Some remarks on modified power entropy

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The aim of this note is to point out some observations concerning modified power entropy of $\mathbb{Z}$- and $\mathbb{N}$-actions. First, we provide an elementary example showing that this quantity is sensitive to transient dynamics, and therefore does not satisfy a variational principle. Further, we show that modified power entropy is not suitable to detect the break of equicontinuity which takes place during the transition from almost periodic to almost automorphic minimal systems. In this respect, it differs from power entropy and amorphic complexity, which are two further topological invariants for zero entropy systems ('slow entropies'). Finally, we construct an example of an irregular Toeplitz flow with zero modified power entropy.

1 Introduction

Given a continuous map $f : X \to X$ on some compact metric space (X, d), the *Bowen-Dinaburg metrics* are given by

$$d_n^f(x,y) \ = \ \max_{i=0}^{n-1} d(f^i(x), f^i(y)) \ .$$

If $S_n(f,\delta)$ denotes the maximal cardinality of a set $S \subseteq X$ which is δ*-separated*[1] with respect to d_n^f, then the *topological entropy* of f can be defined by

$$h_{\text{top}}(f) \ = \ \lim_{\delta\to 0} \limsup_{n\to\infty} \log S_n(f,\delta)/n \ . \tag{1}$$

This quantity measures the 'chaoticity' of a dynamical system and is arguably the most important topological invariant in ergodic theory. If it is either infinite or zero, however, then the complexity of a system has to be described by different means. In the case of infinite entropy, mean dimension has been established as a suitable substitute [LW00].

If the entropy is zero, however, then the situation is less clear. There exist several alternative concepts to describe the complexity of a system in this situation (see, for example, [Mis81, Smí86, MS88, KS91, Fer97, KT97, Fer99, BHM00, HK02, FP07, HPY07, CL10]), and different topological invariants have been proposed for this purpose ([Car97, HK02, HY09, DHP11, Mar13, KC14, FGJ15]).

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[1] Given any function $\rho : X \times X \to \mathbb{R}$, we call a set $S \subseteq X$ δ-separated with respect to ρ if $\rho(x,y) \geq \delta$ for all $x \neq y \in S$. One should think of ρ as a metric, but we will also use the same terminology in more general situations.

Some of them have properties that may be considered as shortcomings, although this partly depends on the viewpoint and the particular purpose one has in mind (we briefly discuss this issue in Section 2 below). In any case, it is not always obvious which one should be considered best in a particular situation, and in general there are still many gaps in the present state of knowledge. At the same time, the issue has considerable relevance, since there exist many system classes of both of theoretical and practical importance in which the topological entropy is zero for structural reasons. Just to mention some examples, these include regular Toeplitz flows [Dow05], circle homeomorphisms [KH97], interval exchange transformations [Via06], certain mathematical quasicrystals [Moo00, BLM07], quasiperiodically forced circle maps [GJS09] or $\mathcal{C}^{1+\alpha}$-surface diffeomorphisms with subexponential growth of periodic orbits [Kat80].

Maybe the most straightforward approach to the problem is to consider subexponential, and in particular polynomial, growth rates instead of exponential ones as in (1). This leads to the notion of *power entropy*[2] h_{pow}, which is also known under the name of *polynomial word complexity* in the context of symbolic systems. One aspect in which this quantity behaves quite differently from topological entropy is the fact that it is very sensitive to transient behaviour. For instance, the existence of a single wandering point[3] of a homeomorphism f implies $h_{\mathrm{pow}}(f) \geq 1$ [Lab13]. In particular, this means that the dynamically trivial Morse-Smale systems have positive power entropy. A direct consequence is the non-existence of a variational principle, which is another decisive difference to the standard notion of topological entropy.

