MODEL SETS WITH POSITIVE ENTROPY IN EUCLIDEAN CUT AND PROJECT SCHEMES

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Abstract. We construct model sets arising from cut and project schemes in Euclidean spaces whose associated Delone dynamical systems have positive topological entropy. The construction works both with windows that are proper and with windows that have empty interior. In a probabilistic construction with randomly generated windows, the entropy almost surely turns out to be proportional to the measure of the boundary of the window.

Resumé. On construit des ensembles de Delone euclidiens obtenus par coupe et projection de sorte que l'entropie des système dynamique associé soit strictement positive. La construction permet d'utiliser une fenêtre propre ou d'intérieur vide. Dans une construction probabiliste, pour presque tout paramêtre, l'entropie est proportionnelle à la mesure de la frontière de la fenêtre.

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1. Introduction

In the last decades, aperiodic order – often referred to as the mathematical theory of quasicrystals – has developed into a broad and highly active field of research, see e.g. [BG, KLS15] for recent books dealing with this topic. In this context, the main attention has been given to models with a strong degree of long-range order. In particular, there is nowadays a fairly good understanding of the relations between pure point diffraction – characterising quasicrystals from the physical viewpoint – and purely discrete dynamical spectrum, which has emerged as one of the major tools in the mathematical analysis of long-range aperiodic order.

In this paper, we have a slightly different focus and construct models that may be considered as intermediate between strong long-range order and disorder. More precisely, we introduce a broad family of model sets, produced by cut and project schemes in Euclidean space, whose associated Delone dynamical systems exhibit a high degree of chaoticity, including positive topological entropy. At the same time, they still inherit a certain degree of long-range order, which is built into the underlying cut and project scheme and manifests itself in a non-vanishing discrete part of the dynamical spectrum as well as in minimality. Although we restrict here to study the basic dynamical properties, we hope that the constructed models may be instrumental in understanding the transition from quasicrystalline to amorphous configurations in solid matter. We note several recent works dealing with similar model sets with 'thick boundary' of the window, based on a variety of different methods [BHS16, BJL15, HP13, HR14, KR15]. The reader may take that as an indication for the timeliness of the endeavor.

We will discuss more specifically how the present paper relates to other works and contributes to the emerging general theory towards the end of this section, after we have introduced the necessary notation. Here, we already note that - to the best of our knowledge - it provides the first examples of model sets with positive entropy based on Euclidean cut and project schemes.

A *cut and project scheme (CPS)* is a triple $(G, H, \mathcal{L})$ consisting of locally compact abelian groups G, called *direct space*, and H, called *internal space*, and a discrete co-compact subgroup (*lattice*) $\mathcal{L} \subseteq G \times H$ such that the canonical projection $\pi_G : G \times H \to G$ is one-to-one and the canonical projection $\pi_H : G \times H \to H$ has dense image. This framework goes back to Meyer's influential book [Mey72] and has later been developed in [Moo97, Moo00, Sch00]. In this paper we will always take $G = \mathbb{R}^N$ and we will assume H to be σ-compact (i.e. a countable union of compact sets) and metrizable. Our main application concerns the case $G = H = \mathbb{R}$. So, the reader may also well think from the very beginning of H as just another Euclidean space $\mathbb{R}^M$ (where $M \neq N$ is possible).

Given a relatively compact subset $W \subseteq H$, which is called a *window* in this context, such a CPS produces a uniformly discrete subset of G via

$$\curlywedge(W) \;=\; \pi_G\left(\mathcal{L} \cap (G \times W)\right).$$

An alternative way to define $\curlywedge(W)$ is to introduce the *star-map*. Set $L := \pi_G(\mathcal{L})$ and $L^* := \pi_H(\mathcal{L})$. Then, the star map $* : L \to L^*$ is given by $\ell \mapsto \ell^*$, where ℓ^* is uniquely defined by $(\ell, \ell^*) \in \mathcal{L}$ due to the injectivity of $\pi_{G|\mathcal{L}}$. Then, we have

$$\curlywedge(W) = \{\ell \in L \mid \ell^* \in W\}.$$

If W has non-empty interior, then $\curlywedge(W)$ is called a *model set*, in the general case it is called a *weak model set*. We will be concerned with model sets whose window has a further 'smoothness' feature: A window $W \subseteq H$ is called *proper* (or sometimes *topologically regular*) if

$$\mathrm{cl}(\mathrm{int}(W)) = W.$$

The associated model set will then also be referred to as *proper model set*. Note that any proper window is compact.

A model set is always Delone set (see the next section for more detailed definitions and a discussion of further facts concerning CPS and model sets).

Given a window $W \subseteq H$ (which will mostly be compact in our considerations below), we can associate a dynamical system to $\curlywedge(W)$ by considering the $\mathbb{R}^N$-action $(s, \Lambda) \mapsto \Lambda - s$ on the *hull* of $\curlywedge(W)$. This hull is given as $\Omega(\curlywedge(W)) = \mathrm{cl}(\{\curlywedge(W) - s \mid s \in \mathbb{R}^N\}$, where the closure is taken in a suitable topology (defined below). The properties of this dynamical system depend crucially on the boundary of the window W.

If W is proper and the boundary of W has Haar measure zero, then the dynamical system $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ is (measurably) isomorphic to the Kronecker flow on the torus $\mathbb{T} = (\mathbb{R}^N \times H)/\mathcal{L}$ defined by $\omega : \mathbb{R}^N \times \mathbb{T} \to \mathbb{T}$, $(s, \xi) \mapsto \xi + [s, 0]_{\mathcal{L}}$ and is therefore uniquely ergodic with purely discrete dynamical spectrum [Sch00] and zero topological entropy [BLR07]. This case has attracted most attention in recent years. In fact, it seems fair to say that *regular model sets*, i.e. sets of the form $\curlywedge(W)$ for proper W whose boundary has measure zero, are the prime examples for quasicrystals. In particular, substantial efforts have been spent over the years to prove pure point diffraction for regular model sets, see e.g. [Hof96, Sch00]. By now this pure pointedness is well understood and three different approaches have been developed: The approach of [Hof96] via Poisson summation formula has recently been extended to a very general framework in [RS15]. The result of [Sch00] can be seen within the context of the equivalence between purely discrete dynamical spectrum and pure point diffraction, proven in this setting in [LMS02] and later generalized in various directions in e.g. [BL04, Gou04, LS03, LM16]. Finally, pure point spectrum can also be shown using almost periodicity [BM04], see also [Stru05].

Conversely, the case of windows with 'thick boundary', in the sense of positive Haar measure, is not as well understood. A general idea in this context is that thickness of the boundary should imply positive topological entropy and failure of unique ergodicity. In fact, corresponding conjectures have been brought forward by Moody, see [HR14] for discussion, and Schlottmann [Sch00]. These conjectures are supported by prominent examples. Indeed, for the well-known example of visible lattice points the associated dynamical system is far from being uniquely ergodic and has positive topological entropy [BMP00, HP13]. This system has still pure point diffraction [BMP00] and pure point dynamical spectrum if it is equipped with a natural ergodic measure [HB14]. Existence of such a canonical ergodic measure for general model sets with thick boundary has received attention recently, see [BHS16] for an approach based on a maximal density condition and [KR15] for an rather structural approach. Quite remarkably, all these model sets with maximal density still have pure point diffraction and pure point dynamical spectrum with respect to the canonical measure [BHS16]. Note, however, that the eigenfunctions will in general not be continuous anymore. In this context, a general upper bound on topological entropy has been established in [HR14]. Given this support for the mentioned conjectures, the recent findings in [BJL15] may seem surprising as they provide examples of proper model sets with thick boundary which are still uniquely ergodic (and minimal) with topological entropy zero. At the same time [BJL15] also provides some examples of proper model sets with minimal dynamical systems of positive entropy lacking unique ergodicity. All examples of [BJL15] are based on Toeplitz systems.

In all examples in the preceding discussion, where the topological entropy was shown to be positive, the internal space H is not an Euclidean space but has a rather more complicated structure (being a p-adic space in the case of the visible lattice points and being an odometer in the case of the Toeplitz systems). In the present paper we provide examples of model sets with positive entropy based on Euclidean internal space.

For the sake of simplicity, we will here restrict to Euclidean CPS with one-dimensional internal space $H = \mathbb{R}$. In principle, similar constructions can be carried out with higher-dimensional internal group, see Section 8 for a brief discussion. Then, a lattice with the above properties is of the form $\mathcal{L} = A(\mathbb{Z}^{N+1})$, where $A \in \mathrm{GL}(N+1, \mathbb{R})$ satisfies the two conditions that $\pi_1 : \mathbb{R}^{N+1} \to \mathbb{R}^N$ is injective on $\mathcal{L}$ and $\pi_2 : \mathbb{R}^{N+1} \to \mathbb{R}$ maps $\mathcal{L}$ to a dense set. Note that this is certainly a generic condition on A this is always satisfied whenever the entries of A are linearly independent over $\mathbb{Q}$. We call such $\mathcal{L}$ an *irrational lattice*. The situation can be summarized in the following diagram.

$$
\begin{array}{ccccc}
\mathbb{R}^N & \xleftarrow{\pi_1} & \mathbb{R}^N \times \mathbb{R} & \xrightarrow{\pi_2} & \mathbb{R} \\
\cup & & \cup & & \cup \\
L & \xleftarrow{1-1} & \mathcal{L} = A(\mathbb{Z}^{N+1}) & \xrightarrow{dense} & L^*
\end{array}
$$

In this setting we construct examples with positive topological entropy (in fact, the maximal entropy possible given the bound in [HR14]) and lack of unique ergodicity. At the same time these examples still are minimal and have a relatively dense set of continuous eigenvalues. So, our examples share positive entropy and lack of unique ergodicity with the examples of [BMP00, HP13] while they differ from these examples by having the additional regularity feature of minimality and a dense set of continuous eigenvalues. On the other hand our examples share minimality and positive entropy with the mentioned examples of [BJL15] but differ from these examples by being based on a Euclidean CPS.

To us, a main achievement of our construction is that it is rather direct and transparent. By this we hope that it can serve as a tool for further investigations as well.

In order to to give a flavor of our results, we will next state one main theorem (an extended version of which is given below in Theorem 5.4), which focuses a probabilistic model with 'random' window. Deterministic constructions are given as well, in Section 6 for the case of model sets and in Section 7 for the case of weak model sets. The latter has started to attract increasing attention due to its relations to number theory, compare discussion above and [HR14, BHS16].

Theorem 1.1 *Suppose $\mathcal{L} \subseteq \mathbb{R}^{N+1}$ is an irrational lattice and C is a Cantor set of positive Lebesgue measure in $[0,1]$. Let $(G_n)_{n\in\mathbb{N}}$ be a numbering of the bounded connected components of $\mathbb{R} \setminus C$ and $\Sigma^+ = \{0,1\}^{\mathbb{N}}$. Denote by $\mathbb{P}$ the Bernoulli distribution on Σ^+ with equal probability $1/2$ for each symbol and define*

$$W(\omega) \;=\; C \cup \bigcup_{n\in\mathbb{N}:\omega_n=1} G_n \ ,$$

where $\omega \in \Sigma^+$. Then for $\mathbb{P}$-almost every $\omega \in \Sigma^+$ the set $W(\omega)$ is proper and the dynamical system $(\Omega(\curlywedge(W(\omega)+\vartheta)),\mathbb{R}^N)$ has positive topological entropy for all $\vartheta \in \mathbb{R}$ and is minimal for ϑ from a residual subset $\Theta \subseteq \mathbb{R}$ (depending on ω).

Remark 1.2 (a) In fact, the topological entropy attains the upper bound provided in [HR14], which is given in terms of the measure of $\partial W(\omega)$ and the density of the lattice $\mathcal{L}$, see Theorem 5.4.

(b) The existence of the residual subset $\Theta \subseteq \mathbb{R}$ such that for all $\vartheta \in \Theta$ the system $(\Omega(\curlywedge(W(\omega)+\vartheta)),\mathbb{R}^N)$ is minimal is a consequence of general (and well-known) theory of model sets and has nothing to do with our (random) setting.

(c) Due to the properness of the window our systems fibre over a torus, i.e. allow for a torus as a factor. This has some consequences: For one thing, by abstract results this then implies that the entropy comes from single fibres (see Remark 2.18 below). In fact, our proof directly exhibits fibres carrying the entropy. Also, having this factor implies that our examples have a relatively dense set of continuous eigenvalues, see Remark 2.11.

(d) Our results also show that if $|C| > 1/2$, then for the set of ω of full measure above and any $\vartheta \in \mathbb{R}$ the dynamical system $(\Omega(\curlywedge(W(\omega)+\vartheta)),\mathbb{R})$ is not uniquely ergodic (see Theorem 5.4).

The reason for the positive toplogical entropy of $(\Omega(\curlywedge(W)+\vartheta),\mathbb{R})$ is the existence of a large 'random component' in the hull, which may be of intrinsic conceptual interest. We say $\Omega(\curlywedge(W))$ *contains an embedded fullshift*, if there exists $S \subseteq \mathbb{R}^N$ of positive asymptotic density and a uniformly discrete $U \subseteq \mathbb{R}^N$ such that for any subset S' of S there exists $\Gamma \in \Omega(\curlywedge(W))$ with $\Gamma \subseteq U$ and

$$S' = \Gamma \cap S.$$

This means that we may think of the elements of S as positions of points (or atoms) which may be switched on or off completely independently of each other, without leaving the hull (but there is no control on what happens outside of S at the same time). Details are discussed in the first part of Section 3. Embedded fullshifts are closely related to the local structure of the window W (or its translate $W+\vartheta$) around the points in L^*. In later parts of Section 3, we also introduce the notion of local independence of W with respect to subsets of L^* to establish criteria for the existence of embedded fullshifts. Depending on the context, either a topological (Lemma 3.12) or a metric version (Lemma 3.14) of this concept can be applied. A discussion of failure of unique ergodicity in the presence of embedded subhifts is given in Section 4.

The proof of Theorem 1.1 is then given in Section 5. In fact, Theorem 5.4 in that section is an extended version of Theorem 1.1 including parts of Remarks 1.2. Section 6 then provides examples of deterministic windows that equally lead to positive entropy. While this construction is slightly more technical, it demonstrates that the randomness in the definition of $W(\omega)$ above is not a key ingredient of the procedure. Moreover, this also sets the ground for the construction of weak model sets (whose window has empty interior) with positive entropy, which is carried out in Section 7.

