Poincaré theory for decomposable cofrontiers

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Abstract

We extend Poincaré's theory of orientation-preserving homeomorphisms from the circle to circloids with decomposable boundary. As special cases, this includes both decomposable cofrontiers and decomposable cobasin boundaries. More precisely, we show that if the rotation number on an invariant circloid A of a surface homeomorphism is irrational and the boundary of A is decomposable, then the dynamics are monotonically semiconjugate to the respective irrational rotation. This complements classical results by Barge and Gillette on the equivalence between rational rotation numbers and the existence of periodic orbits and yields a direct analogue to the Poincaré Classification Theorem for circle homeomorphisms. Moreover, we show that the semiconjugacy can be obtained as the composition of a monotone circle map with a 'universal factor map', only depending on the topological structure of the circloid. This implies, in particular, that the monotone semiconjugacy is unique up to post-composition with a rotation.

If, in addition, A is a minimal set, then the semiconjugacy is almost one-to-one if and only if there exists a biaccessible point. In this case, the dynamics on A are almost automorphic. Conversely, we use the Anosov-Katok method to build a C^∞-example where all fibres of the semiconjugacy are non-trivial.

1 Introduction

Given an orientation-preserving circle homeomorphism φ, the Poincaré Classification Theorem states that the rotation number $\rho(\varphi)$ is well-defined and determines the qualitative dynamical behaviour of φ, in the sense that $\rho(\varphi)$ is rational if and only if φ has a periodic orbit and irrational if and only if φ is monotonically semiconjugate to the respective irrational rotation. This result provides the basis for a rather complete understanding of invertible dynamics on the circle. At the same time the study of cofrontiers, circloids and other classes of circle-like continua, like basin boundaries, co-basin boundaries or pseudocircles, has a long history in plane topology and continuum theory, going back to Kuratowski, Cartwright, Littlewood, Bing and others [Kur28, CL51b, Bin51]. Recently the topic has gained further momentum, since invariant circloids play a crucial role in surface dynamics. It is therefore a natural question to ask whether an analogue to Poincaré's classical result holds for these more general continua. In this article, we extend the Poincaré Classification to circloids with decomposable boundary.[1] In order to define this notion, we let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ and call a continuum $A \subseteq \mathbb{A}$ an *essential annular continuum* if $\mathbb{A} \setminus A$ consists of exactly two connected components, both of which are homeomorphic to $\mathbb{A}$. Further, A is called an essential circloid if it does not contain any strictly smaller essential annular continuum as a subset, and a cofrontier if it is a circloid with empty interior. We refer to Section 2 for further explanations and details.

In the context of the dynamics of surface homeomorphisms, circloids may appear in various situations. For instance, they separate adjacent invariant topological disks or annular domains [BG91, BGM93, Wal91, KY94, FLC03], and any periodic point free continuum of a non-wandering surface homeomorphism is an annular continuum [Kor10] that can further be decomposed into a dense union of invariant circloids and transitive

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[1] We recall that a continuum is called *decomposable* if it can be written as the union of two non-empty proper subcontinua.

annuli [Jäg10]. On the two-torus, the existence of invariant circloids can often be deduced from information on the rotation set [Jäg09a, Dav, GKT14], and invariant "foliations" consisting of circloids play an important role for the problem of linearisation [Jäg09b]. The respective results in topological dynamics have further applications in the theory of C^r-generic diffeomorphisms [FLC03, KN10, KLCN15]. It is thus of vital interest to understand the interplay between the topological structure and the possible dynamical behaviour on such continua. However, while the relation between rational rotation numbers and periodic orbits is quite well-understood [CL51b, BG91, KLCN15], the more intricate question of irrational rotation factors has been left completely open so far.

The problem is complicated by the fact that the rotation number on invariant circloids is not necessarily unique. Non-degenerate rotation intervals have been shown to occur on the Birkhoff attractor [LC88] and, more recently, the pseudocircle [BO]. Such examples can be excluded by adding a mild recurrence assumption [KLCN15], but even in the case of a unique rotation number a semiconjugacy does not have to exist. This was shown by Handel [Han82] and Herman [Her86], who realised the pseudocircle as a minimal set of a smooth surface diffeomorphism. In these examples, the rotation number is irrational, but the dynamics are not semiconjugate to the corresponding rotation. While the pseudocircle is the paradigm example of a circle-like continuum with highly intricate topological structure, a modification of the construction can be used to produce a variety of more 'regular' indecomposable continua with the same behaviour. Hence, *decomposability* of the circloid presents itself as the obvious minimal requirement for a possible analogue to the Poincaré classification. As the following result shows, it turns out to be sufficient as well. Recall that a monotone map is one with connected fibers.

Theorem 1.1. *Suppose $\varphi\colon \mathbb{A} \to \mathbb{A}$ is a homeomorphism homotopic to the identity with an essential φ-invariant circloid A with decomposable boundary. Then every point of A has a well-defined rotation number $\rho \in \mathbb{T}^1$ which is independent of the point, and*

- *ρ is rational if and only if there is a periodic point in A.*
- *ρ is irrational if and only if $\varphi|_A$ is monotonically semiconjugate to the corresponding irrational rotation by ρ on $\mathbb{T}^1$.*

Barge and Gillette showed that the rotation number on a decomposable cofrontier is always unique, and it is rational if and only if there exists a periodic orbit in the cofrontier [BG91]. Theorem 1.1 complements these results to give a direct analogue to the Poincaré Classification Theorem for decomposable cofrontiers and, more generally, to circloids with decomposable boundary. Thus, dynamics with irrational rotation number on an invariant circloid with decomposable boundary are 'linearisable'.

As it further turns out, to a great extent this linearisation does not depend on the dynamics. More precisely, there exists a 'universal factor map' which maps the circloid to a topological circle and semiconjugates the dynamics of *any* homeomorphism preserving the circloid to that of a circle homeomorphism.

Theorem 1.2. *Suppose that $A \subseteq \mathbb{A}$ is an essential circloid with decomposable boundary and there exists a self-homeomorphism of $\mathbb{A}$ leaving A invariant without periodic points in A. Then there exists a continuous and onto map $\Pi : \mathbb{A} \to \mathbb{A}$ with the following properties.*

(i) Π *is monotone and homotopic to the identity;*

(ii) Π *sends A to $\mathbb{T} = \mathbb{T}^1 \times \{0\}$;*

(iii) Π *is injective on $\mathbb{A} \backslash A$;*

(iv) for any homeomorphism $\phi\colon \mathbb{A} \to \mathbb{A}$ leaving A invariant, there exists a homeomorphism $\tilde{\varphi}\colon \mathbb{A} \to \mathbb{A}$ such that $\tilde{\varphi}(\mathbb{T}) = \mathbb{T}$ and $\Pi \circ \varphi = \tilde{\varphi} \circ \Pi$.

(v) If $h\colon A \to \mathbb{T}^1$ is any monotone surjection, then there exists a monotone map $\widetilde{h}\colon \mathbb{T} \to \mathbb{T}^1$ such that $h = \tilde{h} \circ \Pi$. In particular, if h semiconjugates $\varphi_{|A}$ to an irrational rotation R_ρ, then $\tilde{h}$ semiconjugates $\tilde{\varphi}_{|\mathbb{T}}$ to R_ρ.

We give a short self-contained proof in Section 5. It turns out, however, that the family of subcontinua of A given by the fibres of Π coincides with a decomposition of the circloid constructed already by Kuratowski, in a purely topological context [Kur28]. We discuss this in Section 7.

It is well-known that the semiconjugacies in the Poincaré Classification Theorem are unique up to post-composition by a rotation. As an immediate consequence of Theorem 1.2, we obtain the same statement for decomposable circloids.

Corollary 1.3. *The semiconjugacy in Theorem 1.1 is unique up to post-composition by a rotation.*

As should be expected, additional information on the topological structure of the circloid yields further information on the dynamics. We concentrate on the relation between the existence of biaccessible points and almost automorphic dynamics, whose study is a classical topic in abstract topological dynamics [Vee65, Ell69, Aus88]. A homeomorphism $\varphi\colon X \to X$ is almost automorphic if it is semiconjugate to some almost periodic homeomorphism of a space Y in a way that the set of points of Y with a unique preimage under the semiconjugation is dense in Y. The following statement shows how sets of this type appear in surfaces. A point of an essential circloid $A \subset \mathbb{A}$ is called *biaccessible* if it is the unique intersection point of some arc σ with A such that σ intersects both components of $\mathbb{A}\backslash A$. An essential cobasin boundary B is the boundary of an essential circloid, $B = \partial A$, and if A has a biaccessible point x belonging to B, we also say that x is a biaccessible point of B.

Theorem 1.4. *If B is an essential cobasin boundary in $\mathbb{A}$ invariant by a homeomorphism $\varphi\colon \mathbb{A} \to \mathbb{A}$ without periodic points and there is a biaccessible recurrent point in B, then $\varphi|_B$ is almost automorphic.*

Corollary 1.5. *Let $\varphi\colon \mathbb{A} \to \mathbb{A}$ be a homeomorphism and X is an essential φ-invariant continuum such that $\varphi|_X$ is minimal. If X has a biaccessible point, then X is a decomposable cofrontier and $\varphi|_X$ is almost automorphic.*

In fact, to our knowledge all known examples of minimal decomposable cofrontiers are of this type; see for instance [Wal91, Her83]. Therefore, we close with a construction that is based on the well-known Anosov-Katok method [AK70] and demonstrates that all fibres may be non-trivial as well.

Theorem 1.6. *There exists a C^∞ diffeomorphism $\varphi\colon \mathbb{A} \to \mathbb{A}$ leaving invariant a decomposable cofrontier A such that*

(i) the rotation number on A is irrational;

(ii) the dynamics on A are minimal;

(iii) all the fibres of points of $\mathbb{T}^1$ of the semiconjugacy given by Theorem 1.2 are non-trivial continua (i.e. not a single point).

In fact, the fibres of points of $\mathbb{T}^1$ have a diameter uniformly bounded below by a positive constant (see Claim 8.2) and, although we do not give a formal proof, it can be seen from the construction that all these fibres can be given a rich topological structure, reminiscent of the Knaster Buckethandle continuum.

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2 Notation and preliminaries

We denote by $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ the open annulus, and $\pi\colon \mathbb{R}^2 \to \mathbb{A}$ its universal covering map, where $T : (x, y) \mapsto (x + 1, y)$ is a generator of the group of covering transformations.

A subset A of the open annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ is called an *essential annular continuum* if it is compact and connected and its complement $\mathbb{A}\backslash A$ consists of exactly two connected components, both of which are unbounded. Note that in this situation one of the components is unbounded above and bounded below, whereas the other is bounded above and unbounded below, and both of them are homeomorphic to $\mathbb{A}$. Moreover, A is the decreasing intersection of a sequence of closed annuli. We call $C \subseteq \mathcal{U}$ an *essential circloid* if it is a minimal element with respect to inclusion amongst essential annular continua. An essential circloid with empty interior is called an *essential cofrontier*. The boundary of an essential circloid is called an *essential cobasin boundary*. It is the intersection of the boundaries of the two complementary components of the circloid and a minimal element with respect to inclusion amongst essential continua. A subset A of a surface S is called *annular continuum (circloid/cofrontier/cobasin boundary)* if it has a neighbourhood $\mathcal{U}$ homeomorphic to $\mathbb{A}$ such that A as a subset of $\mathcal{U}$ is an essential annular continuum (essential circloid/essential cofrontier/essential cobasin boundary) in the above sense. Note that thus an annular continuum in $\mathbb{A}$ may be non-essential, in which case it is contained in a closed topological disk. From now on, given any annular continuum, circloid, cobasin boundary or cofrontier, we always identify its annular neighbourhood $\mathcal{U}$ with $\mathbb{A}$ and assume implicitly that the objects are essential in $\mathbb{A}$.

A closed subset $\mathcal{A} \subseteq \mathbb{R}^2$ is called *horizontal*, if there exists $M > 0$ with $\mathcal{A} \subseteq \mathbb{R} \times [-M, M]$, and *horizontally separating* if $\mathbb{R} \times (-\infty, -M)$ and $\mathbb{R} \times (M, \infty)$ are contained in different connected components of $\mathbb{R}^2 \backslash A$. It is called a *horizontal strip* if it separates the plane into exactly two connected components, one of them unbounded above and the other unbounded below. A horizontal strip is called *minimal* if it does not strictly contain a smaller horizontal strip. In this case, its boundary is called a *horizontal coplane boundary* and equals the intersection of the boundaries of the two complementary domains of the strip. A horizontal coplane boundary is minimal amongst horizontally separating sets. If A is an essential continuum, we call the set $\mathcal{A} = \pi^{-1}(A)$ its lift. We state the next observation as a lemma, since it will be used repeatedly. Its proof is straightforward and left to the reader.

