Quasiperiodically forced interval maps with negative Schwarzian derivative

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Abstract

In the study of quasiperiodically forced systems invariant graphs have a special significance. In some cases, it was already possible to deduce statements about the invariant graphs of certain classes of systems from properties of the fibre maps. Here, we study quasiperiodically forced interval maps which are monotonically increasing and have negative Schwarzian derivative. First, we derive some basic results which only require monotonicity. Then we give a classification, with respect to the number and to the Lyapunov exponents of invariant graphs, for this class of systems. It turns out, that the possibilities for the invariant graphs are exactly analogous to those for the fixed points of the unperturbed fibre maps.

1 Introduction

Quasiperiodically forced systems occur in various situations in physics. For instance, the differential equations describing a pendulum forced at two incommensurate frequencies give rise to a quasiperiodically forced Poincaré map (see [1, 2] for details and numerical studies). Another example is the so-called Harper map, which is intimately related to certain discrete Schroedinger operators with quasiperiodic potential (e.g. [3, 4]). In addition to this physical motivation, quasiperiodically forced systems provide interesting examples of dynamical behaviour (namely the generic existence of strange non-chaotic attractors, see below) usually not found in uniformly hyperbolic systems, which are much better understood. Therefore they are of interest from a purely mathematical point of view as well.

Due to the aperiodicity of the quasiperiodic forcing there cannot be any fixed points or periodic points for such systems. Therefore invariant graphs are the most simple invariant objects which can occur. Section 2 will point out their significance for the dynamics of the system (all in the case of quasiperiodically driven monotone interval maps): On the one hand, invariant graphs occur generically as boundary lines of invariant compact sets. On the other hand there is a one to one correspondence between the invariant graphs and the ergodic invariant measures of a system.

The stability of an invariant graph is determined by its Lyapunov exponent. If it is negative the graph is attracting, but in which sense depends strongly on whether the graph is continuous or not. This will be specified in section 3, which also contains some other basic observations about invariant graphs and their Lyapunov exponents in systems with monotone fibre maps. Some of the results from this section, as well as from the preceding one, might be considered well-known 'folklore'. But as they seem not to be written down in the literature, proofs are included for the convenience of the reader.

For a map of the real line to itself which is bounded and has strictly negative Schwarzian derivative (e.g. tanh or arctan as typical representatives), there are three possibilities regarding its fixed points: either it has a single fixed point which is stable or neutral, or it has exactly one stable and one neutral fixed point, or it has three fixed points, the inner one of which is unstable and the other two are stable. The main result here is to show, that this carries over directly to quasiperiodically driven systems (see also Theorem 4.2):

> *Suppose that for a system of quasiperiodically forced interval maps, all the fibre maps have strictly negative Schwarzian derivative. Then either there is only one invariant graph with non-positive Lyapunov exponent, or there are two invariant graphs, one of which has strictly negative and the other one zero Lyapunov exponent, or there are three invariant graphs, the inner one of which has strictly positive and the other two strictly negative Lyapunov exponents.*

This is in some way similar to the classification done independently by Keller [6] and Bezhaeva/Oseledets [7] for the class of systems originally proposed by Grebogi et al. [5]. Glendinning [8] has recently generalized parts of these results to a broader class of systems. Their classification also includes a statement about the continuity of the invariant graphs, a question which had to be left open here. However, simulations suggest that non-continuous invariant graphs do occur in the class of systems considered here, too (see Fig. 1.1). It might be worth mentioning, that the statements presented in sections 2 and 3 can be used to reproduce most results from [6] and [7] quite easily.

Non-continuous invariant graphs with negative Lyapunov exponent are usually refered to as examples of strange non-chaotic attractors (SNA's). They have received much interest as a novel phenomenon and have been intensely studied numerically in theoretical physics (e.g. [1, 2, 3, 12, 13, 14]). [4] gives a good overview and further reference. Yet the classification mentioned above, and the results of Herman [11] for quasiperiodically forced fractional linear maps, seem to be the only examples in which the discontinuity of invariant graphs has been proved rigorously.

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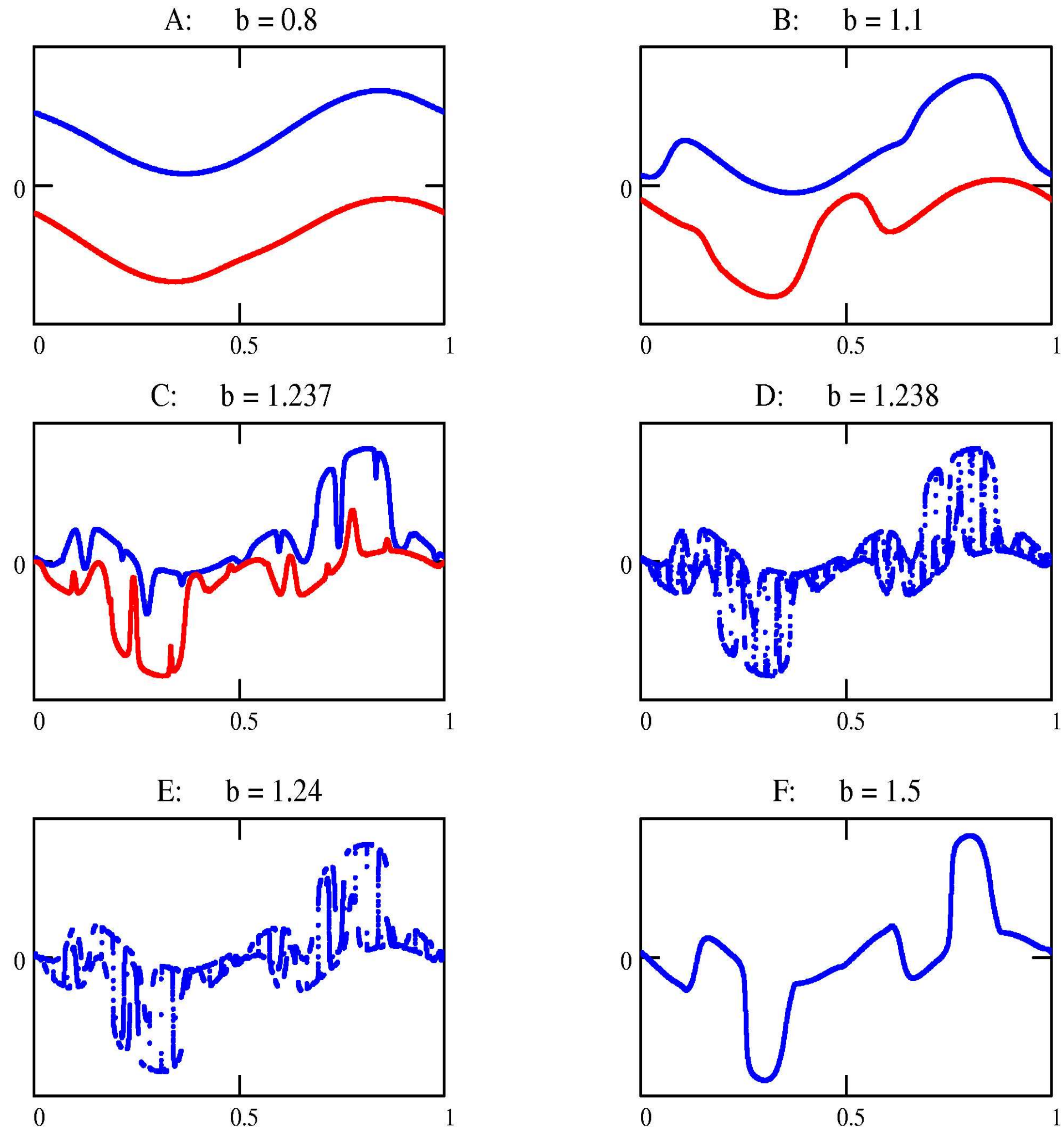


Figure 1.1: Systems of the form $(\theta, x) \mapsto (\theta + \omega, \frac{\arctan(ax)}{\arctan(a)} + b \cdot \sin(2\pi\theta))$ have negative Schwarzian derivative on the fibres. Here 10 000 iterations of the starting points (0,3) and (0,-3) trace out pictures of the upper and lower bounding graphs, respectively. These two graphs are stable, another invariant graph which is unstable lies in between (see Theorem 4.2). The parameter $a = 10$ is the same for all pictures A–F. Upon increasing the parameter b the two distinct graphs come closer together (A–C). When they collide, non-continuous invariant graphs seem to occur over a small parameter range (D,E), until only one continuous invariant graph remains (F). This phenomenon has been observed in other parameter families of quasiperiodically forced systems as well, and is called creation of an SNA via torus collision (see [4]).