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2. Preliminaries

In this section we discuss the necessary background from the theory of point sets and their associated dynamical systems. The material is essentially well-known. For the convenience of the reader we include some proofs.

2.1 Delone sets. A set $\Lambda \subseteq \mathbb{R}^N$ is called *uniformly discrete* if there exists a real number $r > 0$ such that

$$\|x - y\| \geq r \qquad \text{for all } x, y \in \Lambda \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm. The set is called *relatively dense* if there exists a real number $R > 0$ such that

$$B_R(x) \cap \Lambda \neq \emptyset \text{ for all } x \in \mathbb{R}^N , \tag{2}$$

where $B_R(x)$ denotes the closed ball of radius R around x. We call Λ a *Delone set* if it is uniformly discrete and relatively dense in $\mathbb{R}^N$. We say $p \in \mathbb{R}^N$ is a *period* of Λ if $\Lambda + p = \Lambda$ and call Λ *aperiodic* if $p = 0$ is the only period. Given a Delone set Λ, let $x \in \Lambda$ and $\varrho > 0$. Then the pair $(P(\varrho, x), \varrho)$ with

$$P(\varrho, x) := (\Lambda - x) \cap B_\varrho(0)$$

is called a *ϱ-patch of Λ in x*. The *set of all patches* is given by

$$\mathcal{P}(\Lambda) = \{(P(\varrho, x), \varrho) \mid x \in \Lambda, \varrho > 0\}.$$

Note that this definition works also for discrete sets which are not Delone. The set Λ has *finite local complexity* (or *(FLC)* for short) if

$$\sharp\{(\Lambda - x) \cap B_\varrho(0) \mid x \in \Lambda\} < \infty \tag{FLC}$$

for all $\varrho > 0$. This assumption is equivalent to various other properties:

Lemma 2.1 ([Lag98]) *Let Λ be a Delone set. Then the following statements are equivalent:*

(i) Λ has (FLC);
(ii) $\sharp\{(\Lambda - x) \cap B_{2R}(0) \mid x \in \Lambda\} < \infty$, where R is as in (2);
(iii) $\Lambda - \Lambda$ is closed and discrete.

If Λ is a Delone set with $\Lambda - \Lambda$ uniformly discrete, then Λ is called a *Meyer set*. Being a Meyer set is a notably strong property, and in particular implies (FLC) by Lemma 2.1 (iii).

A Delone set Λ is *repetitive* if for all $(P, \varrho) \in \mathcal{P}(\Lambda)$ the set

$$\{x \in \Lambda \mid P(\varrho, x) = P\}$$

is relatively dense in $\mathbb{R}^N$. It has *uniform patch frequencies* (or *(UPF)* for short) if for all patches $(P, \varrho) \in \mathcal{P}(\Lambda)$ the limit

$$\text{(UPF)} \qquad \nu(P,x) \;=\; \lim_{n\to\infty} \frac{\sharp\{y \in (\Lambda - x) \cap B_n(0) \mid P(\varrho, y) = P\}}{\lambda(B_n(0))}$$

exists and the convergence is uniform in $x \in \mathbb{R}^N$. Here, λ denotes the N-dimensional Lebesgue measure.

2.2 Cut and project schemes and model sets. In this section we discuss how Meyer sets arise from CPS. The material of this section is well-known [Mey72, Moo97, Moo00, Sch00]. For the convenience of the reader we include some details and provide precise references.

We adopt the notation introduced in the introduction above and consider a CPS $(G, H, \mathcal{L})$ with $G = \mathbb{R}^N$ and H a locally compact abelian group. We will assume that H is σ-compact and metrizable.[1] As both $\mathbb{R}^N$ and H are σ-compact, the lattice $\mathcal{L}$ must be countable (as it has a compact quotient). The Haar measure of a measurable subset $W \subseteq H$ will be denoted by $|W|$.

Here are the basic properties of sets arising from the CPS.

Lemma 2.2 ([Moo97, Proof of Proposition 2.6 (i)]) *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS and $W \subseteq H$. Then, the following holds:*

- *$\curlywedge(W)$ is uniformly discrete if $\mathrm{cl}(W)$ is compact.*
- *$\curlywedge(W)$ is relatively dense if $\mathrm{int}(W) \neq \emptyset$.*

In particular, $\curlywedge(W)$ is a Delone set with (FLC) (and even Meyer) if W is relatively compact with non-emtpy interior.

Proof. The statement of Proposition 2.6 (i) in [Moo97] deals simultaneously with both uniform discreteness and relatively denseness. However, the proof clearly gives both parts of the present lemma separately. Here, we only discuss how the last statement follows from the first two statements: We have

$$\curlywedge(W) - \curlywedge(W) = \{x - y \mid x^*, y^* \in W\} \subseteq \{z \in L \mid z^* \in W - W\} = \curlywedge(\mathrm{cl}(W) - \mathrm{cl}(W)).$$

Since $\mathrm{cl}(W) - \mathrm{cl}(W)$ is compact, $\curlywedge(\mathrm{cl}(W) - \mathrm{cl}(W))$ is uniformly discrete. Thus, also $\curlywedge(W) - \curlywedge(W)$ is uniformly discrete. □

If $L^* \cap \partial W = \emptyset$, the model set is called *generic*. Here is the fundamental result on proper windows and generic model sets. The result is a consequence of the Baire category theorem.

Lemma 2.3 ([Sch00, Proof of Corollary 4.4]) *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS and $W \subseteq H$. If ∂W has empty interior, then there exists an $h \in H$ such that $W + h$ is generic. In particular, whenever W is proper there exists $h \in H$ such that $W + h$ is generic.*

Proof. As $\mathcal{L}$ is countable and ∂W has empty interior,

$$L^* - \partial W = \bigcup_{l \in L} (l^* - \partial W)$$

can not agree with H by Baire's category theorem. Now, any $h \in H \setminus (L^* - \partial W)$ will have the desired property.

The last statement follows as for any proper window W, clearly, its boundary $\partial W = W \setminus \mathrm{int}(W)$ has empty interior. □

[1] Metrizability of H is only a matter of convenience. It allows us to work with sequences instead of nets. It is clearly met in our specific examples, where we have $G = H = \mathbb{R}$.

A model set $\curlywedge(W)$ is called *regular* if $|\partial W| = 0$.

Lemma 2.4 ([Sch00, Theorem 4.5], [Moo00, Theorem8]) *(a) Let $\curlywedge(W)$ be a regular model set associated to the CPS $(\mathbb{R}^N, H, \mathcal{L})$. Then it has (UPF).*

(b) Let $\curlywedge(W)$ be a generic model set associated to the CPS $(\mathbb{R}^N, H, \mathcal{L})$. Then it is repetitive.

2.3 Delone Dynamical Systems. In this section we show how a uniformly discrete set gives rise to a dynamical system. The dynamical systems arising in this way from Meyer sets are the main object of study in our paper.

Let $\mathcal{F}$ denote the space of all closed subsets of $\mathbb{R}^N$ including the empty set. Let furthermore $\mathcal{U}_r(\mathbb{R}^N)$ be the space of all uniformly discrete sets in $\mathbb{R}^N$ which satisfy (1) with a fixed constant $r > 0$, and $\mathcal{D}_{r,R}$ be the set of all Delone sets with satisfying (1) and (2) with fixed constants $r, R > 0$. We can introduce a metric d on $\mathcal{F}$ as follows: Let

$$j\colon \mathbb{S}^N \longrightarrow \mathbb{R}^N \cup \{\infty\}$$

be the stereographic projection. Here, $\mathbb{S}^N$ denotes the N-dimensional sphere in $\mathbb{R}^{N+1}$ and the point ∞ denotes the additional point in the one-point compactification of $\mathbb{R}^N$, which is the image of the 'north pole' under j. Let d_{H} be the Hausdorff metric on the set of compact subsets of $\mathbb{S}^N$. Then, for any closed $\Lambda \subseteq \mathbb{R}^N$, the set $j^{-1}(\Lambda \cup \{\infty\})$ is a closed and hence compact subset of $\mathbb{S}^N$. Thus, via

$$d(\Lambda_1, \Lambda_2) \;:=\; d_{\mathrm{H}}\big(j^{-1}(\Lambda_1 \cup \{\infty\}), j^{-1}(\Lambda_2 \cup \{\infty\})\big),$$

we obtain a topology on the set of all closed subsets of $\mathbb{R}^N$.

Lemma 2.5 ([LS03]) *The map $d : \mathcal{F}\times\mathcal{F} \to \mathbb{R}^+$ defines a metric on $\mathcal{F}$, which makes $(\mathcal{F}, d)$ into a compact metric space. Further, the sets $\mathcal{U}_r$ and $\mathcal{D}_{r,R}$ are compact in this metric for all $r, R > 0$.*

Proof. Compactness of $(\mathcal{F}, d)$ is discussed in [LS03]. As $\mathcal{U}_r$ and $\mathcal{D}_{r,R}$ are clearly closed, they are also compact. □

Remark 2.6 In the investigation of Delone sets (rather than uniformly discrete sets) another metric may be even more common, see e.g. [LMS02]. However, both metrics induce the same topology, [BL04, LS03].

Let $\Lambda \subseteq \mathbb{R}^N$ be a uniformly discrete set. Then

$$\Omega(\Lambda) \;=\; \mathrm{cl}\left(\left\{\Lambda - s \mid s \in \mathbb{R}^N\right\}\right)$$

is called the *dynamical hull* of Λ. Here, the closure is taken with respect to the topology induced by the metric discussed in Lemma 2.5. Note that this closure may contain the empty set even if Λ was not the empty set. Given the canonical flow $\varphi_s(\Gamma) := \Gamma - s$ on $\Omega(\Lambda)$, we call the pair $(\Omega(\Lambda), \varphi)$ *point set dynamical system* and also write $(\Omega(\Lambda), \mathbb{R}^N)$. Dynamical systems of this form are sometimes called *mathematical quasicrystals.*

Lemma 2.7 ([Sch00, Corollary 3.3 and Proposition 3.1]) *] Let Λ be a Delone set with FLC. Then*

(a) $(\Omega(\Lambda), \varphi)$ is uniquely ergodic if and only if Λ has (UPF);

(b) $(\Omega(\Lambda), \varphi)$ is minimal if and only if Λ is repetitive.

Remark 2.8 Let us note that the equivalence between minimality and a (suitably defined) notion of repetitivity is true in much greater generality as has been known since [Aus88].

Note that (FLC) is always fulfilled for model sets (see Lemma 2.2 above).

The statement of the following proposition is known and discussed within proofs in [Sch00, BLM07].

Proposition 2.9 *Let* $(\mathbb{R}^N, H, \mathcal{L})$ *be a CPS and, as usual,* $L = \pi_G(\mathcal{L})$. *Let* Λ *a Delone set in* $\mathbb{R}^N$ *with* $\Lambda \subseteq L$. *Then, for* $\Gamma \in \Omega(\Lambda)$ *the following assertions are equivalent:*

(i) $\Gamma \subseteq L$.
(ii) Γ *contains one point of* L.

In this case, there exists a sequence (t_n) *in* L *with* $\Lambda + t_n \to \Gamma$.

Proof. (i)$\Longrightarrow$ (ii): This is clear.

(ii)$\Longrightarrow$ (i): Let $x \in \Gamma \cap L$ be given. Consider a sequence (t_n) in $\mathbb{R}^N$ with $\Gamma_n := \Lambda + t_n \to \Gamma$. Without loss of generality we can then assume $x \in \Gamma_n$ for all $n \in \mathbb{N}$. We then have $x \in L$ as well as $x \in L + t_n$ and this implies $t_n \in L$ for all $n \in \mathbb{N}$. This gives, in particular, $\Gamma_n \subseteq L$ for all $n \in \mathbb{N}$. Consider now an arbitrary point $y \in \Gamma$. As $\Gamma_n \to \Gamma$ and $x \in \Gamma_n, \Gamma$, we infer by finite local complexity that $y \in \Gamma_n$ for all sufficiently large n. This then implies $y \in L$.

The last statement has been proven along the proof of (ii)$\Longrightarrow$ (i). $\square$

2.4 Flow morphism and torus parametrisation. The dynamical hull of a Delone set arising from a CPS can be described via the so-called *torus parametrization*. This is discussed in this section.

Consider the CPS $(\mathbb{R}^N, H, \mathcal{L})$ and define the associated *torus* by

$$\mathbb{T} := (\mathbb{R}^N \times H)/\mathcal{L}.$$

Then $\mathbb{T}$ inherits a natural group structure from $\mathbb{R}^N \times H$. We will write $[s,h]_{\mathcal{L}}$ for the element $(s,h) + \mathcal{L} \in \mathbb{T}$. Further, there is a natural $\mathbb{R}^N$-action on $\mathbb{T}$ given by

$$\omega_s(\xi) := \xi + [s,0]_{\mathcal{L}}.$$

For $s \in \mathbb{R}^N$ and $l \in L$, we then find

$$\omega_{s-l}(\xi) = \xi + [s-l,0]_{\mathcal{L}} = \xi + [s,l^*]_{\mathcal{L}}.$$

By the denseness of L^* in H, this shows that the action is minimal, i.e. each orbit is dense. As $\mathbb{T}$ is a group, this gives that the action is uniquely ergodic, i.e. there is only one invariant probability measure (see [Sch00]).

A *flow morphism* or *factor map* between $\mathbb{R}^N$-actions (X,ϕ) and (Y,ψ) is a continuous onto map $\eta : X \to Y$ which satisfies $\eta(\phi_s(x)) = \omega_s(\eta(x))$ for all $x \in X$ and $s \in \mathbb{R}^N$. If such a flow morphism exists, the dynamical system (Y,ψ) is called a *factor* of (X,ϕ).

Proposition 2.10 ([BLM07]) *Let a CPS* $(\mathbb{R}^N H, \mathcal{L})$, *a proper window* $W \subseteq H$ *and* $\Lambda \subseteq \mathbb{R}^N$ *with* $⅄(\mathrm{int}(W)) \subseteq \Lambda \subseteq ⅄(W)$ *be given. Then there exists a unique flow morphism* $\beta : \Omega(\Lambda) \to \mathbb{T}$ *with* $\beta(\Lambda) = 0$. *This flow morphism satisfies*

$$(3) \qquad \beta(\Gamma) = [s,h]_{\mathcal{L}} \iff ⅄(\mathrm{int}(W)+h) - s \subseteq \Gamma \subseteq ⅄(W+h) - s$$

for $\Gamma \in \Omega(\Lambda)$.