Lemma 2.1. *The lift of an essential continuum A is a minimal strip if and only if A is a circloid, and the lift of an essential continuum B is a coplane boundary if and only if B is a cobasin boundary.*

Let $\varphi\colon \mathbb{A} \to \mathbb{A}$ be a homeomorphism homotopic to the identity. Any such map lifts to a homeomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ which commutes with the deck transformation $T : (x, y) \mapsto (x + 1, y)$. If A is a compact invariant subset of φ, the *rotation interval of Φ on A* is defined as

$$\rho_A(\Phi) \;=\; \left\{\rho \in \mathbb{R} \mid \exists z_i \in \pi^{-1}(A), n_i \nearrow \infty : \lim_{i\to\infty} \pi_1\left(\Phi^{n_i}(z_i) - z_i\right)/n_i = \rho\right\} . \tag{2.1}$$

When $\rho_A(\Phi)$ is reduced to a singleton $\{\rho\}$, we say Φ has a unique rotation number ρ in A. In this case, $\lim_{n\to\infty} \pi_1(\Phi^n(z) - z)/n = \rho$ for all $z \in \pi^{-1}(A)$. When this happens we say that φ has a well-defined rotation number $\rho(\varphi) = \rho + \mathbb{Z} \in \mathbb{T}^1$.

Given metric spaces X, Y, a continuous map $\varphi : X \to X$ is *semiconjugate* to $\psi : Y \to Y$ if there exists a continuous onto map $h : X \to Y$ such that $h \circ \varphi = \psi \circ h$. In this situation, we say ψ is a *factor* of φ and h is a *semiconjugacy* or *factor map*. An important case is that of *monotone* semiconjugacies. A continuous map $h : X \to Y$ is called *monotone* if all fibres $h^{-1}(\{y\})$, $y \in Y$, are connected. A set $U \subseteq X$ is called *saturated* with respect to $h : X \to Y$, if $x \in U$ implies $h^{-1}(\{h(x)\}) \subseteq U$. If h is continuous, then it maps saturated open (closed) sets to open (closed) sets. As a direct consequence, we have

Lemma 2.2. *Preimages of connected sets under surjective monotone maps are connected. In particular, preimages of decomposable sets are decomposable.*

A *cellular continuum* in a surface S is one of the form $K = \bigcap_{n\in\mathbb{N}} D_n$ where each D_n is a closed topological disk and $D_{n+1} \subseteq \operatorname{int} D_n$. This is equivalent to saying that K is a continuum and has a neighborhood homeomorphic to $\mathbb{R}^2$ in which K is non-separating.

A partition $\mathcal{F}$ of a metric space X into compact subsets is called an *upper semicontinuous decomposition* if for each open set $U \subseteq X$, the union of all elements of $\mathcal{F}$ contained in U is also open. A *Moore decomposition* of a surface S is an upper semicontinuous decomposition of S into cellular continua. The following version of Moore's theorem is contained in [Dav86, Theorem 25.1] (see also Theorem 13.4 in the same book). It says essentially that the quotient space of a Moore decomposition is the same surface S.

Theorem 2.3. *Given any Moore decomposition $\mathcal{F}$ of a surface S, there exists a map $\Pi : S \to S$ which satisfies the following.*

(i) Π *is continuous and surjective;*

(ii) Π *is homotopic to the identity (and preserves orientation if S is orientable);*

(iii) For all $z \in S$, we have $\Pi^{-1}(z) \in \mathcal{F}$.

The map Π is called the *Moore projection* associated to $\mathcal{F}$.

Finally, we state some basic results from plane topology. We say that a subset $K \subseteq X$ of a topological space separates two points if the two points belong to different connected components of $X \backslash K$.

Lemma 2.4 ([New92, Theorem 14.3])**.** *If two points in the plane are separated by a closed set, then they are also separated by some connected component of that set.*

Lemma 2.5 ([HY61, Theorem 2-28])**.** *In any metric space, a continuum K is homeomorphic to a circle if for any pair $x \neq y$ of points of K, the set $K \backslash \{x, y\}$ is disconnected.*

Lemma 2.6 ([HY61, Theorem 2-16])**.** *If X is a continuum and $Y \subseteq X$ is closed, then the closure of every connected component of $X \backslash Y$ intersects Y.*

3 Minimal generators

Throughout this section, $B \subseteq \mathbb{A}$ denotes a decomposable cobasin boundary and $\mathcal{B} = \pi^{-1}(B)$ its lift. If $G \subseteq \mathcal{B}$ is a continuum such that $\mathcal{B} = \bigcup_{n\in\mathbb{Z}} T^n(G)$, we say that G is a *generator* of $\mathcal{B}$. We say that G is a *minimal generator* if it does not strictly contain a smaller generator. In the same way we may define generators and minimal generators for lifts of circloids. This concept has been used implicitly by Barge and Gillette in [BG91]; the terminology is taken from [JP]. As a consequence of Zorn's Lemma, any generator contains a minimal generator. The aim of this section is to provide a number of basic facts on minimal generators which will be crucial for the later constructions. The main objective is to derive the statements for circloids, but in order to do so first have to consider cobasin boundaries.

Lemma 3.1. *A continuum $G \subseteq \mathcal{B}$ is a generator of $\mathcal{B}$ if and only if $G \cap TG \neq \emptyset$.*

Proof. If $G \cap TG \neq \emptyset$, then $\bigcup_{k\in\mathbb{Z}} T^k G \subsetneq \mathcal{B}$ is horizontally separating and by Lemma 2.1 it has to be equal to $\mathcal{B}$, so G is a generator. To prove the converse, we first note that if G is a generator then $G \cap T^k G \neq \emptyset$ for some $k > 0$, since otherwise G would project injectively onto $B \subseteq \mathbb{A}$ contradicting the fact that B is essential. If $k = 1$ we are done; otherwise assume that k is maximal with the property that $G \cap T^k G \neq \emptyset$, and note that $\bigcup_{i\in\mathbb{Z}} T^{ik} G$ is horizontally separating and so must be equal to $\mathcal{B}$; in particular it contains TG. But $TG \cap T^{ik} G = \emptyset$ for $i < 0$ and $i > 1$ due to the maximality of k. Thus $TG \subseteq G \cup T^k G$, and since G is compact, TG cannot be contained in $T^k G$, so $G \cap TG \neq \emptyset$ as claimed. □

Lemma 3.2. *Suppose that L and R are closed connected subsets of $\mathcal{B}$, with L unbounded to the left and R unbounded to the right. If $L \cap R = \emptyset$, then $\mathcal{B} \backslash (L \cup R)$ is connected, and if $L \cap R \neq \emptyset$ then $L \cup R = \mathcal{B}$.*

Proof. If $L \cap R \neq \emptyset$, then $L \cup R$ is horizontally separating, so by Lemma 2.1 it must be equal to $\mathcal{B}$. Assume that $L \cap R = \emptyset$. Let W^- and W^+ be the connected components of $\mathbb{R}^2 \backslash \mathcal{B}$ which are unbounded below and above, respectively, so $\mathcal{B} = \partial W^- = \partial W^+$. Note that $L \cup R$ cannot be horizontally separating (since this would contradict Lemma 2.1); thus W^- and W^+ are contained in the same connected component U of $\mathbb{R}^2 \backslash (L \cup R)$. Since $E := \mathcal{B} \backslash (L \cup R) \subseteq \partial W^- \cap \partial W^+$, it follows that $E \subseteq U$. Note that U is simply connected, since both L and R are connected and unbounded. Moreover, E is a closed subset in the topology of U, and since $\mathcal{B}$ separates W^- from W^+ in $\mathbb{R}^2$ it follows that $E = \mathcal{B} \cap U$ separates W^- from W^+ in U. By Lemma 2.4 applied to $U \simeq \mathbb{R}^2$, some connected component E_0 of E separates W^- from W^+ in U. Since E_0 is closed in U, we have that $L \cup E_0 \cup R$ is closed and horizontally separating, so by Lemma 2.1 it must be equal to $\mathcal{B}$. This implies that $E_0 = E$, so E is connected. □

Lemma 3.3. *There exists a minimal generator G_0 such that $G_0 \cap T^k G_0 \neq \emptyset$ if and only if $|k| \leqslant 1$. Moreover, $G_0 \backslash TG_0$ and $G_0 \backslash (TG_0 \cup T^{-1} G_0)$ are connected and dense in G.*

Proof. Since B is decomposable, there exists a decomposition $B = X \cup Y$ into proper subcontinua. As B is a cobasin boundary, both X and Y must be inessential in $\mathbb{A}$, which implies that there are open topological disks $D_X \supset X$ and $D_Y \supset Y$. Let $\widetilde{D}_X$ be a connected component of $\pi^{-1}(D_X)$, and $\mathcal{X} = \pi^{-1}(X) \cap \widetilde{D}_X$. Let $\widetilde{D}_Y$ be a connected component of $\pi^{-1}(D_Y)$, and $\mathcal{Y} = \pi^{-1}(X) \cap \widetilde{D}_Y$. The sets $\mathcal{X}$ and $\mathcal{Y}$ project injectively onto X and Y, respectively, so they are continua and $T^i \mathcal{X} \cap \mathcal{X} = \emptyset$ for all $i \neq 0$, and similarly for $\mathcal{Y}$. Since $X \cap Y \neq \emptyset$, there exists $n \in \mathbb{Z}$ such that $T^n\mathcal{Y} \cap \mathcal{X} \neq \emptyset$. The set $\mathcal{X} \cup T^n\mathcal{Y}$ is a generator, so it contains some minimal generator G_0. Let k be the largest integer such that $G_0 \cap T^k G_0 \neq \emptyset$, and suppose for a contradiction that $k > 1$. Then $\bigcup_{i\in\mathbb{Z},\, i\neq 1} T^i G_0 \subseteq \bigcup_{i\in\mathbb{Z},\, i\neq 1} T^i\mathcal{X} \cup T^{n+i}\mathcal{Y}$ is closed and horizontally separating, so in particular it contains $T\mathcal{X}$. Since $T\mathcal{X}$ is disjoint from $T^i\mathcal{X}$ for all $i \neq 1$, it follows that $T\mathcal{X} \subseteq \bigcup_{j\in\mathbb{Z}} T^j\mathcal{Y}$, which implies that $X = \pi(\mathcal{X}) \subset Y$, contradicting our choice of X and Y.

By Lemma 3.2 we have that $E_0 := \mathcal{B}\backslash\bigcup_{k\neq 1} T^k G_0 = G_0\backslash(T^{-1}G_0 \cup TG_0) \neq \emptyset$ is connected. Note that the closure of any connected component of $G_0\backslash TG_0$ intersects TG_0 (see Lemma 2.6), so any connected component of $G_0\backslash TG_0$ must contain E_0 (otherwise it would be contained in $T^{-1}G_0$ which is disjoint from TG_0), so there is only one such component. Thus $G_0\backslash TG_0$ is connected. Since $\overline{\bigcup_{k\in\mathbb{Z}} T^k(G_0\backslash TG_0)} = \mathcal{B}$, it follows that $\overline{G_0\backslash TG_0}$ is a generator and by minimality $G_0 = \overline{G_0\backslash TG_0}$. This implies that $C = G_0 \cap TG_0$ has empty interior in the restricted topology to G_0, and therefore $E_0 = G_0\backslash(T^{-1}G_0 \cup TG_0) = G_0\backslash(C \cup TC)$ is also dense in G_0, completing the proof. □

Lemma 3.4. *If G is any minimal generator of $\mathcal{B}$, then $T^kG \cap G \neq \emptyset$ if and only if $|k| \leqslant 1$. Moreover, $G\backslash TG$ and $G\backslash(TG \cup T^{-1}G)$ are connected and dense in G.*

Proof. Suppose that $T^kG \cap G \neq \emptyset$ for some $k > 1$, so there exists $z \in G$ such that $T^kz \in G$. Let G_0 be as in Lemma 3.3. Replacing G by T^iG for a suitable i, we may assume that $z \in T^{-1}G_0$. This means that G intersects $T^{-1}G_0$ and $T^{k-1}G_0$, where $k-1 \geqslant 1$. Thus the set $G \cup \bigcup_{k\neq 0} T^iG_0$ is closed, connected and horizontally separating, and by Lemma 2.1 it should be equal to $\mathcal{B}$. Thus $G_0\backslash(T^{-1}G_0 \cup TG_0) = G_0\backslash\bigcup_{k\neq 0} T^iG_0 \subseteq G$, implying that $G_0 \subseteq G$. By minimality $G_0 = G$, contradicting the fact that $G \cap T^kG \neq \emptyset$ with $k > 1$.