2 Invariant Graphs

Let $M := \mathbb{T}^1 \times [a,b]$, where $a < b \in \mathbb{R}$. We will study quasiperiodically forced interval maps, i.e. maps of the form

$$T : M \to M \ , \ (\theta, x) \mapsto (\theta + \omega, T_\theta(x)) \ . \tag{2.1}$$

The *fibre map* $T_\theta : [a,b] \to [a,b]$ is given by $T_\theta(x) = \pi_2 \circ T(\theta, x)$ with π_2 the natural projection from M to $[a,b]$. Its derivative with respect to x will be denoted by DT_θ.

We will assume that T is continuous, T_θ is continuously differentiable in x and the derivative $DT_\theta(x)$ depends continuously on (θ, x) and denote the set of all such systems (T, M) by $\mathcal{T}$. Further, we define the class $\mathcal{T}_m$ of systems with weakly monotone fibre maps as

$$\mathcal{T}_m := \{(T,M) \in \mathcal{T} \mid \ \forall \theta \in \mathbb{T}^1 : \ T_\theta(x) \leq T_\theta(y) \ \mathit{if} \ x \leq y \ , \ T_\theta(a) > a, \ T_\theta(b) < b\} \ .$$

The conditions $T_\theta(a) > a$ and $T_\theta(b) > b$ are only required for technical reasons (see proof of Lemma 3.5). In all of the following, the Lebesgue-measure on $\mathbb{T}^1$ will be denoted by m and $r = r_\omega : \mathbb{T}^1 \to \mathbb{T}^1 \ , \ \theta \mapsto \theta + \omega \mod 1$ will always be the rotation with rotation number ω corresponding to the considered system.

Since the irrational rotation is aperiodic, there cannot be any fixed points or periodic points for such systems. Furthermore, any compact invariant set in M will project onto a compact subset of the circle which is invariant under the corresponding rotation. As the irrational rotation is minimal, this must be either the empty set or the whole circle. The simplest invariant objects are therefore invariant graphs:

Definition 2.1 *Let $(T, M) \in \mathcal{T}$. A Lebesgue-measurable function $\varphi : \mathbb{T}^1 \to [a,b]$ is called an invariant graph (with respect to T), if for all $\theta \in \mathbb{T}^1$:*

$$T(\theta, \varphi(\theta)) = (\theta + \omega, \varphi(\theta + \omega)) \ . \tag{2.2}$$

The point set $\Phi := \{(\theta, \varphi(\theta)) : \theta \in \mathbb{T}^1\}$ will be called invariant graph as well, but labeled with the corresponding capital letter.

In the study of dynamical systems, there is generally a focus on compact invariant sets and invariant ergodic measures. As mentioned before, in quasiperiodically forced systems these are closely related to the invariant graphs, at least in the case of monotone fibre maps.

Invariant graphs and compact invariant sets: Suppose $(T, M) \in \mathcal{T}_m$ and $K \subset M$ is a non-empty compact set which is forward invariant under T, i.e. $T(K) = K$. Then $\pi_1(K)$ is a compact subset of $\mathbb{T}^1$ which is invariant under the irrational rotation r. As it is non-empty, it must be the whole circle (minimality of r). Thus, by

$$\varphi^+(\theta) := \sup\{x \in [a,b] : (\theta, x) \in K\} \tag{2.3}$$

an invariant graph φ^+ can be defined (the invariance following from the monotonicity and continuity of the fibre maps), analogously a φ^- via the infimum. As K is closed we get $\limsup_{\theta' \to \theta} \varphi^+(\theta') \leq \varphi^+(\theta)$, which means that φ^+ is upper semi-continuous. In the same way φ^- is lower semi-continuous. This also gives the measurability of the two graphs.

As the *global attractor* $\bigcap_{n \in \mathbb{N}} T^n(M)$ is always a nonempty compact invariant set, there exists at least one invariant graph. The graphs corresponding to this set often have a special significance.

Definition 2.2 *Let $(T, M) \in \mathcal{T}_m$. Then the global attractor of T is the set $K_{max} := \bigcap_{n \in \mathbb{N}} T^n(M)$.*

$$\begin{aligned} \varphi_T^+(\theta) &:= \sup\{x \in [a,b] : (\theta, x) \in K_{max}\} \quad \textit{and} \\ \varphi_T^-(\theta) &:= \inf\{x \in [a,b] : (\theta, x) \in K_{max}\} \end{aligned}$$

are the upper and lower bounding graphs of the system (T, M).

If the fibre maps are not monotone one can still define an upper and lower bounding graph, but then these graphs do not have to be invariant anymore. See [8] for details.

Invariant graphs and invariant ergodic measures: If φ is an invariant graph, by

$$\mu_\varphi(A) := m(\pi_1(A \cap \Phi)) \quad \forall A \in \mathcal{B}(M) \tag{2.4}$$

a T-invariant ergodic measure can be defined (recall the meaning of ergodicity: $T^{-1}(A) = A \Rightarrow \mu_\varphi(A) \in \{0, 1\}$. The following lemma shows, that the converse is true as well. A proof can be found in [17, Thm. 1.8.4(iv)]. Although the statement there is formulated for continuous-time random dynamical systems, the proof literally stays the same.

Lemma 2.3 *Let $(T, M) \in \mathcal{T}_m$ and μ be a T-invariant ergodic measure on M. Then $\mu = \mu_\varphi$ for some invariant graph φ.*

Equivalence classes of invariant graphs and the essential closure: There is a subtle issue in the definition of invariant graphs that has to be addressed: Any invariant graph φ can be modified on a set of measure zero to yield another invariant graph $\tilde{\varphi}$, equal to φ m-a.s. . We usually do not want to distinguish between such graphs. On the other hand, especially when topology is concerned, we sometimes need objects which are well-defined everywhere.

So far, this has not been a problem. The boundary graphs of compact invariant sets are well-defined everywhere, and for the definition of the associated measure (2.4) it does not matter. But as we also want to study the topological properties of invariant graphs, some care has to be taken. We will therefore use the following convention:

We will consider two invariant graphs as equivalent if they are m-a.s. equal and implicitly speak about equivalence classes of invariant graphs, just as functions in $\mathcal{L}^1_\mu$ are identified if they are μ-a.s. equal. If any further assumptions about invariant graphs are made, such as continuity, semi-continuity or inequalities between invariant graphs, we will understand it in the way that there is at least one representative in each of the respective equivalence classes, such that the assumptions are met. These representatives will then be used in the proofs, and all conclusions which are drawn from the assumed properties will be true for all such representatives.

There is one case, where this terminology might cause confusion: It is possible, that an equivalence class contains both an upper and a lower semi-continuous graph, but no continuous graph. To get an idea of what could happen, regard the function $f : x \mapsto \sin\frac{1}{x}\ \forall x \neq 0$. By choosing $f(0) = 1$ we can extend it to an upper semi-continuous function, by choosing $f(0) = -1$ to a lower semi-continuous function, but there is no continuous function in the equivalence class. To avoid ambiguities, we will explicitly mention this case whenever it can occur.