An alternative concept is *modified power entropy* [HK02]. In its definition, the Bowen-Dinaburg metrics are replaced by the corresponding *Hamming metrics*. However, although this is less obvious to see, this notion is equally sensitive to transient dynamics and therefore cannot satisfy a variational principle either. We provide an example to demonstrate this statement in Section 5. Since this question has been left open in the literature so far (see, for example, [HK02, page 92]), the communication of this fact is one of the main motivations for this note.

The second issue we discuss here is the response of power entropy and modified power entropy to the break of equicontinuity, which can be observed during the transition from almost periodic (=equicontinuous) minimal systems to their almost 1-1 extensions. It turns out that power entropy is suitable to detect this change in the qualitative behaviour, whereas modified power entropy is not. In this context, we also introduce and discuss *amorphic complexity*. This is a new topological invariant that equally measures the complexity of zero entropy systems, but is based on an asymptotic rather than a finite-time concept of separation [FGJ15].

Finally, we provide an example of an irregular Toeplitz flow with zero modified power entropy in order to clarify some further aspects of the preceding discussion.

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[2] defined in Section 3

[3] We call $x \in X$ a *wandering point* of f if there exists an open neighbourhood U of x such that $f^n(U) \cap U = \emptyset$ for all $n \in \mathbb{N}$.

2 Some thoughts on slow entropies

In the context of this discussion, we understand *'slow entropy'* in a broad sense of a topological invariant that measures the complexity of dynamical systems in the zero entropy regime. Thereby, our focus lies on $\mathbb{Z}$- and $\mathbb{N}$-actions of low complexity. We note that similar concepts are also used for the description of more general group actions, where the need for considering alternative growth rates stems rather from the fast volume growth of the Følner sequences than from the low complexity of the group action. However, we will not go into any detail in this direction and refer to [KT97, Don14, KKH14] for a discussion and further references. In order to restrict the scope to some degree, we concentrate on real-valued invariants and compact metric spaces. We thus say a slow entropy is a function h defined on the space of pairs (X, f), with X a compact metric space and $f : X \to X$ continuous, which satisfies the following requirements.

- h is real-valued (including ∞) and non-negative;
- h takes the same value for topologically conjugate systems (*topological invariance*);
- If g is a topological factor of f, then $h(g) \leq h(f)$ (*monotonicity*);
- If f has positive topological entropy, then $h(f) = \infty$ (*zero entropy regime*).

Note that in fact topological invariance is a consequence of monotonicity. Beyond these basic assumptions, however, it is not always clear what further properties are desireable for a slow entropy, and which ones are rather not. The reason behind is that this depends to a large extent on the purpose that such a quantity should serve, and there are quite different and sometimes even contradictory aims one could have in mind. We want to discuss this by means of an example.

As it is well-known, one of the most important results about entropy is the variational principle, which states that topological entropy equals the supremum over its measure-theoretic counterparts with respect to all the invariant probability measures of the system. It is one of the main tools in thermodynamic formalism and explains the central role topological entropy plays in this powerful machinery. As a consequence, topological entropy is also independent of transient behaviour and determined by the dynamics on the set of recurrent points only. It is one possible aim for introducing a slow entropy to provide similar tools for the study of zero-entropy systems. Most likely, however, this will require at least some minimal amount of 'chaoticity' in the system. In contrast to this, an alternative task for a slow entropy would be to detect the very onset of complicated dynamical behaviour. For example, one might want it to detect the qualitative change in behaviour when going from equicontinuous systems – to which one would usually assign zero complexity – to non-equicontinuous systems, by taking positive values for the latter. Now, this would mean that, for instance, the slow entropy should give different values to Sturmian subshifts and irrational rotations. However, a Sturmian subshift is uniquely ergodic and measure-theoretically isomorphic to an irrational rotation, so that this immediately contradicts a variational principle.

Hence, it seems obvious that there is not one single notion of slow entropy that fulfills all the possible roles of a topological invariant in the zero entropy regime at the same time. Certainly, this just reflects the great diversity of zero entropy systems, which comprise many classes of quite different complexity. The fact that not all of them can be adequately described with the same concept is not too surprising. A more reasonable aim would be to identify a whole array of useful invariants such that their union allow to cover the zero entropy regime in a reasonable way and distinguish different degrees of complexity. Yet, the present state of knowledge on the topic is still far from this situation, and it will presumably need a lot of further fundamental research in the area in order to get to that point.