The map β from the previous proposition is often called a *torus parametrization* (associated to the CPS $(\mathbb{R}^N, H, \mathcal{L})$ and the window W) and this is how we will refer to it in the remainder of the paper. Note that it satisfies $\beta(\Lambda) = 0$.

Remark 2.11 (Torus parametrization and continuous eigenfunctions) Existence of a torus parametrization has consequences for existence of continuous eigenfunctions. Indeed, in the situation of the preceding proposition we can define for any γ in the dual group of $\mathbb{T}$, i.e. any continuous group homomorphism $\gamma : \mathbb{T} \longrightarrow \{z \in \mathbb{C} : |z| = 1\} =: S^1$, the function $f := f_\gamma := \gamma \circ \beta$ on $\Omega(\Lambda)$. This function satisfies

$$f(\phi_s(\Gamma)) = \gamma((\beta(\Gamma)) + [s,0]_{\mathcal{L}}) = \gamma([s,0]_{\mathcal{L}})f(\Gamma) = \gamma^*(s)f(\Gamma)$$

for all $s \in \mathbb{R}^N$ and $\Gamma \in \Omega(\Lambda)$, where we have defined $\gamma^* : \mathbb{R}^N \longrightarrow S^1$, by $\gamma^*(s) = \gamma([s,0])$. Then, γ^* is an element of the dual group of $\mathbb{R}^N$ and, hence, f is a continuous eigenfunction. Moreover, the general theory of CPS, as discussed in [Mey72, Moo97], shows that the set $\{\gamma^* : \gamma \in \text{dual group of } \mathbb{T}\}$ is relatively dense in $\mathbb{R}^N$. So, we have a relatively dense set of eigenvalues with continuous eigenfunctions.

The structure of fibres of β will be crucial for our further investigation. The following lemma underlines the spirit of the constructions of the next section. Similar arguments can be found e.g. in [BLM07].

Lemma 2.12 *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS and $W \subseteq H$ be a proper window, $\Lambda = \curlywedge(W)$ and β the associated torus parametrization. For given $[0,h]_{\mathcal{L}} \in \mathbb{T}$, the following conditions are equivalent:*

(i) $\Gamma \in \beta^{-1}([0,h]_{\mathcal{L}})$;
(ii) There exists a sequence $h_j \in L^*$ such that $\lim_{j\to\infty} h_j = h$ and

$$\lim_{j\to\infty} \curlywedge(W + h_j) = \Gamma.$$

Proof. (i)⇒(ii): By $\beta(\Gamma) = [0,h]_{\mathcal{L}}$ and Proposition 2.10 we have

$$\Gamma \subseteq \curlywedge(W - h) \subseteq L.$$

Now, from Proposition 2.9 we obtain a sequence $s_j \in L$ with $\Lambda - s_j \to \Gamma$. Due to the continuity of the flow morphism β, we then obtain

$$[0,h]_{\mathcal{L}} = \beta(\Gamma) = \lim_{j\to\infty} \beta(\varphi_{s_j}(\curlywedge(W))) = \lim_{j\to\infty} [0, s_j^*]_{\mathcal{L}}.$$

This easily implies convergence of $h_j := s_j^*$ to $h \in H$ for $j \to \infty$.

(ii)⇒(i): This follows immediately from the continuity of β. □

From the considerations in [BLM07] we obtain the following lemma. As we will need to build on this argument in later sections we include a short proof.

Lemma 2.13 ([BLM07]) *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS and $W \subseteq H$ a proper window and $\Lambda \subseteq \mathbb{R}^N$ with $\curlywedge(\text{int}(W)) \subseteq \Lambda \subseteq \curlywedge(W)$ be given. Then the following dichotomy holds for the torus parametrization:*

(a) If $\emptyset = (\partial W + h) \cap L^*$ then $[0,h]_{\mathcal{L}}$ has exactly one preimage under β.
(b) If there exist an $l \in L$ with $l^* \in \partial W + h$, then $\beta^{-1}([0,h]_{\mathcal{L}})$ contains at least two elements Γ and Γ' which satisfy $l \in \Gamma$ and $l \notin \Gamma'$.

In particular, $[0,h]_{\mathcal{L}}$ has exactly one preimage under β if and only if $W + h$ is generic i.e. $(\partial W + h) \cap L^* = \emptyset$ holds. Moreover, there exists an $h \in H$ such that the fibre $\beta^{-1}([0,h]_{\mathcal{L}})$ has only one element.

Proof. Clearly, $(W+h)\cap L^* = (\mathrm{int}(W)+h)\cap L^*$ if and only if $(\partial W+h)\cap L^* = \emptyset$.

Consider first the case $(\mathrm{int}(W)+h)\cap L^* = (W+h)\cap L^*$. Then, $[0,h]_{\mathcal{L}}$ has exactly one preimage under β by (3).

Consider now the case $l^* \in \partial W + h$ for some $l^* \in L^*$. Since L^* is dense in H and W is proper, we can find elements $s_n, s_n' \in L$, $n \in \mathbb{N}$, such that $h_n = s_n^*$ and $h_n' = (s_n')^*$ satisfy

- $\lim_{n\to\infty} h_n = \lim_{n\to\infty} h_n' = h$,
- $l^* \in \mathrm{int}(W) + h_n$ and $l^* \notin W + h_n'$ for all $n \in \mathbb{N}$.

By going over to subsequences if necessary, we may assume that $\{\varphi_{-s_n}(\Lambda)\}_{n\in\mathbb{N}}$ and $\{\varphi_{-s_n'}(\Lambda)\}_{n\in\mathbb{N}}$ converge to some elements Γ and Γ' of the hull $\Omega(\curlywedge(W))$, respectively. Since

$$\varphi_{-s_n}(\Lambda) = \Lambda + s_n \supset \curlywedge(\mathrm{int}(W)) + s_n = \curlywedge(\mathrm{int}(W) + h_n) \ni l,$$

we obtain $l \in \Gamma$. In a similar way, we can show that at the same time $l \notin \Gamma'$. Hence, we obtain that $\Gamma \neq \Gamma'$. As β is a flow morphism, we have

$$\begin{aligned}\beta(\Gamma) = \beta\left(\lim_{n\to\infty}\varphi_{-s_n}(\Lambda)\right) &= \lim_{n\to\infty}\beta(\varphi_{-s_n}(\Lambda)) = \lim_{n\to\infty}\omega_{-s_n}(\beta(\Lambda))\\ &= \lim_{n\to\infty}\omega_{-s_n}(0) = \lim_{n\to\infty}[-s_n,0]_{\mathcal{L}} = \lim_{n\to\infty}[0,h_n]_{\mathcal{L}} = [0,h]_{\mathcal{L}}.\end{aligned}$$

The same holds for Γ', and hence $\Gamma, \Gamma' \in \beta^{-1}([0,h]_{\mathcal{L}})$.

The 'In particular' statement is immediate from the preceding two statements. The last statement is then clear by Lemma 2.3. □

From Lemma 2.13 and the fact that L^* is countable, we also immediately obtain that regularity of the window W has strong implications for the fibre structure. To state this more precisely we need the following piece of notation: Two measure-preserving $\mathbb{R}^N$-actions (X,ϕ,μ) and (Y,ψ,ν) are *measure-theoretically isomorphic* if there exist full measure sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a measurable bijection $\eta : X_0 \to Y_0$ such that $\eta \circ \phi_s(x) = \psi_s \circ \eta(x)$ for all $x \in X_0$ and $s \in \mathbb{R}^N$.

Corollary 2.14 ([BLM07]) *Consider the situation of the previous lemma and assume $|\partial W| = 0$. Then for λ-almost all $\xi \in \mathbb{T}$ the preimage $\beta^{-1}(\xi)$ is a singleton. In particular, the flow $(\Omega(\Lambda),\varphi)$ is uniquely ergodic and measure-theoretically isomorphic to $(\mathbb{T},\omega)$.*

Remark 2.15 Whenever we consider Delone dynamical systems which arise from proper model sets, the preceding results on the torus parametrization form a basis for our treatment. However, we will also consider the dynamical hull of weak model sets, which are not proper. In this case, we cannot appeal to the previous results. In fact, if W is compact with $\mathrm{int}(W) = \emptyset$ the hull of $\curlywedge(W)$ must contain the empty set and there can not exist a torus parametrization as the empty set is fixed by the action, whereas no point of the torus is fixed by the action.

In the sequel, we will often deal with proper windows W and $\Lambda = \curlywedge(W)$. We will then also need to replace the window W by any of its translates $W+\vartheta$, $\vartheta \in H$. In this case, Proposition 2.10 (applied to $W+\vartheta$ instead of W) yields a unique flow morphism

$$\beta_\vartheta : (\Omega(\curlywedge(W+\vartheta)),\varphi) \longrightarrow (\mathbb{T},\omega), \tag{4}$$

which sends $\curlywedge(W+\vartheta)$ to 0. For $\vartheta = 0 \in H$ we will still write β instead of β_0.

2.5 Uniform distribution and asymptotic densities. Densities of subsets of Euclidean space will play an important role in our considerations. Here, we discuss the necessary tools.

In the following, we use the partial ordering on $\mathbb{R}^N$ which is given by $s \leq t \Leftrightarrow s_i \leq t_i$ for all $i = 1, \ldots, N$. Given $t \in \mathbb{R}$, we let $\bar{t} = (t, \ldots, t) \in \mathbb{R}^N$. Thus

$$F_t := \{s \in \mathbb{R}^N \mid -\bar{t} \leq s \leq \bar{t}\}$$

is a cube of sidelength $2t$ and volume $(2t)^N$.

Whenever S is a uniformly discrete subset of $\mathbb{R}^N$ we define its *asymptotic density* by

$$\nu_S := \limsup_{t\to\infty} \frac{\sharp S \cap F_t}{\lambda(F_t)},$$

where $\sharp A$ denotes the cardinality of A. If the limsup is actually a limit, we call it the *density* of the set S. Model sets provide an instance where densities tend to exist rather generally. This is sometimes discussed under the header 'uniform distribution'. In order to state the corresponding result we will need one more piece of notation: Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS. Then a subset $\mathcal{D}$ of $\mathbb{R}^N \times H$ is called a *fundamental domain* of $(\mathbb{R}^N \times H)/\mathcal{L}$ if it contains exactly one representative of any element in the quotient group. As is well-known, the volume of a (measurable) fundamental domain does not depend on the choice of the actual fundamental domain. We denote this volume by $\mathrm{Vol}(\mathcal{L})$. Note that in the Euclidean case $\mathrm{Vol}(\mathcal{L}) = \det(A)$. We can now recall a result from [Moo02], which in our setting gives the following.

Theorem 2.16 (Uniform distribution for model sets [Moo02]**)** *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS and $W \subseteq H$ measurable. Then, the following holds:*

(a) For almost every $\vartheta \in H$ (with respect to Haar measure on H) the density of $⅄(W + \vartheta)$ exists and is given by $\frac{|W|}{Vol(\mathcal{L})}$.

(b) If W is compact, the inequality

$$\limsup_{t\to\infty} \frac{\sharp ⅄(W + \vartheta) \cap F_t}{\lambda(F_t)} \leq \frac{|W|}{Vol(\mathcal{L})}$$

holds for all $\vartheta \in H$.

(c) If W is open, the inequality

$$\liminf_{t\to\infty} \frac{\sharp ⅄(W + \vartheta) \cap F_t}{\lambda(F_t)} \geq \frac{|W|}{Vol(\mathcal{L})}$$

holds for all $\vartheta \in H$.

Remark 2.17 For recent results of the type presented in the theorem see also [HR14].

Proof. Part (a) of the Theorem is shown in [Moo02]. Inspecting the proof there one can easily infer part (b) and (c) as well. For the convenience of the reader we sketch a proof. This proof can be seen as a variant of the considerations in [Moo02]: Consider $\mathbb{T} = (\mathbb{R}^N \times H)/\mathcal{L}$ and let $\sigma : \mathbb{R}^N \to [0, \infty)$ be a continuous function with compact support and $\int_{\mathbb{R}^N} \sigma ds = 1$. Define the function

$$f : \mathbb{T} \longrightarrow [0, \infty) \quad , \ f(\xi) \ := \sum_{(s,h)\in -\xi} \sigma(s) 1_W(h),$$

where 1_W denotes the characteristic function of W. Then, f is a measurable bounded function. (Note that the sum has only finitely many non-vanishing terms as both σ and 1_W vanish outside compact sets.)

Define for $\xi = [s, h]_{\mathcal{L}}$ the set $⅄(\xi) := ⅄(W + h) - s$ and note that this is indeed well defined. Then a short computation (compare [Moo02]) shows that

$$\left| \frac{1}{\lambda(F_t)} \int_{F_t} f(\omega_s(\xi)) ds - \frac{\sharp ⅄(\xi) \cap F_t}{\lambda(F_t)} \right| \ \to \ 0 \quad \text{ as } t \to \infty$$

for all $\xi \in \mathbb{T}$. Thus, the desired statements (a),(b), (c) will follow from the corresponding statements for the averages

$$a_t(\xi) := \frac{1}{\lambda(F_t)} \int_{F_t} f(\omega_s(\xi)) ds.$$

These statements in turn hold as $(\mathbb{T}, \mathbb{R}^N)$ is uniquely ergodic:

(a) Birkhoff's ergodic theorem directly implies convergence of the averages $a_t(\xi)$ for almost every $\xi \in \mathbb{T}$. As convergence for ξ clearly implies convergence for all $\omega_s(\xi)$, $s \in \mathbb{R}^N$, the almost sure convergence in $\xi \in \mathbb{T}$ implies almost-sure convergence in $\vartheta \in H$.

(b) If we replace 1_W by a continuous function with compact support, then f is continuous and we even have uniform convergence in $\xi \in \mathbb{T}$ by Oxtoby's theorem. Approximating 1_W from above by continuous functions with compact support we obtain the statement (b) uniformly in $\xi \in \mathbb{T}$ and hence also in $\vartheta \in H$.