In order to assign a well defined point set to an equivalence class of invariant graphs, we introduce the *essential closure*:

Definition 2.4 *Let $(T, M) \in \mathcal{T}_m$. If φ is an invariant graph, we define its essential closure (with respect to its associated measure μ_φ) as:*

$$\overline{\Phi}^{ess} := \{(\theta, x) : \mu_\varphi(U \cap \Phi) > 0\ \forall \textit{open neighbourhoods } U \textit{ of } (\theta, x)\} \tag{2.5}$$

Several facts follow immediately from this definition:

- $\overline{\Phi}^{ess}$ is a compact set.
- $\overline{\Phi}^{ess} = \mathrm{supp}(\mu_\varphi)$, which in turn implies $\mu_\varphi(\overline{\Phi}^{ess}) = 1$ (see e.g. [16]).
- Invariant graphs from the same equivalence class have the same essential closure (as they have the same associated measure).
- $\overline{\Phi}^{ess}$ is contained in every other compact set which contains μ_φ-a.e. point of Φ.
- $\overline{\Phi}^{ess}$ is forward invariant under T.

Invariance probably needs a little bit more caution than the other statements:

Suppose $x \in \overline{\Phi}^{ess}$ and U is an open neighbourhood of $T(x)$. Then $T^{-1}(U)$ is an open neighbourhood of x, and therefore $\mu_\varphi(U) = \mu_\varphi \circ T^{-1}(U) > 0$. This means $T(x) \in \overline{\Phi}^{ess}$ and thus $T(\overline{\Phi}^{ess}) \subseteq \overline{\Phi}^{ess}$. On the contrary $T(\overline{\Phi}^{ess})$ is a compact set which contains μ_φ-a.e. point in Φ, therefore $\overline{\Phi}^{ess} \subseteq T(\overline{\Phi}^{ess})$.

Proposition 3.7 will state some topological properties of the essential closure in a special situation. However, despite its topological nature, the proof requires some facts from ergodic theory (namely Lemma 3.2) and must therefore be postponed until Lyapunov exponents have been introduced in the next section.

3 Lyapunov exponents

To study further properties of invariant graphs, we need to define Lyapunov exponents. For simplicity, we denote the fibre maps of T^n by T^n_θ instead of $(T^n)_\theta$.

Definition 3.1 *Let $(T, M) \in \mathcal{T}$, $(\theta, x) \in M$. If the limit*

$$\lambda(\theta, x) := \lim_{n\to\infty} \frac{1}{n} \log |DT^n_\theta(x)|$$

exists, it is called the (normal) Lyapunov exponent in (θ, x). If φ is an invariant graph with $\log |DT_\theta(\varphi(\theta))| \in \mathcal{L}^1_m$, its Lyapunov exponent is defined as

$$\lambda(\varphi) := \int_{\mathbb{T}^1} \log |DT_\theta(\varphi(\theta))| d\theta \ .$$

Note that $\log |DT^n_\theta(\varphi(\theta))| = \sum_{i=0}^{n-1} \log |DT_{\theta+i\omega}(\varphi(\theta + i\omega))|$. Thus, by the Birkhoff ergodic theorem, the Lyapunov exponent of an invariant graph equals that of its points for Lebesgue-a.e. $\theta \in \mathbb{T}^1$.

As mentioned in the introduction, there is a great difference in the meaning of a negative Lyapunov exponent between the continuous and the non-continuous case. If DT_θ is non-singular and the invariant graph φ is continuous, so is the function $\log |DT_\theta(\varphi(\theta))|$. As the irrational rotation is uniquely ergodic, the ergodic sums of continuous functions converge uniformly (see e.g. [16]). If $\lambda = \lambda(\varphi)$ is negative, this means that

$$\exists n_0 \in \mathbb{N}\ \exists \epsilon > 0 : \frac{1}{n_0} \log |DT^{n_0}_\theta(x)| < \frac{\lambda}{2} \ \ \forall (\theta, x) \in U_\epsilon(\varphi) \tag{3.1}$$

where $U_\epsilon(\varphi) := \{(\theta, x) \in M : x \in B_\epsilon(\varphi(\theta))\}$. If DT_θ has singularities the above remains true, which can be seen by approximating $\log |DT_\theta \circ \varphi|$ with the monotonically decreasing sequence of continuous functions $\max\{\log |DT_\theta \circ \varphi|, -N\}$.

Thus, continuous invariant graphs with negative Lyapunov exponent are attracting in the very strong sense that an iterate of T acts uniformly contracting (along the fibres) on a neighbourhood of the graph. From this conclusions regarding the regularity and stability of continuous invariant graphs can be drawn, as done by Stark [9].

On the other hand, if φ is not continuous (3.1) cannot be true anymore. This can be seen from the following lemma, which is just a slightly adapted version of Corollary 1.15 in [10].

Lemma 3.2 *Let $(T, M) \in \mathcal{T}_m$. Suppose φ is an non-continuous invariant graph with negative Lyapunov-Exponent. Then $\overline{\Phi}^{ess}$ contains at least one invariant graph with a non-negative Lyapunov exponent.*

To obtain a general result which also applies to the non-continuous case, the statement in (3.1) must be replaced by the following proposition. In the case where T is a diffeomorphism, the statement should follow from general Pesin theory as well (e.g. supplement in [16] or [17, Thm 7.3.10]), but as we do not assume T to be invertible a few technical difficulties have to be dealt with.

Proposition 3.3 *Suppose $(T, M) \in \mathcal{T}$ and φ is an invariant graph with $\lambda(\varphi) < 0$. Then for m-a.e. $\theta \in \mathbb{T}^1$ there is a $\delta_\theta > 0$, such that*

$$\forall x \in B_{\delta_\theta}(\varphi(\theta)) : \ |T_\theta^n(x) - \varphi(\theta + n\omega)| \to 0 \ (n \to \infty).$$

Proof:
$|DT_\theta|$ may equal zero and is not assumed to be Lipschitz- (or even Hölder-) continuous. In order to deal with these possible irregularities, we choose a bounded, Lipschitz-continuous function $F : M \to [0, \infty)$ with the following properties:

- $\exists c > 0 : \ F(\theta, x) \geq \max\{c, |DT_\theta(x)|\} \ \ \forall (\theta, x) \in M$ (3.2)
- $\displaystyle\int_{\mathbb{T}^1} \log F(\theta, \varphi(\theta)) d\theta < \frac{3}{4}\lambda$

(Exists, according to Stone-Weierstrass and dominated convergence.) Then $\log F$ will be Lipschitz as well, thus $\exists L > 0 \ \forall x, y \in [a, b], \theta \in \mathbb{T}^1$:

$$F(\theta, x) \leq e^{L|x-y|} \cdot F(\theta, y) \ . \tag{3.3}$$

Applying Birkhoff's ergodic theorem to $\theta \mapsto \log F(\theta, \varphi(\theta))$ yields that for m-a.e. $\theta \in \mathbb{T}^1$ there is a constant $K = K_\theta > 1$, such that $\forall n \in \mathbb{N}$:

$$\prod_{i=0}^{n-1} F(\theta + i\omega, \varphi(\theta + i\omega)) \leq K e^{n\frac{\lambda}{2}} \ . \tag{3.4}$$

Now we can choose $\delta > 0$ with $\frac{\lambda}{2} + \delta L < 1$ and $n_0 \in \mathbb{N}$, such that $Ke^{n(\frac{\lambda}{2}+\delta L)} < 1 \ \forall n \geq n_0$. If then $0 < \delta_\theta < \delta$ is chosen such that $\forall x \in B_\theta := B_{\delta_\theta}(\varphi(\theta))$:

$$|T_\theta^n(x) - T_\theta^n(\varphi(\theta))| < \delta \ \ \forall i = 0, \ldots, n_0 \ , \tag{3.5}$$

a straightforward induction yields that $\forall n \in \mathbb{N} \ \forall x \in B_\theta$:

$$|T_\theta^n(x) - \varphi(\theta + n\omega)| < \min\{\delta, \delta \cdot K \cdot e^{n(\frac{\lambda}{2}+\delta L)}\} \tag{3.6}$$

For $n = 0$ this is obvious. Now suppose (3.6) holds for $i = 0, \ldots, n-1$ and let $x \in B_\theta$. Then

$$|T_\theta^n(x) - \varphi(\theta + n\omega)| \leq |x - \varphi(\theta)| \cdot \sup_{z \in B_\theta} |DT_\theta^n(z)| \overset{(3.2)}{\leq} \delta \cdot \sup_{z \in B_\theta} \prod_{i=0}^{n-1} F(\theta + i\omega, T_\theta^i(z))$$

$$\overset{(3.3)}{\leq} \ \delta \cdot \sup_{z \in B_\theta} \prod_{i=0}^{n-1} F(\theta + i\omega, \varphi(\theta + i\omega)) \cdot e^{L|T_\theta^i(z) - \varphi(\theta+i\omega)|} \overset{(3.4) \text{ and } (3.6)}{\leq} \delta \cdot K \cdot e^{n(\frac{\lambda}{2}+\delta L)}$$

Together with (3.5) in the case $n < n_0$, this proves the induction hypothesis and thus the proposition.