Knowing that $G \cap T^kG \neq \emptyset \iff |k| \leqslant 1$, the remaining claims are proved exactly as in the last paragraph of the proof of Lemma 3.3. □

Given a minimal generator G of $\mathcal{B}$, let $\mathcal{L}_n(G) = \bigcup_{k\leqslant n} T^k(G)$ and $\mathcal{R}_n(G) = \bigcup_{k\geqslant n} T^k(G)$. With these notions, we have

Lemma 3.5. *If G and G' are two different minimal generators of $\mathcal{B}$, then either $G' \subseteq \mathcal{B}\backslash\mathcal{L}_{-1}(G) \subseteq \mathcal{R}_0(G)$ or $G' \subseteq \mathcal{B}\backslash\mathcal{R}_1(G) \subseteq \mathcal{L}_0(G)$.*

Proof. Suppose G' intersects both $\mathcal{L}_{-1}(G)$ and $\mathcal{R}_1(G)$. Then $\mathcal{L}_{-1}(G) \cup G' \cup \mathcal{R}_1(G) = \mathcal{B}$ by Lemma 2.1, so the union contains G. Since $G\backslash(TG \cup T^{-1}G)$ is disjoint from $\mathcal{L}_{-1}(G) \cup \mathcal{R}_1(G)$, we have that $G\backslash(TG \cup T^{-1}G) \subseteq G'$, which implies by Lemma 3.4 that $G \subseteq G'$, so by the minimality $G = G'$. □

Corollary 3.6. *If G and G' are minimal generators of $\mathcal{B}$, then G' is contained in two adjacent copies of G and vice versa.*

A *cut* (of $\mathcal{B}$) is a set of the form $G \cap T^{-1}G$ where G is a minimal generator of $\mathcal{B}$. We denote by $\mathcal{C}$ the family of all cuts. Note that by Lemma 3.5, cuts are pairwise disjoint.

Given a cut $C = G \cap T^{-1}(G)$, we let $\widehat{\mathcal{R}}(C) = \mathcal{R}_0(G)$ and $\widehat{\mathcal{L}}(C) = \mathcal{L}_{-1}(G)$, so that $C = \widehat{\mathcal{R}}(C) \cap \widehat{\mathcal{L}}(C)$. Further, we let $\mathcal{L}(C) = \widehat{\mathcal{L}}(C)\backslash C$ and $\mathcal{R}(C) = \widehat{\mathcal{R}}(C)\backslash C$. We write $C \prec C'$ if $C \subseteq \mathcal{L}(C')$, or equivalently if $C' \subseteq \mathcal{R}(C)$. By Lemma 3.5 and its corollary, $\prec$ defines a total order in $\mathcal{C}$. We extend this notation to compare arbitrary subsets $S \subseteq \mathcal{B}$ with cuts by writing $S \prec C$ if $S \subseteq \mathcal{L}(C)$ and $S \preccurlyeq C$ if $S \subseteq \widehat{\mathcal{L}}(C)$. If $z \in \mathcal{B}$ and $\{z\} \prec C$, we simply write $z \prec C$. For two cuts $C \prec C'$, we let $(C, C')_{\mathcal{B}} = \mathcal{R}(C) \cap \mathcal{L}(C') = \{z \in \mathcal{B} \mid C \prec z \prec C'\}$ and $[C, C']_{\mathcal{B}} = \widehat{\mathcal{R}}(C) \cap \widehat{\mathcal{L}}(C') = (C, C')_{\mathcal{B}} \cup C \cup C'$.

We note that cuts need not be connected. However, we have:

Lemma 3.7. *Given two cuts $C \prec C'$, the set $(C, C')_{\mathcal{B}}$ is connected and its closure is $[C, C']_{\mathcal{B}}$.*

Proof. We have $(C, C')_{\mathcal{B}} = \mathcal{B}\backslash(\mathcal{L}_0(G) \cup \mathcal{R}_0(G')) \neq \emptyset$ for some minimal generators G, G'. The connectedness follows from Lemma 3.2, and the fact that its closure is $[C, C']_{\mathcal{B}}$ follows easily from Lemma 3.4. □

Let A be the essential circloid such that $\partial A = B$ (i.e. the union of B with all bounded connected components of $\mathbb{A}\backslash B$), and let $\mathcal{A} = \pi^{-1}(A)$ be its lift. A generator of $\mathcal{A}$ is a continuum $G \subseteq \mathcal{A}$ which satisfies $\bigcup_{n\in\mathbb{Z}} T^nG = \mathcal{A}$. In order to go over from a the decomposable cobasin boundary B to the corresponding circloid A, the following statements will be crucial.

Lemma 3.8. *All connected components of $\mathcal{A}\backslash\mathcal{B}$ are topological disks with diameter bounded by a uniform constant M.*

Proof. Let G be a generator of $\mathcal{B}$, and suppose $\pi_1(G) = [a, b]$. Let $N \in \mathbb{N}$ be such that $a + N > b$. Since $\pi_2(G) = \pi_2(\mathcal{A})$, the latter set has diameter bounded by some constant c. let $M = 2N + c + 1$. If U is a connected component of $\mathcal{A}\backslash\mathcal{B}$ with $\operatorname{diam}(U) > M$, then $\operatorname{diam}(\pi_1(U)) > 2N + 1$ and we may assume $[a - N, b + N] \subseteq \pi_1(U)$ replacing U by T^iU for an appropriate $i \in \mathbb{Z}$. Thus there is a simple arc $\gamma\colon [0, 1] \to U$ such that $\pi_1(\gamma(0)) = a - N$, $\pi_1(\gamma(1)) = b + N$, and $a - N < \gamma(t) < b + N$ for $0 < t < 1$. If $K = \gamma([0, 1])$ and $S := [a - N, b + N] \times \mathbb{R}$, we have that $S\backslash K$ has exactly two connected components S_+ and S_-, the former unbounded above and the later unbounded below. The set $G' = \bigcup_{k=-N}^{N} T^kG$ being connected, disjoint from K and contained in S, must lie entirely in S^- or S^+. Suppose without loss of generality that $G' \subseteq S^-$. Note that $K \subseteq U$ and U is is bounded above. If y is the smallest real such that $\{a\} \times [y, \infty) \subseteq \mathbb{R}^2\backslash\mathcal{B}$, then $\{a\} \times [y, \infty)$ is disjoint from K and thus contained in S^+, and since $z := (a, y) \in \partial\mathcal{B}$ there must exist n such that $z \in T^nG$. Since $T^kG \subseteq S^-$ when $|k| \leqslant N$, it follows that $n > N$, but this is not possible since $\pi_1(T^nG) \subseteq [a + N, \infty) \subseteq (b, \infty)$. □

Given a continuum $S \subseteq \mathbb{R}^2$, the complement $\mathbb{R}^2\backslash S$ consists of one unbounded component and a union of topological disks. We denote the unbounded component by $U_\infty(S)$ and the family of disks by $\mathcal{U}_S$ and let $S^{\text{fill}} = \mathbb{R}^2\backslash U_\infty(S) = S \cup \bigcup_{U\in\mathcal{U}_S} U$. Note that S^{fill} is a nonseparating continuum.

Lemma 3.9. *Suppose U is a connected component of $\mathcal{A}\backslash\mathcal{B}$ and $C_0^- \prec C_0^+$ are cuts such that $(C_0^-, C_0^+)_\mathcal{B} \cap \partial U \neq \emptyset$. If C^-, C^+ are cuts with $C^- \prec C_0^- \prec C_0^+ \prec C^+$, then $\partial U \subseteq [C^-, C^+]_\mathcal{B}$.*

Proof. Suppose for a contradiction that $\partial U \nsubseteq [C^-, C^+]_\mathcal{B}$. Assume without loss of generality that $\partial U \cap \mathcal{L}(C^-) \neq \emptyset$. Since U is bounded, there is $n > 0$ such that $\partial U \subseteq [T^{-n}C^-, T^nC^-]_\mathcal{B}$, and we may assume $T^nC^- \prec C_0^-$ and $C_0^+ \prec T^nC^-$. The sets $L = [T^{-n}C^-, C_0^-]_\mathcal{B}$ and $R = [C^-, T^nC^-]_\mathcal{B}$ are connected, and $L \cap R = [C_0^-, C^-]_\mathcal{B}$ is also connected (by Lemma 3.7). Since ∂U is not contained in either set L or R, we have that $U \subseteq U_\infty(L) \cap U_\infty(R)$. But then by Lemma 2.4 we have that $U \subseteq U_\infty(R \cup L)$, contradicting the fact that U is bounded and $\partial U \subseteq L \cup R$. □

4 Dynamical linearisation: Proof of Theorem 1.1

Throughout this section, we assume that $\varphi\colon \mathbb{A} \to \mathbb{A}$ is a homeomorphism homotopic to the identity and A is a φ-invariant circloid with decomposable boundary. We let $B = \partial B$ and denote the lifts of A, φ and B by $\mathcal{A}$, Φ and $\mathcal{B}$, respectively.

Lemma 4.1. *The rotation number $\rho(\Phi, \mathcal{A}) = \lim_{n\to\infty} \pi_1(\Phi^n(z) - z)/n$ exists and is independent of $z \in \mathcal{A}$.*

Proof. The fact that $\rho = \lim_{n\to\infty} \pi_1(\Phi^n(z_0) - z_0)/n$ exists for some $z_0 \in \mathcal{B}$ follows from the Birkhoff Ergodic Theorem and the existence of an invariant measure for $\varphi|_B$, since $\pi_1(\Phi^n(z_0) - z_0)/n$ is a Birkhoff average for the function $B \ni x \mapsto \pi_1(\Phi(x') - x') \in \mathbb{R}$, where $x' \in \pi^{-1}(x)$ is arbitrary.

Fix a minimal generator G of $\mathcal{B}$ containing z_0. Since $\Phi^n(G)$ is also a minimal generator, by Corollary 3.6 there exists k such that $\Phi^n(G) \subset T^k(G \cup TG)$, so $\operatorname{diam}(\Phi^n(G)) \leqslant 2\operatorname{diam}(G) := M$ for all $n \in \mathbb{N}$. This implies $\lim_{n\to\infty} \pi_1 (\Phi^n(z) - z)/n = \rho$ for all $z \in G$, and since G is a generator the same holds for all $z \in \mathcal{B}$. Finally, we deduce from Lemma 3.8 that the same property holds for $z \in \mathcal{A}$. □

Lemma 4.2. *$\rho(\Phi, \mathcal{A}) = p/q \in \mathbb{Q}$ if and only if there exists $z \in \mathcal{A}$ such that $\Phi^q(z) = T^pz$.*

Proof. The if-part is trivial. For the other implication, note that it is easy to verify that $\rho(T^{-p}\Phi^q, \mathcal{A}) = 0$ if and only if $\rho(\Phi, \mathcal{A}) = p/q$, so it suffices to assume that $\rho(\Phi, \mathcal{A}) = 0$ and show that there is a fixed point in $\mathcal{A}$. Fix a minimal generator G of $\mathcal{B}$. We claim that $\Phi^n(G) \cap G \neq \emptyset$ for all n. Indeed, Corollary 3.6 implies that $\Phi^n(G) \subset T^kG \cup T^{k+1}G$ for some k. If $k > 0$, then $\Phi^n(G) \subset \mathcal{R}_1(G)$ which then implies $\Phi^{nk}(G) \subset \mathcal{R}_k(G)$ and this implies that $\pi_1(\Phi^{nk}(z) - z)/nk \geqslant 1/n$ for all $k \in \mathbb{N}$ and $z \in G$, contradicting our assumption. If $k < -1$, we get a similar contradiction. Thus $k \in \{0, -1\}$, and since G and $\Phi^n(G)$ are minimal generators, $\Phi^n(G)$ must intersect both T^kG and $T^{k+1}G$, so

$\Phi^n(G) \cap G \neq \emptyset$ as claimed. Thus $K = \overline{\bigcup_{n\in\mathbb{Z}} \Phi^n(G)} \subset \mathcal{B}$ is connected and bounded (again due to Corollary 3.6), and $\Phi(K) = K$. Moreover, $K^{\mathrm{fill}} \subset \mathcal{A}$ and $\Phi(K^{\mathrm{fill}}) = K^{\mathrm{fill}}$, so K^{fill} is a non-separating invariant continuum in $\mathbb{R}^2$ and the Cartwright-Littlewood theorem [CL51a] implies that it contains a fixed point of Φ. □

The previous lemma implies the first claim of Theorem 1.1, and one implication of the second claim. To prove the remaining implication, from now on we assume that $\rho = \rho(\Phi, \mathcal{A})$ is irrational, and we fix a minimal generator G_0 of $\mathcal{B}$. Given $x = n + k\rho$ in $Q(\rho) = \{l + \rho m \mid l, m \in \mathbb{Z}\}$, we let $G_x = \Phi^k \circ T^n(G_0)$ and denote by $C_x = G_x \cap T^{-1}(G_x)$ the cut corresponding to G_x.

