Towards a Classification for Quasiperiodically Forced Circle Homeomorphisms

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20th August 2005

Abstract

Poincaré’s classification of the dynamics of homeomorphisms of the circle is one of the earliest, but still one of the most elegant, classification results in dynamical systems. Here we generalize this to quasiperiodically forced circle homeomorphisms homotopic to the identity, which have been the subject of considerable interest in recent years. Herman already showed two decades ago that a unique rotation number exists for all orbits in the quasiperiodically forced case. However, unlike the unforced case, no a priori bounds exist for the deviations from the average rotation. This plays an important role in the attempted classification, and in fact we define a system as ρ-bounded if such deviations are bounded and ρ-unbounded otherwise. For the ρ-bounded case we prove a close analogue of Poincaré’s result: if the rotation number is rationally related to the rotation rate on the base then there exists an invariant strip (the appropriate analogue for fixed or periodic points in this context), otherwise the system is semi-conjugate to an irrational translation of the torus. In the ρ-unbounded case, where neither of these two alternatives can occur, we show that the dynamics are always topologically transitive.

1 Introduction

One of the most fundamental goals of the theory of dynamical systems is to classify systems according to their qualitative dynamical behavior. One of the earliest and most important such result was Poincaré’s classification of homeomorphisms of the circle (e.g. [dMvS93, KH97]). Recall that for any such map f one can define the rotation number ρ, which measures the average speed of rotation of orbits around the circle. Poincaré proved that this was the same for all orbits and hence was an invariant of f. This leads to the following classification:

- If the rotation number is rational ρ = p/q, then f has a periodic orbit of period q.
- If the rotation number is irrational, then f is semi-conjugate to the irrational (rigid) rotation \( R_\rho(\theta) = \theta + \rho \pmod{1} \). Recall that this means that

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there is a continuous surjective \( h \) such that \( f \circ h = h \circ R_\rho \). The map \( h \) is called a semi-conjugacy of \( f \) to \( R_\rho \).

The picture in the irrational case was further completed by Denjoy (e.g. [dMvS93, KH97]):

- If the rotation number of \( f \) is irrational and the derivative of \( f \) has bounded variation, then the semi-conjugacy \( h \) is a homeomorphism. In such a case \( h \) is called a conjugacy and we say that \( f \) is conjugate to \( R_\rho \).

Subsequent work, which formed a key part of the so called KAM theory, then characterized the conditions under which the conjugacy could be guaranteed to be smooth or even analytic (e.g. [dMvS93, KH97]). It is natural to attempt to generalize these results to higher dimensions, that is to homeomorphisms of the torus \( \mathbb{T}^n \). However, even in the case \( n = 2 \) this turns out to be difficult. Firstly one finds that, in general, the rotation vector depends on the orbit. Hence, instead of a unique, well defined rotation number one obtains a rotation set (see [MZ90]). Furthermore, even if this is reduced to a single rotation vector, examples of Furstenberg (see [Mañ87]) and Herman ([Her83]) show that the dynamics may still not be compatible with either of the two cases in Poincaré’s classification.

Here we study a class of systems which are somewhat intermediate between dimension one and two, namely quasiperiodically forced systems, which have recently been the subject of considerable interest. Specifically, we shall consider quasiperiodically forced circle maps, which are continuous maps of the form

\[
T : \mathbb{T}^2 \to \mathbb{T}^2, \ (\theta, x) \mapsto (\theta + \omega, T_\theta(x)),
\]

where the fibre maps \( T_\theta \) (defined by \( T_\theta(x) = \pi_2 \circ T(\theta, x) \)) are all orientation-preserving circle homeomorphisms and \( \omega \) is irrational. Further, we will restrict to the case where \( T \) is homotopic to the identity on \( \mathbb{T}^n \).\(^1\) The class of such systems will be denoted by \( \mathcal{T}_{\text{hom}} \).

On the one hand, the skew product structure implies that such maps have similar order-preserving properties to homeomorphisms of the circle. This is sufficient to ensure that the rotation vector is uniquely defined (see Thm. 2.16 below). On the other hand, the examples in [Fur61] and in [Her83] show that the dynamical phenomena which can be found in \( \mathcal{T}_{\text{hom}} \) are already much richer than those in the class of circle homeomorphisms.

In addition, quasiperiodically forced systems are also interesting in their own right, as they occur in various situations in physics. For example, the quasiperiodically forced Arnold circle map serves as a simplified model for an

\(^1\)Of course, it also makes sense to consider quasiperiodically forced systems which are not homotopic to the identity. However, the fact that it is not possible to work with continuous lifts or to define a rotation number in such a case makes the situation quite different from the one considered here. For example, as the induced map on the fundamental group is not the identity, it is obvious that there cannot be any invariant curves (or, in fact, invariant strips) for such maps, and similarly they cannot be semi-conjugate to a torus translation. In [KKHO03] it is shown that such maps are always topologically transitive. The proof there is given for a specific example system, but works unchanged in the general case of a quasiperiodically forced system which is not homotopic to the identity. In fact, the proof of Theorem 4.1 below is just a slight modification of the same argument. We are not aware of any other rigorous results for systems not homotopic to the identity. We would like to thank an anonymous referee for bringing the reference [KKHO03] to our attention.
oscillator forced at two incommensurate frequencies (e.g. [DGO89]), and the spectral theory of certain discrete Schrödinger operators is intimately related to the dynamical properties of quasiperiodically forced Möbius transformations (e.g. [AK]).

Our aim here is thus to derive a classification for maps of the form (1.1) which is analogous to that for homeomorphisms of the circle. It turns out, however, that a key boundedness property in the one dimensional case no longer necessarily holds for quasiperiodically forced maps. In particular if \( \hat{f} \) is the lift to \( \mathbb{R} \) of a homeomorphism of the circle, then the monotonicity of \( \hat{f} \) and periodicity of \( \hat{f} - \text{Id} \) implies that

\[
|\hat{f}^n(x) - \hat{f}^n(y)| \leq |x - y| + 1 \quad \forall n \in \mathbb{N}, \ x, y \in \mathbb{R}.
\]

This gives a simple proof of the existence of the rotation number \( \rho \) and the following uniform bound for deviations from uniform rotation

\[
|\hat{f}^n(x) - x - n\rho| \leq 1 \quad \forall n \in \mathbb{N}, \ x \in \mathbb{R}.
\]

Interestingly this holds even if \( \hat{f} \) is merely monotone increasing, but not necessarily continuous [RT86, RT91]. In the case of quasiperiodically forced maps, any given fibre map \( T_\theta \) is a homeomorphism of the circle, and hence the analogue of (1.2) holds within each fibre. This is sufficient to prove the existence of a rotation number and fairly elementary arguments from ergodic theory then show that this rotation number is the same for all fibres [Her83, SFGP02]. Unfortunately, although all fibres therefore rotate at the same average rate, the deviations between fibres need not be bounded and hence the analogue of (1.3) no longer need hold, even within a single fibre. Surprisingly, if the analogue of (1.3) does hold for even a single orbit, then [SFGP02] show that it holds for all orbits. Here we call such a system \( \rho \)-bounded. The main aim of this paper is to show that for \( \rho \)-bounded systems we have a classification similar to that for (unforced) circle maps. The details of this are given in Section 3 below. We then go on to discuss the \( \rho \)-unbounded case in Section 4. While we are unable to give an elegant classification according to the rotation number, as in the \( \rho \)-bounded case, we are at least able to show that all systems in this class are topologically transitive.

We begin by collecting a variety of preliminary facts and constructions, which are required in the succeeding sections. We also discuss a result of Furstenberg which can be seen as an analogue to Poincaré’s classification, but in a measure-theoretic rather than a topological sense.