(c) This follows by replacing the approximation from above in (b) by approximation from below. More specifically, by regularity of the Haar measure on H we can chose a compact set $K \subset W$ whose measure is as close to the measure of W as we wish. Now, invoking Urysohn's Lemma we can chose a continuous f with compact support and $1_K \leq f \leq 1_W$. □

2.6 Topological Entropy. In this section we introduce the background from entropy theory. Given an $\mathbb{R}^N$-action ϕ on a compact metric space X (whose metric we denote by d), we say $x, x' \in X$ are (ε, t)-*separated* if

$$\max_{s \in F_t} d(\phi_s(x), \phi_s(x')) \geq \varepsilon.$$

A subset $S \subseteq X$ is called (ε, t)-*separated* if its elements are all pairwise (ε, t)-separated. By $N(\varphi, \varepsilon, t)$ we denote the maximal cardinality of an (ε, t)-separated set. The *topological entropy* of ϕ is defined as

$$h_{\mathrm{top}}(\phi) := \lim_{\varepsilon \to 0} h_\varepsilon(\phi) = \sup_{\varepsilon > 0} h_\varepsilon(\phi),$$

where

$$h_\varepsilon(\phi) = \limsup_{t \to \infty} \frac{1}{\lambda(F_t)} \log N(\phi, \varepsilon, t).$$

We will be particularly interested in the topological entropy of a dynamical system $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ arising from a CPS and a proper W. In this case, there is a torus parametrization

$$\beta : \Omega(\curlywedge(W)) \longrightarrow \mathbb{T}, \tag{5}$$

due to Proposition 2.10 (applied with $\Lambda = \curlywedge(W)$). As $(\mathbb{T}, \mathbb{R}^N)$ is an isometric flow it has entropy zero. This has some consequences for the topological entropy of $(\Omega(\curlywedge(W)), \mathbb{R}^N)$. As it is instructive to our considerations below we discuss next some abstract background in the subsequent two remarks.

Remark 2.18 (Positive entropy comes from fibres) If (Y, ψ) is a factor of (X, ϕ) we can relate the entropy of the two systems. Indeed, we have

$$h_{\mathrm{top}}(\psi) \leq h_{\mathrm{top}}(\phi)$$

(e.g. [KH97]). It is then also possible to obtain an upper bound on $h_{\mathrm{top}}(\phi)$ by considering the "the topological entropy realised in single fibres". In order to be more specific, let $\eta : X \longrightarrow Y$ be the factor map and denote for any $\xi \in Y$, the maximal cardinality of an (ε, t)-separated subset of the fibre $\eta^{-1}(\xi)$ by $N^\xi(\phi, \varepsilon, t)$. Now let

$$h^\xi_{\rm top}(\phi) \ := \ \lim_{\varepsilon\to 0} h^\xi_\varepsilon(\phi) \ , \quad \text{ where } h^\xi_\varepsilon(\phi) \ := \ \limsup_{t\to\infty} \frac{1}{\lambda(F_t)} \log N^\xi(\phi, \varepsilon, t).$$

Then, clearly

$$h^\xi_{\rm top}(\phi) \ \leq \ h_{\rm top}(\phi)$$

for any $\xi \in Y$. As shown in [Bow71] we furthermore have the bound

$$h_{\rm top}(\phi) \ \leq \ h_{\rm top}(\psi) + \sup_{\xi\in Y} h^\xi_{\rm top}(\phi).$$

If $h_{\rm top}(\psi) = 0$ the two preceding inequalities give

$$h_{\rm top}(\phi) \ = \ \sup_{\xi\in Y} h^\xi_{\rm top}(\phi).$$

So, in this case the (positive) topological entropy of ϕ must be realised already in single fibres. Now, this is exactly the situation described in (5). In line with the preceding considerations our approach to positive entropy of $(\Omega(\curlywedge(W)), \varphi)$ below will be based on showing positive entropy already in the fibres. We will do so by exhibiting what we call embedded fullshifts (see below for details).

Remark 2.19 (Positive entropy implies thick boundary) We also note that whenever (X, ϕ) is uniquely ergodic and is measure theoretically isomorphic to a factor (Y, ψ) of zero topological entropy then the topological entropy of (X, ϕ) must vanish as well. The reason is that the metric entropy (which we will not define as we do not need it below) is invariant under measure theoretic isomorphisms. Hence, the metric entropy of (X, ϕ) and (Y, ψ) must agree. As (X, ϕ) is uniquely ergodic its topological entropy agrees with its metric entropy due to a variational principle, see e.g. [TZ91]. In our situation described in (5) we obtain then from Corollary 2.14 that the topological entropy of $(\Omega(W), \mathbb{R}^N)$ must vanish whenever the boundary of W has measure zero. Now, this implies that examples of CPS with positive topological entropy will necessarily have thick boundary and indeed this will feature prominently in our constructions below.

3. Embedded fullshifts and Topological Independence

In this section, we define a simple criterion, namely the existence of 'embedded fullshifts', for positive entropy of the dynamical hull of a uniformly discrete set in $\mathbb{R}^N$. For hulls coming from (weak) model sets, we then relate this to the local structure of the window and introduce the concepts of local topological and metric independence. These will be the main tools to prove positivity of entropy in the constructions in the later sections.

Whenever we meet a CPS $(\mathbb{R}^N, H, \mathcal{L})$ in this and the remaining sections the group H will be abelian, metrizable and σ-compact, compare the discussion on Page 6.

3.1 Embedded fullshifts. Embedded fullshifts are our key concept in providing positive topological entropy.

Definition 3.1 (Embedded fullshift) Let Λ be a uniformly discrete subset of $\mathbb{R}^N$. An *embedded fullshift* in $\Omega(\Lambda)$ is a pair (Ξ, S) consisting of a closed subset Ξ of $\Omega(\Lambda)$ and a subset S of $\mathbb{R}^N$ such that the following holds:

- The set S has positive asymptotic density, i.e.

$$\nu_S = \limsup_{t\to\infty} \frac{\sharp S \cap F_t}{\lambda(F_t)} > 0.$$

- The set
$$U := \bigcup_{\Gamma \in \Xi} \Gamma \subset \mathbb{R}^N$$
is uniformly discrete.
- For any subset S' of S there exists a $\Gamma \in \Xi$ with
$$\Gamma \cap S = S'.$$

The elements of S above are called *free points* of the embedded fullshift. The set U is called *grid* of the embedded fullshift. The quantity ν_S is the *asymptotic density* of the embedded fullshift.

If (Ξ, S) is an embedded fullshift in $\Omega(\Lambda)$ with $\Xi \subseteq \Omega'$ for some $\Omega' \subseteq \Omega(\Lambda)$ we say that Ω' *contains an embedded fullshift*.

Remark 3.2 Consider an embedded fullshift with free points S and grid U.

(a) We clearly have $S \subseteq U$ (and therefore S is uniformly discrete). The points of S are free in the sense that we can choose any subset of S and exactly this will be the subset from S appearing in some $\Gamma \in \Xi$. In later arguments we will not only have to control occurrence of points of S but also non-occurrence of points of S. We will need the set U in order to treat this non-occurrence.

(b) We call an embedded fullshift (Ξ, S) with grid U *maximal* if $(\Xi, S \cup \{u\})$ is not an embedded fullshift for any $u \in U \setminus S$. In this case, we may think of the elements of $U \setminus S$ as points *forced by the embedded fullshift.* It is not hard to see (by an induction procedure) that any embedded fullshift can be extended to a maximal one.

(c) Let (Ξ, S) be an embedded fullshift. Then, $(\tilde{\Xi}, S)$ with
$$\tilde{\Xi} := \mathrm{cl}(\{\Gamma \in \Xi : \Gamma \cap S \neq \emptyset\}),$$
where the closure is taken in the hull of Λ, will also be an embedded subhift (with $\tilde{\Xi} \subseteq \Xi$). Indeed, the only possible difference between $\tilde{\Xi}$ and Ξ are those elements of Ξ, which do not contain any element of S.

(d) Consider a CPS $(\mathbb{R}^N, H, \mathcal{L})$ and a proper window W and $\Lambda = \curlywedge(W)$. Then, for $\xi = [s, h]_{\mathcal{L}}$, all elements of $\beta^{-1}(\xi)$ are contained in the uniformly discrete set $\curlywedge(W + h) - s$ by Proposition 2.10. So, for any subset Ξ of $\beta^{-1}(\xi)$ we have uniform discreteness of $\bigcup_{\Gamma \in \Xi} \Gamma$. So, the uniform discreteness of the grid is automatically satisfied for a fullshift embedded in such a fibre. Also, in this situation if (Ξ, S) is an embedded fullshift in the fibre $\beta^{-1}(\xi)$, then $(\beta^{-1}(\xi), S)$ is an embedded fullshift as well. From Proposition 2.10 and Lemma 2.13 we then infer that the grid for this fullhift is given by $\curlywedge(W + h) - s$.

(e) Whenever the pair (Ξ, S) is an embedded fullshift, then so is the translated pair $(\varphi_s(\Xi), \varphi_s(S))$ for any $s \in \mathbb{R}^N$.

(f) We will be mostly interested in embedded fullshifts contained in either $\Omega(\Lambda)$ or in the fibres $\eta^{-1}(\xi) \subseteq \Omega(\Lambda)$ of some flow morphism $\eta : \Omega(\Lambda) \longrightarrow Y$.

The following provides a simple characterization for existence of an embedded fullshift.

Proposition 3.3 *Let Λ be a uniformly discrete subset of $\mathbb{R}^N$. Then, $\Omega(\Lambda)$ contains an embedded fullshift if and only if there exist $S \subseteq \mathbb{R}^N$ and a uniformly discrete $U \subseteq \mathbb{R}^N$ with the following two properties:*

1. *The set S has positive asymptotic density.*
2. *For all finite $F \subseteq S$ and $a \in \{0,1\}^F$, there exists a $\Gamma \in \Omega(\Lambda)$ with $\Gamma \subseteq U$ and such that for $s \in F$*
$$s \in \Gamma \iff a_s = 1.$$

Proof. If $\Omega(\Lambda)$ contains an embedded fullshift there clearly exist $S \subseteq \mathbb{R}^N$ and a uniformly discrete $U \subseteq \mathbb{R}^N$ satisfying (1) and (2). Conversely, if there exist $S \subseteq \mathbb{R}^N$ and a uniformly discrete $U \subseteq \mathbb{R}^N$ satisfying (1) and (2) we may define

$$\Xi' := \{\Gamma \in \Omega(\Lambda) : \Gamma \cap S \neq \emptyset \text{ and } \Gamma \subseteq U\}.$$

Now, let Ξ be the closure of Ξ'. Then, all elements in Ξ are contained in U and a simple compactness argument shows that for any subset S' of S there exists a $\Gamma \in \Xi$ with $\Gamma \cap S = S'$. Hence, (Ξ, S) is an embedded fullshift contained in $\Omega(\Lambda)$. □

Remark 3.4 Let Λ be a (weak) model set coming from a CPS $(\mathbb{R}^N, H, \mathcal{L})$ such that $\Omega(\Lambda)$ satisfies the conditions (1) and (2) of the preceding proposition. Let (Ξ, S) be the embedded fullshift constructed in the proof of the preceding proposition i.e. $\Xi := \mathrm{cl}(\Xi')$ with $\Xi' := \{\Gamma \in \Omega(\Lambda) : \Gamma \cap S \neq \emptyset \text{ and } \Gamma \subseteq U\}$, and let $U' = \bigcup_{\Gamma \in \Xi'} \Gamma$. Then, the inclusions

$$S \subseteq U' \subseteq t + L$$

turn out to be valid for any $t \in S$. Indeed, shifting S and U by $-t$ for $t \in S$, we may assume without loss of generality $t = 0$ and $0 \in S$. By $0 \in L$, we infer from Proposition 2.9 that any $\Gamma \in \Xi$ containing 0 must be contained in L. Hence, we have that

$$\widehat{U} \;=\; \bigcup_{\Gamma \in \Xi : 0 \in \Gamma} \Gamma \;\subseteq\; L$$

and since $\widehat{U} \subseteq U$ is a discrete set, a simple compactness argument shows $\widehat{U} = U'$. Hence $U' \subseteq L$, and since further any $s \in S$ is clearly contained in U', this shows the claimed statement.

The relevance of embedded fullshifts comes from the following lemma.

Lemma 3.5 (Embedded fullshift implies positive entropy) *Let Λ be a uniformly discrete subset of $\mathbb{R}^N$. If $\Omega(\Lambda)$ contains an embedded fullshift of asymptotic density ν_S, then*

$$h_{\mathrm{top}}(\varphi) \geq \nu_S \cdot \log 2.$$

Proof. Let S be the set of free points and U the grid of the embedded fullshift. Let $r > 0$ such that different points of U have distance at least r. Consider $\Gamma, \Gamma' \in \Omega(\Lambda)$ with $\Gamma, \Gamma' \subseteq U$ and $s \in \Gamma$ and $s \notin \Gamma'$ for some $s \in S$. By uniform discreteness of U the set Γ' then does not contain a point in the ball around s with radius r. This gives

$$d(\varphi_s(\Gamma), \varphi_s(\Gamma')) \geq r.$$

Hence, any pair $\Gamma, \Gamma' \in \Omega(\curlywedge(W))$ which satisfies the above for some $s \in S \cap F_t$ is (r,t)-separated. Consider now an arbitrary $\nu' < \nu_S$. Then, there exist arbitrarily large t with

$$\sharp S \cap F_t \geq \nu' \cdot \lambda(F_t).$$

By the assumption on existence of an embedded fullshift we can chose for any finite subset F of $S \cap F_t$ an element $\Gamma_F \in \Omega(\Lambda)$ with $\Gamma_F \cap S = F$. Then, the elements Γ_F are (t,r) separated by the considerations at the beginning of the proof. Hence, we have

$$N(\varphi, r, t) \geq 2^{\nu' \cdot \lambda(F_t)}.$$

This implies

$$h_r(\varphi) \geq \nu' \cdot \log 2.$$

As $\nu' < \nu_S$ was arbitrary we infer $h_r(\varphi) \geq \nu_S \log 2$. Now, the desired statement follows from $h_{\mathrm{top}}(\varphi) \geq h_r(\varphi)$. □

3.2 Independence of sets and existence of embedded fullshifts. In this section we provide a condition for existence of an embedded subshift.

Consider a CPS $(\mathbb{R}^N, H, \mathcal{L})$ and denote the neutral element of H by 0. We sometimes write $0 \in H$ in order to distinguish it from the origin in $\mathbb{R}^N$.

