\right)_k = u_k + \kappa \cdot t^k(v) \bmod \kappa = u_k ,$$

while at the same time

$$\left(\Phi_v^{\kappa-1} \circ \Phi_{v'}(u)\right)_{\alpha(i)} = u_{\alpha(i)} + \kappa \cdot t^{\alpha(i)}(v) + 1 \bmod \kappa = u_{\alpha(i)} + 1 \bmod \kappa .$$

As $\Phi_v, \Phi_{v'} \in \mathcal{R}\left(\breve{\tau}^{(k_n)}\right)$ and $i \in \{0, \ldots, \ell - 2\}$ was arbitrary, this shows that $\mathcal{R}(\breve{\tau}^{(k_n)})$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$ and thus completes the proof. □

Next, we consider the case $\ell = 3$ and κ odd.

Proposition 4.8. *Suppose that $\kappa \geqslant 2$ is odd. Then $X = \eta_{\kappa,3}^{-1}(X_{\mathrm{pd}})$ is minimal.*

Proof. In this case, the proof is similar to that of Theorem 4.5 for the noble means subshifts. We aim to apply Theorem 1.5(iv) with

$$w^{(n)} = \tau^{(n-1)} \quad \text{and} \quad v^{(n)} = \tau^{(n-2)}\tau^{(n-2)} .$$

Note that thus $\Phi_{\tau^{(n)}} \in \mathcal{R}(w^{(n)})$. Since $|\tau^{(n)}| = 2^n$ for all $n \in \mathbb{N}$ and $\ell - 1 = 2$, we have $r_n = |\tau^{(n)}| \bmod (\ell - 1) = 0$ for all $n \geqslant 1$. The recursive equation (4.21) therefore yields

$$t^0_{n+1} = t^0_n + 2t^0_{n-1} \mod \kappa \quad \text{and} \quad t^1_{n+1} = t^1_n + 2t^1_{n-1} .$$

for all $n \geqslant 2$. If we let $V_n = \left(t^0_n, t^1_n, t^0_{n-1}, t^1_{n-1}\right)^t$, then this means

$$V_{n+1} = A \cdot V_n$$

with

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

Now, the matrix A is invertible over the ring $\mathcal{A}_\kappa = \mathbb{Z}_\kappa$ (note that $\det(A) = 4$ and κ is odd). Hence, similar as in the proof of Theorem 4.5, we obtain periodicity of the sequence $(\Phi_{\tau^{(n)}})_{n \geqslant 1}$ is periodic. (The only difference to the noble means case is the fact that the sequence starts at $n = 1$ instead of $n = 0$.). Thus, in the light of Theorem 1.5(iv), it remains to show that the family of permutations $\{\Phi_{\tau^{(n)}} \mid n \in \mathbb{N}\}$ acts transitively on $\mathcal{A}_\kappa^2$. However, this is already true for the pair

$$\begin{aligned} \Phi_{\tau^{(1)}} &= \Phi_{10} : u_0u_1 \mapsto (u_0+1)u_1 , \\ \Phi^{\tau^{(2)}} &= \Phi_{1011} : u_0u_1 \mapsto (u_0+2)(u_1+1) . \end{aligned}$$

This yields the minimality of X. □

It remains to treat the cases where $\eta_{\kappa,\ell}^{-1}(X_{\rm pd})$ is non-minimal. We start with odd parameters $\ell \geqslant 5$. To that end, we introduce two auxilliary mappings

$$\begin{array}{rcl} \pi_{\rm even} & : & \mathcal{A}_\kappa^{\mathbb{Z}} \to \mathcal{A}_\kappa^{\mathbb{N}_0} \quad , \quad y \mapsto (y_{2n})_{n\in\mathbb{N}_0} \ , \\ \pi_{\rm odd} & : & \mathcal{A}_\kappa^{\mathbb{Z}} \to \mathcal{A}_\kappa^{\mathbb{N}_0} \quad , \quad y \mapsto (y_{2n+1})_{n\in\mathbb{N}_0} \ . \end{array}$$

Proposition 4.9. *Suppose that $\kappa \geqslant 2$ and $\ell \geqslant 5$ is odd. Then $X = \eta_{\kappa,\ell}^{-1}(X_{\rm pd})$ is non-minimal.*

Proof. Let $y \in X_{\rm pd}$ be chosen such that its right side coincides with the period doubling sequence, that is, $y_{[0,|\tau^{(n)}|-1]} = \tau^{(n)}$ for all $n \in \mathbb{N}$. We aim to show that $\xi^{10\ldots0}(y)$ cannot be contained in the forward orbit of $\xi^{0\ldots0}(y)$, thus excluding minimality.

To that end, note that since $y_k = 1$ holds for all $n \in 2\mathbb{N}_0$ (a well-known and easy to prove fact about the period doubling sequence), we have that

$$\pi_{\rm even}\left(\xi^{0\ldots0}(y)\right) = \left(0^{\ell-1}1^{\ell-1}\ldots(\kappa-1)^{\ell-1}\right)^\infty , \tag{4.27}$$

$$\pi_{\rm even}\left(\xi^{10\ldots0}(y)\right) = \left(10^{\ell-2}21^{\ell-2}\ldots0(\kappa-1)^{\ell-2}\right)^\infty . \tag{4.28}$$

Now, let $(n_k)_{k\in\mathbb{N}}$ be any sequence of positive integers such that $n_k \nearrow \infty$ and

$$x = \lim_{k\to\infty} \sigma^{n_k}\left(\xi^{0\ldots0}(y)\right)$$

exists. By going over to a subsequence if necessary, we may assume that $j = n_k \bmod (\ell-1)\kappa$ is constant in k. If j is even, then we obtain

$$\pi_{\rm even}(x) = \sigma^{j/2}\left(\pi_{\rm even}\left(\xi^{0\ldots0}(y)\right)\right) = \sigma^{j/2}\left(0^{\ell-1}1^{\ell-1}\ldots(\kappa-1)^{\ell-1}\right)^\infty .$$

This, however, implies that $x \neq \xi^{10\ldots0}(y)$. If j is odd, then we obtain that

$$\pi_{\rm odd}(x) = \sigma^{(j+1)/2}\left(\pi_{\rm even}\left(\xi^{0\ldots0}(y)\right)\right) = \sigma^{(j+1)/2}\left(0^{\ell-1}1^{\ell-1}\ldots(\kappa-1)^{\ell-1}\right)^\infty .$$

If we had $x = \xi^{10\ldots0}(y)$, then combinded with (4.28) this would imply that the sequence $\xi^{10\ldots0}(y)_{[0,+\infty)}$ is periodic, contradicting the aperiodicity of $y_{[0,+\infty)}$. Altogether, this shows that $\xi^{10\ldots0}(y)$ is not contained in the forward orbit of $\xi^{0\ldots0}(y)$, so that X is non-minimal. □

Finally, we treat the case $\ell = 3$ and $\kappa \geqslant 2$ even.

Proposition 4.10. *If $\kappa \geqslant 2$ is even, then $X = \eta_{\kappa,3}^{-1}(X_{\rm pd})$ is non-minimal.*

Proof. As in the previous proof, we choose y to be the period doubling sequence and consider the points $\xi^{00}(y)$ and $\xi^{10}(y)$. Similar to above, we have

$$\pi_{\rm even}\left(\xi^{00}(y)\right) = (01\ldots(\kappa-1))^\infty , \tag{4.29}$$

$$\pi_{\rm even}\left(\xi^{10}(y)\right) = (1\ldots(\kappa-1)0)^\infty . \tag{4.30}$$

Since $y_k = 0$ holds for all $k \in 4\mathbb{N}_0 + 1$, $\xi^{00}(y)$ is of the form

$$\xi^{00}(y) = 0a_11a_12a_23a_2\ldots a_\kappa(\kappa-1)a_\kappa0a_{\kappa+1}1a_{\kappa+1}\ldots ,$$

with $a_i \in \mathcal{A}_\kappa$ (where we assume $\kappa \geqslant 6$ for better illustration). At the same time,

$$\xi^{00}(y) = 1a_12a_13a_24a_2\ldots a_{\kappa-1}(\kappa-1)a_\kappa0a_\kappa1a_{\kappa+1}\ldots .$$

Hence, if an odd number appears at an even position in $\xi^{00}(y)$, then its left and right neighbours coincide, and the same holds for even numbers at even positions in $\xi^{10}(y)$. Together with (4.29) and (4.30), this implies that $\xi^{10}(y)$ cannot belong to the forward orbit closure of $\xi^{00}(y)$. (A more formal argument can be given along the lines of the proof of Proposition 4.9; we leave the details to the reader.) This yields that X is not minimal, as claimed. □

4.4. **Decomposition of non-minimal lifts into minimal components.** When a permutative lift of a minimal subshift is not minimal, we have the following.

Lemma 4.11. *Suppose that $h \in \mathcal{F}(\kappa, \ell)$ is permutative, $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a minimal subshift and $X = \eta_h^{-1}(Y)$ is not minimal. Then for each minimal set $M \subseteq X$ there exists an integer $r(M) \in \{1, \ldots, \kappa^{\ell-1} - 1\}$ such that, for all $y \in Y$, we have*

$$\# M \cap \eta_h^{-1}(y) \;=\; r(M)\,.$$

We will refer to $r(M)$ as the *rank of M over Y*.