## 2 Preliminaries

### 2.1 Notation

In the following \( m \) will denote Lebesgue measure on \( \mathbb{T}^1 \), \( \lambda \) Lebesgue measure on \( \mathbb{T}^2 \) and \( \pi_i : \mathbb{T}^2 \to \mathbb{T}^1 \ (i = 1, 2) \) the projection to the respective coordinate. We shall denote the \( q \)-fold cover \( \mathbb{R}/q\mathbb{Z} \) of the circle \( \mathbb{T}^1 \) by \( \mathbb{T}^1_q \). If there is no possible ambiguity, all projections from either \( \mathbb{R} \) or \( \mathbb{T}^1_q \) to \( \mathbb{T}^1 \) or from \( \mathbb{R}^2, \mathbb{T}^1 \times \mathbb{R}, \mathbb{T}^1_q \times \mathbb{T}^1_k \) etc. to \( \mathbb{T}^2 \) will be denoted by \( \pi \). Lifts of \( T \) to the covering spaces \( \mathbb{T}^1 \times \mathbb{R} \) or \( \mathbb{R}^2 \) will be denoted by \( \hat{T} \) and variables in the respective covers will be written \( \hat{\theta}, \hat{x} \).
and so on. It will also be useful to define $T^n := \{\{\theta_1, \ldots, \theta_d\} \mid \theta_j \in T^1, \theta_i \neq \theta_j \text{ if } i \neq j\}$.

For any set $A \subseteq T^2$, $T^1 \times \mathbb{R}$ or $\mathbb{R}^2$ let $A_\theta := \pi_2(A \cap \pi_1^{-1}\{\theta\})$ be the restriction of $A$ to the fibre above $\theta$. Thus, either $A_\theta = \{x \in T^1 \mid (\theta, x) \in A\}$ or $A_\theta = \{x \in \mathbb{R} \mid (\theta, x) \in A\}$. For any pair of functions $\varphi, \psi : T^1 \to \mathbb{R}$ we let $[\varphi, \psi]$ be the region $\{(\theta, \bar{x}) \in T^1 \times \mathbb{R} \mid \varphi(\theta) \leq \bar{x} \leq \psi(\theta)\}$ between the graphs of $\varphi$ and $\psi$, and similarly for functions $T^1 \to T^1$ or $\mathbb{R} \to \mathbb{R}$. The notation $\varphi < \psi$ will mean $\varphi(\theta) < \psi(\theta)$ for all $\theta \in T^1$ and similarly for $\varphi \leq \psi, \varphi = \psi, \varphi > \psi$ etc.

Finally, when considering fibre maps of iterates of $T$ or their inverses we use the convention $T^n_\theta := (T^n)_\theta \quad \forall n \in \mathbb{Z}$, so that $T^n_\theta(x) = \pi_2 \circ T^n(\theta, x)$.

### 2.2 Minimal Sets for Monotone Quasiperiodically Forced Maps

We will mainly work with the lift $\hat{T}$ of a map $T \in \mathcal{T}_{\text{hom}}$ rather than with the original map $T$, so we first collect some basic statements about such lifts. The main properties we need are the skew product structure and the monotonicity of the fibre maps. The fact that the lift (or more precisely $\hat{T} - (0, Id)$) is also periodic (in $x$) will only be used in subsequent sections. Therefore, throughout this section we assume that $\hat{T} : T^1 \times \mathbb{R} \to T^1 \times \mathbb{R}$ has skew product structure as in (1.1) and that all fibre maps $\hat{T}_\theta$ are (non-strictly) monotonically increasing, that is $\hat{T}_\theta(x) \leq \hat{T}_\theta(y)$ for all $x \leq y$.

Obviously there cannot be any fixed of periodic points for $\hat{T}$, and due to the minimality of the forcing irrational rotation any compact invariant set must project down to the whole circle. Therefore it is natural to replace fixed points by invariant graphs:

**Definition 2.1 (Invariant Graphs)**

Let $\hat{T}$ be as above and suppose a function $\varphi : T^1 \to \mathbb{R}$ satisfies

$$\hat{T}_\theta(\varphi(\theta)) = \varphi(\theta + \omega).$$

Then $\varphi$ is called a $\hat{T}$-invariant graph.

Note that in the unforced case where $\hat{T}_\theta$ is independent of $\theta$, a fixed point of $\hat{T}_\theta$ becomes a horizontal invariant graph for $\hat{T}$. As long as no ambiguities can arise, the point set $\Phi := \{(\theta, \varphi(\theta)) \mid \theta \in T^1\}$ will also be called an invariant graph. There is a natural relation between compact invariant sets and invariant graphs (see [Sta03]):

**Lemma 2.2**

Let $A \subset T^1 \times \mathbb{R}$ be a compact $\hat{T}$-invariant set. Then

$$\varphi^+_A(\theta) := \sup A_\theta \quad \text{and} \quad \varphi^-_A(\theta) := \inf A_\theta$$

both define invariant graphs. Furthermore $\varphi^+_A$ is upper semi-continuous and $\varphi^-_A$ is lower semi-continuous.

If we apply this procedure to an invariant graph $\varphi$ with compact closure $\overline{\Phi}$, then we simplify notation by writing $\varphi^+ := \varphi^+_{\overline{\Phi}}$ and $\varphi^- := \varphi^-_{\overline{\Phi}}$. Further, we write
\( \varphi^+ \) instead of \( (\varphi^+)^- \), etc. Particularly interesting is the case where \( A \) is a minimal set in the sense of topological dynamics (so that every orbit is dense in \( A \), e.g. \cite{KH97}); see \cite{Sta03} again:

**Lemma 2.3**

Let \( A \) be a minimal set for \( \hat{T} \). Then \( \varphi_A^- = \varphi_A^+ \), \( \varphi_A^+ = \varphi_A^- \) and \( \Phi_A = \Phi_A = A \). Furthermore, the set \( \{ \theta \in T^1 | \varphi_A^+(\theta) = \varphi_A^-(\theta) \} \) is residual.

This motivates the following definition:

**Definition 2.4 (Pinched Sets, Reflexive Graphs)**

(a) A compact set \( A \subseteq T^1 \times \mathbb{R} \) with \( \pi_1(A) = T^1 \) is called pinched if \( \{ \theta \in T^1 | \varphi_A^+(\theta) = \varphi_A^-(\theta) \} \) is residual.

(b) An upper semi-continuous graph \( \varphi : T^1 \to \mathbb{R} \) is called reflexive if \( \varphi^- = \varphi \). Similarly, a lower semi-continuous graph \( \varphi \) is called reflexive if \( \varphi^+ = \varphi \).

We can thus restate Lemma 2.3 in the following way: a compact \( \hat{T} \)-invariant set \( A \) is minimal if and only if it is the closure of a reflexive graph, and the bounding graphs of minimal sets are always reflexive. Furthermore, every minimal set and thus the closure of any reflexive invariant graph is pinched. (In fact, the closure of any semi-continuous graph is pinched, see \cite{Sta03}.)

There is a simple way of producing reflexive graphs from semi-continuous ones:

**Lemma 2.5**

Suppose \( \varphi : T^1 \to \mathbb{R} \) is upper semi-continuous. Then \( \varphi^- \) is reflexive. Similarly, if \( \varphi \) is lower semi-continuous then \( \varphi^+ \) is reflexive.

We prove the case where \( \varphi \) is upper semi-continuous, the lower semi-continuous case is analogous. We have to show that \( \varphi^- = \varphi^+ \). First of all, as \( \varphi^- \leq \varphi \) and due to the semi-continuity of the two graphs, the set \( [\varphi^-, \varphi] \) is compact and contains \( \Phi^- \). Therefore it also contains \( \Phi^+ \), so that \( \varphi^- \leq \varphi^+ \leq \varphi \) and similarly \( \varphi^- \leq \varphi^+ \leq \varphi \). But, as \( \varphi^+ \) is lower semi-continuous, the set \( [\varphi^+, \varphi] \) is compact as well, consequently it contains \( \Phi^- \) and thus also \( \Phi^+ \). This proves the reverse inequality \( \varphi^+ \leq \varphi^- \).