□

In the case of weakly monotone fibres, we get the following

Corollary 3.4 *Let $(T, M) \in \mathcal{T}_m$ and suppose φ is an invariant graph with $\lambda(\varphi) < 0$. Then one of the following is true:*

(i) $\varphi = \varphi_T^+$ *and for m-a.e. $\theta \in \mathbb{T}^1$ and all $x > \varphi(\theta)$:*

$$|T_\theta^n(x) - \varphi(\theta + n\omega)| \to 0 \quad (n \to \infty).$$

(ii) $\varphi < \varphi^+$ *m-a.s., and there is another invariant graph $\psi \geq \varphi$ with $\lambda(\psi) \geq 0$, such that for m-a.e. $\theta \in \mathbb{T}^1$:*

$$\varphi(\theta) \leq x < \psi(\theta) \;\Rightarrow\; |T_\theta^n(x) - \varphi(\theta + n\omega)| \to 0 \quad (n \to \infty)\ .$$

If in case (ii) φ is continuous, then ψ is lower semi-continuous.

Of course a similar statement holds for the region below φ.

Proof:
If φ is not equal to φ_T^+ but m-a.s. below, then

$$\psi(\theta) := \sup\{x \geq \varphi(\theta) : |T_\theta^n(x) - \varphi(\theta + n\omega)| \to 0\} \;\leq\; \varphi_T^+(\theta)$$

defines an invariant graph. Because of the theorem, ψ is m-a.s. not equal to φ. The convergence of orbits between the graphs follows immediately from the definition of ψ, the invariance from the monotonicity and continuity of T. If $\lambda(\psi)$ was negative, the above theorem would yield a contradiction to this convergence behaviour, so $\lambda(\psi)$ must be non-negative. Note that ψ is uniquely determined by the convergence behaviour of all points between φ and ψ.

Now suppose $\varphi \equiv \varphi_T^+$ and let $B := \{\theta : \psi(\theta) = b\}$. Because of $r^{-1}(B) \subseteq B$ and the r-invariance of m, B is m-a.s. invariant. Due to the ergodicity of r its measure must therefore be either 0 or 1. As there is no invariant graph above φ_T^+ it must be 1, which proves the statement in (i).

If in case (ii) φ is continuous ϵ and n_0 can be chosen as in (3.1). To obtain the lower semi-continuity of ψ we show that $\{\theta : \psi(\theta) > s\}$ is open for all $s \in [a, b]$. Suppose $\psi(\theta) > s$. We assume $s \geq \varphi(\theta)$, otherwise we replace s with an arbitrary $t \in [\varphi(\theta), \psi(\theta))$. From $\varphi(\theta) \leq s < \psi(\theta)$ follows $\exists n \in \mathbb{N} : T^n(\theta, s) \in U_{\frac{\epsilon}{2}}(\varphi)$. Then, due to the continuity of T^n, $T^n(\theta', z) \in U_\epsilon(\varphi)$ for all (θ', z) from a small neighbourhood of (θ, s). In particular this is true for all (θ', s) with θ' from a small neighbourhood V of θ. But an orbit which enters $U_\epsilon(\varphi)$ is trapped in there and will finally converge to φ, thus $V \subset \{\theta : \psi(\theta) > s\}$. Note that $V \subset \{\theta : \psi(\theta) > t\} \subset \{\theta : \psi(\theta) > s\}$ if s had to be replaced.

□

The maximal invariant set should not be expected to have repelling boundaries, so the Lyapunov exponents of the bounding graphs should be non-positive. Indeed we have

Lemma 3.5 *Let $(T, M) \in \mathcal{T}_m$. If there exists a function $h : \mathbb{T}^1 \to [0, \infty)$ such that*

$$\inf_{x \in [a,b]} DT_\theta(x) \geq h(\theta) \ \forall \theta \in \mathbb{T}^1 \quad \textit{and} \quad \log h \in \mathcal{L}^1_m(\mathbb{T}^1) \ ,$$

then $\lambda(\varphi^\pm_T) \leq 0$.

Proof:
If φ is an arbitrary graph, let the graph $\widehat{T}(\varphi)$ be defined by $\widehat{T}(\varphi)(\theta) := T_{\theta-\omega} \circ \varphi(\theta - \omega)$. The iterates of the upper bounding line of M are then given by

$$\varphi_1 :\equiv b \ , \ \varphi_{n+1} := \widehat{T}(\varphi_n) \ .$$

This sequence of graphs converges pointwise and monotonically decreasing to φ^+_T. As $h > 0$ m-a.s. and $T_\theta(b) < b$ by definition of $\mathcal{T}_m$, the convergence is strictly monotone on m-a.e. fibre, i.e. $|\varphi_n(\theta) - \varphi^+_T(\theta)| > 0 \ \forall n \in \mathbb{N}$. Thus we have

$$\frac{\varphi_{n+1}(\theta+\omega) - \varphi^+_T(\theta+\omega)}{\varphi_n(\theta) - \varphi^+_T(\theta)} \ \longrightarrow \ DT_\theta(\varphi^+_T(\theta)) \ \ (n \to \infty).$$

As these terms are difference quotients of T_θ, they are bounded from above by a Lipschitz constant L of T and from below by $h(\theta)$. Therefore it is possible to apply dominated convergence after taking the logarithm:

$$\int_{\mathbb{T}^1} \log \frac{\varphi_{n+1}(\theta+\omega) - \varphi^+_T(\theta+\omega)}{\varphi_n(\theta) - \varphi^+_T(\theta)} \, d\theta \ \to \ \lambda(\varphi^+_T) \ \ (n \to \infty)$$

On the other hand these integrals are bounded uniformly from above by zero:

$$\begin{aligned}
&\int_{\mathbb{T}^1} \log \frac{\varphi_{n+1}(\theta+\omega) - \varphi^+_T(\theta+\omega)}{\varphi_n(\theta) - \varphi^+_T(\theta)} \, d\theta = \\
&= \ \int_{\mathbb{T}^1} \log \underbrace{\frac{\varphi_{n+1}(\theta) - \varphi^+_T(\theta)}{\varphi_n(\theta) - \varphi^+_T(\theta)}}_{\leq\ 1,\ \text{as}\ \varphi_n\ \text{is decreasing}} \ + \ \log \frac{\varphi_{n+1}(\theta+\omega) - \varphi^+_T(\theta+\omega)}{\varphi_{n+1}(\theta) - \varphi^+_T(\theta)} \, d\theta \leq \\
&\leq \ \int_{\mathbb{T}^1} \log \frac{\varphi_{n+1}(\theta+\omega) - \varphi^+_T(\theta+\omega)}{\varphi_{n+1}(\theta) - \varphi^+_T(\theta)} \, d\theta
\end{aligned}$$

The last integrand is of the kind $F \circ r - F$, where $F := \log(\varphi_{n+1} - \varphi^+_T)$. At the same time it is bounded from below by a difference quotient, as $(\varphi_{n+1} - \varphi^+_T)(\theta)$ in the denominator is bounded from above by $(\varphi_n - \varphi^+_T)(\theta)$. This means that it has the integrable minorant $\log h$. Therefore we can apply Lemma 3.6, which yields that the integral is zero.

□

Lemma 3.6 *Let $(X, \mathcal{F}, \mu)$ be a probability space, $r : X \to X$ a measurable transformation leaving the measure μ invariant and $F : X \to \mathbb{R}$ a measurable function. If the function $F \circ r - F$ has a minorant or majorant $h \in \mathcal{L}^1_\mu$, then $F \circ r - F \in \mathcal{L}^1_\mu$ and*

$$\int_X (F \circ r - F)\ d\mu = 0\ .$$

A proof can be found in [6], for example. (The lemma itself is probably far older than that.)