Lemma 4.3. *The mapping $Q(\rho) \to \mathcal{C},\ x \mapsto C_x$ is strictly monotonically increasing. In particular, $C_x \cap C_y = \emptyset$ if $x \neq y$. Moreover, the set $[C_x, C_y]_{\mathcal{B}}$ is connected for all $x < y$, decreasing in x and increasing in y.*

Proof. Suppose that $x = n + k\rho < x' = n' + k'\rho$, but $C_{x'} \preccurlyeq C_x$. Then $\Phi^{k'-k} \circ T^{n'-n}(\widehat{\mathcal{L}}(C_0)) \subseteq \widehat{\mathcal{L}}(C_0)$. As a consequence, all orbits in $\widehat{\mathcal{L}}(C_0)$ under the lift $\Psi = \Phi^{k'-k} \circ T^{n'-n}$ of $\varphi^{k'-k}$ are bounded to the right, contradicting the fact that $\rho(\Psi) = x' - x > 0$. This shows the strict monotonicity of $x \mapsto C_x$ and the disjointness. Connectedness of $[C_x, C_y]_{\mathcal{B}}$ is given by Lemma 3.7. □

As in the previous section, given $z \in \mathcal{B}$ and a cut $C \in \mathcal{C}$, we write $z \prec C$ iff $z \in \mathcal{L}(C)$ and $C \prec z$ iff $z \in \mathcal{R}(C)$. In order to extend this notion to all $z \in \mathcal{A}$, note that $\mathcal{A}\backslash\mathcal{B}$ is a union of bounded open topological disks whose boundary is contained in $\mathcal{B}$. Given $z \in \mathcal{A}\backslash\mathcal{B}$, we denote the respective disk containing z by U_z and write $z \prec C$ iff $\partial U_z \subseteq \mathcal{L}(C)$ and $C \prec z$ iff $\partial U_z \subseteq \mathcal{R}(C)$. Equivalently, $z \prec C$ iff $z \in \mathcal{L}(C)^{\mathrm{fill}}$ and $C \prec z$ iff $z \in \mathcal{R}(C)^{\mathrm{fill}}$. Given a subset $S \subseteq \mathcal{A}$, we write $C \prec S$ iff $S \subseteq \mathcal{L}(C)^{\mathrm{fill}}$ and $S \prec C$ iff $S \subseteq \mathcal{R}(C)^{\mathrm{fill}}$. Then, we define $H : \mathcal{A} \to \mathbb{R}$ by

$$H(z) \;=\; \sup\{x \in Q(\rho) \mid C_x \prec z\}\ . \tag{4.1}$$

Lemma 4.4. *The map H is continuous and projects to a monotone semiconjugacy $h : A \to \mathbb{T}^1$ from $\varphi|_A$ to the irrational rotation by ρ.*

Proof. We first show the continuity of the restriction of H to $\mathcal{B}$. By definition, we have that

$$H_{|\mathcal{B}}^{-1}(x,y) \;=\; \bigcup_{\substack{x',y'\in Q(\rho)\\ x<x'<y'<y}} (C_{x'}, C_{y'})_{\mathcal{B}} \;=\; \bigcup_{\substack{x',y'\in Q(\rho)\\ x<x'<y'<y}} [C_{x'}, C_{y'}]_{\mathcal{B}}\ . \tag{4.2}$$

Since the sets $(C_{x'}, C_{y'})_{\mathcal{B}}$ are relative-open in $\mathcal{B}$, this shows that preimages of open sets are open, so that $H_{|\mathcal{B}}$ is continuous.

In order to see that H is continuous on all of $\mathcal{A}$, fix $x < y$ and $z \in H^{-1}(x,y)$. It suffices show that $H^{-1}(x,y)$ contains a neighbourhood of z. If $z \in \mathcal{A}\backslash\mathcal{B}$, then by definition the whole open disk U_z is contained in $H^{-1}(x,y)$. (Note that if $C_x \prec z$, then $C_x \prec z'$ for all $z' \in U_z$.) Thus, suppose that $z \in \mathcal{B}$. Since $H_{|\mathcal{B}}$ is continuous, there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \cap \mathcal{B} \subseteq H^{-1}(x,y)$. If $z' \in B_\varepsilon(z) \cap (\mathcal{A}\backslash\mathcal{B})$, then $B_\varepsilon(z)$ intersects both $\mathcal{B}$ and $U_{z'}$. Consequently, $B_\varepsilon(z)$ intersects $\partial U_{z'}$ and we have $\emptyset \neq \partial U_{z'} \cap B_\varepsilon(z) \subseteq H^{-1}(x,y)$. However, this implies that for some x', y' with $x < x' < y' < y$ we have $\partial U_{z'} \cap (C_{x'}, C_{y'})_{\mathcal{B}} \neq \emptyset$. Therefore Lemma 3.9 yields that $\partial U_{z'} \subseteq [C_{\tilde{x}}, C_{\tilde{y}}]_{\mathcal{B}}$ for any $\tilde{x}, \tilde{y}$ with $x < \tilde{x} < x' < y' < \tilde{y} < x$ and thus $U_{z'} \subseteq H^{-1}(x,y)$. Altogether, we obtain $B_\varepsilon(z) \cap \mathcal{A} \subseteq H^{-1}(x,y)$, which proves the continuity of H on $\mathcal{A}$.

In order to show the further statements, note that since by definition $\Phi(G_x) = G_{x+\rho}$ and $T(G_x) = G_{x+1}$, the same relations hold for C_x and $\mathcal{L}(C_x)$. Using these facts, it is easy to check that H is a semiconjugacy from $\Phi_{|\mathcal{A}}$ to the translation $x \mapsto x + \rho$ on $\mathbb{R}$ and that H commutes with the deck translation T, such that H projects to a semiconjugacy h from $\varphi_{|A}$ to the rotation R_ρ.

It remains to prove the monotonicity of h, which will follow immediately from that of H. We have that

$$H_{|B}^{-1}(x) \;=\; \bigcap_{\substack{x',y'\in Q(\rho)\\ x'<x<y'}} [C_{x'}, C_{y'}]\ . \tag{4.3}$$

This can be seen as a nested intersection of continua and is therefore a continuum itself. The full fibre $H^{-1}(x)$ is obtained by adding the union $\bigcup_{z\in H^{-1}(x)\setminus\mathcal{B}} U_z$ of topological open disks to $H^{-1}_{|B}(x)$. However, if $z \in H^{-1}(x)$, then ∂U_z cannot intersect $\mathcal{L}(C_{x'})$ for any $x' < x$, since otherwise Lemma 3.9 would imply that $\partial U_z \subseteq \widehat{\mathcal{L}}(C_{\tilde{x}})$ for some $\tilde{x} \in (x', x)$ and thus $H(z) \leqslant \tilde{x}$. Similarly, ∂U_z is disjoint from $\mathcal{R}(C_{y'})$ for all $y' > x$, and therefore $\partial U_z \subseteq H^{-1}(x)$. Hence, we have that $H^{-1}(x) = H^{-1}_{\mathcal{B}}(x)^{\mathrm{fill}}$ is the 'fill-in' of a continuum, and hence a continuum itself. □

5 Topological linearisation: a universal factor map

This section essentially contains an alternative proof of an old result by Kuratowski, and we refer to Section 7 for a discussion. As before, we suppose $A \subseteq \mathbb{A}$ is a circloid with decomposable boundary, and $\mathcal{A} = \pi^{-1}(A)$. We let $B = \partial A$ and $\mathcal{B} = \pi^{-1}(B)$. Our aim is to define a Moore decomposition of $\mathbb{A}$ such that the corresponding projection maps A to a topological circle. Similar to before, we start by decomposing B, and we use the family $\mathcal{C}$ of cuts of $\mathcal{B}$ as the main tool. (This is the main difference to the proof of Kuratowski, who concentrates on so-called *layers* instead.)

However, cuts need not be connected, and moreover is easy to give examples where $\bigcup_{C\in\mathcal{C}} C$ does not cover all of $\mathcal{B}$. In order to obtain a decomposition starting from $\mathcal{C}$, we define a strong partial order relation $\lll$ on $\mathcal{C}$ by writing $C \lll C'$ if and only if there exist uncountably many cuts $\tilde{C} \in \mathcal{C}$ such that $C \prec \tilde{C} \prec C'$. Similar to before, we extend this definition to arbitrary subsets $S, S' \subseteq \mathcal{A}$ by writing $S \lll S'$ whenever there exist uncountably many cuts $\tilde{C}$ with $S \prec \tilde{C} \prec S'$.[2] In case of one-point sets $S = \{z\}$, we write $z \lll S'$ instead of $\{z\} \lll S'$. Then, given any $z \in \mathcal{B}$, we define

$$F(z) \;=\; \bigcap_{\substack{C^-,C^+\in\mathcal{C}\\ C^-\lll z\lll C^+}} [C^-, C^+]_{\mathcal{B}} \ . \tag{5.1}$$

We let $\mathcal{F}_{\mathcal{B}} = \{F(z) \mid z \in \mathcal{B}\}$ and call the elements $F \in \mathcal{F}_{\mathcal{B}}$ *fibres* of $\mathcal{B}$. Further, we let $\mathcal{F}_B = \{\pi(F) \mid F \in \mathcal{F}_{\mathcal{B}}\}$. We note that the intersection in (5.1) can be viewed as a nested intersection of continua: by compactness, for every $n \in \mathbb{N}$ there exist $C_n^- \lll z \lll C_n^+$ such that $F(z) \subseteq [C_n^-, C_n^+] \subseteq B_{1/n}(F(z))$. Without loss of generality we may assume that $C_n^- \lll C_{n+1}^-$ and $C_{n+1}^+ \lll C_n^+$ for all $n \in \mathbb{N}$, and we have $F(z) = \bigcap_{n\in\mathbb{N}}[C_n^-, C_n^+]$. By Lemma 3.7 this is a nested intersection of continua, hence a continuum.

Lemma 5.1. *If $S, S' \subseteq \mathcal{B}$ and $S \lll S'$, then there exists $C \in \mathcal{C}$ such that $S \lll C \lll S'$.*

Proof. We first show the following slightly weaker

Claim 5.2. *If $V, V' \subseteq \mathcal{A}$ and $V \lll V'$, then there exist $C, C' \in \mathcal{C}$ such that $V \lll C \prec V'$ and $V \prec C' \lll V'$.*

We will prove the existence of C; that of C' then follows by symmetry. Suppose for a contradiction that for all $C \in \mathcal{A}$ with $V \prec C \prec V'$ we have $V \not\lll C$. Let $R = \bigcap_{C\in\mathcal{C}, C\prec V'} \widehat{\mathcal{R}}(C)$ and choose an increasing sequence of cuts $C_n \prec V'$ with $R = \bigcap_{n\in\mathbb{N}} \widehat{\mathcal{R}}(C_n)$. Note that, for example, it suffices to choose C_n such that $\mathrm{dist}_{\mathcal{H}}(\widehat{\mathcal{R}}(C_n), R) < 1/n$. Then, since $V \not\lll C_n$, there exist at most countably many cuts between V and C_n. However, every cut between V and V' is either equal to C_n for some $n \in \mathbb{N}$, lies between V and C_1 or lies between C_n and C_{n+1} for some $n \in \mathbb{N}$. Altogether, we obtain that there are at most countably many cuts between V and V', contradicting $V \lll V'$. This proves the claim.

Now, suppose for a contradiction that for every cut C between S and S' we either have $S \not\lll C$ or $C \not\lll S'$. Note that both properties cannot hold simultaneously since $S \lll S'$. Let $L' = \bigcap_{C\in\mathcal{C}, S\lll C} \widehat{\mathcal{L}}(C)$ and $R' = \bigcap_{C'\in\mathcal{C}, C'\lll S'} \widehat{\mathcal{R}}(C')$. Then the intersection $E = L' \cap R'$ is non-empty, since for every pair of cuts C, C' with $C' \lll S'$ and $S \lll C$ we have $C' \prec C$ due to our contradiction assumption. Moreover, $G = T(L') \cap R'$ is a continuum, since it can again be represented as a nested intersection of intervals in $\mathcal{B}$.

[2] We note that the requirement of uncountably many intermediate cuts is crucial for the whole construction in this section. We do not elaborate further on this, but just mention that a examples demonstrating why requiring uncountably many intermediate cuts is necessary can be produced by gluing finitely or countably many pseudoarcs together. Otherwise, a statement analogous to Lemma 5.1 does not hold.

Since $G \cap T^{-1}(G) = E \neq \emptyset$, it follows from Lemma 3.1 that G is a generator. Thus, it contains a minimal generator, and consequently the set E contains some cut $\widetilde{C}$. If $S \lll \widetilde{C}$, then Claim 5.2 implies that there exists $C_1 \in \mathcal{C}$ such that $S \lll C_1 \prec \widetilde{C}$, and since $\widetilde{C} \subseteq L'$ it follows that $\widetilde{C} \subseteq \hat{\mathcal{L}}(C_1)$, contradicting the fact that $C_1 \prec \widetilde{C}$. Thus $S \not\lll \widetilde{C}$, and by a similar argument $\widetilde{C} \not\lll S'$. However, this contradicts the fact that $S \lll S'$. □

Lemma 5.3. *Distinct fibres $F, F' \in \mathcal{F}_{\mathcal{B}}$ are disjoint and either $F \lll F'$ or $F' \lll F$.*

Proof. Let $F = F(z)$ and $F' = F(z')$. If $F \neq F'$, then there exists $C \in \mathcal{C}$ such that either $z \lll C$ and $z' \not\lll C$, or $C \lll z$ and $C \not\lll z'$. Assume the former case (the other case is analogous). According to Lemma 5.1, there exist C^-, C^+ with $z \lll C^- \lll C^+ \lll C$. However, this implies that $z \lll C^- \lll C^+ \lll z'$, hence $F(z) \subseteq \hat{\mathcal{L}}(C^-)$ and $F(z') \subseteq \hat{\mathcal{R}}(C^+)$ are disjoint and $F(z) \lll F(z')$. □

We now turn to the decomposition of $\mathcal{A}$. As before, given $z \in \mathcal{A}\backslash\mathcal{B}$, we denote by U_z the connected component of $\mathcal{A}\backslash\mathcal{B}$ containing z.