Consider a (weak) model set arising from the given CPS via a relatively compact $W \subseteq H$. Then the problem of finding an embedded fullshift with set of free points $S \subseteq L$ in the associated dynamical system is actually related to analyzing the local structure of the window W in some neighborhood of the points s^* for $s \in S$. In order to get a first idea on this issue the following observation may be helpful:

Let $F \subseteq L$ be a finite set, $a \in \{0,1\}^F$ arbitrary and $\vartheta \in H$ be given. Now, assume that

$$\emptyset \neq \left(\bigcap_{s \in F : a_s = 1} (W - s^*) \setminus \bigcup_{s \in F : a_s = 0} (W - s^*) \right) \cap (L^* - \vartheta).$$

Then, there exists an $l \in L$ satisfying for any $s \in F$ that $(l^* - \vartheta) \in W - s^* \Longleftrightarrow a_s = 1$. This gives for $s \in F$

$$s \in \curlywedge(W + \vartheta) - l \Longleftrightarrow a_s = 1.$$

Thus, the set $\Gamma := \curlywedge(W + \vartheta) - l$ respects the choice of $F \subseteq L$ given by a. Our dealings below will build on this observation. However, two additional points will come up:

- We have to simultaneously deal with all finite subsets F of a subset S of L. In order to still provide the uniform discrete subset U necessary for an embedded fullshift, we will need to require that the set $S^* = \{s^* \mid s \in S\}$ is relatively compact (see Lemmas 3.7 and 3.14).
- We will allow for one overall shift by $h \in H$.

Motivated by the preceding considerations we give the following definition. The finite index set F appearing in the definition will later be a subset of L or L^*.

Definition 3.6 (Independence with respect to D) Let $\mathrm{D} \subseteq H$ be given. A finite family $(A_s)_{s \in F}$ of subsets of H is *independent with respect to D* if for all $a \in \{0,1\}^F$ we have

$$\emptyset \neq \left(\bigcap_{s \in F : a_s = 1} A_i \setminus \bigcup_{s \in F : a_s = 0} A_i \right) \cap \mathrm{D}.$$

An infinite family of sets is called *independent with respect to D* if the condition above holds for each finite subfamily. We say the window W is *independent in $P \subseteq L^*$ with respect to D*, if the family $W - p, p \in P$, is independent with respect to D.

The following lemma relates these concepts to the existence of embedded fullshifts. The lemma is our main tool to construct embedded fullshifts (and hence, by Lemma 3.5 examples with positive topological entropy). In fact, we will apply the lemma in two situations, namely for proper W and W with empty interior but of positive measure. These situations will then be studied in the two subsequent sections. The lemma is formulated in a general version that includes two parameters ϑ and h. However, for a first reading it might be helpful to set them both to zero.

Lemma 3.7 (Basic criterion for embedded fullshifts) *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS, $W \subseteq H$ relatively compact and $\vartheta, h \in H$. If $\curlywedge(W + h)$ possesses a subset S of positive asymptotic density such that $S^* = \{s^* : s \in S\}$ is relatively compact* *and $W+h$ is independent in S^* with respect to $L^*+(h-\vartheta)$, then $\Omega(⅄(W+\vartheta))$ contains an embedded fullshift.*

Remark 3.8 Note that we do not require that W has non-empty interior. So, the sets $⅄(W+v)$ may not be relatively dense. However, they are uniformly discrete and this is all we need to consider the hull.

Proof. We will show that the conditions (1) and (2) of Proposition 3.3 for existence of an embedded fullshift are met for S as in the statement of the lemma and

$$U := ⅄(W+h-S^*).$$

Note that U is indeed uniformly discrete as $W+h-S^*$ is relatively compact, compare Lemma 2.2.

Condition (1) is met by assumption. To show condition (2) fix a finite subfamily $F \subseteq S$ and $a \in \{0,1\}^F$. Then, by independence of $W+h$ in S^* with respect to $L^*+(h-\vartheta)$, we have

$$\emptyset \neq \left(\bigcap_{s\in F: a_s=1} (W+h-s^*) \setminus \bigcup_{s\in F: a_s=0} (W+h-s^*) \right) \cap (L^*+h-\vartheta) \ .$$

Thus there exists an

$$\bar{m}^* \in (L^*+h-\vartheta) \tag{6}$$

such that

$$\bar{m}^* \in W+h-s^* \ \Leftrightarrow\ a_s = 1 \tag{7}$$

for all $s \in F$. Further, by the symmetry $L^* = -L^*$ we have

$$\bar{m}^* = h-\vartheta-m^* \tag{8}$$

for some $m \in L$. Combining this with (7) we obtain

$$s \in ⅄(W+\vartheta)+m \quad \text{if and only if} \quad a_s = 1 \ . \tag{9}$$

Moreover, we have

$$\Gamma := ⅄(W+\vartheta)+m = ⅄(W+\vartheta+m^*) \overset{(8)}{=} ⅄(W+h-\bar{m}^*) \subseteq U,$$

where we used that $\bar{m}^*$ belongs to $W+h-S^*$ by (7) to obtain the last inclusion. Thus Γ belongs to $\Omega(⅄(W+\vartheta))$ with $\Gamma \subseteq U$, and due to (9) we have

$$s \in \Gamma \quad \text{if and only if} \quad a_s = 1 \ .$$

This finishes the proof. □

A slightly more specific notion of independence is given in the following definition. It will be needed in particular to obtain further information about embedded fullshifts in the case of proper model sets.

Definition 3.9 (Local independence in $0 \in H$) Let $\mathrm{D} \subset H$ be given. An infinite family $(A_s)_{s\in P}$ of subsets of H is said to be *locally independent in* $0 \in H$ with respect to D if

$$0 \ \in\ \mathrm{cl}\left(\left(\bigcap_{s\in F: a_s=1} A_s \setminus \bigcup_{s\in F: a_s=0} A_s \right) \cap D \right)$$

for any finite subset $F \subseteq P$ and any $a \in \{0,1\}^F$. The window W is said to be *locally independent in $P \subseteq L$ with respect to D* if the the family $W-p$, $p \in P$, is locally independent in $0 \in H$ with respect to D. Hence, the window W is locally independent in P with respect to D if and only if

$$0 \in \operatorname{cl}\left(\left(\bigcap_{s\in F:a_s=1}(W-s^*)\setminus\bigcup_{s\in F:a_s=0}(W-s^*)\right)\cap \mathrm{D}\right) \tag{10}$$

for any finite family $F \subseteq P$ and any $a \in \{0,1\}^F$.

Corollary 3.10 *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS, $W \subseteq H$ relatively compact, proper and $\vartheta, h \in H$. Assume that $⅄(W+h)$ possesses a subset S of positive asymptotic density such that $W+h$ is locally independent in $S^* = \{s^* : s \in S\}$ with respect to $L^* + (h-\vartheta)$. Let β_ϑ denote the flow morphism described in* (4)*. Then there is an embedded fullshift contained in $\beta_\vartheta^{-1}([0, h-\vartheta]_{\mathcal{L}})$.*

Proof. This follows by extending the proof of the previous lemma. Here are the details: Fix a finite subfamily $F \subseteq S$ of and $a \in \{0,1\}^S$. Then, due to (10) we can choose $\bar{m}^*$ such that it satisfies (6) and (7) and additionally require that $\bar{m}^*$ is arbitrarily close to 0. This means that we can find a sequence of $\bar{m}_j^*$ such that (6) and (7) hold for all $j \in \mathbb{N}$ and at the same time

$$\lim_{j\to\infty} \bar{m}_j^* \;=\; 0\ . \tag{11}$$

Further, we can fix a relatively compact neighbourhood V of 0 and assume without loss of generality that $\bar{m}_j^* \in V$ for all $j \in \mathbb{N}$.

If now $m_j \in L$ are chosen such that $\bar{m}_j^* = h - \vartheta - m_j*$, analogous to (8), then we obtain

$$s \in \Gamma_j := ⅄(W+\vartheta) + m_pj \quad \text{if and only if} \quad a_j = 1\ . \tag{12}$$

Moreover, we have

$$\Gamma_j \;=\; ⅄(W+\vartheta+m_j^*) \;=\; ⅄(W+h-\bar{m}_j^*) \;\subseteq\; ⅄(W+h-V) \;=:\; U_1\ ,$$

where U_1 is discrete since $W+h-V$ is relatively compact (compare Lemma 2.2).

Now, (Γ_j) is a sequence in the compact space $\Omega(⅄(W+\vartheta))$. Hence, it possesses an accumulation point $\Gamma \in \Omega(⅄(W+\vartheta))$. As Γ_j is a subset of U_1 and U_1 is uniformly discrete, convergence of the $\Gamma_j \to \Gamma$ and (12) yield

$$s \in \Gamma \quad \text{if and only if} \quad a_s = 1.$$

As W is proper, β_ϑ is continuous. This gives

$$\begin{aligned}
\beta_\vartheta(\Gamma) &= \lim_{j\to\infty} \beta_\vartheta(⅄(W+\vartheta)+m_j)\\
&= \lim_{j\to\infty} \beta_\vartheta(⅄(W+\vartheta+m_j^*)\\
&= \lim_{j\to\infty} [0, m_j^*]_{\mathcal{L}}\\
\text{(by (8))} \quad &= \lim_{j\to\infty} [0, h-\vartheta-\bar{m}_j^*]_{\mathcal{L}}\\
\text{(by (11))} \quad &= [0, h-\vartheta]_{\mathcal{L}}.
\end{aligned}$$

This shows that the Γ constructed above are all contained in the fibre $\beta_\vartheta^{-1}([0, h-\vartheta]_{\mathcal{L}})$. Thus, we obtain an embedded fullshift in that fibre. □

3.3 Local topological independence and proper W. In this section we consider the case that W is proper. We provide a sufficient condition for applicability of Corollary 3.10. This condition is given in Lemma 3.12. Our application to the random model in Theorem 1.1 will be based on that lemma.

We say a finite family of sets A_s, $s \in F$, of subsets of H is *locally topologically independent in* $0 \in H$ if for all $a \in \{0,1\}^F$ we have

$$0 \in \mathrm{cl}\left(\mathrm{int}\left(\bigcap_{s\in F: a_s=1} A_s \setminus \bigcup_{s\in F: a_s=0} A_s\right)\right).$$

An infinite family of sets is called *locally topologically independent in* 0 if the condition above holds for each finite subfamily. A window W is *locally topologically independent in* $P \subseteq L^*$, if the family $W - p, p \in P$, is locally topologically independent in 0.

Lemma 3.11 *Any family of subsets of H, which is locally topologically independent in $0 \in H$, is locally independent in 0 with respect to any dense $D \subseteq H$.*

Proof. Consider an arbitrary finite subfamily A_s, $s \in F$, of the original family and let $a \in \{0,1\}^F$ be given. Define

$$A(a) = \bigcap_{s\in F: a_s=1} A_s \setminus \bigcup_{s: a_s=0} A_s.$$

By assumption we have $0 \in \mathrm{cl}(\mathrm{int}(A(a)))$. Since D is dense in H, the intersection $\mathrm{int}(A(a)) \cap \mathrm{D}$ is dense in $\mathrm{int}(A(a))$. Thus, we can choose a sequence $(h_j)_{j\in\mathbb{N}}$ in $\mathrm{int}(A(a)) \cap \mathrm{D}$ such that $\lim_{j\to\infty} h_j = 0$. □

Lemma 3.12 (Topological criterion for embedded fullshifts) *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS, $W \subseteq H$ a proper window and $h \in H$. Assume that there exists a subset S of $\curlywedge(W + h)$ of positive asymptotic density such that $W + h$ is locally topologically independent in $S^* = \{s^* : s \in S\}$. Then, the fibre $\beta_\vartheta^{-1}([0, h - \vartheta]_{\mathcal{L}})$ contains an embedded fullshift for every $\vartheta \in \mathbb{R}$.*

Proof. As $W + h$ is locally topologically independent in S^* and $L^* + (h - \vartheta)$ is dense in $\mathbb{R}^N$ for all $\vartheta \in H$, the preceding lemma gives that $W + h$ is locally independent in S^* with respect to $L^* + (h - \vartheta)$ for all $\vartheta \in \mathbb{R}$. As W is proper, we can now apply Corollary 3.10 to obtain that the fibre $\beta_\vartheta^{-1}([0, h - \vartheta]_{\mathcal{L}})$ contains an embedded fullshift. □

3.4 Metric independence and general W. The aim of this section is to adapt the above concepts for the case of weak model sets, that is, to compact windows with empty interior. In this case, we need to replace open sets by sets of positive measure and invoke uniform distribution in order to prove analogous statements. As a result we will obtain a criterion for embedded fullshifts for general relatively compact W with positive measure. This criterion is given in Lemma 3.14.

Definition 3.13 A finite family $(A_s)_{s\in F}$ of subsets of H is *metrically independent* if for all $a \in \{0,1\}^F$ we have

$$0 < \left|\left(\bigcap_{s\in F: a_s=1} A_s \setminus \bigcup_{s\in F: a_s=0} A_s\right)\right|.$$

An infinite family of subsets of H is called *metrically independent* if the condition above holds for each finite subfamily. Further, we say the window W is *metrically independent in* $P \subseteq L^*$, if the family $W - p, p \in P$, is metrically independent.

Lemma 3.14 (Metric criterion for embedded fullshifts) *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS, $W \subseteq H$ a relatively compact window and $h \in H$. Assume that there exists a subset S of $\curlywedge(W + h)$ of positive asymptotic density such that $S^* = \{s^* : s \in S\}$ is* *relatively compact and $W+h$ is metrically independent in S^*. Then $\Omega(\curlywedge(W+\vartheta))$ contains an embedded fullshift for almost every $\vartheta \in H$.*

Proof. Let F be a finite subset of S and let $a \in \{0,1\}^F$ be given. Consider the family $W+h-s^*$, $s \in F$, and define

$$\mathcal{W}(a) \;=\; \bigcap_{s\in F:a_s=1} (W+h-s^*) \setminus \bigcup_{s\in F:a_s=0} (W+h-s^*).$$

Since $W+h$ is metrically independent in S^*, we have

$$0 < |\mathcal{W}(a)|.$$

By uniform distribution, Theorem 2.16, we thus obtain that the density of

$$\curlywedge(\mathcal{W}(a)-h+\vartheta)$$

is positive for almost every $\vartheta \in H$. By excluding a set of measure zero, we therefore obtain a set $\Theta(a) \subseteq H$ of full measure such that for every $\vartheta \in \Theta(a)$ the set $L^*+h-\vartheta$ intersects $V \cap \mathcal{W}(a)$. Intersecting over the countable family of all finite $F \subseteq S$ and $a \in \{0,1\}^F$ we obtain a set $\Theta \subseteq H$ of full measure such that for each $\vartheta \in \Theta$ the set $L^*+h-\vartheta$ intersects $V \cap \mathcal{W}(a)$ for arbitrary $F \subseteq S$ and $a \in \{0,1\}^F$. Hence, $W+h$ is locally independent around V in S^* with respect to $L^*+h-\vartheta$ for each $\vartheta \in \Theta$. Given this, Lemma 3.7 implies the assertion. □

4. Embedded fullshifts and unique ergodicity

In this section we study how existence of a embedded fullshifts of sufficiently high density prevents unique ergodicity.