Proof. Let $y, \hat{y} \in Y$ and assume that $\#\left(\eta_h^{-1}(y) \cap M\right) = m$ and $\eta_h^{-1}(y) \cap M = \left\{x^{(1)}, \ldots, x^{(m)}\right\}$. By minimality of Y, compactness of M and continuity of η_h, we can choose $n_k \nearrow \infty$ such that $\lim_{k\to\infty} \sigma^{n_k}(y) = \hat{y}$ and $\lim_{k\to\infty} \sigma^{n_k}(x^{(i)}) = \hat{x}^{(i)} \in \eta_h^{-1}(\hat{y}) \cap M$. Since the $x^{(i)}$ are pairwise $(\ell-1)$-separated by Proposition 1.1, so are the limits points $\hat{x}^{(i)}$. This shows $\#\left(\eta_h^{-1}(\hat{y}) \cap M\right) \geqslant \#\left(\eta_h^{-1}(y) \cap M\right)$. As $y, \hat{y}$ were arbitrary, we obtain that $\#\left(\eta_h^{-1}(y) \cap M\right)$ is independent of $y \in Y$. □

However, it is important to note that the ranks of two distinct minimal subsets of the lift may well be different. An easy example is given by the lift of the fixed point $0^{\mathbb{Z}}$ under $\eta_{2,3}^{-1}$, which is given by the two fixed points $0^{\mathbb{Z}}$ and $1^{\mathbb{Z}}$ and the two-periodic orbit $\left\{(01)^{\mathbb{Z}}, (10)^{\mathbb{Z}}\right\}$. It turns out that all a priori possible combinations of ranks may occur. In order to see this, the following basic observation is needed.

Lemma 4.12. *Let $\kappa \in \mathbb{N}$, and π be a permutation of $\mathcal{A}_\kappa$. Then there exists a permutative block code $h \in \mathcal{F}(\kappa, 2)$ such that $\Phi_0^{(h)} = \pi$.*

Proof. It suffices to choose $h(ab) = b - \pi(a) \bmod \kappa$. □

Given a permutation π of $\mathcal{A}_\kappa$, we denote by $\mathcal{P}(\pi)$ the partition of $\mathcal{A}_\kappa$ into π-orbits. Note that each element $a \in \mathcal{A}_\kappa$ is periodic under π with some period $p(a) \leqslant \kappa$ and its orbit is given by $\mathcal{O}_\pi(a) = \{\pi^i(a) \mid i = 0, \ldots, p(a) - 1\}$. Now, choose integers $1 \leqslant r_1, \ldots, r_n \leqslant \kappa$ such $\sum_{i=1}^n r_i = \kappa$. Let π be a permutation of $\mathcal{A}_\kappa$ such that $\mathcal{P}(\pi)$ can be written as $\mathcal{P}(\pi) = \{P_1, \ldots, P_n\}$ with $\#P_i = r_i$ for all $i = 1, \ldots, \kappa$. Note that thus $p(a) = r_i$ for all $a \in P_i$.

Now, choose $h \in \mathcal{F}(\kappa, 2)$ such that $\Phi_0^{(h)} = \pi$. Then for any $a \in \mathcal{A}_\kappa$, we have that

$$\xi^a\left(0^{\mathbb{Z}}\right) \;=\; \left(a\pi(a)\ldots\pi^{p(a)-1}(a)\right)^\infty$$

which is a periodic point of period $p(a)$. As a consequence, $\eta_h^{-1}\left(0^{\mathbb{Z}}\right)$ partitions into n different minimal sets (periodic orbits) $M_1, \ldots, M_n$, each of cardinality $\#M_i = r(M_i) = r_i$. This provides an easy demonstration for the fact that in general there are no combinatorial restrictions on how a permutative lift of a minimal subshift is partitioned into minimal sets of different rank. More sophisticated examples based on Toeplitz flows will be given in [Kan]. These will be based on the following generalisation of Theorem 1.5.

Theorem 4.13. *Suppose that $h \in \mathcal{F}(\kappa, \ell)$ is permutative, $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a minimal subshift and $X = \eta_h^{-1}(Y)$. Further, let $\mathcal{Q}$ be a partition of $\mathcal{A}_\kappa^{\ell-1}$ and assume that there exists a sequence of words $(w^{(n)})_{n\in\mathbb{N}}$ such that the following hold (for all $n \in \mathbb{N}$ where applicable):*

(i) $|\tau^{(n)}| \nearrow +\infty$ as $n \to \infty$;
(ii) $\tau^{(n+1)}$ starts with $\tau^{(n)}$;
(iii) $\mathcal{R}\left(\tau^{(n)}\right)$ leaves each $Q \in \mathcal{Q}$ invariant, that is, $\pi(Q) = Q$ for all $\pi \in \mathcal{R}\left(\tau^{(n)}\right)$;
(iv) $\mathcal{R}\left(\tau^{(n)}\right)$ acts transitively on each $Q \in \mathcal{Q}$.

Then $X = \biguplus_{Q\in\mathcal{Q}} M_Q$, where each M_Q is a minimal set given by

$$M_Q \;=\; \left\{x \in X \mid v \lhd x \Rightarrow \exists n \in \mathbb{N},\ u \in \mathcal{Q} : v \lhd \xi^u\left(w^{(n)}\right)\right\}\,. \tag{4.31}$$

Moreover, we have $r(M_Q) = \#Q$ for all $Q \in \mathcal{Q}$.

Proof. Since $w^{(n+1)}$ starts with $w^{(n)}$, there exists a sequence $y \in Y$ such that $y_{[0,|w^{(n)}|-1]} = w^{(n)}$ for all $n \in \mathbb{N}$. Fix $\tilde{u} \in \mathcal{A}_\kappa^{\ell-1}$ and let $M \subseteq X$ be the minimal component that contains $\xi^{\tilde{u}}(y)$, that is, $M = \overline{\mathcal{O}_\sigma(\xi^{\tilde{u}}(y))}$. Choose $n \in \mathbb{N}$ such that

$$\mathcal{W}_{|\omega^{(n)}|+\ell-1}(M) \cap \mathcal{W}_{|\omega^{(n)}|+\ell-1}(M') \;=\; \varnothing\ , \tag{4.32}$$

where $M' = X \backslash M$. Now, suppose that $u, \tilde{u}$ belong the the same partition element $Q \in \mathcal{Q}$ and there is $v \in \mathcal{A}_\kappa^*$ such that $w^{(n)}vw^{(n)} \in \mathcal{W}(Y)$ and $\Phi_{w^{(n)}v}(\tilde{u}) = u$. Then

$$\xi^{\tilde{u}}\left(w^{(n)}vw^{(n)}\right)_{[|w^{(n)}v|,|w^{(n)}vw^{(n)}|-1]} \;=\; \xi^u\left(w^{(n)}\right)\ .$$

Due to (4.32), this entails that $\xi^u(y) \in M$. Similar as in the proof of Theorem 1.5 in Section 4.1, the fact that $\mathcal{R}\left(w^{(n)}\right)$ acts transitively on Q then yields that $\xi^u(y) \in M$ for all $u \in Q$. Using minimality, we can conclude that $M = M_Q$.

It remains to prove that the sets M_Q, $Q \in \mathcal{Q}$, are pairwise disjoint. To that end, it suffices to show that $\xi^{u'}(y) \notin M_Q$ whenever $u' \notin Q$. Assume for a contradiction that $\xi^{u'}(y) \in M_Q$, so that in particular $\xi^{u'}\left(w^{(n)}\right) \in \mathcal{W}\left(M_Q\right)$. By minimality, this yields $\xi^{u'}\left(w^{(n)}\right) \lhd \xi^u(y)$ for $u \in Q$. However, this is only possible if there exists some $v \in \mathcal{A}_\kappa^*$ such that y starts with $w^{(n)}vw^{(n)}$ and $\Phi_{w^{(n)}v}(u) = u'$. As $\Phi_{w^{(n)}v} \in \mathcal{R}\left(w^{(n)}\right)$ in this case, this contradicts (iii). □

5. Unique ergodicity

5.1. **Unique ergodicity of substitution subshifts.** Before we turn to discuss the unique ergodicity for lifts of arbitrary strictly ergodic subshifts, we first want to point out in this section that for all the substitution examples discussed in Section 4.3, unique ergodicity is a more or less direct consequence of minimality. The reason is the fact that the linear recurrence of primitive substitution subshifts is preserved by lifts under permutative sliding block codes, in the sense that every minimal component of the lift is linearly recurrent (and therefore uniquely ergodic). This is true in general for all primitive substitutions and will be discussed in [GJK]. However, in order to demonstrate the underlying principle, we want to provide a simplified argument for the case of the noble means substitutions. Recall that all primitive substitution subshifts are linearly recurrent [DL06] and all linearly recurrent subshifts are uniquely ergodic (see, for instance, [Dur00, Bru22]).

Now, for given parameters $p \in \mathbb{N}$ and $\kappa, \ell \geqslant 2$, let $X = \eta_{\kappa,\ell}^{-1}\left(X_{\mathrm{nm},p}\right)$. Due to the minimality of X by Theorem 4.5, there exists some $k \in \mathbb{N}$ such that for all $u, u' \in \mathcal{A}_\kappa^{\ell-1}$, the lift $\xi^u\left(\tau^{(k)}\right)$ of $\tau^{(k)}$ contains $\xi^{u'}(0)$. Moreover, as shown in Section 4.3.1, the sequence of prefix permutations $(\Phi_{\tau^{(n)}})_{n\in\mathbb{N}}$ is periodic with some period $q \in \mathbb{N}$. In particular, we have $\Phi_{\tau^{(n-1)}} = \Phi_1$ and $\Phi_{\tau^{(n)}} = \Phi_0$ for all $n \in q\mathbb{Z}$. This immediately implies that

$$\Phi_{\Gamma^n_{\mathrm{nm},p}(w)} \;=\; \Phi_w \tag{5.1}$$

holds for all $w \in \mathcal{A}_\kappa^*$. Further, we therefore obtain that

$$\xi^{u'}\left(\tau^{(n)}\right) \lhd \xi^u\left(\tau^{(n+k)}\right) \textit{ holds for all } u, u' \in \mathcal{A}_\kappa^{\ell-1} \textit{ and } n \in q\mathbb{Z}\ . \tag{5.2}$$

Note also that for any $n \in \mathbb{N}$, any $y \in X_{\mathrm{nm},p}$ can be written as an infinite concatenation of substitutions words $\tau^{(n)}$ and $\tau^{(n-1)}$. This implies that any $w \in \mathcal{W}\left(X_{\mathrm{nm},p}\right)$ with $|w| \leqslant |\tau^{(n-2)}|$ is contained in $\tau^{(n)}$ as a subword.