Lemma 5.4. *Suppose $F \in F_{\mathcal{B}}$ intersects ∂U_z for some $z \in \mathcal{A}\backslash\mathcal{B}$. Then $\partial U_z \subseteq F$.*

Proof. Suppose $F = F(z')$ intersects ∂U_z. Then given any cuts $C^- \lll z \lll C^+$, Lemma 5.1 yields cuts C_0^-, C_0^+ with $C^- \lll C_0^- \lll F \lll C_0^+ \lll C^+$ and $\partial U_z \cap (C_0^-, C_0^+)_{\mathcal{B}} \neq \emptyset$. By Lemma 3.9, this implies that $\partial U_z \subseteq [C^-, C^+]_{\mathcal{B}}$. Since this is true for all pairs of cuts satisfying $C^- \lll z \lll C^+$, we obtain $\partial U_z \subseteq F$. □

Bounded connected components of the complement of a fibre $F \in \mathcal{F}_{\mathcal{B}}$ are also bounded connected components of $\mathcal{A}\backslash\mathcal{B}$. Therefore, the preceding lemma implies that $F^{\text{fill}} = F \cup \bigcup_{\partial U_z \cap F \neq \emptyset} U_z$. This allows to define a decomposition of $\mathcal{A}$ by $\mathcal{F}_{\mathcal{A}} = \{F^{\text{fill}} \mid F \in \mathcal{F}_{\mathcal{B}}\}$. We denote fibres of $\mathcal{F}_{\mathcal{A}}$ by $\widehat{F}$, and given $z \in \mathcal{A}$ we let $\widehat{F}(z)$ be the unique fibre in $\mathcal{F}_{\mathcal{A}}$ which contains z. Note that $F(z) = \partial \widehat{F}(z)$, and so $\widehat{F}(z) = (\partial \widehat{F}(z))^{\text{fill}}$.

Remark 5.5. An alternative way to define the fibres of $\mathcal{A}$ is the following. Recall that we write $C \prec z$ for a cut C and $z \in \mathcal{A}$ iff $z \in \mathcal{R}(C)^{\text{fill}}$, and $z \prec C$ iff $z \in \mathcal{L}(C)^{\text{fill}}$. The notions $C \lll z$ and $z \lll C$ in $\mathcal{F}_{\mathcal{A}}$ can then be defined as before by the existence of uncountably many intermediate cuts. Using this, the fibres of $\mathcal{A}$ can be defined exactly in the same way as those of $\mathcal{B}$ in (5.1). Equivalence of the two definitions is provided by the following statement.

Lemma 5.6. *For any $z \in \mathcal{A}$, we have*

$$\widehat{F}(z) \;=\; \bigcap_{\substack{C^-, C^+ \in \mathcal{C} \\ C^- \lll z \lll C^+}} [C^-, C^+]_{\mathcal{B}}^{\mathit{fill}} \,. \tag{5.2}$$

Proof. If $z \in \mathcal{A}\backslash\mathcal{B}$ and $z' \in \partial U_z$, then by definition $\widehat{F}(z) = \widehat{F}(z')$, and the intersection on the right hand side of (5.2) coincides as well. Thus, we may assume $z \in \mathcal{B}$. However, in this case the right side is just $F(z)^{\text{fill}} = \widehat{F}(z)$ by Lemma 5.4. □

Given $F^-, F^+ \in \mathcal{F}_{\mathcal{B}}$, we let $(F^-, F^+)_{\mathcal{B}} = \bigcup_{F^- \prec F \prec F^+} F = \{z \in \mathcal{B} \mid F^- \prec z \prec F^+\}$ and $[F^-, F^+]_{\mathcal{B}} = F^- \cup (F^-, F^+)_{\mathcal{B}} \cup F^+$. Note that these intervals are relative-open, respectively relative-closed, in $\mathcal{B}$. If $\widehat{F}$ and $\widehat{F}'$ are distinct fibres in $\mathcal{F}_{\mathcal{A}}$, then $F = \partial \widehat{F}$ and $F' = \partial F'$ are fibres in $\mathcal{F}_{\mathcal{B}}$. According to Lemma 5.3, we always have either $F \prec F'$ or $F' \prec F$, so either $\widehat{F} \prec \widehat{F}'$ or $\widehat{F}' \prec \widehat{F}$, and the notions $\prec$ and $\lll$ coincide for the pairs $\widehat{F}, \widehat{F}'$ and F, F'. As before, given $\widehat{F}^- \prec \widehat{F}^+$, we let $(\widehat{F}^-, \widehat{F}^+)_{\mathcal{A}} = \bigcup_{\widehat{F}^- \prec \widehat{F} \prec \widehat{F}^+} \widehat{F}$ and $[\widehat{F}^-, \widehat{F}^+]_{\mathcal{A}} = \widehat{F}^- \cup (\widehat{F}^-, \widehat{F}^+)_{\mathcal{A}} \cup \widehat{F}^+$. As a consequence of Lemma 5.4, we obtain that $(\widehat{F}^-, \widehat{F}^+)_{\mathcal{A}} = (\partial \widehat{F}^-, \partial \widehat{F}^+)_{\mathcal{B}}^{\text{fill}}$ and $[\widehat{F}^-, \widehat{F}^+]_{\mathcal{A}} = [\partial \widehat{F}^-, \partial \widehat{F}^+]_{\mathcal{B}}^{\text{fill}}$. In particular, this implies the following observation, which we state for further use.

Lemma 5.7. *Given F^-, F^+, we have that $(\widehat{F}^-, \widehat{F}^+)_{\mathcal{A}}$ is relative-open and $[\widehat{F}^-, \widehat{F}^+]_{\mathcal{A}}$ is relative-closed in $\mathcal{A}$.*

Moreover, we have

Lemma 5.8. *For all $\widehat{F}^- \prec \widehat{F}^+ \in \mathcal{F}_{\mathcal{A}}$, the set $[\widehat{F}^-, \widehat{F}^+]_{\mathcal{A}}$ is a non-separating continuum.*

Proof. If $\widehat{F}^- = \widehat{F}(z^-)$ and $\widehat{F}^+ = \widehat{F}(z^+)$ with $z^-, z^+ \in \mathcal{B}$, then

$$\partial[\widehat{F}^-, \widehat{F}^+]_{\mathcal{A}} \;=\; \bigcap_{\substack{C^- \lll z^- \\ z^+ \lll C^+}} [C^-, C^+]_{\mathcal{B}} \,. \tag{5.3}$$

Hence, as a nested intersection of continua the set $\partial[\widehat{F}^-, \widehat{F}^+]_{\mathcal{B}}$ is connected, and so $[\widehat{F}^-, \widehat{F}^+]_{\mathcal{A}} = \left(\partial[\widehat{F}^-, \widehat{F}^+]_{\mathcal{B}}\right)^{\mathrm{fill}}$ is a filled – and hence non-separating – continuum. □

Lemma 5.9. *$\mathcal{F}_{\mathcal{A}}$ is an upper semicontinuous decomposition of $\mathcal{A}$.*

Proof. Let $U \subseteq \mathcal{A}$ be an open set. We need to show that $V = \{z \in \mathcal{A} : \widehat{F}(z) \subseteq U\}$ is open in $\mathcal{A}$. Let $z \in V$, so that $\widehat{F}(z) \subseteq U$. From Lemma 5.6 it is easy to verify that $\widehat{F}(z)$ is the intersection of all sets of the form $[\widehat{F}^-, \widehat{F}^+]_{\mathcal{A}}$ with $\widehat{F}^-, \widehat{F}^+ \in \mathcal{F}_{\mathcal{A}}$ and $\widehat{F}^- \lll \widehat{F}(z) \lll \widehat{F}^+$, and since this can be seen as a decreasing intersection, such $\widehat{F}^+$ and $\widehat{F}^-$ may be chosen satisfying $[\widehat{F}^-, \widehat{F}^+]_{\mathcal{A}} \subset U$. Since $(\widehat{F}^-, \widehat{F}^+)_{\mathcal{A}}$ is open in $\mathcal{A}$ and contains $z \in V$, which was chosen arbitrarily, it follows that V is open in $\mathcal{A}$ as claimed. □

Lemma 5.10. *$\mathcal{F}_{\mathcal{A}}$ projects to an upper semicontinuous decomposition $\mathcal{F}_A = \{\pi(F) \mid F \in \mathcal{F}_{\mathcal{A}}\}$ of A into cellular continua, and it has uncountably many elements.*

Proof. Since by hypothesis there exists a self-homeomorphism φ of $\mathbb{A}$ leaving A invariant without periodic points in A, and we may assume that φ is orientation-preserving (replacing it by φ^2 if necessary), Theorem 1.1 yields the existence of a monotone map $h : A \to \mathbb{T}^1$, which lifts to a monotone map $H : \mathcal{A} \to \mathbb{R}$ that commutes with the translation T. It is easily checked that for every $x \in \mathbb{R}$ the set $G_x = H^{-1}([x, x+1])$ is a generator of $\mathcal{A}$, and consequently every fibre $H^{-1}(x) = T^{-1}(G_x) \cap G_x$ contains a cut. This further implies that every fibre of H contains (at least) one element of $\mathcal{F}_{\mathcal{A}}$.

In particular, the above yields that $T^k F \cap F = \emptyset$ for all $F \in \mathcal{F}_{\mathcal{A}}$ and $k \in \mathbb{Z}\backslash\{0\}$, hence $\mathcal{F}_{\mathcal{A}}$ projects to a decomposition of A with uncountably many elements, and each $F \in \mathcal{F}_{\mathcal{A}}$ projects injectively into A. Since F is a cellular continuum, this implies that $\pi(F)$ is a cellular continuum as well. Upper semicontinuity of $\mathcal{F}_A$ then follows directly from that of $\mathcal{F}_{\mathcal{A}}$. □

Let $\mathcal{F}$ be the decomposition of A consisting of all elements of $\mathcal{F}_A$ together with all sets of the form $\{z\}$ with $z \notin A$. Note that $\mathcal{F}$ is a Moore decomposition of $\mathcal{A}$, and therefore Theorem 2.3 applies.

Proposition 5.11. *The Moore projection $\Pi : \mathbb{A} \to \mathbb{A}$ provided by Theorem 2.3 maps A to an essential simple closed curve.*

Proof. Due to Lemma 2.5 it suffices to show that if $x, y \in \Pi(A)$ are different points, then $\Pi(A)\backslash\{x, y\}$ is disconnected. By Lemma 2.2, this is the same as saying that for any pair of distinct elements F_1, F_2 of $\mathcal{F}_A$, the set $A\backslash(F_1 \cup F_2)$ is disconnected. To show this, let $\widehat{F}_1, \widehat{F}_2 \in \mathcal{F}_{\mathcal{A}}$ be connected components of $\Pi^{-1}(F_1)$ and $\Pi^{-1}(F_2)$. Note that since $T^{-1}F \lll F \lll TF$ for all $F \in \mathcal{F}_{\mathcal{A}}$, the projection Π is injective on $(\widehat{F}_1, T\widehat{F}_1)_{\mathcal{A}}$, and there is exactly one $k \in \mathbb{Z}$ such that $\widehat{F}_1 \lll T^k\widehat{F}_2 \lll T\widehat{F}_1$. If $U = (\widehat{F}_1, T^k\widehat{F}_2)_{\mathcal{A}}$ and $V = (T^k\widehat{F}_2, T\widehat{F}_1)_{\mathcal{A}}$, we have that $\Pi(U)$ and $\Pi(V)$ are disjoint nonempty open subsets of A, and $\Pi(U) \cup \Pi(V) = A\backslash(F_1 \cup F_2)$, proving that the latter set is disconnected. □

Lemma 5.12. *Suppose $h : A \to \mathbb{T}^1$ is a monotone map. Then the fibres $h^{-1}(x)$, $x \in \mathbb{T}^1$, are saturated with respect to $\mathcal{F}_A$.*