The particular contribution of the present paper in this context is a modest one. As mentioned, we concentrate mostly on modified power entropy and clarify some of the mentioned aspects concerning this particular notion. A short summary will be given in Section 8.

3 Power entropy, modified power entropy and amorphic complexity

As mentioned above, power entropy measures the polynomial growth rate of orbits distinguishable by the Bowen-Dinaburg metrics d_n^f. In analogy to (1), it is defined as

$$h_{\text{pow}}(f) \;=\; \lim_{\delta\to 0}\lim_{n\to\infty}\frac{\log S_n(f,\delta)}{\log n}\;,$$

whenever the limits with respect to $n\to\infty$ exists. If this is not the case, then one defines upper and lower power entropy $\overline{h}_{\text{pow}}$ and $\underline{h}_{\text{pow}}$ by taking the limit superior and limit inferior, respectively. Note that due to the monotonicity in δ, the existence of the second limit is automatic. We refer to [Mar13] for more information about this quantity.

In the definition of *modified power entropy* (MPE), the Bowen-Dinaburg metrics are replaced by the *Hamming metrics*

$$\hat{d}_n^f(x,y) \;=\; \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(x),f^i(y))\;.$$

If $\hat{S}_n(f,\delta)$ denotes the maximal cardinality of a set $S\subseteq X$ that is δ-separated with respect to $\hat{d}_n^f$, then the modified power entropy of f is defined as

$$h_{\text{mod}}(f) \;=\; \lim_{\delta\to 0}\lim_{n\to\infty}\frac{\log \hat{S}_n(f,\delta)}{\log n}\;,$$

provided the limit as $n\to\infty$ exists. If not, then one can again define upper and lower versions $\overline{h}_{\text{mod}}$ and $\underline{h}_{\text{mod}}$. The fact that $\hat{d}_n^f\leq d_n^f$ implies $\hat{S}_n(f,\delta)\leq S_n(f,\delta)$ and hence $h_{\text{mod}}(f)\leq h_{\text{pow}}(f)$. We also note that $h_{\text{top}}(f)>0$ implies $h_{\text{mod}}(f)=\infty$.[4]

In both cases, the concept of separation that is used in the first step is one in finite time: both metrics d_n^f and $\hat{d}_n^f$ depend only on the first n iterates of the considered points. The limit for $n\to\infty$ is then taken in a second step. However, since asymptotic notions like proximality, distality or Li-Yorke pairs play a central role in topological dynamics, it seems natural to also consider topological invariants that are directly based on an asymptotic concept of separation. This is true for the following notion.

Given $x,y\in X$ and $\delta>0$, we let

$$M_{\delta,n}^f(x,y) \;=\; \#\left\{0\leq k<n \mid d(f^k(x),f^k(y))\geq\delta\right\}$$

[4]This is well-known folklore, but for the convenience of the reader we provide a short direct proof in the next section.

and

$$\nu_\delta^f(x,y) \ = \ \overline{\lim_{n\to\infty}} \frac{M_{\delta,n}^f(x,y)}{n} \ .$$

We say that x and y are (f,δ,ν)-*separated* if

$$\nu_\delta^f(x,y) \ \geq \ \nu \ .$$

Given $\nu > 0$, we denote the maximal cardinality of a set $S \subseteq X$ which is ν-separated with respect to ν_δ^f by $S_\nu^*(f,\delta)$. Then, the *amorphic complexity* of f is defined as

$$\mathrm{ac}(f) \ = \ \lim_{\delta\to 0}\lim_{\nu\to 0} \frac{\log S_\nu^*(f,\delta)}{-\log\nu} \ .$$