In order to see that X is linearly recurrent, let $\hat{w} = \xi^{\hat{u}}(w) \in \mathcal{W}(X)$, with $\hat{u} \in \mathcal{A}_\kappa^{\ell-1}$ and $w \in \mathcal{W}\left(X_{\mathrm{nm},p}\right)$. Let $n \in q\mathbb{Z}$ be minimal with the property that $|\tau^{(n-2)}| \geqslant |w|$, so that $w \lhd \tau^{(n)}$. Since we choose n minimal, we have

$$|\tau^{(n)}| \;\leqslant\; 2^{q+2} \cdot |w|\ . \tag{5.3}$$

Now, suppose that $v \in \mathcal{W}\left(X_{\mathrm{nm},p}\right)$ satisfies $|v| \geqslant 2^{q+k+4} \cdot |w|$. Then $|v| \geqslant 4|\tau^{(n+k)}|$, which implies $\tau^{(n+k)} \lhd v$. By (5.2), we obtain that $\xi^{u'}\left(\tau^{(n)}\right) \lhd \xi^u(v)$ holds for all $u, u' \in \mathcal{A}_\kappa^{\ell-1}$. Consequently, we also have $\xi^{u'}(w) \lhd \xi^u(v)$ for all $u, u' \in \mathcal{A}^{\ell-1}$, and in particular $\xi^{\hat{u}}(w) \lhd \xi^u(v)$. Since $u \in \mathcal{A}_\kappa^{\ell-1}$ and $v \in \mathcal{W}\left(X_{\mathrm{nm},p}\right)$ with $|v| \geqslant 2^{q+k+4}$ were arbitrary, we obtain that $\xi^{\hat{u}}(w)$ is contained in any word of length $\geqslant 2^{q+k+4} \cdot |w| + \ell - 1$ in $\mathcal{W}(X)$. As $|\xi^{\hat{u}}(w)| = |w| + \ell - 1$, the same holds in particular for any word of length $\geqslant (\ell - 1) \cdot 2^{q+k+4} \cdot |\xi^{\hat{u}}(w)|$. Therefore X is linearly recurrent with constant $L = (\ell - 1) \cdot 2^{q+k+4}$, and hence uniquely ergodic.

Together with Theorem 4.5, this proves Theorem 1.6. For the Tribonacci and period doubling substitutions, we refer to the mentioned result in [GJK].

5.2. **Unique ergodicity via uniform frequencies of return permutations.** We now turn to the characterisation of unique ergodicity for general permutative lifts, without relying on linear recurrence or any substitution structure. Thereby, the skew product representation provided by Theorem 1.3 allows to see that there always exists a canonical invariant Borel measure for the lift of a uniquely ergodic subshift by a permutative sliding block code. The main task will then be to show the uniqueness of this particular invariant measure.

Proposition 5.1. *Suppose that $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a subshift, μ_Y is a shift-invariant Borel measure on Y, $h \in \mathcal{F}(\kappa, \ell)$ is a permutative block code and $X = \eta_h^{-1}(Y)$. Then there exists a shift-invariant Borel measure μ_X on X which satisfies*

$$\mu_X \left([\xi^u(w)]^+_{|w|+\ell-2} \right) \;=\; \frac{1}{\kappa^{\ell-1}} \cdot \mu_Y \left([w]^+_{|w|-1} \right) \tag{5.4}$$

for all $w \in \mathcal{W}(Y)$ and $u \in \mathcal{A}_\kappa^{\ell-1}$ and is uniquely determined by these equations.

Proof. First of all, we note that

$$\mathcal{W}(X) \;=\; \left\{ \xi^u(w) \mid w \in \mathcal{W}(Y),\; u \in \mathcal{A}_\kappa^{\ell-1} \right\} \cup \bigcup_{n=1}^{\ell-1} \mathcal{A}_\kappa^n \;.$$

Therefore, any Borel measure on X is determined by its values on the cylider sets of the form $[\xi^u(w)]^+_{|w|+\ell-1}$. In particular, μ_X is uniquely determined by (5.4).

In order to see that μ_X is indeed a shift-invariant Borel measure, note that the skew product transformation $T : Y \times \mathcal{A}_\kappa^{\ell-1} \to Y \times \mathcal{A}_\kappa^{\ell-1}$ defined in (1.3) has the invariant measure $\tilde{\mu}_Y = \mu_Y \times \theta$, where θ denotes the equidistribution on $\mathcal{A}_\kappa^{\ell-1}$. By definition, we have that

$$\tilde{\mu}_Y \left([w]^+_{|w|} \times \{u\} \right) \;=\; \frac{1}{\kappa^{\ell-1}} \cdot \mu_Y \left([w]^+_{|w|} \right) \tag{5.5}$$

for any $w \in \mathcal{W}(Y)$ and $u \in \mathcal{A}_\kappa^{\ell-1}$. Moreover, if γ is the conjugacy between (X, σ) and $(Y \times \mathcal{A}_\kappa^{\ell-1}, T)$ given by (1.2), then

$$\gamma \left([w]^+_{|w|} \times \{u\} \right) \;=\; [\xi^u(w)]^+_{|w|+\ell-1} \;. \tag{5.6}$$

Therefore, μ_X equals the measure $(\gamma^{-1})_* \tilde{\mu}_Y = \tilde{\mu}_Y \circ \gamma$ on X. As γ is a conjugacy and $\tilde{\mu}_Y$ is T-invariant, μ_X is a shift-invariant Borel measure as claimed. □

We say $x \in \mathcal{A}_\kappa^{\mathbb{Z}}$ has *uniform word frequencies*[1] if, for all $w \in \mathcal{A}_\kappa^*$, there exist numbers $\bar{\nu}(x, w)$ such that

$$\lim_{n \to \infty} \sup_{k \in \mathbb{Z}} \left| \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_{[w]^+_{|w|}} \circ \sigma^{k+j}(x) - \bar{\nu}(x, w) \right| \;=\; 0 \;. \tag{5.7}$$

The following statement is well-known (see, for instance, [Oxt52, Sch00]).

Proposition 5.2. *If $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a minimal subshift, then the following are equivalent.*

(i) (Y, σ) is uniquely ergodic;

(ii) all $y \in Y$ have uniform word frequencies;

(iii) there exists some $y \in Y$ with uniform word frequencies.

[1]Oxtoby [Oxt52] calls this property *strict transitivity*, whereas in the context of aperiodic order and the study of Delone sets it is usually referred to as *uniform patch frequencies* [Sch00].

Remark 5.3. *If the supremum in (5.7) is only taken over $k \in \mathbb{N}$ (instead of $k \in \mathbb{Z}$), then this still yields unique ergodicity on the ω-limit set of y. However, if y is almost periodic, then the ω-limit set equals the orbit closure. Hence, the statement of Proposition 5.2 remains valid if the supremum in (5.7) is only taken over $k \in \mathbb{N}$.*

In order to adapt this statement to our situation, we need some more notation. Given a permutative block code $h \in \mathcal{F}(\kappa, \ell)$, a strictly ergodic subshift $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$, $w, v \in \mathcal{W}(Y)$ with $|w| \leqslant |v|$ and $u, u' \in \mathcal{A}_\kappa^{\ell-1}$, we define

$$\mathrm{Ap}(w,v) \;=\; \left\{0 \leqslant j \leqslant |v|-|w| \mid v_{[j,j+|w|-1]} = w\right\} \quad \text{and} \tag{5.8}$$

$$\mathrm{Ap}_{u\mapsto u'}(w,v) \;=\; \left\{j \in \mathrm{Ap}(w,v) \mid \Phi_{v_{[0,j-1]}}(u) = u'\right\} \ . \tag{5.9}$$

Note that we keep the dependence of these quantities on h implicit. Let further

$$\nu^{\mathrm{ap}}(w,v) \;=\; \frac{\#\mathrm{Ap}(w,v)}{|v|-|w|+1} \ , \tag{5.10}$$

$$\nu^{\mathrm{ap}}_{u\to u'}(w,v) \;=\; \frac{\#\mathrm{Ap}_{u\mapsto u'}(w,v)}{|v|-|w|+1} \ , \tag{5.11}$$

as well as

$$D_u^{\mathrm{ap}}(w,v) \;=\; \max_{u',u''\in\mathcal{A}^{\ell-1}} \left|\nu^{\mathrm{ap}}_{u\mapsto u'}(w,v) - \nu^{\mathrm{ap}}_{u\mapsto u''}(w,v)\right| \ , \tag{5.12}$$

$$D^{\mathrm{ap}}(w,v) \;=\; \max_{u\in\mathcal{A}^{\ell-1}} D_u^{\mathrm{ap}}(w,v) \ . \tag{5.13}$$

Then we obtain the following.