Another nice property of minimal sets is that they are strictly ordered. We introduce the following notation to describe the order of sets: If \( A, B \) are bounded subsets of \( T^1 \times \mathbb{R} \) with \( \pi_1(A) = \pi_1(B) = T^1 \), then

\[
A \preccurlyeq B :\Rightarrow \quad \varphi_A^- \leq \varphi_B^- \quad \text{and} \quad \varphi_A^+ \leq \varphi_B^+
\]

\[
A \prec B :\iff \quad \varphi_A^+ < \varphi_B^-. \quad \text{We then have (see \cite{Sta03} again):}
\]

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2In order to see that the closure of a reflexive invariant graph is minimal, suppose \( \varphi^+ \) is an upper semi-continuous reflexive graph and \( \varphi^- := \varphi^+ \). As there cannot be any other semi-continuous invariant graph between \( \varphi^+ \) and \( \varphi^- \) (by reflexiveness), the bounding graphs of any compact invariant set \( B \) contained in \( \Phi^+ \) must be \( \varphi^+ \) and \( \varphi^- \) again, such that \( B = \Phi^+ \).
Lemma 2.6
If \( A, B \subseteq \) are minimal w.r.t. \( \hat{T} \), then either \( A \prec B, A = B \) or \( A \succ B \).

Using Lemma 2.5, it is now easy to see that for any compact invariant set \( A \) the 'highest minimal set' it contains is given by \( \Phi_A^- \).

Note that if \( A \) is a compact invariant set then so is \([\varphi_A^-, \varphi_A^+]\), but the two sets are not necessarily equal (examples of this for so-called pinched skew products can be found in [Jág04]). As it will sometimes be convenient to work with the second, filled-in set rather than with the first, we make the following definition:

Definition 2.7 (Filled-In Sets, Minimal Strips)
If \( A \subseteq \mathbb{T}^1 \times \mathbb{R} \) is a compact set with \( \pi_1(A) = \mathbb{T}^1 \), we denote the filled-in set by
\[ A^\text{fill} := [\varphi_A^-, \varphi_A^+] \].

\( A \) is called a strip if it consists of one interval on every fibre (i.e. \( A = A^\text{fill} \)). If the two bounding graphs are a complementary pair of reflexive graphs (that is \( \varphi_A^+ = \varphi_A^- \) and \( \varphi_A^+ = \varphi_A^- \)), \( A^\text{fill} \) is called a minimal strip.

Remark 2.8
(a) As there can be no other semi-continuous graph between a pair of reflexive graphs, a minimal strip cannot contain any smaller strip.

(b) Note that a priori minimality of strips is a purely topological property and not necessarily related to any dynamics.

(c) Similarly to minimal sets, the intersection of two invariant minimal strips is either empty or the two strips are equal. However, in the case of strips we do not even have to invoke the dynamics: two minimal strips are already equal if their intersection projects down to the whole circle. This follows from the fact that the intersection is a strip in this case, and a minimal strip cannot strictly contain any smaller strip.

(d) Similar to minimal sets, two disjoint strips are strictly ordered. This can be seen as follows: If \( A, B \) are both strips, then due to the semi-continuity of the bounding graphs both \( \{\theta \in \mathbb{T}^1 \mid \varphi_A^-(\theta) < \varphi_B^-(\theta)\} \) and \( \{\theta \in \mathbb{T}^1 \mid \varphi_B^+(\theta) < \varphi_A^+(\theta)\} \) are open. As both sets together cover the whole circle, one of them must be empty.

(e) Below we will also work with lifts to \( \mathbb{R}^2 \) instead of \( \mathbb{T}^1 \times \mathbb{R} \). We call a set \( A \subseteq \mathbb{R}^2 \) a strip if it consists of one interval on every fibre, projects down to all of \( \mathbb{R} \) and \( \pi_1^{-1}(K) \cap A \) is compact for every compact set \( K \subseteq \mathbb{R} \). The above definitions of bounding graphs, reflexivity, filled-in sets, minimal strips and the order relations between sets can all be applied to strips in \( \mathbb{R}^2 \) in the same way.

2.3 Invariant Graphs and Invariant Strips
Invariant graphs are the natural analogue of fixed points for quasiperiodically forced monotone maps. When we come to study quasiperiodically forced circle maps we also need to generalize the concept of periodic orbits to that of periodic graphs. Since such graphs may wrap around the torus more than once in the \( \theta \)-direction, the following definition is a little bit more complicated than Def. 2.1.
Definition 2.9 (p, q-Invariant Graphs)
Let $T \in T_{\text{hom}}$, $p, q \in \mathbb{N}$. A p,q-invariant graph is a measurable function $\varphi : T^1 \to T^p_q \theta \mapsto \{\varphi^i_j(\theta) \mid 1 \leq i \leq p, 1 \leq j \leq q\}$ with

$$T\varphi(\{\varphi^1_1(\theta), \ldots, \varphi^p_q(\theta)\}) = \{\varphi^1_1(\theta + \omega), \ldots, \varphi^p_q(\theta + \omega)\}$$ \hspace{1cm} (2.2)

for m.a.e. $\theta \in T^1$, which satisfies the following:

(i) $\varphi$ cannot be decomposed (in a measurable way) into disjoint subgraphs $\varphi^1, \ldots, \varphi^m$ ($m \in \mathbb{N}$) which also satisfy (2.2).

(ii) $\varphi$ can be decomposed into $p$ periodic $q$-valued subgraphs $\varphi^1, \ldots, \varphi^p$.

(iii) The subgraphs $\varphi^1, \ldots, \varphi^p$ cannot further be decomposed into invariant or periodic subgraphs.

If $p = q = 1$, $\varphi$ is called a simple invariant graph. The point set $\Phi := \{\{\theta, \varphi^i_j(\theta)\} \mid \theta \in T^1, 1 \leq i \leq p, 1 \leq j \leq q\}$ will also be called an invariant graph, but labeled with the corresponding capital letter. An invariant graph is called continuous if it is continuous as a function $T^1 \to T^p_q$.

The weakne ss of this classification is that it does not for example distinguish between a continuous invariant curve and an invariant graph which is dense in the whole torus. These are clearly different situations from a topological point of view. On the other hand, it would be far too restrictive to consider only continuous invariant graphs. Thus, for instance, Herman’s examples show that it is possible to have $\rho$-bounded systems with a rational rotation number, but no continuous invariant graph ([Her83], Proposition 4.6 and Section 4.14). In such cases the invariant graph appears to have been replaced by a more complex set, which often has bifurcated from a
continuous invariant graph. We have already seen a good candidate for such a set in the previous section in the form of an invariant strip, with its related semi-continuous invariant bounding graphs. This is exactly the concept we want to use now in order to complement Furstenberg’s result with a topological classification. Note that there is no straightforward way to define semi-continuity for a function $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$. In order to obtain some kind of semi-continuity property for the bounding graphs of a strip in $\mathbb{T}^2$ we therefore include the existence of an appropriate ‘reference curve’ in the definition of an invariant strip. Such a curve also ensures the lifting properties to the cover that we shall require. An appropriate definition is given by

**Definition 2.11 (q-Curves)**

Let $T \in \mathcal{T}_{\text{hom}}$.

(a) An open $T$-invariant set $U$ is called a 1, $q$-invariant open tube if there exists a q-curve $\gamma$ such that for all $\theta \in \mathbb{T}$ $U_\theta$ consists of $q$ disjoint open and nonempty intervals $U_{\theta,j}$ and $\gamma_j(\theta) \in U_{\theta,j} \forall j = 1, \ldots, q$.

(b) When $U^1$ is a 1, $q$-invariant open tube for $T^p$ containing the q-curve $\Gamma^1$ and the sets $U^i := T^{i-1}(U^1) \ (i = 1, \ldots, p)$ are pairwise disjoint, then $U := \bigcup_{i=1}^p U^i$ is called a $p, q$-invariant open tube, each of its connected components $U^i$ containing the q-curve $\Gamma^i := T^{i-1}(\Gamma^1)$.