Proof. The map h lifts to a monotone map $H : \mathcal{A} \to \mathbb{R}$ that commutes with the deck translation T. As argued in the proof of Lemma 5.10, every fibre $H^{-1}(x)$ contains at least one cut. If $z \in H^{-1}(x) \cap \mathcal{B}$ and $z' \in H^{-1}(y) \cap \mathcal{B}$ with $x < y$, this implies that $z \lll z'$ and thus $F(z) \neq F(z')$. Hence, no two points of the same fibre of $\mathcal{F}_{\mathcal{B}}$ can be contained in different fibres of H. In other words, the fibres of $H_{|\mathcal{B}}$ are saturated with respect to $\mathcal{F}_{\mathcal{B}}$. By monotonicity, $F \subseteq H^{-1}(x)$ implies $\widehat{F} = F^{\mathrm{fill}} \subseteq H^{-1}(x)$, such that the fibres of H are also saturated with respect to $\mathcal{F}_{\mathcal{A}}$. □

5.1 Proof of Theorem 1.2 Denote by $\Pi : \mathbb{A} \to \mathbb{A}$ the Moore projection associated to $\mathcal{F}_{\mathcal{A}} \cup \{x \in \mathbb{A} \mid x \notin A\}$, so it satisfies assertion (i) of Theorem 1.2. Assertions (ii) and (iii) follow from Proposition 5.11 and Schönflies' theorem (by post-composing with a homeomorphism of $\mathbb{A}$ which maps $\Pi(A)$ to $\mathbb{T}$). Since the lift Φ of any homeomorphism φ of $\mathbb{A}$ leaving A invariant has to map minimal generators to minimal generators, it follows from the above constructions that Φ permutes the elements of $\mathcal{F}_{\mathcal{A}}$. This allows to define $\tilde{\varphi} : \mathbb{A} \to \mathbb{A}$ by requiring that $\Pi^{-1}(\tilde{\varphi}(z)) = \varphi(\Pi^{-1}(z))$. If $U \subseteq \mathbb{A}$ is open, then $\tilde{\varphi}^{-1}(U) =$

$\Pi(\varphi^{-1}(\Pi^{-1}(U))$ is open as well, since Π maps saturated open sets to open sets. Hence, $\tilde{\varphi}$ is continuous and assertion (iv) holds. To prove assertion (v), suppose that $h\colon A \to \mathbb{T}^1$ is a monotone surjection, then by Lemma 5.12, $h^{-1}(x)$ is saturated with respect to $\mathcal{F}_A$, and thus h induces a map $\widetilde{h}\colon \mathbb{T} \to \mathbb{T}^1$ by $\widetilde{h}(x) = h(x')$ where $x' \in \Pi^{-1}(x)$ is arbitrary. Note that $\widetilde{h}$ is continuous since for any open set $U \subset \mathbb{T}^1$ we have $\widetilde{h}^{-1}(U) = \Pi(h^{-1}(U))$ which is again open since it is the image of a saturated open set. If h is a semiconjugation from φ to R_ρ, then given $x \in \mathbb{T}^1$ and $x' \in \Pi^{-1}(x)$ we have

$$\tilde{h} \circ \tilde{\varphi}(x) = \tilde{h} \circ \tilde{\varphi} \circ \Pi(z) = \tilde{h} \circ \Pi \circ \varphi(z) = h \circ \varphi(z) = R_\rho \circ h(z) = R_\rho \circ \tilde{h}(x).$$

completing the proof if (v). □

5.2 Proof of Corollary 1.3 Suppose h_1, h_2 are two semiconjugacies from $\varphi_{|A}$ to the irrational rotation R_ρ, where ρ is the rotation number of φ on A. By post-composing with a rotation, we may assume that there exists $z_0 \in A$ with $h_1(z_0) = h_2(z_0)$.

Let $\tilde{\varphi}$ be the homeomorphism of $\mathbb{A}$ from part (iv) of Theorem 1.2, so that $\Pi \circ \varphi = \tilde{\varphi} \circ \Pi$ and $\tilde{\varphi}(\mathbb{T}) = \mathbb{T}$, and let $\widetilde{h}_i$ the maps such that $h_i = \widetilde{h} \circ \Pi$ from part (v) of the same theorem. Then $\widetilde{h}_i$ semiconjugates $\widetilde{\varphi}|_{\mathbb{T}}$ to the irrational rotation R_ρ of $\mathbb{T}$. Since the semiconjugacies in the Poincaré Classification Theorem are unique up to post-composition by a rotation, this yields that $\widetilde{h}_1 = \widetilde{h}_2$ and thus $h_1 = h_2$. □

6 Almost automorphic minimal continua

A homeomorphism $f\colon X \to X$ of a metric space is called almost automorphic if there exists $x \in X$ such that, whenever the limit $\widetilde{x} = \lim_{k\to\infty} f^{n_k}(x)$ exists for some sequence $n_k \to \infty$, then $x = \lim_{k\to\infty} f^{-n_k}(\widetilde{x})$. Further, f is almost periodic if for every $\epsilon > 0$, the set $\{n \in \mathbb{Z} : \forall x \in X,\ d(f^n(x), x) < \epsilon\}$ is syndetic (i.e. has uniformly bounded gaps). The Veech Structure Theorem [Vee65] asserts that f is almost automorphic if and only if f is semiconjugate to an almost periodic map of some space Y by means of an almost 1-1 continuous surjection, i.e. a continuous surjection $h\colon X \to Y$ for which the set of points of Y with a unique preimage is dense.

6.1 Proof of Theorem 1.4 Let $\varphi\colon \mathbb{A} \to \mathbb{A}$ be a homeomorphism, $B \subset \mathbb{A}$ is an essential φ-invariant cobasin boundary without periodic points, and $x_0 \in B$ a recurrent bi-accessible point. Since $x_1 = \varphi(x_0)$ is also bi-accessible, we may find an inessential simple closed curve $\gamma \subset \mathbb{A}$ intersecting B exactly at x_0 and x_1. If V_0 and V_1 are the two connected components of $\mathbb{A}\backslash\gamma$ and $B_i = V_i \cap B$, it is easy to verify that $B_i \cup \{x_0, x_1\}$ is a continuum for $i \in \{0, 1\}$. Thus B is decomposable, and Theorem 1.2 implies that there exists $\Pi\colon \mathbb{A} \to \mathbb{A}$ mapping the circloid A whose boundary is B to the circle $\mathbb{T} = \mathbb{T}^1 \times \{0\}$ and inducing a map $\tilde{\varphi}\colon \mathbb{A} \to \mathbb{A}$ which satisfies $\tilde{\varphi} \circ \Pi = \Pi \circ \varphi$ and preserves $\mathbb{T}$. Let U^- and U^+ denote the connected components of $\mathbb{A}\backslash B$ which are unbounded below and above, respectively, and let C be a simple arc joining a point of U^- to a point of U^+ such that $C \cap B = \{x_0\}$. Then $\Pi(\widetilde{C})$ is an arc joining a point below $\mathbb{T}$ to a point above $\mathbb{T}$ and intersecting $\mathbb{T}$ exactly at $\widetilde{x}_0 = \Pi(x_0)$. By continuity, the compact set $\Pi^{-1}(\widetilde{C})$ is contained in the closure of $\Pi^{-1}(\widetilde{C}\backslash\{\widetilde{x}_0\})$, and since $\Pi|_{U^-\cup U^+}$ is an injective map onto $\mathbb{A}\backslash\mathbb{T}$ we have that $\Pi^{-1}(\widetilde{C}\backslash\{\widetilde{x}_0\}) = C\backslash\{x_0\}$. Thus $\Pi^{-1}(\widetilde{C}) \subset C$, and $\Pi^{-1}(\widetilde{x}_0) \subset C \cap B = \{x_0\}$.

Hence, $\Pi^{-1}(\Pi(x_0)) = \{x_0\}$, and the same is true for any point in the orbit of x_0. Since $\widetilde{\varphi}\colon \mathbb{T} \to \mathbb{T}$ is a homeomorphism of the circle without periodic points, the classic Poincaré theory implies that it is semi-conjugate to an irrational rotation R_ρ of the circle by means of a continuous monotone surjection $h\colon \mathbb{T} \to \mathbb{T}^1$ which is injective in the nonwandering set of $\widetilde{\varphi}$. In particular, since $\widetilde{x}_0$ is recurrent, $h^{-1}(h(\widetilde{x}_0)) = \{\widetilde{x}_0\}$, and so $h' = h \circ \Pi$ is an almost 1-1 semiconjugation between $\varphi|_B$ and R_ρ, showing that $\varphi|_B$ is almost-automorphic. □

6.2 Proof of Corollary 1.5 Let U^- and U^+ be the connected components of $\mathbb{A}\backslash X$ unbounded below and above, respectively. Then $\partial U^- \cup \partial U^+$ is a compact invariant subset of X and therefore is equal to X. In particular $\partial U^- \cap \partial U^+ \neq \emptyset$, and since that is also a compact invariant set we deduce $X = \partial U^- \cap \partial U^+$. Therefore, X is a cofrontier, and the Corollary follows from Theorem 1.4. □

7 The Kuratowski decomposition

Let Λ be a continuum on the sphere $\mathbb{S}^2$ which is the common boundary of two open simply connected sets. Using our terminology, this is the same as saying that Λ is a cobasin boundary.

In [Kur28], Kuratowski studied the following topological problem: when can we find a monotone surjection from Λ onto the circle? Following the method used in [Kur27] to study a similar question for irreducible continua, he defined a decomposition into *layers*[3] as follows: A fundamental layer of Λ is any subset of Λ which is maximal with the property of being a continuum which is the union of at most countably many subcontinua of Λ, each of which either is indecomposable or has empty interior in Λ. When there is a unique fundamental layer, the continuum Λ is called *monostratic.*

The main results of [Kur28] implies that Λ is non-monostratic if and only if there exists some monotone continuous surjection from Λ onto the circle. In fact, if Λ is non-monostratic then the fundamental layers form a monotone upper semicontinuous decomposition of Λ, and the quotient space by this decomposition is a simple closed curve. In addition, the decomposition into fundamental layers is the finest upper semicontinuous monotone decomposition of Λ into subcontinua with the property of having the circle as a quotient space, in the sense that any other such decomposition has its elements saturated by fundamental layers.

This can be stated in terms of maps as follows:

Theorem 7.1 (Kuratowski)**.** *If Λ is a non-monostratic cobasin boundary in $\mathbb{S}^2$, then there exists a monotone continuous surjection $\Pi\colon \Lambda \to \mathbb{T}^1$ such that for any other monotone continuous surjection $P\colon \Lambda \to \mathbb{T}^1$, there exists a map $\phi\colon \Lambda \to \Lambda$ such that $P = \phi \circ \Pi$.*

There are many works which extend to a more general setting the concept of finding a finest monotone upper semicontinuous decomposition with the property of having a quotient space with a given property (what is often called the *core* decomposition with the given property); see for instance [Cha73, FS67, Rak77, Vou74] and references therein.

Using the previous theorem together with Moore's theorem, Kuratowski obtains a result which can be reworded as follows [Kur28, Theorem II]:

Theorem 7.2. *If Λ is a circloid in $\mathbb{S}^2$ with non-monostratic boundary, then there exists a monotone continuous surjection $\Pi\colon \mathbb{S}^2 \to \mathbb{S}^2$ which maps Λ onto the equator and is injective on $\mathbb{S}^2\backslash\Lambda$.*

In fact, it can easily be verified that the decomposition of Λ into fibers of Π is the core decomposition with respect of having the circle as quotient space. The map Π can be characterized by considering the decomposition of $\mathbb{S}^2$ into points of $\mathbb{S}^2\backslash\Lambda$ together with the "filled" layers of $\partial\Lambda$ (i.e. the union of each layer L with all the connected components of $\mathbb{S}^2\backslash L$ which are disjoint from Λ). This decomposition turns out to be the same given in the proof of Theorem 1.2.

In essence, the arguments given in Section 5 can be viewed as an alternative proof of Kuratowskis result and could be replaced by an application of Theorems 7.1 and 7.2. The only fact which is used in order to prove Theorem 1.2 is the existence of uncountably many elements in the decomposition $\mathcal{F}_{\mathcal{A}}$. This is obtained via Lemma 5.10 from the existence of the semiconjugacy in Theorem 1.1. Since it can be shown without too much effort that uncountability of $\mathcal{F}_{\mathcal{A}}$ is equivalent to non-monostraticity, the starting points for Kuratowski's proof and our proof Theorem 1.2 are almost the same.

Thus, in this context, the main contribution of Theorem 1.1 could also be stated as follows: if an invariant circloid has decomposable boundary and no periodic points, then its boundary is non-monostratic.

8 An example with large fibers

In this section we prove Theorem 1.6. More specifically, we will prove that there exists a C^∞ diffeomorphism $f\colon \mathbb{A} \to \mathbb{A}$ and a monotone continuous surjection $\Pi\colon \mathbb{A} \to \mathbb{A}$ such that:

- f leaves invariant an essential decomposable cofrontier Λ;
- $f|_\Lambda$ is minimal;

[3]Translation of the original term *tranches* (in french).