As before, this assumes that the limits with respect to ν exist. Otherwise, it is again possible to define an upper and a lower amorphic complexity. Basic properties of this quantity, like topological invariance, factor relations, power invariance and a product rule, as well as the application to a number of example classes are discussed in [FGJ15]. Somewhat surprisingly, amorphic complexity turns out be very well applicable and accessible to explicit computations in various system classes like regular Toeplitz flows, Sturmian shifts and Denjoy type circle homeomorphisms or cut and project quasicrystals. The reason behind is the fact that the asymptotic nature of the employed separation concept allows to obtain bounds on the separation numbers $S_\nu^*(f,\delta)$ by applying suitable ergodic theorems. We refer to [FGJ15] again for details.

The main reason for treating amorphic complexity here is to complete the discussion in [FGJ15, Section 3.7] by showing that there are no direct relations, in terms of an inequality, between amorphic complexity and the other two notions. Thereby, for one of the directions, we will have to rely on the same example as for the non-existence of a variational principle for modified power entropy. Hence, we come back to this issue at the end of the next section.

In all of the above, we have considered polynomial growth rates, which turn out to be the appropriate scale for many important example classes. In general, however, it is certainly possible to take into account more or less arbitrary growth rates. We say $a : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ is a *scale function* if a is strictly increasing in both arguments. Then, in analogy to the power entropy, the *upper a-entropy* of f is defined as

$$\overline{h}_a(f) \ = \ \sup_{\delta>0}\sup\left\{ s>0 \ \middle| \ \overline{\lim_{n\to\infty}} \frac{S_n(f,\delta)}{a(s,n)} > 0 \right\} \ ,$$

and the lower one accordingly. In order to obtain good properties, one usually assumes that the scale functions are O-regularly varying, that is, $\overline{\lim}_{n\to\infty} \frac{a(s,mn)}{a(s,n)} < \infty$ for all $m \in \mathbb{N}$. Since this definition allows to capture any rates of asymptotic growth in the subexponential regime, one of the most important distinctions on the qualitative level is whether $\sup_{n\in\mathbb{N}} S_n(\delta,f)$ is bounded for all $\delta > 0$, or infinite for all sufficiently small δ. We will mainly focus on this aspect in our discussion of almost 1-1 extensions in Section 6. Of course, all these comments on the use of different scale functions equally apply to modified power entropy and amorphic complexity. For the latter, scale functions need to have the separation frequency as the second argument, so in this case a is a positive real-valued function on $\mathbb{R}^+ \times (0,1]$ and O-regularly varying means that $\overline{\lim}_{\nu\to 0} \frac{a(s,c\nu)}{a(s,\nu)}$ is finite for all $c > 0$.

4 Modified power entropy and topological entropy

In many cases, modified power entropy is strictly smaller than power entropy (see e.g. [HK02]). However, on an exponential scale (that is, using the scale function $a(s,n) = \exp(sn)$), this difference disappears. Since we do not know an appropriate reference, we include a proof of this well-known result.

Lemma 4.1. *Suppose X is a compact metric space and $f : X \to X$ is continuous. Then*

$$\lim_{\delta\to 0} \varliminf_{n\to\infty} \frac{\log \hat{S}_n(f,\delta)}{n} \;=\; \lim_{\delta\to 0} \varlimsup_{n\to\infty} \frac{\log \hat{S}_n(f,\delta)}{n} \;=\; h_{\text{top}}(f) \ .$$

In particular, $h_{\text{top}}(f) > 0$ implies $h_{\text{mod}}(f) = \infty$.