(c) A compact invariant set $A$ is called a $p, q$-invariant strip if its complement is a $p, q$-invariant open tube. If $p = q = 1$ then $A$ is called a simple invariant strip.

**Remark 2.13**

Suppose $A$ is a $p, q$-invariant strip. Then we can define a pair of $p, q$-invariant graphs by

$$\varphi^i_j(\theta) = \begin{cases} \inf \{ x \in [0, 1) \mid \gamma^{i+1}_j(\theta) - x \in A_\theta \} & \text{if } i \in \{1, \ldots, p-1\} \\
\gamma^{i+1}_j(\theta) & \text{if } i = p \end{cases}$$
and
\[ \psi^i_j(\theta) = \gamma^i_j(\theta) + \inf\{x \in [0,1) \mid \gamma^i_j(\theta) + x \in U^i_\theta \}, \]
where \( U = \bigcup_{i=1}^p U^i \) is the invariant open tube complementary to \( A \) and the \( \gamma^i \) are the \( q \)-curves contained in \( U^i \). In analogy to Lemma 2.2, one may call \( \varphi \) and \( \psi \) are the upper and lower bounding graphs of \( A \), respectively. Of course, a continuous invariant graph is a special case of such a bounding graph, and also of an invariant strip itself.

As mentioned already, using the \( q \)-curves contained in an open tube we can lift these objects to the covering spaces \( \mathbb{T}^1 \times \mathbb{R} \) or \( \mathbb{R}^2 \): First of all, if \( \gamma \) is a \( q \)-curve then there are \( q \) different lifts \( \hat{\gamma}^1 < \ldots < \hat{\gamma}^q \) such that \( \hat{\gamma}^i(0) \in [0,1) \). From these we can produce all possible lifts by the addition of an integer, i.e. \( \hat{\gamma}^{i+q} := \hat{\gamma}^i + n \). If \( \gamma \) belongs to the 1, \( q \)-invariant tube \( U \) we can lift \( U \) ‘around’ the respective lifts of \( \gamma \) and obtain, up to addition of integers, \( q \) different lifts \( \hat{U}^1, \ldots, \hat{U}^q \) of \( U \). The important thing now is to see that these also have a certain invariance property: they might not be invariant under a lift of \( T \) itself, but at least it is possible to choose a lift of \( T^q \) which leaves them invariant. Of course, the same holds for 1, \( q \)-invariant strips.

In order to see this we choose a lift \( \hat{T} \) of \( T \) such that \( \hat{T}(0, \hat{\gamma}_1(0)) \in \hat{U}^i \) for some \( i \in \{1, \ldots, q\} \) and claim that \( \hat{T}(\hat{U}^1) \subseteq \hat{U}^i \), which then immediately implies \( \hat{T}(\hat{U}^1) = \hat{U}^i \). Otherwise let \( \hat{\theta}^* := \inf\{\hat{\theta} \geq 0 \mid \hat{T}(\hat{\theta}, \hat{\gamma}_1(\hat{\theta})) \notin \hat{U}^i \} \). Then \( \hat{T}(\hat{\theta}^*, \hat{\gamma}_1(\hat{\theta}^*)) \) is contained in some other lift \( \hat{U}^* \) of \( U \). But as \( \hat{U}^* \) is open this means that the set contains \( \hat{T}(\hat{\theta}, \hat{\gamma}_1(\hat{\theta})) \) for all \( \hat{\theta} \) from a whole neighborhood of \( \hat{\theta}^* \), contradicting the definition of \( \hat{\theta}^* \). Now the monotonicity of the fibre maps implies \( \hat{T}(\hat{U}^n) = \hat{U}^{n+i-1} \forall n \in \mathbb{N} \) and consequently \( \hat{T}(\hat{U}^1) = \hat{U}^1 + (0,i) \). But this means we can choose another lift of \( T^q \) such that the \( \hat{U}^n \) are left invariant.

Now we can prove two elementary statements about invariant strips, which will simplify dealing with these objects later on.

**Lemma 2.14**

Let \( T \in \mathcal{T}_{\text{hom}} \). Then every compact invariant set \( A \) which consists of exactly one non-trivial interval on every fibre (i.e. \( A_\theta \neq \emptyset \) or \( \mathbb{T}^1 \)) is a simple invariant strip.

**Proof:**
Obviously \( U := A^c \) is an open set which also consists of exactly one non-trivial interval on every fibre. It remains to show that there exists a 1-curve in \( U \).

To that end, note that as \( U \) is open and \( U_\theta \neq \emptyset \) for any \( \theta \in \mathbb{T}^1 \) we can choose an open box \( W_\theta \subseteq U \) with \( \theta \in I_\theta := \pi_1(W_\theta) \). By compactness, finitely many intervals \( I_{\theta_1}, \ldots, I_{\theta_n} \) cover the whole circle. Let \( I_j \subseteq I_{\theta_j} \) be disjoint open intervals with \( \mathbb{T}^1 \subseteq \bigcup_{j=1}^n I_j \). Over any of these intervals there is a constant line segment contained in \( U \), and as \( A \) only consists of one interval on every fibre we can always join two adjacent segments by a vertical line in one of the two directions on the circle. That way we get a closed line in \( U \) which wraps once around the torus in the \( \theta \)-direction. Due to the vertical parts this line is not yet the graph of a continuous function, but as \( U \) is open we can slightly tilt them without leaving \( U \) to obtain the 1-curve we require.

\( \square \)
Lemma 2.15

\( T \in \mathcal{T}_{\text{hom}} \) has a \( p,q \)-invariant strip if and only if there exist numbers \( n,l,k \in \mathbb{N} \) such that a lift of \( T^n \) to \( T_1^l \times T_k^1 \) has a simple invariant strip.

Proof:

\( \Rightarrow \): As \( T \) has a \( p,q \)-invariant strip there exists a \( 1,q \)-invariant open tube \( U \) for \( T^p \).

As argued above, a lift \( \hat{U} \) of \( U \) to \( \mathbb{R}^2 \) is invariant under a suitable lift of \( T^{pq} \). If \( \hat{U} \subseteq \hat{U}^1 \) is a lift of the corresponding \( q \)-curve and \( \hat{\gamma}(\hat{\theta} + q) = \hat{\gamma}(\hat{\theta}) + k \), then \( \hat{U}_{\hat{\theta} + q} = \hat{U}_{\hat{\theta}} + k \). Consequently the projection of \( U \) onto \( T_1^l \times T_k^1 \) is a simple invariant open tube with respect to a suitable lift of \( T^{pq} \).

\( \Leftarrow \): Let \( \hat{F} : T_1^l \times T_k^1 \) be the lift of \( T^n \) and \( \hat{A} \) the corresponding simple invariant strip. W.l.o.g. we can assume that \( \hat{A} \) does not contain any other invariant strip. (Otherwise we lift \( \hat{A} \) to \( \mathbb{R}^2 \), take the highest minimal strip it contains as described in Section 2.2 and project it down again. This new strip will have the required property then.) As \( \hat{A} + (0,1) \) is a minimal invariant strip as well, it is disjoint from \( \hat{A} \) (unless \( k = 1 \)). Hence the projection of \( \hat{A} \) to \( T_1^l \times T_k^1 \) consists of one non-trivial interval on every fibre, and using Lemma 2.14 we see that it is a simple invariant strip for the corresponding lift of \( T^n \). Therefore we can assume \( k = 1 \). Likewise we can assume that \( l \) is minimal, i.e. that \( \hat{A} + (i,0) \) is disjoint from \( \hat{A} \) for all \( i = 1, \ldots, l - 1 \), otherwise the two strips are equal and we can project down to \( T_1^l \times T^1 \).