- Π semi-conjugates f with an irrational rotation of $\mathbb{A}$: $\Pi f = R_\rho \Pi$ for some $\rho \notin \mathbb{Q}$;
- Each fiber of $\Pi|_\Lambda$ is a continuum with diameter at least 1/4.

In what follows, intervals in $\mathbb{T}^1$ are assumed to be positively oriented, so for $a, b \in \mathbb{T}^1$ the interval (a, b) is the component of $\mathbb{T}^1 \backslash \{a, b\}$ which is positively oriented and (b, a) the remaining one (and similarly for closed intervals).

8.1 The Anosov-Katok method We will use the Anosov-Katok method, which we describe here briefly (and refer to [FK04] for more details and further references). The map f will be obtained as a limit of maps f_n, each of which is C^∞-conjugate to a rational rotation $R_{p_n/q_n} : (\theta, y) \mapsto (\theta + p_n/q_n, y)$, so

$$f_n = H_n R_{p_n/q_n} H_n^{-1}$$

where $H_n \in \mathrm{Diff}^\infty(\mathbb{A})$. The maps H_n are successive compositions of maps:

$$H_n = h_1 \circ \cdots \circ h_{n-1} \circ h_n.$$

The maps $h_n \in \mathrm{Diff}^\infty(\mathbb{A})$ are chosen inductively alongside with the numbers p_n/q_n with the following condition: If h_n and p_n/q_n is already chosen, h_{n+1} can be chosen arbitrarily, with the only restriction that it commutes with R_{p_n/q_n}:

$$h_{n+1} R_{\frac{p_n}{q_n}} = R_{\frac{p_n}{q_n}} h_{n+1}.$$

This guarantees that for any τ,

$$H_{n+1} R_\tau H_{n+1}^{-1} = H_n h_{n+1} R_\tau h_{n+1}^{-1} H_n^{-1} = H_n \left(h_{n+1} R_{(\tau - \frac{p_n}{q_n})} h_{n+1}^{-1} \right) R_{\frac{p_n}{q_n}} H_n^{-1},$$

which means that if $\tau = p_{n+1}/q_{n+1}$ is chosen close enough to p_n/q_n (but different from it), the map f_{n+1} can be made arbitrarily C^∞-close to f_n, and in particular one can make the C^n-distance $\xi_n = d_{C^n}(f_{n+1}, f_n) + d_{C^n}(f_{n+1}^{-1}, f_n^{-1})$ is as small as desired. Repeating this process, if the numbers ξ_n are chosen such that $\sum_{n \in \mathbb{N}} \xi_n < \infty$, the maps f_n converge in the C^∞-topology to an element of $\mathrm{Diff}^\infty(\mathbb{A})$.

Since there is a great degree of freedom in the choice of h_n as well as p_n/q_n at each step, additional restrictions may be placed in order to guarantee that the limit map has special properties. To outline the construction that we will follow, our choice of these maps will be such that there exists a decreasing sequence of essential annuli A_n such that h_{n+1} is the identity outside A_n and maps A_{n+1} into A_n in such a way that every horizontal circle in A_{n+1} becomes ϵ-dense in A_n (with an appropriate choice of ϵ depending on n) and every vertical segment in A_{n+1} is mapped to a set with diameter greater than 1/4. This is achieved by first using a map that "twists" vertical segments inside A_{n+1} so that any such segment becomes horizontally large, and then composing with a map that maps A_{n+1} to something that oscillates vertically inside A_n (see Figure 8.1).

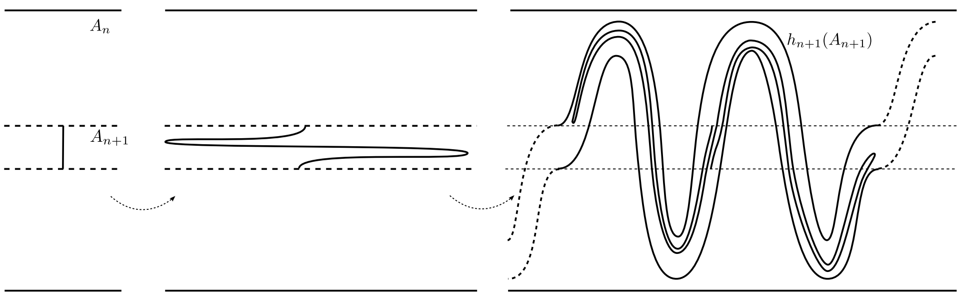

Figure 8.1: How h_{n+1} maps a vertical segment in A_{n+1} (in two steps).

The cofrontier Λ will be the intersection of the sets $\Lambda_n = H_n(A_n)$, which is a decreasing intersection. An appropriate choice of p_{n+1}/q_{n+1} and the fact that h_{n+1} spreads ϵ-densely in A_n every horizontal circle will guarantee that $f|_\Lambda$ is minimal and p_n/q_n converges to an irrational number α. Moreover, although the maps H_n are not required to converge to a homeomorphism, we will guarantee that H_n^{-1} does converge to a continuous surjection Π in the C^0 topology. Due to the way in which these maps are defined, this automatically implies that Π conjugates f with the rotation R_α, and the conditions on h_n will guarantee that the preimage of any point of Λ by Π has diameter at least $1/4$.

Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ be decreasing sequences of positive real numbers such that $\alpha_1 < 1$, $\alpha_n \to 0$ as $n \to \infty$, and $\sum_{n\in\mathbb{N}} \epsilon_n < \infty$.

Claim 8.1. *There exist sequences $(h_n)_{n\in\mathbb{N}}$ in $\mathrm{Diff}^\infty(\mathbb{A})$, $(p_n/q_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ (where p_n, q_n are relatively prime integers, $q_n > 0$), and $(M_n)_{n\geqslant 2}$ in $\mathbb{N}$ such that, letting*

- $H_n = h_1 h_2 \cdots h_n$,
- $f_n = H_n R_{p_n/q_n} H_n^{-1}$, *item* $A_n = \mathbb{T}^1 \times [-\alpha_n, \alpha_n]$,
- $\Lambda_n = H_n(A_n)$,

the following properties hold for $k \geqslant 1$:

(1) $h_k(A_k) = A_k$, *and* $h_k(z) = z$ *for all* $z \in \mathbb{A}\backslash A_k$;

(2) $\mathrm{diam}(H_k(\{\theta\} \times [-\alpha_k, \alpha_k])) > 1/4$ *for all* $\theta \in \mathbb{T}^1$;

(3) $q_k > 13^k$;

and for $k \geqslant 2$,

(4) $h_k R_{p_{k-1}/q_{k-1}} = R_{p_{k-1}/q_{k-1}} h_k$;

(5) $H_k(\mathbb{T}^1 \times \{\alpha\})$ *is* ϵ_{k-1}*-dense in* Λ_{k-1} *for all* $\alpha \in [-\alpha_k, \alpha_k]$;

(6) $d(\pi_1(h_k(z)), \pi_1(z)) < 3/q_{k-1}$ *for all* $z \in \mathbb{A}$;

(7) $d_{C^k}(f_k, f_{k-1}) < \epsilon_{k-1}$ *and* $d_{C^k}(f_k^{-1}, f_{k-1}^{-1}) < \epsilon_{k-1}$;

(8) $|p_k/q_k - p_{k-1}/q_{k-1}| < \epsilon_{k-1}$;

(9) $\{f_k^j(z) : 0 \leqslant j \leqslant M_{i+1}\}$ *is* ϵ_i*-dense in* Λ_i *for all* $z \in \Lambda_k$ *and* $1 \leqslant i \leqslant k-1$;

Before explaining how to obtain such sequences, let us prove that they lead to a map with the required properties.

Claim 8.2. *Using the maps defined in Claim 8.1, there exist $f \in \mathrm{Diff}^\infty(\mathbb{A})$, $\Pi \in C^0(\mathbb{A}, \mathbb{A})$, and $\rho \in \mathbb{T}^1$ such that, if $\Lambda = \bigcap_{n\in\mathbb{N}} \Lambda_n$, then*

- $f_n \to f$ *in the C^∞ topology as $n \to \infty$;*
- $H_n^{-1} \to \Pi$ *in the C^0 topology as $n \to \infty$;*
- Π *is a monotone semiconjugation between f and R_ρ;*
- Λ *is an essential f-invariant decomposable cofrontier and* $\Pi(\Lambda) = \mathbb{T}^1 \times \{0\}$;
- $\mathrm{diam}(\Pi^{-1}(\Pi(z))) \geqslant 1/4$ *for all $z \in \Lambda$, and $\Pi|_{\mathbb{A}\backslash\Lambda}$ is injective;*
- $f|_\Lambda$ *is minimal.*

Proof. Note that

$$d_{C^0}(H_n^{-1}, H_{n+m}^{-1}) = d_{C^0}(H_q^{-1} n H_{n+m}, H_{n+m}^{-1} H_{n+m}) = d_{C^0}(h_{n+1}h_{n+2}\cdots h_{n+m}, \mathrm{Id}).$$

By (6) one has

$$|\pi_1(h_{n+1}h_{n+2}\cdots h_{n+m}(z)) - \pi_1(z)| \leqslant \sum_{i=n}^{n+m-1} \frac{3}{q_i},$$

and by (1), since each h_{n+i} leaves A_n invariant and is the identity outside A_n,

$$|\pi_2 h_{n+1}h_{n+2}\cdots h_{n+m}(z) - \pi_2(z)| \leqslant 2\alpha_n.$$

Since $\alpha_n \to 0$ and $\sum_i 1/q_i < \infty$, we see that the two coordinates of the map

$$z \mapsto h_{n+1}h_{n+2}\cdots h_{n+m}(z) - z$$

are uniformly small if n, m are large enough, and therefore $(H_i^{-1})_{i\in\mathbb{N}}$ is a Cauchy sequence in $C^0(\mathbb{A}, \mathbb{A})$, so it converges to some continuous (and surjective) map Π. Note also that $H_i^{-1}(z)$ is eventually constant if $z \notin \Lambda$. Since each H_i^{-1} is a homeomorphism onto its image, it follows easily that $\Pi|_{\mathbb{A}\backslash\Lambda}$ is injective, and $\Pi(\Lambda) = \mathbb{T}^1 \times \{0\}$.

By (7) and the fact that $\sum_n \epsilon_n < \infty$, the sequences $(f_n)_{n\in\mathbb{N}}$ and $(f_n^{-1})_{n\in\mathbb{N}}$ are Cauchy in the C^r-topology for any r, the sequence $(f_n)_{n\in\mathbb{N}}$ converges to some C^∞ diffeomorphism f in the C^∞-topology.

By (8), the sequence $(p_n/q_n)_{n\in\mathbb{N}}$ is Cauchy and therefore has some limit $\rho \in \mathbb{T}^1$. oreover, since

$$H_n^{-1} f_n = H_n^{-1}(H_n R_{p_n/q_n} H_n^{-1}) = R_{p_n/q_n} H_n^{-1},$$

taking limits as $n \to \infty$ one deduces that $\Pi f = R_\rho \Pi$. The fact that Π is a uniform limit of homeomorphisms implies that Π is monotone (see for instance [Why55]). Moreover, note that $H_n(\{\theta\} \times [-\alpha_n, \alpha_n])$ is connected and has diameter at least $1/4$ due to (2). Choosing $z_n, z_n' \in H_n(\{\theta\} \times [-\alpha_n, \alpha_n])$ such that $d(z_n, z_n') \geqslant 1/4$ and taking convergent subsequences of (z_n) and (z_n') one finds two points $z, z' \in \Lambda$ such that $d(z, z') \geqslant 1/4$ and $\Pi(z) = \Pi(z') = (\theta, 0)$, showing that $\operatorname{diam}(\Pi^{-1}(\theta, 0)) \geqslant 1/4$. This means that $\operatorname{diam}(\Pi^{-1}(\Pi(z))) \geqslant 1/4$ for all $z \in \Lambda$.

The minimality of $f|_\Lambda$ follows immediately from (9). Since the set $\Lambda = \bigcap_{n\in\mathbb{N}} \Lambda_n$ is a decreasing intersection of essential closed topological annuli, it is an essential annular continuum, and using the fact that $f|_\Lambda$ is minimal we deduce that Λ is an essential cofrontier (by an argument already used in the proof of Corollary 1.5).