Topology of semi-continuous invariant graphs: As mentioned before, one reason for the interest in quasiperiodically forced systems is the occurrence of non-continuous invariant graphs. On the other hand, semi-continuous invariant graph seem to play a special role, as we have seen in Section 2. We can now study the topological properties of such graphs. Actually, in the situation of the proposition below, their essential closures resemble Cantor sets, with the only difference that they are not totally disconnected, but still nowhere dense. The intersection with any horizontal line $\{(\theta, s) : \theta \in \mathbb{T}^1\}$ $(s \in [a, b])$ is indeed a Cantor set.

Proposition 3.7 *Let $(T, M) \in \mathcal{T}_m$. If $\varphi^- \leq \varphi^+$ are two distinct, non-continuous invariant graphs with negative Lyapunov exponents, φ^- is lower semi-continuous and φ^+ is upper semi-continuous, and if there is no other semi-continuous invariant graph in between, then the following is true:*

- *(i)* *$K := \overline{\Phi^-}^{ess} = \overline{\Phi^+}^{ess}$ is minimal.*
- *(ii)* *There exists another invariant graph ψ, with non-negative Lyapunov exponent and $\overline{\Psi}^{ess} = K$, between φ^- and φ^+.*
- *(iii)* *$B := \{\theta : \varphi^-(\theta) = \varphi^+(\theta)\} = \{\theta : \varphi^-$ and φ^+ are both continuous in $\theta\}$ is dense in $\mathbb{T}^1$ and $m(B) = 0$.*
- *(iv)* *K is a perfect set (i.e. each of its points is an accumulation point of the set) and has empty interior.*

Proof:
Before really starting the proof, we want to exclude one rather pathological possibility. As mentioned in Section 2, it is a priori conceivable that there is a lower semi-continuous invariant graph $\tilde{\varphi}$ in the equivalence class of φ^+. Due to the semi-continuity of the two graphs, $\tilde{\varphi}$ must be strictly below φ^+. Therefore the set $\{(\theta, x) : \tilde{\varphi}(\theta) \leq x \leq \varphi^+(\theta)\}$ is compact and invariant. As it contains all points in Φ^+ it also contains $\overline{\Phi^+}^{ess}$, but this contradicts Lemma 3.2, because $\lambda(\varphi^+) < 0$. Thus, this situation cannot occur in the case of negative Lyapunov exponents.

(i) Due to the semi-continuity of the two graphs $A := \{(\theta, x) : \varphi^-(\theta) \leq x \leq \varphi^+(\theta)\}$ is a compact and invariant set. As it contains Φ^+, it also contains its essential closure. This is compact and invariant as well and therefore has a semi-continuous lower bounding graph. As A contains no other lower semi-continuous invariant graph, this has to be φ^-. Thus we get that $(\theta, \varphi^-(\theta)) \in \overline{\Phi^+}^{ess}$ m-a.s. and therefore $\overline{\Phi^-}^{ess} \subseteq \overline{\Phi^+}^{ess}$. The other way around we get that $\overline{\Phi^+}^{ess} \subseteq \overline{\Phi^-}^{ess}$, i.e. $\overline{\Phi^-}^{ess} = \overline{\Phi^+}^{ess}$. The same argument for the ω-limit gives that $\omega((\theta, x)) = K \ \forall(\theta, x) \in K$, so K is minimal.

(ii) The existence of ψ is a direct consequence of Lemma 3.2, the fact $\overline{\Psi}^{ess} = K$ then follows from the minimality of K.

(iii) $\varphi^+ - \varphi^-$ is upper semi-continuous, so $B_n := \{\theta : |\varphi^+(\theta) - \varphi^-(\theta)| < \frac{1}{n}\}$ is open. We show that B_n is dense in $\mathbb{T}^1$ as well, in order to apply Baire's theorem. From any open set $U \in \mathbb{T}^1$, take a θ such that $(\theta, \varphi^-(\theta)) \in \overline{\Phi^+}^{ess}$. Then, by definition of the essential closure, any neighbourhood of $(\theta, \varphi^-(\theta))$ contains infinitely many points of Φ^+. This means there is a sequence $(\theta_n)_{n\in\mathbb{N}}$ with $\theta_n \to \theta$ and $\lim_{n\to\infty} \varphi^+(\theta_n) = \varphi^-(\theta)$. From the semi-continuity of φ^- and the fact that $\varphi^- \leq \varphi^+$, we get that $\lim_{n\to\infty} \varphi^-(\theta_n) = \varphi^-(\theta)$ as well, and therefore B_n intersects U. As U was arbitrary, B_n is dense in $\mathbb{T}^1$. Now Baire's theorem yields, that the residual set $\bigcap_{n\in\mathbb{N}} B_n = B$ is dense in $\mathbb{T}^1$, too.

The continuity on B follows from the semi-continuity of both graphs. On the other hand let $\theta \notin B$. If $(\theta, \varphi^-(\theta)) \in \overline{\Phi^+}^{ess}$, then there is always a sequence $(\theta_n, \varphi^+(\theta_n))$ converging to $(\theta, \varphi^-(\theta))$. Therefore if $\varphi^+(\theta) \neq \varphi^-(\theta)$, then φ^+ cannot be cts. in θ. If $(\theta, \varphi^-(\theta)) \notin \overline{\Phi^+}^{ess}$, then φ^- can surely not be cts. in θ as $(\theta, \varphi^-(\theta))$ lies outside of the compact set K, which contains almost all points of Φ^-. Thus both graphs cannot be cts. outside of B at the same time.

B is an r-invariant set and cannot have full measure (as the two graphs are distinct), thus by the ergodicity of r its measure is zero.

(iv) As K is minimal, it is equal to the ω-limit of each of its points. But the ω-limits only consist of accumulation points of the orbits, and due to the invariance of K these orbits will be inside of K, too.

The statement that K has empty interior follows immediately from the fact that B is dense in $\mathbb{T}^1$.

□

4 Systems with negative Schwarzian derivative on the fibres

Now we turn to study systems with negative Schwarzian derivative on the fibres. First, we recall the following known facts:

The Schwarzian derivative of a function $F \in \mathcal{C}^3([a,b])$ with $DF > 0$ is defined as

$$SF := \frac{D^3F}{DF} - \frac{3}{2}\left(\frac{D^2F}{DF}\right)^2 .$$

It is strongly connected with the concept of cross ratios. The cross ratio of four points $x < y < z < w$ from $[a,b]$ is defined as

$$CR[x,y,z,w] := \frac{(w-x)\cdot(z-y)}{(y-x)\cdot(w-z)} ,$$

the cross ratio distortion of F with respect to these points is

$$D(F,[x,y,z,w]) := \frac{CR[F(x),F(y),F(z),F(w)]}{CR[x,y,z,w]} .$$

Note that this can be regarded as a quotient of difference quotients, which compares the product of the average slope and the one in the middle with the product of the slopes on the right and left part of the interval $[x,w]$. This geometric interpretation will be fundamental for the proof of Theorem 4.2 . The following two known facts will be used (see e.g. [15]):

- $S(G \circ F) = (SG \circ F) \cdot (DF)^2 + SF$ (4.1)
- $SF < 0$ on $[a,b]$ $\Leftrightarrow$ $D(F,[x,y,z,w]) > 1$ $\forall x < y < z < w \in [a,b]$ (4.2)

(4.2) will be used to show that the properties of the fibre maps carry over to the forced systems, as mentioned in the introduction. In addition, in order not to lose the strictness of the inequality while taking limits, the following lemma is needed:

Lemma 4.1 *Let $I \subset \mathbb{R}$ be a compact interval. Then $\forall \gamma > 0$ $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\forall a < b < c < d \in I$ with $\min\{|c-a|,|d-c|\} > \gamma$ or $|c-b| > \gamma$ $\forall F \in \mathcal{C}^3(I)$ with $DF > 0$ and $SF < -\epsilon$:*

$$D(F,[a,b,c,d]) > 1 + \delta .$$

Proof:
The proof will use the fact, that for the composition of F with another function g

$$D(F \circ g,[x,y,z,w]) = D(F,[g(x),g(y),g(z),g(w)]) \cdot D(g,[x,y,z,w]) .$$

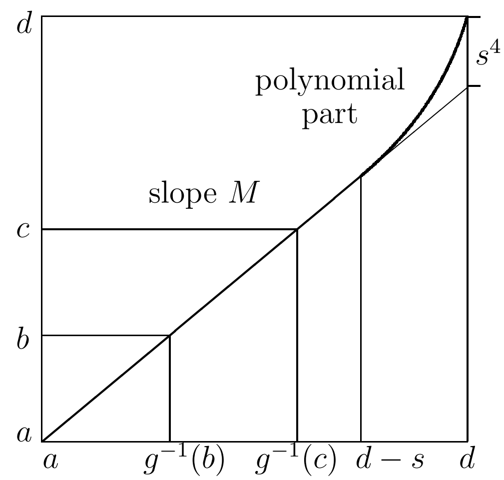
The strategy will be to compose F with a function g of positive Schwarzian derivative. This function is chosen such that on the one hand it reduces the cross ratio by a fixed factor, on the other hand the composition still has negative Schwarzian derivative. Thus the decrease in the cross ratio has to be compensated by F.

Suppose $SF < -\epsilon$ and $\min\{|c-a|, |d-c|\} > \gamma$. Then g is chosen as

$$g : [a,d] \to [a,d] \ , \ x \mapsto \begin{cases} M \cdot (x-a) + a & x \in [a, d-s] \\ M \cdot (x-a) + a + (x-(d-s))^4 & x \in [d-s, d] \end{cases}$$

where $M := 1 - \dfrac{s^4}{d-a}$. It is easy to show that g is $\mathcal{C}^3$, monotonically increasing and maps the endpoints of the interval to themselves. If $g^{-1}(c)$ is to the left of $d-s$, then the two parts to the left are contracted, only $[g^{-1}(c), d]$ is expanded. Thus the cross ratio is reduced. Now for given γ and ϵ, s has to be chosen independently of F and a, b, c, d.

Construction of g:



First $S(F \circ g)$ must be negative. As $S(F \circ g) = (SF \circ g) \cdot (Dg)^2 + Sg$ and $SF < -\epsilon$, this is surely the case if

$$\frac{Sg}{(Dg)^2} \leq \epsilon \ . \tag{4.3}$$

We will choose

$$s \leq \sqrt[4]{\gamma} \ . \tag{4.4}$$

From this, as $2\gamma \leq d-a$, we get that $s \leq \sqrt[4]{\frac{d-a}{2}}$, which implies $M > \frac{1}{2}$ and therefore $Dg > \frac{1}{2}$ (see definitions of M and g). Then (4.3) will hold if $Sg \leq \frac{\epsilon}{4}$. On $[a, d-s]$ $Sg = 0$ and on $[d-s, s]$ we have (still assuming (4.4) and therefore $M > \frac{1}{2}$)

$$Sg = \frac{24(x-d+s)}{M + 4(x-d+s)^3} - \frac{3}{2}(\frac{D^2 g}{Dg})^2 \leq \frac{24s}{M} < 48s$$

Hence $Sg \leq \frac{\epsilon}{4}$, if in addition to (4.4)

$$s \leq \frac{\epsilon}{200} \ . \tag{4.5}$$

Thus $S(F \circ g) < 0$ can be ensured by (4.4) and (4.5). Secondly, we want the linearly contracting part of g to extend at least until $g^{-1}(c)$, such that only the interval

$[g^{-1}(c), d]$ is expanded and the other intervals are uniformly contracted. Therefore we request $g^{-1}(c) \le d - s$, which is equivalent to $g(d-s) \ge c$. If we chose

$$s \le \frac{\gamma}{2}\ , \tag{4.6}$$

then (using $\gamma \le d - c$) we get that $g(d-s) \ge g(c + \frac{d-c}{2}) \ge M \cdot (c + \frac{d-c}{2} - a) + a$. Now

$$M \cdot (c - a + \tfrac{d-c}{2}) + a \ge c$$

is equivalent to

$$M \ge \frac{c-a}{c-a+\frac{1}{2}(d-c)} = \left(1 + \tfrac{(d-c)}{2(c-a)}\right)^{-1},$$

and the last term on the right is bounded from above:

$$\left(1 + \frac{(d-c)}{2(c-a)}\right)^{-1} \ \le\ \left(1 + \frac{\gamma}{2|I|}\right)^{-1}.$$

Using $\gamma < d - a$, we can ensure $M \ge \left(1 + \frac{\gamma}{2|I|}\right)^{-1}$ by choosing

$$s \le \sqrt[4]{\gamma \cdot \left(1 - \frac{1}{1 + \frac{\gamma}{2|I|}}\right)}\ . \tag{4.7}$$

Then (4.6) and (4.7) together imply $g(d-s) \ge c$.

Altogether, s can be chosen independently of F and a, b, c, d as the minimum of the upper bounds in (4.4)–(4.7). M will still vary with $d - a$, but as this is bounded by $|I|$ we have

$$M \le 1 - \frac{s^4}{|I|} =: p < 1 \ \text{ for any choice of } a, b, c, d \text{ and } F. \tag{4.8}$$

Therefore, we can give an upper bound for the cross ratio distortion caused by g:

The interval $[d, a]$ as a whole is invariant. $[a, g^{-1}(b)]$ and $[g^{-1}(b), g^{-1}(c)]$ are contracted by the same factor M, the corresponding difference quotients in the cross ratio distortion cancel each other out. Thus $D(g, [a, g^{-1}(b), g^{-1}(c), d])$ depends only on the change of the length of $[g^{-1}(c), d]$:

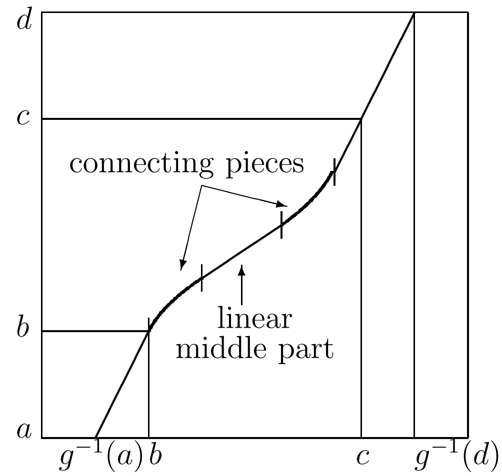
$$\begin{aligned} D(g, [a, g^{-1}(b), g^{-1}(c), d]) \ &=\ \frac{d - g^{-1}(c)}{d-c} \ =\ \frac{(d-a) - \frac{1}{M}(c-a)}{(d-c)} \ = \\ =\ \frac{d - c - (\frac{1}{M} - 1)(c-a)}{d-c} \ &=\ 1 - (\frac{1}{M} - 1) \cdot \frac{c-a}{d-c} \ \le \\ \le\ 1 - \underbrace{(\frac{1}{p} - 1) \cdot \frac{\gamma}{|I|}}_{\in(0,1)} \ &=:\ q \ <\ 1 \end{aligned}$$

If $\delta := \frac{1}{q} - 1 > 0$, then g reduces the cross ratio at least by the factor $q = \frac{1}{1+\delta}$. As F has to compensate this decrease, we get

$$D(F, [a, b, c, d]) > 1 + \delta \ .$$

In the case $c - b > \gamma$ the strategy is exactly the same, only g has to be constructed differently. Here it is simpler to leave the two points b and c in the middle invariant, and to expand the right and the left part of the interval uniformly. In order to do this, different linear parts have to be connected three times differentiable. In principle, any function with the third derivative point symmetric to the middle of the respective interval could be used to construct these connecting pieces, but trigonometric functions are probably the most easy to handle. Let therefore be

Construction of g in the 2. case:



$$h : [0, 1] \to [0, 1] \quad , \quad x \mapsto \frac{1}{2\pi^2} \left(\frac{1}{2}(2\pi x)^2 + \cos(2\pi x) - 1 \right) \ .$$