We now need to show that the projection \( A := \pi(\hat{A}) \) of \( \hat{A} \) to \( T^2 \) is a \( 1,l \)-invariant strip for \( T^n \). The problem is that we cannot simply project the \( q \)-curve in \( T_1^l \times T_k^1 \) down, as it might then intersect \( A \). Therefore we have to construct a new \( q \)-curve in the complement of \( A \). In order to do so, we choose a lift \( \hat{A} \) of \( \hat{A} \) in \( \mathbb{R}^2 \). As \( \hat{U} := (\varphi_{\hat{A}}^\pi,\varphi_{\hat{A}}^\rho + 1) \) is a fundamental domain of \( \hat{A}^\rho \), there exists a unique lift of every set \( \hat{A} + (i,0) \) in \( \hat{U} \). All these lifts are minimal strips, and hence they are strictly ordered (see Remark 2.8(b)). Furthermore, their union equals \( \pi^{-1}(A) \cap \hat{U} \). Obviously there exists a continuous curve \( \hat{\gamma} \) between \( \hat{A} \) and the first of these strips above \( \hat{A} \), and we can choose \( \hat{\gamma} \) such that it is the lift of a \( q \)-curve in the complement of \( A \).

\[ \square \]

2.4 Fibrewise Rotation Numbers and \( \rho \)-Bounded Orbits

As already indicated in the introduction, the monotonicity of each fibre map combined with the unique ergodicity of the forcing rotation ensure the existence of a fibrewise rotation number:

Theorem and Definition 2.16 (Herman, [Her83])

Let \( T \in \mathcal{T}_{\text{hom}} \) and \( \hat{T} : T^1 \times \mathbb{R} \to T^1 \times \mathbb{R} \) be a lift of \( T \). Then the limit

\[ \rho_{\hat{T}} := \lim_{n \to \infty} \frac{1}{n}(\hat{T}_{\hat{\theta}}^n(\hat{x}) - \hat{x}) \quad (2.4) \]

exists and is independent of \( \hat{x} \) and \( \hat{\theta} \), the convergence in (2.4) is uniform on \( T^1 \times \mathbb{R} \) and in addition

\[ \rho_{\hat{T}} = \lim_{n \to \infty} \frac{1}{n} \int_{T^1} \hat{T}_{\hat{\theta}}^n(0) d\hat{\theta} \quad (2.5) \]

10
Furthermore, $\rho_T := \rho_{\hat{T}} \mod 1$ is independent of the choice of the the lift $\hat{T}$. It is called the fibrewise rotation number of $T$.

However, there is a crucial difference between the quasiperiodically forced case and an unforced homeomorphism of the circle. As highlighted in the introduction, in the latter case there is a bound (1.3) on the possible deviations of any orbit from the average rotation. For a general quasiperiodically forced map such a bound need not exist, motivating the following definition:

**Definition 2.17 ($\rho$-Bounded Orbits)**

(a) If $\hat{T}$ is a lift of $T \in T_{\text{hom}}$ and $\rho \in \mathbb{R}$, we say that the orbit of $(\theta, \hat{x})$ is $\rho$-bounded if there exists a constant $C > 0$ such that

$$|\hat{T}_n^\theta(\hat{x}) - \hat{x} - n\rho| \leq C \forall n \in \mathbb{N}. \quad (2.6)$$

(b) Similarly, we say that the orbit of $(\theta, \hat{x})$ is $\rho$-bounded above (below) if there exists $C > 0$ such that $\hat{T}_n^\theta(\hat{x}) - \hat{x} - n\rho \leq C \; (\geq -C) \forall n \in \mathbb{N}$.

(c) If the orbits of $\hat{T}$ are $\rho$-bounded, then the same is true for any other lift of $T$. Thus we say $T$ has $\rho$-bounded orbits if the lifts have bounded orbits in the above sense.

More informally we will also speak of the *deviations from the constant rotation* when referring to the quantities $|\hat{T}_n^\theta(\hat{x}) - \hat{x} - n\rho|$. The following result about $\rho$-boundedness of orbits is taken from [SFGP02]:

**Theorem 2.18**

Let $T \in T_{\text{hom}}$. If there exists one $\rho$-bounded orbit, then all orbits are $\rho$-bounded and the constant $C$ in (2.6) can be chosen uniformly for all $(\theta, x) \in \mathbb{T}^2$. If there is no $\rho$-bounded orbit, there exists at least one orbit which is $\rho$-bounded above and one which is $\rho$-bounded below, but for a residual set of $\theta$’s all orbits on the $\theta$-fibre $\{\theta\} \times \mathbb{T}^1$ are both $\rho$-unbounded above and below.

Note that the uniformity of $C$ is neither explicitly stated nor proved in [SFGP02]. However, Lemma 7 of this reference does show that there is a uniform constant for all $(\theta, x) \in U \times \mathbb{T}^1$ where $U$ is a residual set, and extending this to the whole of $\mathbb{T}^2$ is elementary. As a consequence of this statement, it makes sense to define

**Definition 2.19 ($\rho$-bounded Systems)**

A map $T \in T_{\text{hom}}$ is called $\rho$-bounded if all orbits are $\rho$-bounded. Otherwise it is called $\rho$-unbounded.

As we shall see in the next section, to $\rho$-bounded systems a direct analogue of Poincaré’s classification for circle maps applies. Finally, we relate the existence of a $p,q$-invariant strip to the properties of the rotation number.

**Definition 2.20 (Rational Dependence)**

Two numbers $\omega, \rho \in \mathbb{R}$ (or $\mathbb{T}^1$) are said to be rationally dependent if there exists $(l,k,q) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$ such that $l + k\omega + q\rho = 0$. Otherwise they are called rationally independent.
Definition 2.21 (Irrational Torus Translation)
For \( \omega, \rho \in \mathbb{T}^1 \) let \( R_{\omega,\rho} : (\theta, x) \mapsto (\theta + \omega, x + \rho) \). We call \( R_{\omega,\rho} \) an irrational torus translation if \( \omega \) and \( \rho \) are not rationally dependent.

Note that as we assumed our rotation number \( \omega \) to be irrational, rational dependence of \( \omega \) and \( \rho \) is equivalent to the existence of \( l,k,q \in \mathbb{Z}, \ q \neq 0 \) such that \( \rho_T = \frac{k}{q}\omega + \frac{l}{q} \). The following result, which relates the concepts of invariant strips and fibrewise rotation numbers, can be found in [JK03] (Lemma 3.9).

Proposition 2.22
Let \( T \in \mathcal{T}_{\text{hom}} \) and suppose there exists a \( p,q \)-invariant strip. Then \( \omega \) and \( \rho_T \) are rationally dependent. Furthermore, the orbits of \( T \) are \( \rho_T \)-bounded.

3 The \( \rho \)-bounded Case

For \( \rho \)-bounded systems the following statement is in perfect analogy with Poincaré’s classical result:

Theorem 3.1
Suppose \( T \in \mathcal{T}_{\text{hom}} \) is \( \rho \)-bounded. Then one of the following holds:

(a) \( \rho_T \) and \( \omega \) are rationally dependent and there exists a \( p,q \)-invariant strip for \( T \).

(b) \( \rho_T \) and \( \omega \) are rationally independent and \( T \) is semi-conjugate to the irrational torus-translation \( R_{\omega,\rho_T} \). Furthermore, the semi-conjugacy \( h \) can be chosen so that it is fibre-respecting (i.e. \( \pi_1 \circ h = \pi_1 \)) and all fibre maps \( h_\theta \) are order-preserving circle maps.