To see that Λ is decomposable, let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$ be the upper and lower half-circles, respectively, and let $\Lambda_n^j = H_n(I_j \times [-\alpha_n, \alpha_n])$ for $j \in \{1, 2\}$. For each j, let Λ^j be the Hausdorff limit of some convergent subsequence of $(\Lambda_n^j)_{n\in\mathbb{N}}$. Then Λ^1 and Λ^2 are compact connected nonempty sets, and from the fact that $\Lambda_n = \Lambda_n^1 \cup \Lambda_n^2$ it is easy to verify that $\Lambda = \Lambda^1 \cup \Lambda^2$. On the other hand, note that from (3) one has $\mu := \sum_{i=1}^\infty 3/q_i < 3\sum_{i=1}^\infty 1/13^i = 1/4$, and (6) implies that $d(\pi_1(H_n(z)), \pi_1(z)) < \mu$ (as noted in the beginning of the proof). Thus $\pi_1(\Lambda_n^j) = \pi_1(H_n(I_j \times [-\alpha_n, \alpha_n]))$ is an interval of length at most $1/2 + 2\mu$. Therefore $\pi_1(\Lambda^j)$ has length at most $1/2 + 2\mu < 1$, and since Λ is essential this means that $\Lambda^j \neq \Lambda$ for $j \in \{1, 2\}$, showing that $\Lambda = \Lambda^1 \cup \Lambda^2$ is a decomposition of Λ into nonempty proper subcontiua. □

Proof of Claim 8.1. We define the sequences recursively. Start with $p_1 = 0$ and $q_1 = 1/13^2$. To simplify the construction we will assume $\alpha_1 > 1/4$, so that letting $h_1 = \mathrm{Id}_{\mathbb{A}}$ conditions (1))-((3) automatically hold for $k = 1$.

Assuming that h_k, p_k/q_k, and M_k are already defined for $1 < k \leqslant n$ and satisfy the required properties, we will define them for $k = n + 1$. As explained in the introduction of this section, as long as p_{n+1}/q_{n+1} is chosen close enough to p_n/q_n we can guarantee that f_{n+1} will be arbitrarily close to f_n in the C^n-topology. Therefore we will first choose h_{n+1} accommodating to our needs, and later we will choose p_{n+1}/q_{n+1} close enough to p_n/q_n (but different from it) so as to guarantee conditions (7), (9) and (3).

In order to define our map h_{n+1} we first introduce two auxiliary maps. Given $\delta > 0$ and positive integers K, n, q the first map $V = V_{n,q,\delta} \in \mathrm{Diff}^\infty(\mathbb{A})$ will have the following properties:

- (V1) $V(z) = z$ for $z \in \mathbb{A} \backslash (\mathbb{T}^1 \times A_n)$;
- (V2) If $I \subset \mathbb{T}^1$ is any interval of length greater than δ, then there is a subinterval of I with endpoints θ_0, θ_1 such that V maps each set $\{\theta_0\} \times [-\alpha_{n+1}, \alpha_{n+1}]$ and $\{\theta_1\} \times [-\alpha_{n+1}, \alpha_{n+1}]$ into a different connected component of $Aq_n \backslash (\mathbb{T}^1 \times [-\alpha_n + \delta, \alpha_n - \delta])$.
- (V3) $d(\pi_1(V(z)), \pi_1(z)) < \delta$ for all $z \in \mathbb{A}$.
- (V4) V commutes with $R_{\frac{1}{q}}$.

There are several ways to define such a map. We describe one way[4], depicted in Figure 8.2: let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function such that $\phi(t) = 1$ if $|t| < \alpha_{n+1}$, $\phi(t) = 0$ if $|t| > \alpha_n$ and $0 \leqslant \phi(t) \leqslant 1$ for all $t \in \mathbb{R}$, and consider the vector field

$$V(\theta, y) = \left(0, \sin\left(2\pi \frac{\delta x}{4q}\right)\phi(y)\right)$$

[4]An alternative definition (which we will not develop further) leads to an area-preserving map V with similar properties; roughly, this is done by using a map which expands A_{n+1} horizontally, contracts vertically, and folds it back into A_n while keeping all the required properties. By doing this, one could guarantee that the maps f_n (and therefore limit map f in Claim 8.2) are area-preserving.

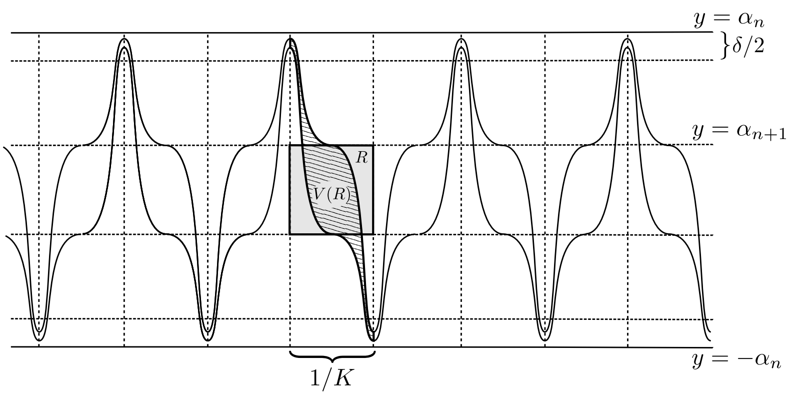

Figure 8.2: A possible map $(\theta, y) \mapsto V(\theta, y)$

on $\mathbb{A}$. If $t \in \mathbb{R}$ is large enough, the time-t map of the flow induced by V is a diffeomorphism satisfying the requred properties (see Figure 8.2).

The second map $T = T_{n,q} \in \mathrm{Diff}^\infty(\mathbb{A})$ is such that:

(T1) $T(z) = z$ for all $z \in \mathbb{A}\backslash A_n$ and all $z \in \mathbb{T}^1 \times \{0\}$;

(T2) $T(A_{n+1}) = A_{n+1}$;

(T3) $[\theta - 1/q, \theta + 1/q] \subset \pi_1(T(\{\theta\} \times [-\alpha_{n+1}, \alpha_{n+1}]))$ for all $\theta \in \mathbb{T}^1$;

(T4) $d(\pi_1(T(z)), \pi_1(z))) \leqslant 2/q$ for all $z \in \mathbb{A}$;

(T5) T commutes with $R_{1/q}$.

Such a map may be defined explicitly as

$$(\theta, y) \mapsto \left(\theta + \frac{\psi(y)y}{\alpha_{n+1}q_n}, y\right),$$

where $\psi\colon \mathbb{R} \to \mathbb{R}$ is a C^∞ bump function such that $0 \leqslant \psi(z) \leqslant 1$ for all $z \in \mathbb{R}$, ψ is equal to 1 on $[-\alpha_{n+1}, \alpha_{n+1}]$ and 0 outside $[-\alpha_n, \alpha_n]$ and $|\psi(y)y| \leqslant 2\alpha_{n+1}$ for all $y \in \mathbb{R}$.

Let $s = \sup_{\theta\in\mathbb{T}^1} \mathrm{diam}(H_n(\{\theta\} \times [-\alpha_n, \alpha_n])) - 1/4$. By continuity of H_n and property (2) it follows that $s > 0$. Let $\epsilon = \min\{s, \epsilon_{n+1}\}$, and choose $0 < \delta$ so small that $d(H_n(z), H_n(z')) < \epsilon/2$ whenever $d(z, z') < 3\delta$, and in addition $\delta < 1/q_n$ and $\delta < (\alpha_n - \alpha_{n+1})/4$.

We now claim that the map $h_{n+1} = V \circ T$ satisfies the required properties using $q = q_n$. First, note that properties (1) and (4) for $k = n + 1$ follow directly from (V1), (T1), (V4) and (T5).

Let us further note that whenever $\gamma \subset A_{n+1}$ is an arc such that $\mathrm{diam}(\pi_1(\gamma)) > \delta$, then $V(\gamma)$ is 3δ-dense in $\pi_1(\gamma) \times [-\alpha_n, \alpha_n]$. Indeed, the properties of V imply that $[-\alpha_n + \delta, \alpha_n - \delta] \subset \pi_2(V(\gamma))$ and $\pi_1(V(\gamma))$ lies in an δ-neighborhood of $\pi(\gamma)$. Thus, given $z \in \pi_1(\gamma) \times [-\alpha_n + \delta, \alpha_n - \delta]$, there exists $z' \in \gamma$ such that $\pi_2(z') = \pi_2(z)$ and $d(z', z) < 2\delta$. From this it follows easily that $V(\gamma)$ is 3δ-dense in $\pi_1(\gamma) \times [-\alpha_n, \alpha_n]$ as claimed.

Since, for any $\alpha \in [-\alpha_{n+1}, \alpha_{n+1}]$, the set $T(\mathbb{T}^1 \times \{\alpha\})$ is contained in A_{n+1}, the previous remarks imply that $h_{n+1}(\mathbb{T}^1 \times \{\alpha\}) = V(T(\mathbb{T}^1 \times \{\alpha\}))$ is 3δ-dense in A_n, anqd since $H_{n+1}(\mathbb{T}^1 \times \{\alpha\}) = H_n(h_{n+1}(\mathbb{T}^1 \times \{\alpha\}))$, the choice of δ implies that $H_{n+1}(\mathbb{T}^1 \times \{\alpha\})$ is ϵ-dense in $H_n(A_n) = \Lambda_n$. Since $\epsilon < \epsilon_{n+1}$, this shows that property (5) holds for $k = n+1$.

Given $\theta \in \mathbb{T}^1$, let $\gamma = T(\{\theta\} \times [-\alpha_{n+1}, \alpha_{n+1}])$. Then $[\theta - 1/q_n, \theta + 1/q_n] \subset \pi_1(\gamma)$, and in particular $\mathrm{diam}(\pi_1(\gamma)) > 2/q_n > \delta$, so that $V(\gamma)$ is 3δ-dense in $\pi_1(\gamma) \times [-\alpha_n, \alpha_n]$. Recall from our choice of ϵ that the diameter of $H_n(\{\theta\} \times [-\alpha_n, \alpha_n])$ is at least $1/4 + \epsilon$. Thus we can find $z_1, z_2 \in \{\theta\} \times [-\alpha_n, \alpha_n]$ such that $d(H_n(z_1), H_n(z_2)) \geqslant 1/4 + \epsilon$. But then there exist $z_1', z_2' \in V(\gamma)$ such that $d(z_i, z_i') < 3\delta$, and our choice of δ implies that $d(H_n(z_i), H_n(z_i')) < \epsilon/2$ for $i \in \{1, 2\}$. Thus $d(H_n(z_1'), H_n(z_2')) > 1/4$, and it follows that $H_n(V(\gamma))$ has diameter greater than $1/4$. Since $V(\gamma) = h_{n+1}(\{\theta\} \times [-\alpha_{n+1}, \alpha_{n+1}])$, we see that property ((2)) holds for $k = n + 1$.

To verify property ((6)) for $k = n+1$, note that, using the properties of V and T and the fact that $\delta < 1/q_n$,

$$d(\pi_1(h_{n+1}(z), \pi(z)) = d(\pi_1(V(T(z))), \pi_1(z)) \leqslant \delta + 2/q_n < 3/q_n.$$

The remaining properties are guaranteed by choosing p_{n+1}/q_{n+1} close enough to p_n/q_n. Indeed, as noted at the beginning of the proof, properties (7) and (9) are guaranteed in this way, and property (3) will hold if p_{n+1}/q_{n+1} is close to p_n/q_n but not equal to it. Therefore there is an open interval I containing p_n/q_n such that properties (7) (8) and (3) hold as long as $p_{n+1}/q_{n+1} \in I \backslash \{p_n/q_n\}$.

By our induction assumption, property (9) holds for f_n, so that $\{f_n^k(z) : 0 \leqslant k \leqslant M_i\}$ is ϵ_i-dense in Λ_i for each $z \in \Lambda_n$ and $1 \leqslant i \leqslant n-1$. However, this condition is C^0-open, meaning that it still holds if one replaces f_n by a map which is C^0-close enough to f_n. Therefore, reducing the interval I we may assume that whenever $p_{n+1}/q_{n+1} \in I \backslash \{p_n/q_n\}$, the set $\{f_{n+1}^k(z) : 0 \leqslant k \leqslant M_{i+1}\}$ is ϵ_i-dense in Λ_i when $1 \leqslant i \leqslant n-1$ (noting that $\Lambda_{n+1} \subset \Lambda_n$). It remains to verify that there exists M_{n+1} such that the latter also holds when $i = n+1$.

Choose an irrational $\theta \in I$ and note that by property (5) the set $H_{n+1}(\mathbb{T}^1 \times \{\alpha\})$ is ϵ_{n+1}-dense in Λ_n for any $\alpha \in [-\alpha_{n+1}, \alpha_{n+1}]$, so the map $g = H_{n+1} R_\theta H_{n+1}^{-1}$ has the property that every g-orbit of a point in Λ_{n+1} is ϵ_{n+1}-dense in Λ_n. By compactness, we may choose M_{n+1} such that $\{g^k(z) : 0 \leqslant k \leqslant M_{n+1}\}$ is ϵ_{n+1}-dense in in Λ_n for any $z \in \Lambda_{n+1}$. Since this is an open condition, if $p_{n+1}/q_{n+1} \in I \backslash \{p_n/q_n\}$ is chosen close enough to θ we have that $\{f_{n+1}^k(z) : 0 \leqslant k \leqslant M_{n+1}\}$ is ϵ_{n+1}-dense in Λ_n, as required. Thus property (9) holds, completing the proof.

□

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