Theorem 5.4. *Suppose that $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a strictly ergodic subshift with unique invariant probability measure μ_Y and $h \in \mathcal{F}(\kappa,\ell)$ is a permutative block code of length ℓ. Then $X = \eta_h^{-1}(Y)$ is strictly ergodic if and only if there exists $y \in Y$ such that*

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} D^{\mathrm{ap}}\left(w, y_{[k,k+n]}\right) \;=\; 0 \tag{5.14}$$

holds for all $w \in \mathcal{W}(Y)$. Moreover, (5.14) is equivalent to

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \max_{u,u'\in\mathcal{A}_\kappa^{\ell-1}} \left(\nu^{\mathrm{ap}}_{u\mapsto u'}\left(w, y_{[k,k+n]}\right) - \frac{\mu_Y\left([w]^+_{|w|}\right)}{\kappa^{\ell-1}}\right) \;=\; 0 \ . \tag{5.15}$$

Proof. We first show that unique ergodicity is equivalent to (5.15). To that end, fix $\hat{u} \in \mathcal{A}^{\ell-1}$ and $k \in \mathbb{N}$ and let $u = \Phi_{y_{[0,k-1]}}(\hat{u})$. Then, for any $u' \in \mathcal{A}^{\ell-1}$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{j=0}^{n-1} \mathbf{1}_{[\xi^{u'}(w)]^+_{|w|+\ell-1}} \circ \sigma^{k+j}\left(\xi^{\hat{u}}(y)\right) \;&=\; \frac{1}{n}\sum_{j=0}^{n-1} \mathbf{1}_{[\xi^{u'}(w)]^+_{|w|-\ell-1}} \circ \sigma^{j}\left(\xi^{u}\left(\sigma^k(y)\right)\right) \\
&=\; \nu^{\mathrm{ap}}_{u\mapsto u'}\left(w, y_{[k,k+n+|w|-2]}\right) \;\overset{n\to\infty}{\longrightarrow}\; \frac{\mu_Y\left([w]^+_{|w|}\right)}{\kappa^{\ell-1}} \ ,
\end{aligned}$$

where (5.15) implies that the convergence is uniform in $k \in \mathbb{N}$. Hence, the sequence $\xi^u(y)$ has uniform word frequencies and Proposition 5.2 yields the unique ergodicity of (X,σ). Note that (5.15) in combination with the uniform ergodic theorem implies that unique invariant measure must be the measure μ_X defined by (5.4).

It remains to show the equivalence between (5.14) and (5.15). The fact that (5.15) implies (5.14) is obvious. Conversely, in order to see that (5.14) implies (5.15), note that the unique ergodicity of (Y,σ) yields

$$\nu^{\mathrm{ap}}\left(w, y_{[k,k+n]}\right) \;\overset{n\to\infty}{\longrightarrow}\; \mu_Y\left([w]^+\right) \ ,$$

and the convergence is uniform in $k \in \mathbb{N}$. However, we also have that

$$\nu^{\mathrm{ap}}\left(w, y_{[k,k+n]}\right) \;=\; \sum_{u'\in\mathcal{A}^{\ell-1}} \nu^{\mathrm{ap}}_{u\mapsto u'}\left(w, y_{[k,k+n]}\right) \ ,$$

where $u \in \mathcal{A}^{\ell-1}$ is arbitrary. In combination with (5.14), this implies (5.15). □

In order to estimate the quantities $D^{\mathrm{ap}}(w,v)$, the following statement can be useful.

Lemma 5.5. *In the situation of Theorem 5.4, suppose that $v,w \in \mathcal{W}(Y)$ and v is a concatenation of words $v = v^{(0)} \dots v^{(M)}$ and $|w| \leqslant |v^{(j)}|$ for all $j = 0,\dots,M$. Then*

$$D^{\mathrm{ap}}(w,v) \;\leqslant\; \sum_{j=0}^{M} \frac{|v^{(j)}|-|w|+1}{|v|-|w|+1} D^{\mathrm{ap}}\left(w,v^{(j)}\right) + \frac{M\cdot(|w|-1)}{|v|-|w|+1} \,. \tag{5.16}$$

Proof. Let $k_i = \sum_{j=0}^{i-1} |v^{(j)}|$, so that $v_{[k_i,k_{i+t}-1]} = v^{(i)}$. Then, since

$$\Phi_{v_{[0,k_i+k]}} \;=\; \Phi_{v^{(i)}_{[0,k]}} \circ \Phi_{v_{[0,k_i-1]}}$$

for all $k = 0,\dots,|v^{(i)}|-1$, we have

$$\mathrm{Ap}_{u\mapsto u'}(w,v) \cap [k_i, k_{i+1}-|w|] \;=\; \mathrm{Ap}_{\Phi_{v_{[0,k_i-1]}}(u)\mapsto u'}\left(w,v^{(i)}\right) - k_i \,.$$

Combined with the definition of $D^{\mathrm{ap}}(v,w)$, this yields (5.16). □

6. Permutative sliding block codes and Toeplitz flows

6.1. **Basic notions for Toeplitz flows.** A point $\xi \in \mathcal{A}^{\mathbb{Z}}$ is called a *Toeplitz sequence* if it is not periodic and has the property that

$$\text{for all } n \in \mathbb{Z} \text{ there exists } p \in \mathbb{N} \text{ such that } \xi_n = \xi_{n+pk} \text{ holds for any } k \in \mathbb{Z}. \tag{6.1}$$

By $p(\xi,n)$ we denote the least integer p that satisfies (6.1). Given a point $\xi \in \mathcal{A}_\kappa^{\mathbb{Z}}$, $a \in \mathcal{A}_\kappa$ and any $p \in \mathbb{N}$, we let $\mathrm{Per}(\xi,p,a) = \{n \in \mathbb{Z} \mid \xi_{n+kp} = a \text{ for all } k \in \mathbb{Z}\}$, and define the set of p-periodic positions of ξ as

$$\mathrm{Per}(\xi,p) \;=\; \bigcup_{a\in\mathcal{A}_\kappa} \mathrm{Per}(\xi,p,a) \,.$$

The set of all periodic positions of ξ is then given as $\mathrm{Per}(\xi) = \bigcup_{p\in\mathbb{N}} \mathrm{Per}(\xi,p)$. We call $p \in \mathbb{N}$ an *essential period* of ξ if $\mathrm{Per}(\xi,p) \neq \varnothing$ and there exists no $q < p$ such that $\mathrm{Per}(\xi,p,a) \subseteq \mathrm{Per}(\xi,q,a)$ holds for all $a \in \mathcal{A}_\kappa$.

Remark 6.1. *Note that if $\mathrm{Per}(\xi,p,a) \subseteq \mathrm{Per}(\xi,q,a)$, then we actually have $\mathrm{Per}(\xi,p,a) = \mathrm{Per}(\xi,\gcd(p,q),a)$, where $\gcd$ denotes the greatest common divisor. Hence, in order to see that p is essential, it suffices that $\mathrm{Per}(\xi,p,a) \neq \mathrm{Per}(\xi,q,a)$ for all $q < p$ with $q|p$.*

For any $\xi \in \mathcal{A}_\kappa^{\mathbb{Z}}$ and $p \in \mathbb{N}$, we further define the p-skeleton $\hat{\xi}^p \in \mathcal{A}_{\kappa+1}^{\mathbb{Z}}$ of ξ by

$$\hat{\xi}^p_n \;=\; \begin{cases} \xi_n \text{ if } n \in \mathrm{Per}(\xi,p) \\ \kappa \text{ if } n \notin \mathrm{Per}(\xi,p) \end{cases} \quad , \quad n \in \mathbb{Z} \,. \tag{6.2}$$

Then $\hat{\xi}^p$ is p-periodic and p is an essential period of ξ if it is the least period of $\hat{\xi}^p$. A strictly increasing sequence of positive integers $P = (p_k)_{k\in\mathbb{N}}$ is called a *scale* if $p_k \mid p_{k+1}$ for all $k \in \mathbb{N}$. It is called a *period structure* of a Toeplitz sequence $\xi \in \mathcal{A}_\kappa^{\mathbb{Z}}$ if

- for all $k \in \mathbb{N}$, p_k is an essential period of ξ;
- $\bigcup_{k\in\mathbb{N}} \mathrm{Per}(\xi,p_k) = \mathbb{Z}$.

Given a Toeplitz sequence $\xi \in \mathcal{A}_\kappa^{\mathbb{Z}}$, we call X_ξ a *Toeplitz subshift*. We have

Theorem 6.2 ([JK69]). *Every Toeplitz subshift is minimal.*

For a Toeplitz sequence $\xi \in \mathcal{A}_\kappa^{\mathbb{Z}}$ and a period structure $P = (p_k)_{k\in\mathbb{N}}$ of ξ, we define the density of periodic positions of ξ with respect to P as

$$\overline{D}(\xi,P) \;=\; \lim_{k\to\infty} \frac{\#\,\mathrm{Per}(\xi,p_k) \cap [0,p_k-1]}{p_k}.$$

Note that $\frac{\#\,\mathrm{Per}(\xi,p_k)\cap[0,p_k-1]}{p_k}$ is non-decreasing in k and bounded from above by 1, so that the limit always exists (and can be replaced by the supremum). Suppose $Q = (q_k)_{k\in\mathbb{N}}$ is another period structure of ξ. Then for any $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $p_k \mid q_m$ and $q_k \mid p_m$ for $m \geqslant N$. Hence $\overline{D}(\xi, P) = \overline{D}(\xi, Q)$ and we can define the density of ξ as $\overline{D}(\xi) = \overline{D}(\xi, P)$, where P is any period structure of ξ. A Toeplitz sequence $\xi \in \mathcal{A}_\kappa^{\mathbb{Z}}$ is called *regular* if $\overline{D}(\xi) = 1$ and *irregular* if $\overline{D}(\xi) < 1$. A Toeplitz subshift $X \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is then called regular if it contains a regular Toeplitz sequence and irregular otherwise.