Proof:
(a) Suppose \( T \in \mathcal{T}_{\text{hom}} \) is \( \rho \)-bounded and \( \rho = \rho_T = \frac{k}{q}\omega + \frac{l}{q} \) for some \( k,l,q \in \mathbb{N} \).
Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be a lift of \( T^q \) with \( \rho_F = k\omega \). Note that the rotation number of \( F \) on the base is \( q\omega \), i.e. \( F : (\hat{\theta}, \hat{x}) \mapsto (\hat{\theta} + q\omega, F_\hat{\theta}(\hat{x})) \). Now let \( \gamma_0(\hat{\theta}) := \frac{\hat{\theta}}{q} \) and define
\[
A := \bigcup_{n \in \mathbb{Z}} F^n(\Gamma_0) .
\]
Obviously \( A \) is \( F \)-invariant, and it projects down to all of \( \mathbb{R} \) (as this is already true for \( \Gamma_0 \)). Further, \( A \) is contained in the strip \( K := \{ (\hat{\theta}, \hat{x}) \mid \hat{x} \in [\frac{k}{q}\hat{\theta} - C, \frac{k}{q}\hat{\theta} + C] \} \), where \( C \) is a suitable constant as in Def. 2.17. Thus, for any \( \hat{\theta} \in \mathbb{R} \) the set \( A_{\hat{\theta}} \) is compact. W.l.o.g. we can assume \( k > 2C + 1 \), otherwise we replace \( k \) and \( q \) by sufficiently large multiples \( kl \) and \( lq \).

Now let \( \hat{B} := A^{\mathbb{N}} \). Due to the periodicity of \( F \) and the definition of \( A \) and \( \hat{B} \) we have \( \hat{B}_{\hat{\theta}+q} = \hat{B}_{\hat{\theta}} + k \). Thus \( \hat{B} \) projects down to a simple strip \( B \) in \( \mathbb{T}^1_k \times \mathbb{T}^1_k \). As \( B \) is contained in the projection of \( K \) and \( k > 2C + 1 \), it is easy to see that there exists a \( 1 \)-curve in the complement. \( B \) is a simple invariant strip with respect to the projection of \( F \) to \( \mathbb{T}^1_k \times \mathbb{T}^1_k \), which completes the proof via Lemma 2.15.
(b) Now suppose the two rotation numbers are rationally independent. In order to show that $T$ and $R_{ω,ρT}$ are semi-conjugate we will proceed in two steps. First, we show that there exists a family of disjoint minimal strips $(B_r)_{r \in \mathbb{R}}$ in $T^1 \times \mathbb{R}$ with $\tilde{T}(B_r) = B_{r+\rho T}$, $B_{r+n} = B_r + (0,n)$, and such that $r \mapsto B_r$ is strictly order-preserving. Then we use this to construct a semi-conjugacy $H$ between $T$ and $R_{ω,ρT}$, which projects down to a semi-conjugacy $h$ between the original systems.

**Step 1:** Let $\tilde{T} : T^1 \times \mathbb{R} \to T^1 \times \mathbb{R}$ be a lift of $T$ with $\rho_{\tilde{T}} = \rho_T =: \rho$. As $|\tilde{T}_n(x) - x - n\rho| \leq C \forall n \in \mathbb{N}$, $(\theta, \tilde{x}) \in T^1 \times \mathbb{R}$, the sets

$$A_r := \bigcup_{n \in \mathbb{Z}} \tilde{T}^n(T^1 \times \{r-n\rho\}) \quad (3.1)$$

are all contained in $T^1 \times [r-C, r+C]$ and therefore compact. As a direct consequence of the definition $A_{r+n\rho} = \tilde{T}^n(A_r) \forall n \in \mathbb{Z}$, the periodicity of $\tilde{T}_\theta - \text{Id}$ implies $A_{r+n} = A_r + (0,n)$.

Furthermore, it follows directly from the monotonicity of the fibre maps $\tilde{T}_\theta$ that the $A_r$ are ordered, i.e. $A_r \ll A_s$ whenever $r \leq s$. However, this order need not necessarily be strict. Therefore let

$$B_r := (\Phi_{\tilde{T}_\theta}^+)^{\text{fill}}. \quad (3.2)$$

This means that we take the highest minimal strip contained in $A^{\text{fill}}$, compare Section 2.2. These sets inherit the ordering property of the sets $A_r$, as all the actions in (3.2) preserve the order of sets. We now aim to show that they are all disjoint, such that the ordering is strict (i.e. $B_r \ll B_s$ whenever $r < s$).

To that end we claim that whenever $B_r \cap B_s \neq \emptyset$ for some $r < s$ there exists a $p,q$-invariant strip for $T$, contradicting the rational independence of $\omega$ and $\rho_T$.

Suppose $\exists \theta \in T^1 : (B_r)_\theta \cap (B_s)_\theta \neq \emptyset$ and choose $r', s' \in \mathbb{R}$ such that $r + \delta \leq r' < s' \leq s - \delta$ for some $\delta > 0$. If we let

$$\varphi_r^+ := \varphi_{B_r}^+ \quad \text{and} \quad \varphi_r^- := \varphi_{B_r}^- \quad \forall r \in \mathbb{R}$$

then $(B_r)_\theta \cap (B_s)_\theta \neq \emptyset$ implies $\varphi_r^+(\theta) \geq \varphi_s^-(\theta)$. Now, for any $n \in \mathbb{Z}$ with $n\rho \in [-\delta, \delta] \mod 1$ we have $B_{r-k} \ll B_{r'-n\rho} \ll B_{s'-n\rho} \ll B_{s-k}$ for some $k \in \mathbb{Z}$. This means

$$\varphi_{r'-n\rho}^+(\theta) \geq \varphi_{r-k}^+(\theta) = \varphi_r^+(\theta) - k \geq \varphi_s^-(\theta) - k = \varphi_{s-k}^-(\theta) \geq \varphi_{s'-n\rho}^-(\theta)$$

and thus $(B_{r'-n\rho})_\theta \cap (B_{s'-n\rho})_\theta \neq \emptyset$. As $\tilde{T}^n(B_{r'-n\rho}) = B_{r'}$ and $\tilde{T}^n(B_{s'-n\rho}) = B_{s'}$, this means $(B_{r'})_\theta + n\omega \cap (B_{s'})_\theta + n\omega \neq \emptyset$.

Consequently $\pi_1(B_r \cap B_s)$ contains the set $\{\theta + n\omega \mid n\rho \mod 1 \in [-\delta, \delta]\}$, which is dense in $T^1$ as it is the projection of $\{R_{\omega,\rho T}^n(0,0) \mid n \in \mathbb{Z}\} \cap (T^1 \times [-\delta, \delta])$ onto the circle and $R_{\omega,\rho T}$ is minimal. By the minimality of
the two strips $B_{r'}$ and $B_{s'}$ this implies $B_{r'} = B_{s'}$. Now, as we can take $r' = n\rho \mod 1$ and $s' = k\rho \mod 1$ for some $n, k \in \mathbb{Z}$, this yields $\tilde{T}^{n-k}(B_{r'}) = B_{r'} + l$ for some $l \in \mathbb{Z}$. Therefore $B_{r'}$ projects down to an invariant strip for $T$. This completes Step 1.

**Step 2:** Now define $H : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ by $H(\theta, \tilde{x}) := (\theta, H_{\theta}(\tilde{x}))$ and

$$H_{\theta}(\tilde{x}) := \sup \{ r \in \mathbb{R} | \varphi_+^r(\theta) \leq \tilde{x} \}.$$ 

From the properties of the sets $B_{r'}$ and their bounding graphs $\varphi_+^r$, it follows easily that $H \circ \tilde{T} = R_{\omega, \rho} \circ H$, all $H_{\theta}$ are order-preserving (i.e. monotone) and that $H_{\theta}(\tilde{x} + n) = H_{\theta}(\tilde{x}) + n \ \forall n \in \mathbb{Z}$, $\tilde{x} \in \mathbb{R}$. It remains to show that $H$ is continuous, i.e. that preimages of open sets are open.

To that end consider an open set $U \subseteq \mathbb{T}^1 \times \mathbb{R}$ and $(\theta_0, \tilde{y}) \in U$. There is a closed box $W := B_3(\theta_0) \times [r, s]$ with $(\theta_0, \tilde{y}) \in \text{Int}(W) \subseteq U$. But this means $H^{-1}(U)$ contains the set $\{ (\theta, \tilde{x}) | \theta \in B_3(\theta_0), \varphi_+^r(\theta) < \tilde{x} < \varphi_-^r(\theta) \}$, which is an open neighborhood of $H^{-1}(\theta_0, \tilde{y})$ (openness following from the semi-continuity of the graphs $\varphi_+^r$ and $\varphi_-^r$).