It is easy to check that $h(0) = h'(0) = h''(0) = h'''(0) = 0$, $h(1) = 1$, $h'(1) = 2$ and $h''(1) = h'''(1) = 0$. Now, for given $\epsilon > 0$, $\gamma > 0$ and $a < b < c < d$ (where we can assume that $\epsilon < 1$ and $\gamma < 1$), let

$$l \ := \ \frac{\gamma}{4} \ , \ \alpha \ := \ \frac{\epsilon l^3}{32\pi} \ , \ M \ := \ 1 + \frac{\alpha}{l} \ , \ M' \ := \ 1 - \frac{\alpha}{l} \ , \ a' \ := \ b - \frac{b-a}{M} = g^{-1}(a) \ ,$$

$$d' \ := \ c + \frac{d-c}{M} = g^{-1}(d) \ , \ h_{\gamma,\epsilon} \ : \ [0, l] \to \mathbb{R} \ , \ x \mapsto \alpha \cdot h(\frac{x}{l}) \ .$$

Again an easy calculation gives that $h_{\gamma,\epsilon}(l) = \alpha$ and $h'_{\gamma,\epsilon}(l) = \frac{2\alpha}{l}$. Now we can define g as

$$g : [a', d'] \to [a, d] \ , \ x \mapsto \begin{cases} a + M(x - a') & x \in [a', b] \\ b + \alpha + M'(x - b) - h_{\gamma,\epsilon}(b + l - x) & x \in [b, b + l] \\ b + \alpha + M'(x - b) & x \in [b + l, \frac{c+b}{2}] \\ b + \alpha + M'(x - b) + h_{\gamma,\epsilon}(x - \frac{c+b}{2}) & x \in [\frac{c+b}{2}, \frac{c+b}{2} + l] \\ d - M(d' - x) & x \in [\frac{c+b}{2} + l, d'] \end{cases}$$

From $\frac{\alpha}{l} < \frac{1}{2}$ follows $M' > \frac{1}{2}$ and therefore $Dg > \frac{1}{2}$. Combined with

$$\max_{x \in [a', b']} |g'''(x)| = \max_{x \in [0, l]} |h'''_{r,\epsilon}(x)| = \frac{\alpha}{l^3} \cdot \max_{x \in [0,1]} |h'''(x)| = \frac{\epsilon}{8} \ ,$$

this gives $Sg \le \frac{\epsilon}{4}$, so that again we have $S(F \circ g) < 0$. As in the first case g reduces the cross ratio:

$$D(g, [a', b, c, d']) = \underbrace{\frac{d-a}{d'-a'}}_{\le M} \cdot \frac{c-b}{c-b} \cdot \underbrace{\frac{b-a'}{b-a}}_{=\frac{1}{M}} \cdot \underbrace{\frac{d'-c}{d-c}}_{=\frac{1}{M}} \le \frac{1}{M}$$

The choice of M was independent of F and a, b, c, d. This completes the proof, just as in the first case.

$\square$

Now we turn to the class of systems

$$\mathcal{T}_s := \{(T, M) \in \mathcal{T}_m : T_\theta \in \mathcal{C}^3([a, b]),\ DT_\theta > 0,\ ST_\theta < 0\ \forall \theta \in \mathbb{T}^1\}.$$

As mentioned before, we have the following classification.

Theorem 4.2 *Let $(T, M) \in \mathcal{T}_s$. Then there are three possible cases:*

(i) *There exists one invariant graph φ with $\lambda(\varphi) \leq 0$.*

(ii) *There exist two invariant graphs φ and ψ with $\lambda(\varphi) < 0$ and $\lambda(\psi) = 0$.*

(iii) *There exist three invariant graphs $\varphi^- \leq \psi \leq \varphi^+$ with $\lambda(\varphi^-) < 0$, $\lambda(\psi) > 0$ and $\lambda(\varphi^+) < 0$.*

Regarding the topology of the invariant graphs, there are the following possibilities:

(i) *If the single invariant graph has negative Lyapunov-Exponent, it is always continuous. Otherwise the equivalence class contains at least an upper and a lower semi-continuous representative.*

(ii) *The upper invariant graph is upper semi-continuous, the lower invariant graph lower semi-continuous. If φ is not continuous and ψ (as an equivalence class) is only semi-continuous in one direction, then $\overline{\Phi}^{ess} = \overline{\Psi}^{ess}$.*

(iii) *ψ is continuous if and only if φ^+ and φ^- are continuous. Otherwise φ^- is at least lower semi-continuous and φ^+ is at least upper semi-continuous. If ψ is neither upper nor lower semi-continuous, then $\overline{\Phi^-}^{ess} = \overline{\Psi}^{ess} = \overline{\Phi^+}^{ess}$.*

For the asymptotic behaviour on m-a.e. fibres the following is true:

All points above the upper bounding graph converge to it (in the sense of $|T^n_\theta(x) - \varphi^+(\theta + n\omega)| \to 0\ (n \to \infty)$), the analogue is true for points below the lower bounding graph. Points between two graphs always converge to the one which has negative Lyapunov exponent.

If the system has an additional symmetry, this reduces the number of possibilities further:

Corollary 4.3 *Let $(T, M) \in \mathcal{T}_s$. If there exists an even integer k, such that $T_\theta(-x) = -T_{\theta+\frac{1}{k}}(x)$, then one of the following is true:*

(i) *There exists one invariant graph φ, which satisfies*

$$\varphi(\theta) = -\varphi(\theta + \frac{1}{k}) \quad m\text{-a.s.}\ .$$

(Continuity as in the theorem.)

(ii) There are three invariant graphs $\varphi^- \leq \psi \leq \varphi^+$, which satisfy

$$\varphi^+(\theta) = -\varphi^-(\theta + \frac{1}{k}) \quad \textit{and} \quad \psi(\theta) = -\psi(\theta + \frac{1}{k}) \quad m\text{-}a.s. \ .$$

If one of the three graphs is continuous, then so are the other two, if they are all non-continuous then $\overline{\Phi^-}^{ess} = \overline{\Psi}^{ess} = \overline{\Phi^+}^{ess}$.

Proof of theorem 4.2:
Obviously there exists at least one invariant graph, so first we show that there are at the most three of them:

Suppose there are four distinct invariant graphs $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \varphi_4$. Let $F(\theta) := \log CR[\varphi_1, \varphi_2, \varphi_3, \varphi_4](\theta)$, wherever all four graphs take different values. F is m-a.s. well-defined, and as we are only interested in integrals of F this is sufficient. For the cross ratio distortion of T_θ with respect to the four graphs we have

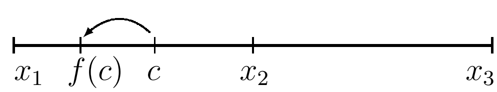
$$\frac{CR[\varphi_1, \varphi_2, \varphi_3, \varphi_4](\theta + \omega)}{CR[\varphi_1, \varphi_2, \varphi_3, \varphi_4](\theta)} > 1 \quad m\text{-a.s.} \ ,$$

hence the logarithm $F \circ r - F$ of this quotient is m.-a.s. positive and therefore has positive integral as well. As it has the constant function 0 as an integrable minorant, Lemma 3.6 yields that the integral is zero, giving a contradiction.

Now we turn to the Lyapunov exponents. Before going into detail, we sketch the crucial argument in the easier case of an unperturbed interval map:

Suppose $f : I \to I$ is a monotonically increasing interval map with fixed points $x_1 < x_2 < x_3$. If f has negative Schwarzian derivative, then this already determines the stability of the three fixed points, i.e. the signs of $\log Df(x_i)$:

The definition of the cross ratio distortion together with (4.2) immediately imply, that any point c between x_1 and x_2 is shifted toward x_1 by the action of f, i.e. $\frac{f(c)-f(x_1)}{c-x_1} < 1$. In the limit $c \to x_1$ this gives $Df(x_1) < 1$, where Lemma 4.1 is needed to obtain the strictness of the inequality, as mentioned before. If there are only two fixed points $x_1 < x_2$, the same argument still gives $Df(x_1) \cdot Df(x_2) < 1$.
Obviously, when considering quasiperiodically forced maps, this argument cannot be applied to a single fibre map, as the values of invariant graphs do of course not have to be fixed.