Theorem 6.3 ([JK69])**.** *A regular Toeplitz subshift is uniquely ergodic.*

6.2. **Toeplitz subshifts from one-sided Toeplitz sequences.** Examples of Toeplitz subshifts are often constructed via one-sided Toeplitz sequences. Since this is usually done only implicitely in the literature, we provide some details for the convenience of the reader. In analogy to (6.1), a point $\xi \in \mathcal{A}_\kappa^{\mathbb{N}_0}$ is called a *one-sided Toeplitz sequence* if it is not periodic and has the property that

$$\begin{gathered}\textit{for all } n \in \mathbb{N}_0, \textit{ there exists } p \in \mathbb{N} \textit{ such that } \xi_n = \xi_{n+pk} \\ \textit{holds for any } n \in \mathbb{Z} \textit{ with } n + pk \geqslant 0.\end{gathered} \tag{6.3}$$

By $p(\xi, n)$, we denote the least integer p that satisfies (6.3) for a given $n \in \mathbb{N}_0$. Given $\xi \in \mathcal{A}_\kappa^{\mathbb{N}_0}$, we again define its associated subshift by X_ξ. When ξ is almost periodic, we alternatively can choose any $z \in \mathcal{A}_\kappa^{\mathbb{Z}}$ with $z_{[0,+\infty]} = \xi$ and define X_ξ as the ω-limit of z.

Given a one-sided sequence $\xi \in \mathcal{A}_\kappa^{\mathbb{N}_0}$ and any $p \in \mathbb{N}$, we define the p-periodic positions of ξ as $\mathrm{Per}^+(\xi, p) = \{n \in \mathbb{N}_0 \mid p(\xi, n)|p\}$. Again, p is an essential period of ξ if $\mathrm{Per}^+(\xi, p) \neq \varnothing$ and there exists no $q < p$ such that $\mathrm{Per}^+(\xi, p, a) \subseteq \mathrm{Per}^+(\xi, q, a)$ for any $a \in \mathcal{A}_\kappa$. If we define the p-skeleton $\hat{\xi}^p$ of a one-sided Toeplitz sequence as in (6.2), with $\mathrm{Per}(\xi, p)$ replaced by $\mathrm{Per}^+(\xi, p)$, then this means that p is essential if and only if it is the least period of $\hat{\xi}^p$. Similar to before, a scale $P = (p_k)_{k \in \mathbb{N}}$ is a period structure of a Toeplitz sequence $\xi \in \mathcal{A}_\kappa^{\mathbb{N}_0}$ if

- for all $k \in \mathbb{N}$, p_k is an essential period of ξ;
- $\bigcup_{k \in \mathbb{N}} \mathrm{Per}^+(\xi, p_k) = \mathbb{N}_0$.

The density of the periodic positions of ξ can then be defined as above, with $\mathrm{Per}(\xi, p_k)$ replaced by $\mathrm{Per}^+(\xi, p_k)$, and will be denoted by $\overline{D}^+(\xi, P)$ and $\overline{D}^+(\xi)$, respectively. Then ξ is called regular if $\overline{D}^+(\xi) = 1$ and irregular otherwise.

We omit the proof of the following elementary statement.

Lemma 6.4. *If $X \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ is a minimal subshift and $\xi, \zeta \in X$. Then, for any $p \in \mathbb{N}$, there exists $q \in \{0, \ldots, p-1\}$ such that $\mathrm{Per}(\zeta, p, a) = \mathrm{Per}(\xi, p, a) + q$ holds for all $a \in \mathcal{A}_\kappa$. In particular, we have $\mathrm{Per}(\zeta, p) = \mathrm{Per}(\xi, p) + q$. Moreover, if ξ, ζ are Toeplitz sequences, then any period structure of ξ is also a period structure of ζ and vice versa.*

As a consequence, we can speak of a period structure of a Toeplitz subshift.

Proposition 6.5. *If $\xi \in \mathcal{A}_\kappa^{\mathbb{N}}$ a one-sided Toeplitz sequence, then*

- *X_ξ is a Toeplitz subshift;*
- *if P is a period structure of ξ, then P is a period structure of X_ξ;*
- *X_ξ is regular if and only if ξ is regular.*

Proof. Let $\xi \in \mathcal{A}_\kappa^{\mathbb{N}_0}$ be a one-sided Toeplitz sequence with period structure $P = (p_k)_{k \in \mathbb{N}}$. We first note that since ξ is almost periodic, the subshift X_ξ is minimal.

In order to see hat P is a period structure for X_ξ, we have to show that it is a period structure for some Toeplitz sequence $x \in X_\xi$. To that end, let $(n_j)_{j \in \mathbb{N}}$ and $(r_j)_{j \in \mathbb{N}}$ be defined by recursion on j as follows. For $j = 0$ we let $n_0 = 0$ and $r_0 = p(\xi, 0)$. If n_j, r_j are given for some $j \geqslant 0$, we let $n_{j+1} = n_j + r_j$ and

$$r_{j+1} = \mathrm{lcm}\,\{r_j, p(\xi, n_{j+1} - j - 1), p(\xi, n_{j+1} + j + 1)\} \ ,$$

where lcm denotes the least common multiple. By induction on j, one sees that all positions in $[n_j - j, n_j + j]$ are r_j-periodic within ξ. Hence, we have that

$$\xi_{[n_{j+1} - j, n_{j+1} + j]} = \xi_{[n_j + r_j - j, n_j + r_j + j]} = \xi_{[n_j - j, n_j + j]} \ .$$

As a consequence, we obtain a well-defined sequence $x \in X_\xi$ by setting

$$x_{[-j,j]} \; = \; \xi_{[n_j-j,n_j+j]} \tag{6.4}$$

for all $j \in \mathbb{N}$. If we choose $z \in \mathcal{A}_\kappa^{\mathbb{Z}}$ with $z_{[0,\infty)} = \xi$, then it follows by construction that $\lim_{j\to\infty} \sigma^{n_j}(z) = x \in X_\xi$. We claim that x is Toeplitz.

By going over to a subseqeuence if necessary, there exist $a_k \in \{0,\dots,p_k-1\}$, $k \in \mathbb{N}$, such that $n_j \bmod p_k \equiv a_k$ for all but finitely many j. Consequently,

$$\mathrm{Per}^+(\xi,p_k) - a_k + p_k\mathbb{Z} \; \subseteq \; \mathrm{Per}(x,p_k) \; .$$

Due to (6.4), the backwards iterates $\sigma^{-n_j}(x)$ converge to a point that coincides with ξ on $\mathbb{N}_0$ as $j \to \infty$. Hence, we have

$$(\mathrm{Per}(x,p_k) + a_k) \cap \mathbb{N}_0 \; \subseteq \; \mathrm{Per}^+(\xi,p_k) \; .$$

As p_k is an essential period of ξ, this implies that it is also an essential period of x. Since P is a period structure of ξ, we have that for any $j \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $r_j \mid p_k$. Hence

$$\mathbb{Z} \; = \; \bigcup_{j\in\mathbb{N}} \mathrm{Per}(x,r_j) \; \subseteq \; \bigcup_{k\in p_k} \mathrm{Per}(x,p_k) \; ,$$

so that x is indeed a two-sided Toeplitz sequence, with period structure P. As X_ξ is minimal, we have $X_\xi = \overline{\{\sigma^n(x) \mid n \in \mathbb{Z}\}}$. Thus, we obtain that X_ξ is a Toeplitz subshift. Moreover, it follows from the above construction that

$$\#\mathrm{Per}(x,p_k) \cap [0,p_k-1] \; = \; \#\mathrm{Per}^+(\xi,p_k) \cap [0,p_k-1]$$

for all $k \in \mathbb{N}$. Therefore x is regular if and only if ξ is a regular. □

Remark 6.6. *We note that while in the situation of Proposition 6.5, X_ξ is a Toeplitz flow, the sequence ξ may not necessarily extend to a two-sided Toeplitz sequence. This is the case, for instance, for the period doubling sequence.*

6.3. **Construction of one-sided Toeplitz sequences.** A standard method to construct a one-sided Toeplitz sequence in $\mathcal{A}_\kappa^{\mathbb{N}_0}$ is to enlarge the alphabet $\mathcal{A}_\kappa$ by an additional letter (which in our case will be κ) and to approximate ξ with p_k-periodic sequences $\xi^{(k)} \in \mathcal{A}_{\kappa+1}^{\mathbb{N}_0}$ in such a way that $\xi^{(k)}$ and $\xi^{(k+1)}$ only differ at positions $n \in \mathbb{N}_0$ with $\xi_n^{(k)} = \kappa$. Thus, the additional letter κ serves as a placeholder in the construction, which is successively replaced by letters from the original alphabet. We refer to [Wil84] or [Dow05] and references within for more background and the construction of a wide array of specific examples.