This means $H$ is the required semi-conjugacy between the lifts of $T$ and $R_{\omega, \rho}$, and as $H_{\theta}(\tilde{x} + n) = H_{\theta}(\tilde{x}) + n$ we can project it down to $\mathbb{T}^2$ to obtain a semi-conjugacy $h$ between the original systems.

\[\square\]

**Remark 3.2**

(a) In case (a) it is also possible to show that the rotation numbers determine the numbers $p$ and $q$ in Def. 2.12, as well as some further combinatorial properties of the invariant strips. A detailed description of this can be found in [JK03].

(b) In case (b) the semi-conjugacy $h$ is uniquely determined up to constant rotation in the direction of the fibres. This can be seen as follows: suppose that $h$ and $\tilde{h}$ have the properties mentioned in the theorem. Then, since both must map the unique $T$-invariant measure $\mu$ to the Lebesgue measure on $\mathbb{T}^2$, we have $h_{\theta} = \tilde{h}_{\theta} + c_{\theta}$ m-a.s., where the constant $c_{\theta}$ may a priori depend on $\theta$. But as $c_{\theta+\omega} = h_{\theta+\omega}(T_{\theta}(x)) - \tilde{h}_{\theta+\omega}(T_{\theta}(x)) = h_{\theta}(x) + \rho_T - \tilde{h}_{\theta}(x) - \rho_T = c_{\theta}$ and the rotation on the base is ergodic, $c$ is m-a.s. constant. By continuity this extends to all fibres.

Now we turn to the existence of wandering open sets. In dimension one, the classical examples of circle homeomorphisms with irrational rotation number and wandering open sets were given by Denjoy (see [dMvS93]). They are constructed by “blowing up” one or more orbits of the respective irrational rotation. The following lemma shows that this is also the only possibility of producing such examples in the quasiperiodically forced case, with orbits of points being replaced by orbits of line segments. It suffices to consider the $\rho$-bounded case, since as we shall see below, $\rho$-unbounded systems are always topologically transitive and therefore cannot have any wandering open sets.
Lemma 3.3
Suppose $T \in \mathcal{T}_{\text{hom}}$ is semi-conjugate to the irrational torus translation $R_{\omega,\rho}$ by a semi-conjugacy $h$ as in Thm. 3.1 and let $U$ be a connected wandering open set. Then $h(U)$ is a straight line segment. Further more, the image $h(V)$ of any other wandering open set $V$ must have the same slope as $h(U)$.

Proof:
Let $I \subseteq \pi(U)$ and $\psi : I \to \mathbb{T}^1$ be any continuous curve contained in $U$ and as usual denote its graph by $\Psi$. As $h$ is only a semi-conjugacy and therefore not necessarily one to one, the curves $R^n_{\omega,\rho} \circ h(\Psi)$ ($n \in \mathbb{N}$) may not be pairwise disjoint. However, as $h$ preserves the order on each fibre they may touch, but not cross each other. Obviously, the only curves in $\mathbb{T}^2$ which never cross any of their images under an irrational translation are straight line segments. Thus any such curve contained in $U$ is mapped to a line segment. Suppose that there are two curves $\psi_1, \psi_2$ in $U$ which are mapped to different line segments. Then there exists a third curve $\varphi$ in $U$ which coincides with $\psi_1$ on one interval and with $\psi_2$ on another. But then $h(\varphi)$ cannot be contained in a line segment, leading to a contradiction. Therefore any curve, and in fact any point contained in $U$, must be mapped into the same line segment by $h$.

Now suppose $V$ is another wandering open set. There are two possibilities: Either the orbits of $U$ and $V$ are disjoint. In this case the orbits of $h(U)$ and $h(V)$ under the irrational torus translation must be disjoint as well, and this is only possible if both line segments have the same slope. On the other hand, if $T^nV \cap U \neq \emptyset$, then $R^n_{\omega,\rho}(h(V))$ and $h(U)$ must coincide on all fibres in the open set $\pi_1(T^nV \cap U \neq \emptyset)$, and this is again only possible if both line segments have the same slope.

We close this section with a few remarks concerning the generalization of Denjoy’s Theorem (e.g. [dMvS93, KH97]) to quasiperiodically forced systems. The existence of wandering open sets can be excluded in a similar fashion to that for circle homeomorphisms by requiring that the map is sufficiently smooth. The condition needed here is

$$V(T) := \int_{\mathbb{T}^1} \text{Var}DT_\theta \, d\theta < \infty,$$

(3.3)

where $DT_\theta$ denotes the derivative of the fibre maps and $\text{Var}(f)$ is the variation of the function $f$. However, unlike the one-dimensional case, excluding the possibility of wandering sets is not enough to ensure the existence of a conjugacy to the irrational translation. Therefore the following Denjoy-like statement, which is contained in [JK03], is somewhat weaker in nature:

Theorem 3.4
Let $T \in \mathcal{T}_{\text{hom}}$ with $V(T) < \infty$ and suppose there exists no $T$-invariant strip. Then $T$ is topologically transitive.

An important step in the proof given in [JK03] is to analyze the combinatorial behavior of wandering open sets. However, as $\rho$-unbounded systems are always transitive (see next section) and it therefore suffices to consider just the $\rho$-bounded case, Lemma 3.3 can be used to simplify this part of the proof significantly.
4 The \( \rho \)-unbounded Case

We first remark that the \( \rho \)-unbounded case is non-empty. The simplest examples, which are due to Furstenberg, can be given by a skew translation of the torus, i.e. maps of the form

\[
A : T^2 \to T^2, \quad (\theta, x) \mapsto (\theta + \omega, x + a(\theta))
\]

(4.1)

were \( a : T^1 \to T^1 \) is continuous and homotopic to a constant. Now there exist continuous (if \( \omega \) is Liouvillean even smooth) functions \( \hat{a} : T^1 \to \mathbb{R} \) with

\[
\int_{T^1} \hat{a}(\theta) \, d\theta = 0,
\]

such that the cohomological equation

\[
\hat{a}(\theta) = \hat{\varphi}(\theta + \omega) - \hat{\varphi}(\theta)
\]

(4.2)

has a measurable solution, but no continuous solution (see [Mań87], [KR01]). This implies that the corresponding map \( A \) given by (4.1) with \( \hat{a} = \pi \circ \hat{\varphi} \) is minimal (see [KH97], Prop. 4.2.6). Thus, there cannot be an invariant strip. On the other hand, as \( \varphi := \pi \circ \hat{\varphi} \) gives an invariant graph, \( A \) cannot be semi-conjugated to an irrational translation. Consequently, such a map must be \( \rho \)-unbounded.

A slight modification of this construction also gives an example of an \( \rho \)-unbounded system without invariant graphs (Case II B below): If \( \hat{a} \) is replaced by \( \hat{a} + \rho \) with some \( \rho \in [0,1] \setminus \mathbb{Q} \), then it is easily checked that \( h(\theta, x) := (\theta, x - \varphi(\theta)) \) satisfies \( h^{-1} \circ R_{\omega,\rho} \circ h = T \) and therefore gives an isomorphism between \( T \) and the irrational translation. Thus \( T \) is uniquely ergodic with respect to the Lebesgue measure on \( T^2 \) and therefore cannot have any invariant graphs.