Recall that we have defined the asymptotic density of a subset Γ of $\mathbb{R}^N$ by

$$\nu_\Gamma := \limsup_{t\to\infty} \frac{\sharp\Gamma \cap F_t}{\lambda(F_t)}.$$

Let now $(\mathbb{R}^N, H, \mathcal{L})$ be a cut and project scheme and $W \subseteq H$ be relatively compact and $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ the associated dynamical system. If this system is uniquely ergodic, then, by Lemma 2.7(a), the density $\lim_{t\to\infty} \frac{\sharp\Gamma\cap F_t}{\lambda(F_t)}$ exists for every $\Gamma \in \Omega(\curlywedge(W))$ and is independent of Γ (as this density is just the patch frequency of the patch $(\{0\}, r/2)$, where r is the minimal distance between points in Γ). Based on this observation we can now show that $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ can not be uniquely ergodic if it contains an embedded fullshift with set of free points S and grid U such that ν_S is large compared to ν_U.

Proposition 4.1 *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS and W a relatively compact window and suppose that $\Omega(\curlywedge(W))$ contains an embedded fullshift with set of free points S and grid U and $\nu_S > \nu_U/2$. Then $(\Omega(\curlywedge(W)), \varphi)$ is not uniquely ergodic. This applies in particular if W is proper and $\Omega(\curlywedge(W))$ contains a fullshift embedded in a fibre with asymptotic density $\nu_S > |W|/2\,Vol(\mathcal{L})$.*

Proof. Let (Ξ, S) be the embedded fullshift in question. Let $\Gamma_1, \Gamma_0 \in \Xi$ be given with $\Gamma_1 \cap S = S$ and $\Gamma_0 \cap S = \emptyset$. Then

$$\limsup_{t\to\infty} \frac{\sharp\Gamma_1 \cap F_t}{\lambda(F_t)} \geq \limsup_{t\to\infty} \frac{\sharp S \cap F_t}{\lambda(F_t)} = \nu_S > \frac{\nu_U}{2}$$

but at the same time

$$\liminf_{t\to\infty} \frac{\sharp\Gamma_0 \cap F_t}{\lambda(F_t)} \leq \liminf_{t\to\infty} \frac{\sharp(U \setminus S) \cap F_t}{\lambda(F_t)} \leq \nu_U - \nu_S < \nu_U/2.$$

This contradicts the existence of uniform patch frequencies discussed above and thus excludes unique ergodicity.

To show the statement for the case of a fullshift in a fibre, note that for an embedded fullshift in a fibre the grid U is contained in $\curlywedge(W+\vartheta)-s$ for some $\vartheta \in H$ and $s \in \mathbb{R}^N$ (compare Remark 3.2 (d)). By uniform distribution, Theorem 2.16 (b) , we then have $\nu_U \leq |W+\vartheta|/\mathrm{Vol}(\mathcal{L}) = |W|/\mathrm{Vol}(\mathcal{L})$. Now, the statement follows from the considerations in the first part of the proof. □

5. Random windows and positive entropy

In this section we will provide a proof of the main theorem, Theorem 1.1, presented in the introduction. In fact, we will provide a strengthening of that result. Up to here our discussion involved fairly general CPS. In this section, we will restrict attention to Euclidean CPS. More specifically, we will consider the following situation (S):

- $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ is a Euclidean CPS with $\mathcal{L} = A(\mathbb{Z}^{N+1})$, where $A = \mathrm{GL}(N+1, \mathbb{R})$ is such that $\pi_1 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is injective on $\mathcal{L}$ and $\pi_2 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ maps $\mathcal{L}$ to a dense set $L^* = \pi_2(\mathcal{L}) \subseteq \mathbb{R}$.
- $C \subseteq \mathbb{R}$ is a Cantor set of positive measure in $[0,1]$. Let $(G_n)_{n\in\mathbb{N}}$ a numbering of the bounded connected components in $\mathbb{R} \setminus C$.

We then define for $\omega \in \Sigma^+ = \{0,1\}^{\mathbb{N}}$ the set

$$W(\omega) \;=\; C \cup \bigcup_{n:\omega_n=1} G_n. \tag{13}$$

Let $\mathbb{P}$ be the Bernoulli distribution on Σ^+ with probability $p \in (0,1)$ (i.e. $\mathbb{P}$ is the product measure $\prod_{n\in\mathbb{N}} \mu$, where μ is the measure on $\{0,1\}$ which assigns the value p to $\{0\}$ and $1-p$ to $\{1\}$).

Lemma 5.1 *For $\mathbb{P}$-almost every $\omega \in \Sigma^+$, the window $W(\omega)$ is proper.*

Proof. First, the complement of $W(\omega)$ in $\mathbb{R}$ consists of a union of connected components of $\mathbb{R} \setminus C$ (those G_n with $\omega_n = 0$ and the two unbounded components in the complement of C). Since these are all open, $W(\omega)$ is compact. Further, as $\bigcup_{n:\omega_n=1} G_n$ belongs to the interior of $W(\omega)$, we have

$$\partial W(\omega) \subseteq C.$$

Next, we are going to show the reverse inclusion (almost surely). Suppose $x \in C$. Since C is perfect, there exists a sequence of gaps $\{G_{n_k}\}_{k\in\mathbb{N}}$ such that $\inf G_{n_k} \to x$ for $k \to \infty$. By definition of the window, only intervals G_{n_k} with $a_{n_k}(\omega) = 1$ are included in $W(\omega)$. Since all random variables are independent, the Borel-Cantelli-Lemma implies

$$\mathbb{P}(\{\text{ for infinitely many } k,\ G_{n_k} \text{ is included in } W(\omega)\}) \;=\; 1\ ,$$
$$\mathbb{P}(\{\text{ for infinitely many } k,\ G_{n_k} \text{ is not included in } W(\omega)\}) \;=\; 1\ .$$

Thus, for $\mathbb{P}$-almost every ω there exist subsequences $G_{n_{k_j}} \subseteq W(\omega)$ and $G_{n_{k'_j}} \subseteq H \setminus W(\omega)$ such that $\lim_{j\to\infty} \inf G_{n_{k_j}} = \lim_{j\to\infty} \inf G_{n_{k'_j}} = x$. Hence, we have $x \in \partial W(\omega)$ $\mathbb{P}$-almost surely for every fixed $x \in C$. Now, let $M \subseteq C$ be a countable and dense subset of C. Then for any $x \in M$ the argument above shows $x \in \partial W(\omega)$ $\mathbb{P}$-almost surely. Hence, the countable set M is contained in $\partial W(\omega)$ $\mathbb{P}$-almost surely. Consequently, we also have

$$\mathrm{cl}(M) = C \subseteq \partial(W(\omega))$$

$\mathbb{P}$-almost surely. Together with the converse inclusion shown above, this yields

$$\partial(W(\omega)) = C$$

$\mathbb{P}$-almost surely. From this we obtain

$$\mathrm{int}(W(\omega)) = W(\omega) \setminus \partial(W(\omega)) = \bigcup_{n:\omega_n=1} G_n$$

$\mathbb{P}$-almost surely. Using this equality and going again through the argument giving $C \subseteq \partial W(\omega)$, we then find $\mathbb{P}$-almost surely

$$C \subseteq \partial(\mathrm{int}(W(\omega)).$$

From this we then obtain

$$\mathrm{cl}(\mathrm{int}(W(\omega))) = \mathrm{int}(W(\omega)) \cup \partial(\mathrm{int}(W(\omega))) \supset \mathrm{int}(W(\omega)) \cup C = W(\omega)$$

and hence

$$\mathrm{cl}(\mathrm{int}(W(\omega))) = W(\omega)$$

for $\mathbb{P}$-almost all $\omega \in X$. □

In the next step, we need to find a suitable $h \in \mathbb{R}$ and a respective subset S of $\curlywedge(C+h)$ of positive asymptotic density. In order to avoid some technicalities later, it turns out convenient to work with $\tilde{C} = C \setminus \left(\bigcup_{n\in\mathbb{N}} \partial G_n \cup \{\inf C, \sup C\}\right)$. Note that $|\tilde{C}| = |C|$, since the difference $C \setminus \tilde{C}$ is just the countable set of endpoints of the intervals G_n together with the two extremal points of C.

Lemma 5.2 *For Lebesgue-almost all $h \in \mathbb{R}$, the sequence $\curlywedge(\tilde{C}+h)$ has asymptotic density given by $|C|/|\det A|$.*

Proof. This is a direct consequence of uniform distribution, Theorem 2.16. Note that in the case at hand the measure of a fundamental domain is just given by $|\det A|$. □

It remains to prove that the random window $W(\omega)$ is $\mathbb{P}$-almost surely locally topologically independent (as defined in Section 3.3) in the sequence $L^* \cap (\tilde{C} + h)$.

Lemma 5.3 *Let C be a Cantor set with positive measure and let $W(\omega)$ be defined as in (13). Choose $h \in \mathbb{R}$. Then for $\mathbb{P}$-almost every ω the window $W(\omega) + h$ is locally topologically independent in $L^* \cap (\tilde{C} + h)$.*

Proof. Let F be an arbitrary finite subset of $L^* \cap (\tilde{C} + h)$. Let

$$\delta_1 \ = \ \frac{1}{2} \cdot \min_{x \neq y \in F} |x - y| \ .$$

Since any Cantor set is nowhere dense and perfect, there exist gaps $I_1^x \subseteq (0, \delta_1)$ of $C + h - x$, $x \in F$, such that

$$\bigcap_{x\in F} I_1^x \ \neq \ \emptyset \ .$$

By the choice of δ_1, we have $I_1^x + x \neq I_1^y + y$ if $x \neq y \in F$. Further, if we let $\delta_2 = \min\{1, \min_{x\in F} (\inf I_1^x)\}$, then by the same argument there exist gaps $I_2^x \subseteq (0, \delta_2)$ of $C + h - x$ such that

$$\bigcap_{x\in F} I_2^x \ \neq \ \emptyset$$

and $I_2^x + x \neq I_2^y + y$ for $x \neq y \in F$. Proceeding inductively with this construction, in the $(n + 1)$-st step we define

$$\delta_{n+1} = \min\left\{\frac{1}{n}, \min_{x\in F} (\inf I_n^x)\right\}$$

and choose gaps $I_{n+1}^x \subseteq (0, \delta_{n+1})$ of $C + h - x$ such that

$$\bigcap_{i \in \mathcal{C}} I_{n+1}^x \neq \emptyset$$

and $I_{n+1}^x + x \neq I_{n+1}^y + y$ whenever $x \neq y \in F$. Now, let $(G_n)_{n\in\mathbb{N}}$ be a labeling of all gaps of $C + h$. Then by construction, we have $I_j^x = G_{n_j^x} - x$ for some $n_j^x \in \mathbb{N}$. Moreover, the choice of the δ_n and $I_n^x \subseteq (0, \delta_n)$ ensures that $n_j^x \neq n_{j'}^{x'}$ if $(x, j) \neq (x', j')$. In particular, this means that $(\omega_{n_j^x})_{j\in\mathbb{N}}^{x\in F}$ is a two-parameter family of identically distributed independent random variables. Therefore, we obtain that for any $a \in \{0, 1\}^F$ the set

$$\Omega(a) \;=\; \{\omega \in \Sigma^+ \mid \exists \text{ infinitely many } j \in \mathbb{N} :\; \omega_{n_j^x} = 1 \text{ iff } a_x = 1\}$$

has full measure $\mathbb{P}(\Omega(a)) = 1$. However, for all $\omega \in \Omega(a)$, we have that

$$I_j \;=\; \bigcap_{x\in F} I_j^x \;\subseteq\; \left(\bigcap_{x\in F : a_x = 1} W(\omega) + h - x\right) \setminus \left(\bigcup_{x \in F : a_x = 0} W(\omega) + h - x\right) .$$

Since the intervals I_j are all open and $\lim_{j\to\infty} \inf I_j = 0$, this shows the local topological independence of $W(\omega)$ in F. As this works for any finite subfamily F of $L^* \cap (\tilde{C} + h)$ and there exist only countably many such subfamilies, we obtain local topological independence of $W(\omega) + h$ in $L^* \cap (\tilde{C} + h)$ for $\mathbb{P}$-almost every $\omega \in \Sigma^+$. □

We can now summarize the preceding considerations in the following theorem.

Theorem 5.4 *Assume the situation (S) described at the beginning of this section. Then, there exists a subset Σ_0^+ of Σ^+ of full $\mathbb{P}$-measure such that the following holds:*

(a) For all $\omega \in \Sigma_0^+$ and $\vartheta \in \mathbb{R}$ there exists a set $\Xi(\omega) \subseteq \mathbb{T}$ of full measure such that the Delone dynamical system $(\Omega(\curlywedge(W(\omega) + \vartheta)), \mathbb{R})$ contains an embedded fullshift in $\beta_\vartheta^{-1}(\xi)$ for every $\xi \in \Xi(\omega)$.

(b) For all $\omega \in \Sigma_0^+$ and $\vartheta \in \mathbb{R}$ the Delone dynamical system $(\Omega(\curlywedge(W(\omega) + \vartheta)), \mathbb{R})$ has positive topological entropy $h_{\mathrm{top}}(\varphi) = \frac{|C| \log 2}{|\det A|}$.

(c) For every $\omega \in \Sigma_0^+$ there exists a residual set Θ in $\mathbb{R}$ such that the Delone dynamical system $(\Omega(\curlywedge(W(\omega) + \vartheta)), \mathbb{R})$ is minimal for every $\vartheta \in \Theta$.

(d) For every $\omega \in \Sigma_0^+$ and every $\vartheta \in \mathbb{R}$ the Delone dynamical system $(\Omega(\curlywedge(W(\omega)+ \vartheta)), \mathbb{R})$ is not uniquely ergodic provided C additionally satisfies $|C| > 1/2$.

Proof. By Lemma 5.1 there exists a set Σ_1^+ of full measure in Σ^+ such that $W(\omega)$ is proper for every $\omega \in \Sigma_1^+$. By Lemma 5.3 and Fubini's theorem, there exists a set Σ_2^+ of full measure in Σ^+ such that for every $\omega \in \Sigma_2^+$ the window $W(\omega) + h$ is locally topologically independent in $L^* \cap (\tilde{C} + h)$ for almost every $h \in \mathbb{R}$. Set $\Sigma_0^+ := \Sigma_1^+ \cap \Sigma_2^+$. Now, consider an arbitrary $\omega \in \Sigma_0^+$.