In our context, we will implement this idea in the following way. Given a scale $P = (p_k)_{k\in\mathbb{N}}$ with $p_1 > 1$, we choose a sequence of words $v^{(k)} \in \mathcal{A}_{\kappa+1}^{p_k}$ of length p_k and consider the corresponding p_k-periodic sequences $\xi^{(k)} \in \mathcal{A}_{\kappa+1}^{\mathbb{N}_0}$ defined by $\xi^{(k)}_{[0,p_k-1]} = v^{(k)}$. Thereby, we require that the following conditions hold:

(T1) $\xi_j^{(k+1)} = \xi_j^{(k)}$ whenever $\xi_j^{(k)} \neq \kappa$;

(T2) for any $k \in \mathbb{N}$ there exists $k' > k$ such that $\xi^{(k')}_{[0,p_k)} \in \mathcal{A}_\kappa^{p_k}$;

(T3) if $\xi_j^{(k)} = \kappa$ for some $k \in \mathbb{N}$ and $j \in \mathbb{N}_0$, then there exists $k' > k$ and $m \in \mathbb{N}$ such that $\xi_j^{(k')}, \xi_{j+mp_k}^{(k')} \in \mathcal{A}_\kappa$ and $\xi_j^{(k')} \neq \xi_{j+mp_k}^{(k')}$;

(T4) for any $k \in \mathbb{N}$, p_k is the least period of $\xi^{(k)}$.

The precise role of these conditions will become clear in the proof of

Proposition 6.7. *Suppose that (T1)–(T4) hold. If we let*

$$k(n) \; = \; \min\left\{k \in \mathbb{N} \mid \xi_n^{(k)} \neq \kappa\right\}$$

and define $\xi \in \mathcal{A}_\kappa^{\mathbb{N}}$ by $\xi_n = \xi_n^{(k(n))}$, then ξ is a one-sided Toeplitz sequence. Further, the scale P used in the construction is a period structure of ξ, the p_k-skeletons $\hat{\xi}^{p_k}$ coincide with the *sequences $\xi^{(k)}$ for all $k \in \mathbb{N}$ and we have*

$$\overline{D}^{+}(\xi) \;=\; \lim_{k\to\infty} \frac{\#\{0 \leqslant j < p_k - 1 \mid v_j^{(k)} \neq \kappa\}}{p_k} \,.$$

Proof. Note that due to (T2), we have that $k(n)$ is finite for all $n \in \mathbb{N}_0$, and by $(T1)$ we obtain that $\xi_n^{(k)} = \xi_n^{(k(n))}$ for all $k \geqslant k(n)$. Further, we have $n \in \mathrm{Per}^+(\xi, p_{k(n)})$ by construction, while (T3) yields $n \notin \mathrm{Per}^+(\xi, p_{k'})$ for any $k' < k(n)$. Consequently

$$\mathrm{Per}^+(\xi, p_k) \;=\; \left\{ n \in \mathbb{N}_0 \mid \xi_n^{(k)} \neq \kappa \right\}$$

and therefore $\hat{\xi}^{p_k} = \xi^{(k)}$. Since p_k is the least period of $\xi^{(k)}$ by (T4), this implies that P is a period structure of ξ. In particular, we obtain that ξ is Toeplitz, since every symbol is repeated with some period p_k and ξ is aperiodic since the p_k are not uniformly bounded. □

Remark 6.8. *We note that (T4) can be ensured by requiring the stonger condition*

(T4') for every $k \in \mathbb{N}$ and $q < p_k$ with $q|p_k$, there exist $n, n' \in \mathbb{N}$ such that $q|(n - n')$, $\xi_n^{(k)}, \xi_{n'}^{(k)} \neq \kappa$ and $\xi_n^{(k)} \neq \xi_{n'}^{(k)}$.

In this case, the fact that p_k is the least period of $\xi^{(k)}$ is already ensured by those positions with $\xi_n^{(k)} \neq \kappa$. We will use this observation in some constructions below.

6.4. **Minimality of permutative lifts of Toeplitz flows.** We now aim to explore the relations between Toeplitz constructions and the lifting procedure via permutative block codes. Our first observation is the following.

Theorem 6.9. *For any $\ell, \kappa \in \mathbb{N}$ there exists a Toeplitz flow $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ such that for any permutative block code $h \in \mathcal{F}(\kappa, \ell)$ the lift $X = \eta_h^{-1}(Y)$ is minimal.*

Proof. Fix any permutative block code $h \in \mathcal{F}(\kappa, \ell)$. Consider a scale $P = (p_k)_{k\in\mathbb{N}_0}$ with $p_1 > \ell - 1$, $q_k = \frac{p_{k+1}}{p_k} > \kappa^{\ell-1}$ for any $k \in \mathbb{N}_0$. By recursion on k, we construct two sequences of words $w^{(k)}$ and $v^{(k)} = w^{(k)}\kappa^{\ell-1}$ of length $p_k - \ell + 1$ and p_k, respectively, so that the p_k-periodic sequences $\xi^{(k)}$ with $\xi^{(k)}_{[0,p_k-1]} = v^{(k)}$, $k \in \mathbb{N}$, satisfy properties (T1)–(T4) above.

We first choose a word $w^{(1)} \in \mathcal{A}_\kappa^{p_0-\ell+1}$ such that (T4) is satisfied for $v^{(1)} = w^{(1)}\kappa^{\ell-1}$. Then, if $v^{(k)} = w^{(k)}\kappa^{\ell-1}$ is already given for some $k \geqslant 1$, we choose $u^{(k,0)}, \ldots, u^{(k,q_k-2)} \in \mathcal{A}_\kappa^{\ell-1}$ such that

$$\left\{ u^{(k,i)} \mid i = 0, \ldots, \kappa^{\ell-1} - 1 \right\} \;=\; \mathcal{A}_\kappa^{\ell-1} \tag{6.5}$$

(and $u^{(k,\kappa^{\ell-1})}, \ldots, u^{(k,q_k-2)}$ are arbitrary). Now, we define

$$w^{(k+1)} \;=\; w^{(k)}u^{(k,0)}, \ldots, w^{(k)}u^{(k,q_k-2)}w^{(k)} \quad \text{and} \quad v^{(k+1)} = w^{(k+1)}\kappa^{\ell-1} \,.$$

Then (T1) and (T2) hold by construction. Further, $\xi^{(k+1)}$ differs from $\xi^{(k)}$ only on intervals of the form $[mp_k - (\ell-1), mp_k - 1]$ for $m \in \mathbb{Z}\backslash q_k\mathbb{Z}$. Since we insert all the different words in $\mathcal{A}_\kappa^{\ell-1}$ into these intervals by (6.5), none of the newly filled positions can be p_k-periodic. This yields (T3). Since κ only appears in the last $\ell - 1$ positions in $v^{(k)}$, p_k is the least period of $\xi^{(k)}$. This shows (T4). As a consequence, Proposition 6.7 yields that the sequence ξ resulting from this construction is a one-sided Toeplitz sequence with period structure P. If we let $Y = X_\xi$, it remains to see that the lift of Y is minimal. However, it is a direct consequence of (6.5) and Lemma 3.2 that $\mathcal{R}(w^{(k)})$ acts transitively on $\mathcal{A}_\kappa^{\ell-1}$ for each $k \in \mathbb{N}$. Hence, $\eta_h^{-1}(X_\xi)$ is minimal by Theorem 1.5. □

Remark 6.10. *The Toeplitz flows resulting from the above construction will be regular. However, with some straightforward modifications, it is also possible to obtain irregular examples.*

As the above result shows, a Toeplitz flow may admit a minimal extension when lifted by any permutative sliding block code of a fixed length. However, the situation changes when the length of the permutative block code is allowed to vary. The following result shows that no Toeplitz flow can have a minimal lift under all permutative sliding block codes on the same alphabet.

Theorem 6.11. *Consider a Toeplitz flow $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$. Then there exist some $\ell \in \mathbb{N}$ and a permutative block code $h \in \mathcal{F}(\kappa, \ell)$ such that $\eta_h^{-1}(Y)$ is not minimal.*

Proof. We fix a Toeplitz sequence $y \in Y$, let $\ell = \min\{p(y,n) \mid n \in \mathbb{Z}\} + 1$, choose $n \in \mathrm{Per}(y, \ell-1)$, let $c = y_n$ and consider the permutative block code $h \in \mathcal{F}(\kappa, \ell)$ given by

$$h : \mathcal{A}_\kappa^\ell \to \mathcal{A}_\kappa \quad , \quad w_0 \ldots w_{\ell-1} \mapsto w_{\ell-1} - w_0 + c \bmod \kappa \ .$$

Note that given any lift $\xi^u(y)$, $u \in \mathcal{A}_\kappa^{\ell-1}$, this implies $\xi^u(y)_{k+\ell-1} = \xi^u(y)_k + y_k - c$ for any $k \in \mathbb{Z}$. Since $\ell - 1$ is the least period that occurs for any symbol in y, we obtain that

$$\mathrm{Per}\left(\xi^u(y), \ell-1\right) \ = \ \mathrm{Per}(y, \ell-1, c) \ .$$

Further, given $a \in \mathcal{A}_\kappa$, we obtain

$$\mathrm{Per}\left(\xi^{a\ldots a}(y), \ell-1\right) \ = \ \mathrm{Per}\left(\xi^{a\ldots a}(y), \ell-1, c\right) \ ,$$

and due to Lemma 6.4 we have $\mathrm{Per}\left(x, \ell-1\right) = \mathrm{Per}\left(x, \ell-1, c\right)$ for any x in the orbit closure of $\xi^{a\ldots a}(y)$. However, this implies that the orbit closures of $\xi^{a\ldots a}(y)$ and $\xi^{b\ldots b}(y)$ with $a \neq b$ are disjoint, so that $\eta_h^{-1}(Y)$ is not minimal. □

Finally, we show that for any fixed permutative sliding block code, there exists a Toeplitz flow with non-minimal lift.