The second interesting class of examples comes from the study of matrix cocycles, given as maps

\[
T : T^1 \times \mathbb{R}^2 \to T^1 \times \mathbb{R}^2, \quad (\theta, v) \mapsto (\theta + \omega, M(\theta)v)
\]

(4.3)

where \( M : T^1 \to SL(2,\mathbb{R}) \) is continuous and homotopic to \( \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \). Using the action of \( M \) on the projective space \( P(\mathbb{R}^2) \), such cocycles can be identified with quasiperiodically forced Möbius-transformations. In [Her83] Herman showed that for each pair \( (\omega, \rho) \in T^2 \), \( \omega \in T^1 \setminus (\mathbb{Q}/\mathbb{Z}) \) there exists a matrix cocycle over the rotation by \( \omega \) with fibrewise rotation number \( \rho \) and a positive Lyapunov exponent. The later ensures the existence of two measurable invariant graphs, such that the system cannot be semi-conjugated to an irrational translation. On the other hand, if \( \omega \) and \( \rho \) are rationally independent this excludes the existence of an invariant strip. Thus the system must be \( \rho \)-unbounded. In contrast to Furstenberg’s, these examples are analytic even if the rotation number on the base is diophantine. They also show that the existence of measurable invariant graphs does not require the rational dependence of the rotation numbers.

The following theorem gives at least some information about the topological dynamics in the \( \rho \)-unbounded case:

**Theorem 4.1**

If \( T \in \mathcal{T}_{\text{hom}} \) is \( \rho \)-unbounded then it is topologically transitive.
Proof:
First we need to present a number of notations and conventions for continuous
curves \( \varphi : I \to \mathbb{T}^1 \) where \( I = [a, b] \subseteq \mathbb{T}^1 \) is a compact interval. The point set
which is given by the graph of such a curve will be denoted by the corresponding
capital letter (as for invariant graphs). If \( \varphi : I \to \mathbb{T}^1 \) is a continuous curve the
set \( T(\Phi) \) is the point set of a continuous curve as well, which we will denote
by \( T \varphi : I + \omega \to \mathbb{T}^1 \). Given any point \( \hat{x} \in \pi^{-1}(\varphi(a)) \) and any lift \( \hat{I} = [\hat{a}, \hat{b}] \)
of the interval \( I \), there exists a unique lift \( \hat{\varphi} : \hat{I} \to \mathbb{R} \) of \( \varphi \) which satisfies
\( \hat{\varphi}(\hat{a}) = \hat{x} \). Using any such lift we can define the \textit{winding number}
of the curve \( \varphi \) by \( k(\varphi) := \hat{\varphi}(\hat{b}) - \hat{\varphi}(\hat{a}) \). Further let \( v(\varphi) := \max_{\theta \in I} \hat{\varphi}(\theta) - \min_{\theta \in I} \hat{\varphi}(\theta) \). It is
easy to see that if \( \psi : I' \to \mathbb{T}^1 \) is another curve with \( I' \subseteq I \) and \( k(\psi) \geq v(\varphi) + 1 \),
then \( \Psi \) must intersect \( \Phi \).

We have to show that given any two open sets \( U, V \) there exists some \( n \in \mathbb{N} \),
such that \( T^n(U) \cap V \neq \emptyset \). It suffices to consider the case where \( U \) and \( V \) are
open rectangles, i.e. \( U = I_1 \times J_1 \) and \( V = I_2 \times J_2 \) for some open intervals
\( I_1, I_2, J_1, J_2 \subseteq \mathbb{T}^1 \). Now take a compact interval \( I \subseteq I_2 \) and any continuous
curve \( \varphi : I \to \mathbb{T}^1 \) with \( \Phi \subseteq \mathbb{T}^1 \). For some \( n \in \mathbb{N} \), the interiors of the intervals
\( I, T^{-1}(I), \ldots, T^{-n}(I) \) cover the whole circle. Consequently there is some \( \delta > 0 \)
such that any interval \( I' \) of length smaller or equal to \( \delta \) is fully contained in one
of these intervals. By Thm. 2.18 there exists both an orbit which is \( \rho_T \)-bounded
above as well as one which is \( \rho_T \)-unbounded above. Therefore we can choose an
interval \( I' = [\theta_1, \theta_2] \subseteq I_1 \) of length smaller or equal to \( \delta \), such that the orbits
on the \( \theta_1 \)-fibre are \( \rho_T \)-bounded above whereas the orbits on the \( \theta_2 \)-fibre are not.
This means that if we take any continuous curve \( \psi : I' \to \mathbb{T}^1 \) with \( \Psi \subseteq U \), we
can find some \( m \in \mathbb{N} \), such that \( k(T^m \psi) \geq \max_{i \in \mathbb{N}} v(T^i \varphi) + 1 \). As \( T^m(I') \) is
contained in \( T^{-j}(I) \) for some \( j \in \{0, \ldots, n\} \), this implies that \( T^m(\Psi) \) intersects
\( T^{-j}(\Phi) \). But as \( \Psi \subseteq U \) and \( \Phi \subseteq V \), this gives \( T^{m+j}U \cap V \neq \emptyset \).

\( \square \)

Combining the results from the last section with Thm. 2.10, we obtain the
following basic classification:
Let us examine Case II(B) more closely: suppose \( T \in \mathcal{T}_{\text{hom}} \) is isomorphic to the uniquely ergodic skew translation \( A \) via the isomorphism \( h \). In general, an isomorphism of two systems on the torus cannot be lifted to \( T^1 \times \mathbb{R} \) in any obvious way. However, although not explicitly mentioned in [Fur61], the construction of \( h \) in the proof of Thm. 2.10 gives a lot of additional information about \( h \). It is the projection of a function \( \hat{h}: T^1 \times \mathbb{R} \to T^1 \times \mathbb{R} \), which is an isomorphism between a lift \( \hat{T} \) of \( T \) and a skew translation \( \hat{A} \) which projects down to \( A \). Further, it is of the form \( \hat{h}(\theta, x) = (\theta, \hat{h}_\theta(x)) \) with all fibre maps \( \hat{h}_\theta \) being monotone and continuous, and finally \( \hat{h} \) leaves all integer lines \( T^1 \times \{n\} \) invariant. This last property ensures that \( \hat{h} \) preserves the rotation number, in the sense that \( \rho_{\hat{T}} = \int_{T^1} \hat{a}(\theta) \ d\theta \). This leaves the following possibilities:

(i) \( \hat{a} \) is cohomologous to a constant, which is necessarily equal to \( \rho_T \), i.e.

\[
\hat{a}(\theta) = \hat{\varphi}(\theta + \omega) - \hat{\varphi}(\theta) + \rho_T
\]

for some measurable function \( \hat{\varphi}: T^1 \to \mathbb{R} \). As we have seen in the discussion of Furstenberg’s examples above, this means that \( T \) is isomorphic to \( R_{\omega,\rho_T} \) by a fibre-respecting isomorphism \( h \). As isomorphy preserves ergodicity and \( h \) leaves \( \lambda \) invariant, \( R_{\omega,\rho_T} \) must be ergodic w.r.t. Lebesgue measure as well, and therefore irrational. In particular, \( \omega \) and \( \rho_T \) cannot be rationally dependent in this case.

(ii) There exists no solution to the cohomological equation (4.4). In this case it is hard to say anything further, see questions (a) and (b) below.
We close with three questions related to this discussion, which we have to leave open here:

Questions 4.2

(a) Is it possible that a system \( T \in T_{\text{hom}} \) is isomorphic to an irrational translation \( R_{\omega, \rho} \) with \( \rho \) not being contained in the module \( \{ \rho_T + k\omega \mid k \in \mathbb{Z} \} \)? Note that two irrational translations \( R_{\omega, \beta} \) and \( R_{\omega, \beta + k\omega} \) from the same module are always isomorphic by \( h(\theta, x) = (\theta, x + k\theta) \).

(b) Is it possible to have rationally related rotation numbers in Case II B?

(c) Is it possible to have transitive but not minimal dynamics in the \( \rho \)-unbounded case (or in Case I B)?

Acknowledgments. We would like to thank Sylvain Crovisier for interesting discussions on the subject. Tobias Jaeger acknowledges support from the German Science Foundation (DFG) and also profited from a Marie Curie Fellowship during a visit to the University of Surrey.

References