As due to Lemma 5.2 the set $\curlywedge(\tilde{C} + h)$ has asymptotic density $|C|/|\det A|$ for almost every $h \in \mathbb{R}$, we obtain that for almost every $h \in \mathbb{R}$ the assumptions of Lemma 3.12 are satisfied for $W = W(\omega) + h$ and $S = \curlywedge(\tilde{C} + h)$. Therefore, we then obtain a full measure set $\Xi'(\omega) \subseteq \mathbb{R}$ such that for any $h \in \Xi'(\omega)$ and any $\vartheta \in \mathbb{R}$ there exists an embedded fullshift in the fibre $\beta_\vartheta^{-1}(\xi)$ for $\xi = [0, h - \vartheta]_{\mathcal{L}}$. Since the existence of an embedded subshift in a fibre is a property that is invariant under translation by t, we then obtain an embedded fullshift for all $[t, h - \vartheta]_{\mathcal{L}}$ with $(t, h) \in \mathbb{R}^N \times \Xi'(\omega)$. The projection of the latter set to $\mathbb{T}$ gives the required full measure set $\Xi(\omega)$ that satisfies the assertion (a).

As for (b) we note that the proven part (a) together with Lemma 3.5 directly gives $h_{\mathrm{top}} \geq \frac{|C|\log 2}{|\det A|}$. On the other hand by the general results of [HR14] we know that $h_{\mathrm{top}} \leq \frac{|C|\log 2}{\mathrm{Vol}(\mathcal{L})}$. Combining these inequalities and using $\mathrm{Vol}(\mathcal{L}) = |\det A|$, we arrive at the statement (b).

Statement (c) then follows from general well-known theory. In fact, it is a direct consequence of Lemma 2.3 combined with (b) of Lemma 2.4 and (b) of Lemma 2.7.

Finally, it remains to show (d). The preceding considerations give almost surely an embedded fullshift with set of free points S satisfying $\nu_S = \frac{|C|}{\mathrm{Vol}(\mathcal{L})}$. Clearly, the grid U must be contained in $\curlywedge([0,1]+h)-t$ for some $h \in \mathbb{R}$ and $t \in \mathbb{R}$ and hence satisfies

$$\nu_U \leq \nu_{\curlywedge([0,1]+h)-t} \leq \frac{1}{\mathrm{Vol}(\mathcal{L})} ,$$

where the last inequality follows by uniform distribution (Theorem 2.16). This shows

$$\nu_S \ > \ \frac{\nu_U}{2} ,$$

and Proposition 4.1 gives the desired statement. □

Remark 5.5 Whenever $W(\omega)$ is proper, the dynamical system $(\Omega(\curlywedge(W(\omega)+\vartheta)),\mathbb{R})$ has the torus $\mathbb{T}$ as its maximal equicontinuous factor and a relatively dense set of continuous eigenvalues for any $\vartheta \in \mathbb{R}$, compare Remark 2.11.

6. A deterministic construction

In order to prepare for the construction of weak model sets with positive entropy in the next section, we first provide a deterministic construction of proper model sets with positive entropy. The starting point of our construction will be the construction of an initial Cantor set C_0 that is adapted to the respective CPS. To that end, we need to introduce some further notation.

We assume without loss of generality that the matrix $A \in \mathrm{GL}(N+1,\mathbb{R})$ that defines the lattice $\mathcal{L} = A(\mathbb{Z}^{N+1})$ is of the form $A = (a_{i,j})_{i,j=1}^{N+1}$, where $a_{N+1,j} \in (0,1)$ for all $j = 1,\ldots,N$ and $a_{N+1,N+1} = 1$. In this case, given any $v = (v_1,\ldots,v_N) \in \mathbb{Z}^N$, there exists a unique $v_{N+1} \in \mathbb{Z}$ such that

$$l_v^* \ := \ \pi_2\left(A \cdot \begin{pmatrix} v \\ v_{N+1} \end{pmatrix}\right) \ = \ \sum_{j=1}^{N+1} a_{N+1,j} v_j \ \in \ [0,1) \cap L^* .$$

Note that thus $l_v^* = \sum_{j=1}^N a_{N+1,j} v_j \bmod 1$. Given $v \in \mathbb{Z}^N$, let $\|v\|_\infty = \max_{j=1}^N |v_j|$ and fix a numbering $(v(n))_{n\in\mathbb{N}}$ of $\mathbb{Z}^N$ such that $\|v(n)\|_\infty$ is non-decreasing in n. Note that this implies that if we let $R_t = \{1,\ldots,(2t+1)^N\}$ for $t \in \mathbb{N}$, then $v(R_t) = \{v(n) \mid n \in R_t\} = F_t$ for all $t \in \mathbb{N}$.

Lemma 6.1 *There exists an increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ such that the open intervals $I_k = (l_{v(n_k)}^*, l_{v(n_k)}^* + \varepsilon_k)$ satisfy*

(i) $I_j \cap I_k = \emptyset$ *for all* $j \neq k$,
(ii) $\mathrm{cl}\left(\bigcup_{k\in\mathbb{N}} I_k\right) = [0,1]$,
(iii) $\lim_{k\to\infty} \frac{k}{n_k} > 1/2$.

Proof. For simplicity, we work in the additive group $\mathbb{R}/\mathbb{Z}$ and omit to write mod 1. In other words, by slighly abusing notation we automatically interpret real numbers as elements of the circle. In particular, we denote by $d(x,0)$ the distance of $x \in \mathbb{R}$

to the nearest integer. We choose a strictly increasing sequence of integers $(\kappa(t))_{t\in\mathbb{N}}$ that satisfies

$$\sum_{t\in\mathbb{N}} \frac{\sharp F_{2t}}{\sharp F_{\kappa(t)}} \;\leq\; \frac{1}{2\cdot 5^N} \tag{14}$$

and let

$$\eta_v \;=\; \min\left\{ d(l_u^*,0)/2 \mid u \in F_{\kappa(\|v\|_\infty)} \cap \mathbb{Z}^N \right\}$$

and $J_n := \left[l^*_{v(n)}, l^*_{v(n)} + \eta_{v(n)} \right) \cap [0,1]$. Then, we define

$$B := \{ n \in \mathbb{N} \mid J_n \cap J_j \neq \emptyset \text{ for some } j < n\} \ .$$

We now want to estimate the cardinality of $B \cap R_t$. To that end, note that if $J_n \cap J_j \neq \emptyset$ and $J_{n'} \cap J_j \neq \emptyset$ for some $n, n' > j$, then

$$d(l^*_{v(n)}, l^*_{v(n')}) \;=\; d(l^*_{v(n)-v(n')}, 0) \;<\; 2\eta_{v(j)}$$

and therefore $v(n) - v(n') \notin F_{\kappa(\|v(j)\|_\infty)}$. Similarly, $v(n) - v(j) \notin F_{\kappa(\|v(j)\|_\infty)}$, and the same for $v(n') - v(j)$. Covering $F_t \setminus \left(F_{\kappa(\|v(j)\|_\infty)} + v(j)\right)$ by at most $\sharp F_{2t}/\sharp F_{\kappa(\|v(j)\|_\infty)}$ translates of $F_{\kappa(\|v(j)\|_\infty)}$ for each j leads to the following rough estimate.

$$\begin{aligned} \sharp(B\cap R_t) \;&\leq\; \sum_{j=1}^{\sharp F_t} \sharp\left\{ n \in \{j+1,\ldots,\sharp F_t\} \mid J_n \cap J_j \neq \emptyset \right\} \\ &\leq\; \sum_{j=1}^{\sharp F_t} \frac{\sharp F_{2t}}{\sharp F_{\kappa(\|v(j)\|_\infty)}} \;\leq\; \sharp F_{2t} \sum_{k=1}^{t} \frac{\sharp F_k}{\sharp F_{\kappa(k)}} \;\overset{(14)}{\leq}\; \frac{\sharp F_{2t}}{2\cdot 5^N} \;\leq\; \frac{\sharp F_t}{2} \ . \end{aligned} \tag{15}$$

Now let $n_1 = 0$ and define

$$n_{k+1} = \min\{ n > n_k \mid J_n \cap J_{n_j} = \emptyset \text{ for all } j \leq k\}.$$

Then by defining $\varepsilon_k = \min\{\eta_{v(n_k)}, 1 - l^*_{v(n_k)}\}$ and thus $I_k = J_{n_k}$, property (i) follows by construction. Likewise, it is clear that the union of the I_k is dense in $[0,1]$. Otherwise, there would be some interval $(a,b) \subseteq [0,1]$ which does not intersect any of the I_k. However, in this case any interval J_n that is contained in (a,b) would have to appear as some I_k by the above construction, leading to a contradiction. Note here that the image of $\mathbb{Z}^N$ under the mapping $v \mapsto l_v^*$ is dense in $[0,1]$, so that eventually one of the intervals J_n needs to be contained in (a,b).

Further, we have that

$$\widehat{B} \;:=\; \mathbb{N} \setminus \{ n_k \mid k \in \mathbb{N}\} \;\subseteq\; B.$$

Hence, if we let $\mathcal{N} = \{n_k \mid k \in \mathbb{N}\}$, then this implies that

$$\sharp\left(\mathcal{N} \cap R_t\right) \;\geq\; \sharp R_t - \sharp(R_t \cap B) \;\geq\; \sharp R_t/2 \ .$$

If we use in addition that $\lim_{t\to\infty} \sharp(R_t \setminus R_{t-1})/\sharp R_t = \lim_{t\to\infty} \sharp(F_t \setminus F_{t-1})/\sharp F_t = 0$, this yields that

$$\lim_{m\to\infty} \sharp(\mathcal{N} \cap \{1,\ldots,m\})/m \;\geq\; 1/2$$

which in turn implies property (iii). □

Note that

$$C_0 \;=\; [0,1] \setminus \bigcup_{k\in\mathbb{N}} I_k$$

is a Cantor set, since all intervals I_k are pairwise disjoint and their union is dense in the circle. It should also be pointed out that property (iii) of the preceding lemma implies that C_0 has positive measure, but we will not make explicit use of this fact.

Lemma 6.2 *Let C be a Cantor set in $[0,1]$ such that $\{0,1\} \subseteq C$. Then there exists a sequence of open sets $A_j \subseteq [0,1]$ such that*

(i) for all $j \in \mathbb{N}$ the set A_j is a union of gaps of C,
(ii) $\partial A_j = C$ for all $j \in \mathbb{N}$,
(iii) the family $(A_j)_{j\in\mathbb{N}}$ is locally topologically independent in 0.

Proof. For any two Cantor sets $C, C' \subseteq [0,1]$ with $\{0,1\} \subseteq C \cap C'$ exists an orientation-preserving homeomorphism of $[0,1]$ which maps C to C'. So without loss of generality, we may assume C that is the middle third Cantor set. Then we can write

$$C = \left\{ \sum_{n=1}^{\infty} 2a_n 3^{-n} \,\middle|\, a \in \{0,1\}^{\mathbb{N}} \right\}.$$

Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ and denote by $|a|$ the length of $a \in \mathcal{A}$. Then

$$G_a = \left(\sum_{n=1}^{|a|} 2a_n 3^{-n} + 3^{-n}, \sum_{n=1}^{|a|} 2a_n 3^{-n} + 2 \cdot 3^{-|a|} \right) \tag{16}$$

are exactly the gaps of C. We will construct the sets A_j such that they all contain

$$A = \bigcup_{a\in\mathcal{A}:|a|\in 4\mathbb{N}} G_a$$

but no G_a with $|a| \in 4\mathbb{N}+1$. Since all points of C are approximated by gaps of both types, we always have $\partial A_j = C$. Thus, properties (i) and (ii) hold.

Let $a^{(n)} = 0^{2n+1}1 \in \{0,1\}^{2n+2}$. Choose a countable partition $(S_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ into infinite sets. Further, let $((M_j, N_j))_{j\in\mathbb{N}}$ be a numbering of all pairs of disjoint finite sets of integers. Then let

$$V_j = \bigcup_{n\in\mathbb{N}:j\in M_n} S_n$$
$$A_j = A \cup \bigcup_{l\in V_j} G_{a^{(l)}} .$$

For any $n \in \mathbb{N}$ the set S_n is a subset of all V_j with $j \in M_n$ and disjoint from all V_j with $j \in N_n$. Thus, the set

$$\bigcap_{j\in M_n} A_j \setminus \bigcup_{j\in N_n} A_j$$

contains $\bigcup_{l\in S_n} G_{a^{(l)}}$. Since S_n is infinite, this shows the local topological independence required in condition (iii). □

Now let $C_0 = [0,1] \setminus \bigcup_{k\in\mathbb{N}} I_k$ as above and define a window W by

$$W = C_0 \cup \bigcup_{k\in\mathbb{N}} (I_k \cap \mathrm{cl}(A_k + \inf(I_k))) . \tag{17}$$

Note that $\inf(I_k) = l^*_{n_k}$ by construction. Due to

$$W = \mathrm{cl}\left(\bigcup_{k\in\mathbb{N}} I_k \cap W \right) = \mathrm{cl}\left(\bigcup_{k\in\mathbb{N}} I_k \cap \mathrm{cl}(A_k + \inf(I_k)) \right) = \mathrm{cl}(\mathrm{int}(W))$$

the window is proper.

Theorem 6.3 *Let $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ be a CPS and W as in (17). Suppose $\beta_\vartheta : \Omega(\curlywedge(W + \vartheta)) \to \mathbb{T}$ is the corresponding flow morphism from* (3) *and* (4)*. Further, choose $S := (l_{v(n_k)})$, where $(n_k)_{k\in\mathbb{N}}$ is chosen as in Lemma 6.1 and $l_{v(n_k)}$ is defined by $(l_{v(n_k)}, l^*_{v(n_k)}) \in \mathcal{L}$. Then the following holds:*

(a) For all $\vartheta \in \mathbb{R}$, the pair (Ξ, S) with $\Xi = \beta_\vartheta^{-1}([0,-\vartheta]_{\mathcal{L}})$ is an embedded fullshift, with the set U from Definition 3.1 given by $U = \Lambda(W)$. In particular, $(\Omega(\curlywedge(W+\vartheta)), \mathbb{R}^N)$ has positive topological entropy for all $\vartheta \in \mathbb{R}$.
(b) The system $(\Omega(\curlywedge(W+\vartheta)), \mathbb{R}^N)$ is not uniquely ergodic.