The action of a map f with negative Schwarzian derivative and three fixed points: Any point in the left interval is shifted to the left, analogously a point in the right interval is shifted to the right. This gives information about the derivative in the fixed points. In the case of quasiperiodically forced maps it determines the stability of the invariant graphs.

However, as above in the proof of the non-existence of four invariant graphs, we can integrate over the resulting inequalities, again after taking the logarithms. The reader should keep this simple picture in mind when contemplating the long inequalities in (ii) and (iii).

(i) A single invariant graph is, at the same time, upper and lower bounding graph. Lemma 3.5 therefore gives $\lambda(\varphi) \leq 0$.

(ii) Without loss of generality we can assume $\psi \leq \varphi$. As $ST_\theta(x)$ is strictly negative and continuous in (θ, x), there is an $\epsilon > 0$ such that $ST_\theta(x) < -\epsilon\ \forall (\theta, x) \in M$. Let $\gamma > 0$ be such that $m(A) > 0$ where $A := \{\theta : |\varphi(\theta) - \psi(\theta)| > 2\gamma\}$. Then, by Lemma 4.1, $\exists \delta > 0\ \forall \theta \in A\ \forall h < \frac{\gamma}{2}$:

$$\frac{T_\theta(\psi(\theta)+h) - T_\theta(\psi(\theta))}{h} \cdot \frac{T_\theta(\varphi(\theta)) - T_\theta(\varphi(\theta)-h)}{h} \leq$$
$$\leq \frac{1}{1+\delta} \cdot \frac{T_\theta(\varphi(\theta)) - T_\theta(\psi(\theta))}{\varphi(\theta) - \psi(\theta)} \cdot \frac{T_\theta(\varphi(\theta)-h) - T_\theta(\psi(\theta)+h)}{\varphi(\theta) - \psi(\theta) - 2h} .$$

(Compare the above inequality with the definition of the cross ratio distortion of T_θ with respect to the points $\psi(\theta) < \psi(\theta)+h < \varphi(\theta)-h < \varphi(\theta)$! To apply Lemma 4.1, note that the distance in the middle between $\psi(\theta)+h$ and $\varphi(\theta)-h$ is bounded from below by γ.)

Let $p := \log(1+\delta) > 0$. Applying the logarithm after taking the limit $h \to 0$ on both sides yields (using the invariance of the two graphs)

$$\log DT_\theta(\psi(\theta)) + \log DT_\theta(\varphi(\theta)) \leq 2 \cdot \log\left(\frac{\varphi(\theta+\omega) - \psi(\theta+\omega)}{\varphi(\theta) - \psi(\theta)}\right) - p .$$

Outside of A we can still use the negative Schwarzian derivative and obtain the same inequality without the additional term $-p$ (at least wherever the two graphs are no equal, i.e. m-almost surely). The function on the right is of the form $F \circ r - F$ again, moreover it is bounded, as the argument of the logarithm is a difference quotient. Integrating both sides gives (again by using Lemma 3.6)

$$\lambda(\varphi) + \lambda(\psi) \leq -p \cdot m(A) < 0 .$$

Two invariant graphs which are next to each other cannot both have negative Lyapunov exponents, a consequence of Corollary 3.4. On the other hand neither of the two Lyapunov exponents can be positive, as both graphs are bounding graphs (Lemma 3.5). Thus, one of them must equal zero.

(iii) As in case (ii), the sum of two Lyapunov exponents is always strictly negative. Therefore, it suffices to show that the inner graph always has positive Lyapunov exponent.

Let therefore $\gamma > 0$ be such that $m(A) > 0$ where $A := \{\theta : |\varphi^+(\theta) - \psi(\theta)| > 2\gamma$ and $|\varphi^-(\theta) - \psi(\theta)| > 2\gamma\}$. Then, again by Lemma 4.1, $\exists \delta > 0\ \forall \theta \in A\ \forall h < \gamma$:

$$\frac{T_\theta(\varphi^+(\theta)) - T_\theta(\varphi^-(\theta))}{\varphi^+(\theta) - \varphi^-(\theta)} \cdot \frac{T_\theta(\psi(\theta)+h) - T_\theta(\psi(\theta))}{h} \geq$$
$$\geq (1+\delta) \cdot \frac{T_\theta(\varphi^+(\theta)) - T_\theta(\psi(\theta)+h)}{\varphi^+(\theta) - \psi(\theta) - h} \cdot \frac{T_\theta(\psi(\theta)) - T_\theta(\varphi^-(\theta))}{\psi(\theta) - \varphi^-(\theta)} .$$

After taking the limits, applying the logarithm and integrating both sides again, we obtain

$$\lambda(\psi) \geq p \cdot m(A) > 0 \ ,$$

where $p := \log(1+\delta)$. All other terms are of the kind $F \circ r - F$ and have integral zero by Lemma 3.6 .

Topology of the invariant graphs:

(i) As φ is at the same time upper and lower bounding graph, it is semi-continuous in both directions (as an equivalence class). As mentioned before (proof of Prop. 3.7), this means that φ is cts. if $\lambda(\varphi) < 0$.

(ii) As φ and ψ are the only invariant graphs, they are semi-continuous as bounding graphs. The statement about the essential closures follows directly from the arguments used in the proof of Prop. 3.7 .

(iii) The system (T, M) can be extended to a diffeomorphism on $\mathbb{T}^1 \times \mathbb{R}$. If the inner invariant graph is continuous, then it is a stable invariant graph with respect to the inverse of this extension. Thus the two outer graphs are semi-continuous in one direction by Corollary 3.4, and in the other as bounding graphs. If on the other hand the two bounding graphs are continuous, the continuity of the inner graph follows again from Corollary 3.4 . (The possibility that there is no continuous invariant graph, but only two semi-continuous graphs in the equivalence class is excluded by the negative Lyapunov exponent again.)

If ψ is not semi-continuous in any direction, then the bounding graphs cannot be continuous (Corollary 3.4 once more). Therefore, the essential closures are equal by Prop. 3.7 .

The statement about the convergence behaviour follows directly from Corollary 3.4 .

□

Proof of corollary 4.3:
If φ is an invariant graph of a system with the given symmetry, then

$$\tilde{\varphi}(\theta) := -\varphi(\theta + \tfrac{1}{k})$$

defines another invariant graph:

$$\begin{aligned} T_\theta(\tilde{\varphi}(\theta)) \ &= \ T_\theta(-\varphi(\theta + \frac{1}{k})) \ = \\ &= \ -T_{\theta+\frac{1}{k}}(\varphi(\theta + \frac{1}{k})) \ = \ -\varphi(\theta + \omega + \frac{1}{k}) \ = \ \tilde{\varphi}(\theta + \omega) \end{aligned}$$

Due to the symmetry, one bounding graph must be the mirror image of the other in this sense. As their Lyapunov exponents must therefore be equal, case (ii) from the theorem can be excluded. The inner invariant graph can only be symmetric to itself (i.e. to a graph in the same equivalence class), just as in the case of a single invariant graph. The statement about the continuity follows from the theorem and the fact that, due to the symmetry again, either both or none of the bounding graphs are continuous. $\square$

Remark 4.4 The assumptions of the theorem (and the corollary) can be weakened to some extend:

Suppose T has only weakly monotone fibre maps, but the function $\theta \mapsto \inf_{x\in[a,n]} \log DT_\theta(x)$ is still integrable. Further assume $ST_\theta(x) < -\epsilon\ \forall x \in [a, b]$ holds on a set $A \subseteq \mathbb{T}^1$ of positive measure and $ST_\theta(x) \leq 0\ \forall x \in [a, b]$ holds m-a.s. .

The proof above (and thus the statement of the theorem) will then stay true with only slightest modifications needed. The only difference is, that the continuity of the inner invariant graph does not imply continuity of the bounding graphs anymore. In fact the classical example from Grebogi et al. in [5], to which the theorem then applies, provides a counterexample to this.

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