Proposition 6.12. *Let $h \in \mathcal{F}(\kappa, \ell)$ be permutative. Then there exists a Toeplitz subshift $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ such that $X = \eta_h^{-1}(Y)$ is non-minimal.*

Proof. We again use the same construction scheme as before and choose a scale $(p_k)_{n\in\mathbb{N}_0}$ with $p_1 > \ell$ and let $q_k = \frac{p_{k+1}}{p_k}$. Further, we let

$$w^{(1)} \ = \ 1^{2\ell}0^{2\ell} \quad \text{and} \quad v^{(0)} \ = \ w^{(1)}\kappa^\ell \ .$$

Note that our construction scheme then implies that the word $w^{(1)}$ appears in the resulting sequence ξ (defined by $\xi_{[0,p_k-1]} = v^{(k)}$) exactly at positions in $p_1\mathbb{N}_0$. Fix some $u \in \mathcal{A}_\kappa^{\ell-1}$. Then Lemma 3.2 implies that there exist two prefixes $u^{(0)}, u^{(1)} \in \mathcal{A}_\kappa^{\ell-1}$ such that

$$\Phi_{w^{(1)}0u^{(0)}}(u) \ = \ \Phi_{w^{(1)}1u^{(1)}}(u) \ = \ u \ .$$

We now recursively define the words $w^{(k)}$ and $v^{(k)}$ of our construction scheme by

$$w^{(k+1)} \ = \ \begin{cases} \left(w^{(k)}0u^{(0)}\right)^{q_k-1} w^{(k)}\kappa^\ell & \text{if } k \text{ is odd} \\ \left(w^{(k)}1u^{(1)}\right)^{q_k-1} w^{(k)}\kappa^\ell & \text{if } k \text{ is even} \end{cases} \ .$$

Then the fact that conditions (T1)–(T4) hold is straightforward to chack. Hence, the construction yields a Toeplitz subshift $Y = X_\xi$. Moreover, we obtain that $\Phi_{w^{(1)}vw^{(1)}}(u) = u$ holds whenever $w^{(1)}vw^{(1)} \in \mathcal{W}(Y)$. This implies that $\mathcal{R}\left(w^{(1)}\right)$ does not act transitively on $\mathcal{A}_\kappa^{\ell-1}$, so that Y is non-minimal by Theorem 1.5. □

6.5. **Unique ergodicity of permutative lifts of Toeplitz flows.**

We first show that unique ergodicity is not a direct consequence of minimality in our setting. This can already be demonstrated by using the simple block code $h_{2,2}$ from Section 4.2.

Theorem 6.13. *There exists a strictly ergodic Toeplitz subshift $Y \subseteq \mathcal{A}_2^{\mathbb{Z}}$ such that $\eta_{2,2}^{-1}(Y)$ is minimal, but not uniquely ergodic.*

Proof. We use the construction theme introduced in Section 6.3 and first choose

$$w^{(1)} \ = \ 110^{p_1-3} \quad \text{and} \quad v^{(1)} \ = \ w^{(1)}2 \ .$$

with $p_1 \geqslant 6$, so that $|w^{(1)}| = p_1 - 1$ and $|v^{(1)}| = p_1$. Note that according procedure described in Section 6.3 and due to the specific choice of $v^{(1)}$, the word $w^{(1)}$ occurs exactly at positions $n \in p_1\mathbb{Z}$ in the resulting sequence $\xi \in \mathcal{A}_2^{\mathbb{Z}}$:

$$\xi_{[n,n+p_1-2]} \ = \ w^{(1)} \quad \Leftrightarrow \quad n \in p_1\mathbb{Z} \ . \tag{6.6}$$

Further, we fix a sequence of integers $(q_k)_{k\in\mathbb{N}}$ with $q_k \geqslant 3$ and recursively define

$$w^{(k+1)} \;=\; \left(w^{(k)}0\right)^{q_k-3} w^{(k)}1w^{(k)}1w^{(k)} \quad \text{and} \quad v^{(k+1)} \;=\; w^{(k+1)}2\ .$$

Then conditions (T1)–(T4) with scale $(p_k)_{k\in\mathbb{N}}$ given by $p_k = p_1 \cdot \prod_{j=1}^{k-1} q_k$ are straightforward to check, so that Propositions 6.7 and 6.5 yields that $Y = X_\xi$ is a regular Toeplitz subshift. Moreover, the following statements hold:

(i) the words $w^{(k)}$ contain an even number of 1s (follows by induction on k);

(ii) consequently, we have

$$\Phi_{\left(w^{(k)}0\right)^j} \;=\; \mathrm{Id}_{\mathcal{A}_2} \;=\; \Phi_{\left(w^{(k)}0\right)^{q_k-3}w^{(k)}1w^{(k)}1}\ ;$$

for all $k \in \mathbb{N}$ and $j = 1, \ldots, q_k - 3$;

(iii) at the same time

$$\Phi_{\left(w^{(k)}0\right)^{q_k-3}w^{(k)}1} \;=\; \bar{\cdot}\ ,$$

where $\bar{\cdot} : \mathcal{A}_2 \to \mathcal{A}_2, a \mapsto 1 - a$, and this permutation belongs to $\mathcal{R}\left(w^{(k)}\right)$.

By construction, we have $\xi_{[0,|w^{(k)}|-1]} = w^{(k)}$. In particular, any word $w \in \mathcal{W}(Y)$ is contained in $w^{(k)}$ for some $k \in \mathbb{N}$. Therefore (iii), in combination with Theorem 1.5(iii), yields the minimality of $X = \eta_{2,2}^{-1}(Y)$.

In order to see that X is not uniquely ergodic, note that due (6.6) and properties (ii) and (iii), we have

$$D^{\mathrm{ap}}\left(w^{(1)}, w^{(2)}\right) \;=\; \frac{q_1 - 1}{p_1(q_1-1)+1} \tag{6.7}$$

as well as

$$\begin{aligned}
&\left(\#\mathrm{Ap}_{0\mapsto 0}\left(w^{(1)}, w^{(k+1)}\right) - \#\mathrm{Ap}_{0\mapsto 1}\left(w^{(1)}, w^{(k+1)}\right)\right)\\
&\quad=\; (q_k - 1)\cdot\left(\#\mathrm{Ap}_{0\mapsto 0}\left(w^{(1)}, w^{(k)}\right) - \#\mathrm{Ap}_{0\mapsto 1}\left(w^{(1)}, w^{(k)}\right)\right)\\
&\qquad\;\; - \left(\#\mathrm{Ap}_{0\mapsto 0}\left(w^{(1)}, w^{(k)}\right) - \#\mathrm{Ap}_{0\mapsto 1}\left(w^{(1)}, w^{(k)}\right)\right)\\
&\quad=\; (q_k - 2)\cdot\left(\#\mathrm{Ap}_{0\mapsto 0}\left(w^{(1)}, w^{(k)}\right) - \#\mathrm{Ap}_{0\mapsto 1}\left(w^{(1)}, w^{(k)}\right)\right)\ .
\end{aligned}$$

This entails that

$$\begin{aligned}
D^{\mathrm{ap}}\left(w^{(1)}, w^{(k+1)}\right) \;&=\; (q_k-2)\cdot\frac{|w^{(k)}| - |w^{(1)}| + 1}{w^{(k+1)} - w^{(1)} + 1}\cdot D^{\mathrm{ap}}\left(w^{(1)}, w^{(k)}\right)\\
&\geqslant\; \frac{(q_k-2)\cdot(q_{k-1}-1)}{q_k\cdot q_{k-1}}\cdot D^{\mathrm{ap}}\left(w^{(1)}, w^{(k)}\right)\ .
\end{aligned}$$

holds for all $k \geqslant 2$. As a consequence, we obtain that for all $k \in \mathbb{N}$

$$D^{\mathrm{ap}}\left(w^{(1)}, w^{(k+1)}\right) \;\geqslant\; \left(\prod_{j=2}^{k}\frac{(q_j-2)\cdot(q_{j-1}-1)}{q_j\cdot q_{j-1}}\right)\cdot\frac{q_1-2}{p_1(q_1-1)+1}\ .$$

Clearly, we can now choose the parameters p_1 and q_k with $k \in \mathbb{N}_0$ such that

$$\inf_{k\in\mathbb{N}} D^{\mathrm{ap}}\left(w^{(1)}, w^{(k)}\right) \;>\; 0\ .$$

Due to Theorem 5.4, this shows that $X = \eta_{2,2}^{-1}(Y)$ cannot be uniquely ergodic. □

Our final aim is to modify the construction of the Toeplitz sequences in the proof of Theorem 6.9 in order to obtain not only minimal, but strictly ergodic lifts.

Theorem 6.14. *Let $h \in \mathcal{F}(\kappa, \ell)$ be permutative. Then there exists a Toeplitz subshift $Y \subseteq \mathcal{A}_\kappa^{\mathbb{Z}}$ such that $X = \eta_h^{-1}(Y)$ is strictly ergodic.*

To prove this statement, we first recall a classical result from finite graph theory. A finite directed graph is a pair $D = (V, E)$, where V is a finite set of vertices and $E \subseteq V \times V$ is a set of directed edges. It is called *connected* if for every $v, w \in V$ there are vertices $v = v_0, v_1, \dots, v_n = w$ such that $(v_i, v_{i+1}) \in E$ for $i \in \{0, \dots, n-1\}$. For $v \in V$, define the *indegree* of v as $\deg^-(v) = \#\{(u, v) \in E\}$, and the *outdegree* of v as $\deg^+(v) = \#\{(v, u) \in E\}$. The graph is *balanced* if $\deg^-(v) = \deg^+(v)$ for every $v \in V$. An *Euler circuit* in D is a finite sequence of edges $(v_0, v_1), \dots, (v_{n-1}, v_n)$ in which every edge of D appears exactly once. We will use the following standard criterion for the existence of an Euler-circuit.