Proof. By construction, the local topological independence of W in S^* is equivalent to the local topological independence of the sets $(A_k)_{k\in\mathbb{N}}$ and thus follows from Lemma 6.2(iii). Hence, by Lemma 3.12, $\beta^{-1}([0,-\vartheta]_{\mathcal{L}})$ contains an embedded fullshift. This proves (a).

To prove statement (b), observe that with the notation introduced before Lemma 6.1 we have that $U' := \{l_v \mid v \in \mathbb{Z}^N\} = \Lambda([0,1])$. Further $\nu_{U'} = 1/\det(A)$ by Theorem 2.16 (where part (b) is applied to the window $(0,1)$ to obtain a lower estimate). As $U \subseteq U'$, we have that $\nu_U \leq 1/\det(A)$. At the same time, it follows directly from Lemma 6.1(iii) that

$$\nu_S \;\geq\; \nu_{U'}/2 = \frac{1}{2\det(A)} \;\geq\; \nu_U/2\ .$$

Hence, $(\Omega(\curlywedge(W+\vartheta)), \mathbb{R}^N)$ cannot be uniquely ergodic by Proposition 4.1. □

7. Weak model sets with positive entropy

In this section, we will modify the construction of the previous section 6 such that the resulting window W has an empty interior, but the dynamical system $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ still has positive topological entropy. Note that in this case we are not dealing with Delone sets.

Lemma 7.1 *Let $C \subseteq [0,1]$ be the middle third Cantor set. Then there exists a sequence of sets $A_j \subseteq [0,1]$ such that*
(i) $C \subseteq \partial A_j$ for all $j \in \mathbb{N}$,
(ii) $\mathrm{int}(A_j) = \emptyset$ for all $j \in \mathbb{N}$,
(iii) the family $(A_j)_{j\in\mathbb{N}}$ is locally metrically independent in 0.

Proof. We can write

$$C = \left\{ \sum_{n=1}^{\infty} 2a_n 3^{-n} \,\middle|\, a \in \{0,1\}^{\mathbb{N}} \right\}.$$

As before, let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ and denote by $|a|$ the length of $a \in \mathcal{A}$ and by G_a the gap of C corresponding to a. Let K be another Cantor set in $[0,1]$ such that $\{0,1\} \subseteq K$, $|K| > 0$ and $0 \in \mathrm{cl}_{\mathrm{ess}}(K) = \{x \in \mathbb{R} \mid |B_\varepsilon(0) \cap K| > 0 \text{ for all } \varepsilon > 0\}$. We will construct the sets A_j such that each set contains C and, to ensure metric independence, we insert K into the gaps of C. Thus, let again $a^{(n)} = 0^{2n+1}1 \in \{0,1\}^{2n+2}$ and choose a countable partition $(S_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ into infinite sets. Further, let $(M_j, N_j)_{j\in\mathbb{N}}$ be a numbering of all pairs of disjoint finite sets of integers. Then let

$$V_j := \bigcup_{n\in\mathbb{N}: j\in M_n} S_n$$

and

$$A_j := C \cup \bigcup_{l\in V_j} \left(G_{a^{(l)}} \cap (M + \inf(G_{a^{(l)}}))\right).$$

Then conditions (i) and (ii) follow again by construction. Further, for any $n, j \in \mathbb{N}$ the set S_n is a subset of V_j whenever $j \in M_n$ and disjoint from V_j whenever $j \in N_n$.

Since S_n is infinite, for any $\varepsilon > 0$ there exists $l \in S_n$ such that $G_{a^{(l)}} \subseteq B_\varepsilon(0)$. Since $0 \in \mathrm{cl}_{\mathrm{ess}}(K)$, the set $G_{a^{(l)}} \cap (K + \inf G_{a^{(l)}})$ has positive measure. Thus, as

$$G_{a^{(l)}} \cap (K + \inf(G_{a^{(l)}})) \subseteq B_\varepsilon(0) \cap \left(\bigcap_{j \in M_n} A_j \setminus \bigcup_{j \in N_n} A_j \right),$$

the set on the right has positive measure. Since this holds for all $\varepsilon > 0$ and the pair (M_n, N_n) was arbitrary, this shows the metric independence of family $(A_j)_{j \in \mathbb{N}}$. □

Now, let $(n_k)_{k \in \mathbb{N}}$ and the intervals I_k be as in Lemma 6.1. As in the previous section, let $C_0 = [0,1] \setminus \bigcup_{k \in \mathbb{N}} I_k$. Define a window W of empty interior by

$$W \ = \ C_0 \cup \bigcup_{k \in \mathbb{N}} (I_k \cap (\inf(I_k) + A_k)) \ . \tag{18}$$

Note, that we have $\inf(I_k) = l^*_{v(n_k)}$ by construction of the I_k.

Theorem 7.2 *Let $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ be a CPS and W as in (18). Further, choose $S = (l_{n_k})$ as in Theorem 6.3.*

Then for almost all $\vartheta \in \mathbb{R}$ the hull $\Omega(\curlywedge(W + \vartheta))$ contains an embedded fullshift and $(\Omega(\curlywedge(W + \vartheta)), \mathbb{R})$ has positive topological entropy.

Note that as the window has empty interior in this case, the hull $\Omega(\Lambda + \vartheta)$ contains the empty set and therefore the fact that the action cannot be uniquely ergodic is obvious.

Proof. By construction, the metric independence of W in S^* is equivalent to the metric independence of the sets $(A_k)_{k \in \mathbb{N}}$ and thus follows from Lemma 7.1(iii). Hence, by Lemma 3.14, $\Omega(\curlywedge(W + \vartheta)$ contains an embedded fullshift for almost all $\vartheta \in \mathbb{R}$ (compare the proof of Theorem 6.3). □

Remark 7.3 Similar as in Lemma 3.12, one may show that the embedded fullshift Ξ which is obtained is contained in $\curlywedge(W + \vartheta)$ (that is, $\Gamma \subseteq \curlywedge(W + \vartheta)$ for all $\Gamma \in \Xi$). In the case of proper model sets, this was used further to conclude that Ξ is contained in the fibre $\beta^{-1}([\vartheta, 0]_{\mathcal{L}})$. However, for weak model sets there is not analogous statement to that, since a torus parametrisation does not exist in this case.

8. Remarks on higher-dimensional internal groups

In the previous sections, we have concentrated on examples of positive entropy model sets with one-dimensional internal group $H = \mathbb{R}$. While this makes the constructions easier on a technical level and allows to avoid heavy notation, it is also possible to produce similar examples with higher-dimensional internal groups. There is also a certain motivation for this. The eigenvalues of the continuous dynamical eigenfunctions in Remark 2.11 are those of the underlying Kronecker flow on the torus $(G \times H)/\mathcal{L}$. Hence, increasing the dimension of the internal group leads to a richer spectrum of continuous eigenfunctions, while keeping the dimension of the direct space constant.

The analysis in the case of a one-dimensional internal group $H = \mathbb{R}$ is simplified by the fact that in this situation the boundary of a proper window is always a Cantor set. In higher dimensions, this boundary also needs to contain non-trivial connected components, and there is a much greater variety of possible structures. For this reasons, general statements as the one in Theorem 5.4 (which starts with an arbitrary Cantor set) may be more difficult to make. However, when it comes to the construction of specific examples, the arguments employed in the previous sections can be adapted with only minor modifications. For instance, a random construction analogous to that in Theorem 5.4 may be carried out by starting with a Sierpinski carpet of positive measure, labelling the squares which were removed in the construction of the carpet and including each of them in the window independently with probability 1/2. The proof of Theorem 5.4 could then easily be modified to show that the Delone dynamical system on the hull of the resulting model set almost surely has positive entropy.

Deterministic constructions as in Sections 6 and 7 can equally be carried out with higher-dimensional H. In this case, one would have to start with the projection of a fundamental domain of the lattice $\mathcal{L}$ to H and the remove neighbourhoods of rapidly decreasing size around points in L^* to obtain an initial Cantor set C_0 (compare Lemma 6.1). Pasting in locally topologically independent sets into these 'holes' will then again lead to (weak) model sets with positive entropy.

References

[Aus88] J. Auslander. *Minimal flows and their extensions*. North-Holland Mathematical Studies, vol 153, North-Holland Publishing Co., Amsterdam, 1988.

[Auj14] J.B. Aujogue. On embedding of repetitive Meyer multiple sets into model multiple sets *Ergodic Theory Dyn. Syst.* **36**(6):1679–1702, 2016.

[ABKL15] J.B. Baptiste, M. Barge, D. Lenz, J. Kellendonk. Equicontinuous factors, proximality and Ellis semigroup for Delone sets. *Mathematics of aperiodic order*, 137–194, Prog. Math., **309**, Birkhäuser/Springer, Basel, 2015.

[BG] M. Baake, U. Grimm. *Aperiodic Order. Vol. 1: A Mathematical Invitation*, Cambridge Univ. Press, Cambridge, 2013.

[BHS16] M. Baake, C. Huck, N. Strungaru. On weak model sets of extremal density. *Indag. Math.* **28**(1):3–31, 2017.

[BJL15] M. Baake, T. Jaeger, D. Lenz. Toeplitz flows and model sets. *Bull. Lond. Math. Soc.* **48**(4):691–698, 2016.

[BL04] M. Baake and D. Lenz. Dynamical systems on translation bounded measures: Pure point dynamical and diffraction spectra, *Ergod. Th. & Dynam. Syst.* **24**(6):1867–93, 2004.

[BLM07] M. Baake, D. Lenz, and R.V. Moody. Characterization of model sets by dynamical systems. *Ergodic Theory Dyn. Syst.*, **27**(2):341–382, 2007.

[BLR07] M. Baake, D. Lenz and C. Richard. Pure point diffraction implies zero entropy for Delone sets with uniform cluster frequencies. *Lett. Math. Phys.* **82**:61–77, 2007.

[BMP00] M. Baake, R.V. Moody, P.A.B. Pleasants. Diffraction from visible lattice points and k-th power free integers. ‚extitDiscr. Math. **221**:3–42, 2000.

[BM04] M. Baake, R.V. Moody. Weighted Dirac combs with pure point diffraction. *J. reine angew. Math. (Crelle)* **573**:61–94, 2004

[Bow71] R. Bowen. Entropy for group endomorphisms and homogeneous spaces *Trans. Am. Math. Soc., **153**:401–413, 1971.*

[Gou04] J.-B. Gouéré. Quasicrystals and almost periodicity. *Commun. Math. Phys.***255**:651–681, 2005.

[Hof96] A. Hof. On diffraction by aperiodic structures, *Commun. Math. Phys.* **169**:25–43, 1995.

[HB14] C. Huck, M. Baake. Dynamical properties of k -free lattice points. *Acta Phys. Polon.* A **126**:482–485, 2014.

[HP13] C. Huck, P. Pleasants. Entropy and diffraction of the k-free points in n-dimensional lattices. *Discr. Comput. Geom.* **50**:39–68, 2013.

[HR14] C. Huck and C. Richard. On pattern entropy of weak model sets. *Discr. Comput. Geom.* **54**:741–757, 2015.

[KH97] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, 1997.

[KLS15] J. Kellendonk, D. Lenz, J. Savinien (eds). Mathematics of aperiodic order. *Progress in Mathematics*, **309** Birkhäuser/Springer, Basel, 2015.

[KS14] J. Kellendonk, L. Sadun. Meyer sets, topological eigenvalues, and Cantor fiber bundles. *J. Lond. Math. Soc.* (2) **89**:114–130, 2014.

[KR15] G. Keller, C. Richard. Dynamics on the graph of the torus parametrisation Preprint 2015. arXiv:1511.06137.

[Lag98] J.C. Lagarias. Geometric Models for Quasicrystals I. Delone Sets of Finite Type. *Discrete Comput. Geom.* **21**(2):161–191, 1999.

[LMS02] J.-Y. Lee, R.V. Moody, and B. Solomyak. Pure Point Dynamical and Diffraction Spectra. *Annales Henri Poincaré* **3**(5):1003–1018, 2002.

[LM16] D. Lenz and R.V. Moody. Stationary processes and pure point diffraction. *Ergod. Th. & Dynam. Syst.* **37**(8):2597–2642, 2017.

[LS03] D. Lenz, P. Stollmann. Delone dynamical systems and associated random operators. Conference Proceedings, Constanta (Romania), July 2-7, 2001, J.-M. Combes, J. Cuntz, G.A. Elliott, G. Nenciu, H. Siedentop, S. Stratila (eds.), Theta Foundation.

[LS09] D. Lenz, N. Strungaru. Pure point spectrum for measure dynamical systems on locally compact Abelian groups. *J. Math. Pures Appl.* ***92**:323–341*, 2009.

[Mey72] Y. Meyer. Algebraic Number Theory and Harmonic Analysis. North Holland, Amsterdam (1972).

[Moo97] R.V. Moody. Meyer sets and their duals, in: R.V. Moody (ed.), *The Mathematics of Long-Range Aperiodic Order*, NATO ASI Series C 489, Kluwer, Dordrecht, 403–441, 1997.

[Moo00] R.V. Moody. Model sets: A survey. in: F. Axel, F. Dénoyer and J.P. Gazeau (eds.) *From Quasicrystals to More Complex Systems*, Springer, Berlin and EDP Sciences, Les Ulis, 145–166, 2000.

[Moo02] R. V. Moody. Uniform Distribution in Model Sets. *Can. Math. Bull.*, **45**:123–130, 2002.

[RS15] C. Richard, N. Strungaru. Pure point diffraction and Poisson summation *Ann. Henri Poincaré* **18**(12):3903–3931, 2017.

[Sch00] M. Schlottmann. Generalized Model Sets and Dynamical Systems. *Directions in Mathematical Quasicrystals*, M. Baake and R.V. Moody (eds.), CRM Monograph Series vol. 13, AMS, Providence, RI, 143–159, 2000.

[Stru05] N. Strungaru. Almost periodic measures and long range order in Meyer sets. *Disc. Comput. Geom.* **33**:483–505, 2005.

[TZ91] A.T. Tagi-Zade. A variational characterization of the topological entropy of continuous groups of transformations. The case of $\mathbb{R}^n$-actions. *Mat. Zametki* **49**:114–123, 1991. (Translation in Math. Notes 49:305–311, 1991.)

[Wal82] P. Walters. *An Introduction to Ergodic Theory.* Springer Verlag, 1982.