Theorem 6.15. [BJG09, Theorem 1.7.2]. *A finite directed graph admits an Euler circuit if it is connected and balanced.*

For our construction, it will be useful to consider colored edges. An *edge-colored directed graph* is a directed graph together with a finite set of colors C and a coloring map $\chi : E \to C$. An Euler circuit of an edge-colored case will be denoted as

$$v_0, \chi(v_0, v_1), v_1, \chi(v_1, v_2), \dots, v_{n-1}, \chi(v_{n-1}, v_n),$$

thus keeping track of the colors. Further, we will consider the case where G is a finite group and $S \subseteq G$. The *left Cayley graph* of G induced by S has vertex set G and the set of edges $\{(g, sg) \mid g \in G,\ s \in S\}$. Every vertex has indegree and outdegree $|S|$, so this graph is balanced. If S generates G, then it is connected. Given a finite set A, we denote by $\mathrm{Per}(A)$ the group of permutations of A.

Now we are ready to prove Theorem 6.14. As mentioned, we will use a refined version of the construction in Theorem 6.9 and mainly concentrate on the required modifications.

Proof of Theorem 6.14. Fix a permutative block code $h \in \mathcal{F}(\kappa, \ell)$. In order to start the inductive construction, we first choose a word $w^{(1)} \in \mathcal{A}_\kappa^*$ and let $v^{(1)} = w^{(1)}\kappa^{\ell-1}$ and $p_1 = |v^{(1)}|$. Now, suppose that the words $v^{(1)}, \dots, v^{(k)}$ with $v^{(j)} = w^{(j)}\kappa^{\ell-1}$ and $|v^{(j)}| = p_j$ have already been defined. Due to Lemma 3.2, the mapping

$$\pi_k : \mathcal{A}_\kappa^{\ell-1} \to \mathrm{Per}\left(\mathcal{A}_\kappa^{\ell-1}\right) \quad , \quad u \mapsto \Phi_{w^{(k)}u}$$

is injective. Consider the finite set of permutations

$$S^{(k)} = \left\{\Phi_{w^{(k)}u} \;\middle|\; u \in \mathcal{A}_\kappa^{\ell-1}\right\} .$$

Denote by G_k the group generated by $S^{(k)}$. Consider the left Cayley graph (G_k, E_k) of G_k induced by S_k (so that $E_k = \{(g, sg) \mid g \in G_k, s \in S^{(k)}\}$), with the edge $(g, \Phi_{w^{(k)}u} \circ g)$ colored by u. In other terms, we choose $\mathcal{A}_\kappa^{\ell-1}$ to be the set of colors and use the mapping $\chi(g, h) = \pi_k^{-1}(hg^{-1})$ as the coloring map. We set

$$q_k = \sum_{g \in G_k} \deg^+(g) = \kappa^{\ell-1} \cdot \#G_k .$$

The graph (G_k, E_k) is connected and balanced, so Theorem 6.15 gives an Euler circuit. We assume without loss of generality that it starts at the identity and write it as

$$g^{(k,0)}, u^{(k,0)}, \dots, g^{(k,q_k-1)}, u^{(k,q_k-1)} \tag{6.8}$$

where $g^{(k,0)} = \mathrm{Id}_{\mathcal{A}_\kappa^{\ell-1}} \in \mathrm{Per}\left(\mathcal{A}_\kappa^{\ell-1}\right)$ and $\chi\left(g^{(k,i)}, g^{(k,i+1)}\right) = u^{(k,i)}$ for $i \in \{0, \dots, q_k - 2\}$. Now choose

$$w^{(k+1)} = w^{(k)}u^{(k,0)} \dots w^{(k)}u^{(k,q_k-2)}w^{(k)} \quad , \quad v^{(k+1)} = w^{(k+1)}\kappa^{\ell-1}$$

and $p_{k+1} = p_k \cdot q_k$. Note that here we did not use the last edge in the Euler circuit.

The verification of (T1)–(T4) is the same as in Theorem 6.9. We obtain a one-sided Toeplitz sequence ξ with period structure $P = (p_k)_{k \in \mathbb{N}}$ and set $Y = X_\xi$. Since the density of holes in the p_k-skeleton is $\frac{\ell-1}{p_k} \to 0$, the Toeplitz flow Y is regular and hence strictly ergodic. Similar to the situation in Theorem 6.9, $S^{(k)}$ acts transtively on $\mathcal{A}_\kappa^{\ell-1}$ and $S^{(k)} \subseteq \mathcal{R}(w^{(k)})$. Since $|w^{(k)}| \to \infty$, Theorem 1.5(iii) shows that $X = \eta_h^{-1}(Y)$ is minimal.

Now we show that X is uniquely ergodic. It follows by construction that

$$\Phi_{w^{(k)}u^{(k,0)}\ldots w^{(k)}u^{(k,i)}} = g^{(k,i+1)}$$

for $i \in \{0,\ldots,q_k-2\}$. Transitivity of $S^{(k)}$ acting on $\mathcal{A}_\kappa^{\ell-1}$ in combination with the orbit-stabilizer theorem give that, for any $v,v' \in \mathcal{A}_\kappa^{\ell-1}$,

$$\#\left\{g \in G_k \;\middle|\; g(v) = v'\right\} = \frac{\#G_k}{\kappa^{\ell-1}} .$$

Meanwhile, since every $g \in G_k$ occurs $\kappa^{\ell-1}$ times among the source vertices of the Euler circuit (6.8), it follows that

$$\#\left\{i \in \{0,\ldots,q_k-1\} \;\middle|\; g^{(k,i)}(v) = v'\right\} = \#G_k \tag{6.9}$$

for any $v,v' \in \mathcal{A}_\kappa^{\ell-1}$. Now, in order to apply Theorem 5.4, choose $y \in Y$ with $y_{[0,\infty)} = \xi$, and fix $w \in \mathcal{W}(Y)$. Take k such that $|w| \leqslant |w^{(k)}|$. Consider $n,m \in \mathbb{N}$. We aim to estimate the quantity $D^{\mathrm{ap}}(w, y_{[n,n+m-1]})$ as defined in (5.13). First, we consider the aligned p_{k+1}-blocks. Let n_1,n_2 be the smallest and largest integers in $[n,n+m-1]$, respectively, that are a multiple of p_{k+1}. Then by construction

$$y_{[n_1,n_2-1]} = w^{(n+1)}v_1w^{(n+1)}v_2\ldots w^{(n+1)}v_t ,$$

where $v_1,\ldots,v_t \in \mathcal{A}_\kappa^{\ell-1}$ and $t \leqslant \left\lfloor \frac{m}{p_{k+1}} \right\rfloor$. Suppose that $w \lhd w^{(k)}$, that is, $w^{(k)}_{[j,j+|w|-1]} = w$ for some $j \leqslant |w^{(k)}| - |w|$. Consider any prefix $u \in \mathcal{A}_\kappa^{\ell-1}$. Then the prefixes $\xi^u(y)_{[k,k+\ell-2]}$ that appear in $\xi^u(y)$ at positions $k = j, j+p_k, \ldots, j+(q_k-1)p_k$ (over the the corresponding p_k-separated occurrences of w in $w^{(k+1)}$) are

$$\Phi_{w^{(k)}_{[0,j-1]}}\left(g^{(k,i)}(u)\right), \qquad i = 0,\ldots,q_k-2,$$

where $\Phi_{w^{(k)}_{[0,j-1]}} = \mathrm{Id}_{\mathcal{A}_\kappa^{\ell-1}}$ for $j=0$. Due to (6.9), these prefixes are uniformly distributed over $\mathcal{A}_\kappa^{\ell-1}$, independent of the prefix u. Thus, all occurrences of w contained entirely in one copy of $w^{(k+1)}$ do not contribute to $D^{\mathrm{ap}}\left(w, y_{[n,n+m-1]}\right)$. While occurences of w at other positions may contribute, the interval $[n_1, n_2-1]$ contains only $tq_k \cdot (|w|+\ell-2) - |w|$ such positions. If we add the at most $2p_{k+1}$ remaining positions in $[n,n+m-1]\backslash[n_1,n_2-1]$, this yields

$$\begin{aligned} D^{\mathrm{ap}}(w, y_{[n,n+m-1]}) &\leqslant \frac{tq_k \cdot (|w|+\ell-2) + 2p_{k+1} - |w|}{m-|w|} \\ &\leqslant \frac{tq_k \cdot (|w|+\ell) + 2p_{k+1}}{tp_{k+1}} = \frac{|w|+\ell}{p_k} + \frac{2}{t} \overset{m\to\infty}{\longrightarrow} 0 . \end{aligned}$$

If follows from Theorem 5.4 that X is uniquely ergodic. □

Remark 6.16. *The Toeplitz flow constructed above is actually linearly recurrent, but this is not essential. Indeed, the numbers $q_k = \kappa^{\ell-1}\#G_k$ are uniformly bounded because G_k is a subgroup of the finite group $\mathrm{Per}\left(\mathcal{A}_\kappa^{\ell-1}\right)$. At the k-th stage, however, we may traverse the chosen Euler circuit c_k times before constructing $w^{(k+1)}$. This would result in multipliers $q_k = c_k\kappa^{\ell-1}\#G_k$ in the Toeplitz construction, without affecting the balance argument. By choosing c_k to grow sufficiently fast, one obtains similar examples that are not linearly recurrent.*

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