Proof. The $\leq$-inequalities are obvious. Further, it is well-known that

$$h_{\text{top}}(f) = \lim_{\varepsilon\to 0} \varliminf_{n\to\infty} \frac{\log S_n(f,\varepsilon)}{n}$$

(e.g. [KH97]). Therefore, it suffices to show that

$$\lim_{\delta\to 0} \varliminf_{n\to\infty} \frac{\log \hat{S}_n(f,\delta)}{n} \;\geq\; \lim_{\varepsilon\to 0} \varliminf_{n\to\infty} \frac{\log S_n(f,\varepsilon)}{n} \ . \tag{2}$$

To that end, fix $\varepsilon > 0$ and $\alpha > 0$ and choose $\delta \in (0, \varepsilon\alpha/2)$. Further, let $U_1, \ldots, U_K$ be a finite partition of X into sets of diameter $< \varepsilon$.

Given $n \in \mathbb{N}$, let $N = \hat{S}_n(f,\delta)$ and choose a partition of X into sets $P_1, \ldots, P_N$ with the property that $\hat{d}_n^f(x,y) \leq \delta$ for all $x, y \in P_j$, $j = 1, \ldots, N$. From each of the P_j, we select one point $x_j \in P_j$. (Note that all of the P_j are non-empty due to the definition of N.) Then, given $x \in P_j$, we define $\omega(x) \in \{0, \ldots, K\}^n$ by

$$\omega_i(x) \;=\; \begin{cases} 0 & \text{if } d(f^i(x), f^i(x_j)) \;<\; \varepsilon/2 \ , \\ k & \text{if } d(f^i(x), f^i(x_j)) \;\geq\; \varepsilon/2 \text{ and } x \in U_k \ . \end{cases}$$

Note that if $x, y \in P_j$ and $\omega(x) = \omega(y)$, then $d_n^f(x,y) < \varepsilon$. Hence, the maximal cardinality of a subset of P_j which is ε-separated with respect to d_n^f is at most $\#\{\omega(x) \mid x\$nP_j\}$. Moreover, we have that for each $x \in P_j$

$$\#\{0 \leq i \leq n-1 \mid \omega_i(x) \neq 0\} \;\leq\; \alpha n \ ,$$

since otherwise $\hat{d}_n^f(x, x_j) \geq \alpha n \varepsilon/2 \;>\; \delta$, contradicting the choice of the P_j. Hence, using that $\binom{n}{\lfloor n\alpha \rfloor} \leq \exp(-\alpha \log(\alpha) n)$ for sufficiently small α, we obtain that

$$\#\{\omega(x) \mid x \in P_j\} \;\leq\; \binom{n}{\lfloor n\alpha \rfloor} \cdot K^{\lfloor n\alpha \rfloor} \;\leq\; \exp\left(\alpha(\log(K) - \log(\alpha))n\right) \ .$$

Altogether, this yields that

$$S_n(f,\varepsilon) \;\leq\; \hat{S}_n(f,\delta) \cdot \exp\left(\alpha(\log(K) - \log(\alpha))n\right) \ .$$

Since $\lim_{\alpha\to 0} \alpha(\log(K) - \log(\alpha)) = 0$ and $\alpha > 0$ was arbitrary, this proves (2). □

5 A counterexample to the existence of a variational principle for MPE

Let $I = [0,1]$ and $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. The main aim in this section is the construction of an example of the following type.

Theorem 5.1. *There exists a skew product map of the form*

$$f : I \times \mathbb{T}^1 \to I \times \mathbb{T}^1 \quad , \quad f(x,y) = (\tau(x), y + \beta(x) + \rho) \ ,$$

where

- $\rho \in \mathbb{R} \setminus \mathbb{Q}$,
- τ *is a diffeomorphism of* I *with exactly two fixed points at* 0 *and* 1,
- $\beta : I \to \mathbb{T}^1$ *is a differentiable function with* $\beta_{|\{0\}\cup B_\varepsilon(1)} = 0$ *for some* $\varepsilon > 0$,

such that f *satisfies*

$$\underline{h}_{\mathrm{mod}}(f) \geq 1/2 \ .$$

Before we turn to the proof, we first draw the following conclusion.

Corollary 5.2. *There is no real-valued isomorphism invariant of measure-preserving dynamical systems that satisfies a variational principle with modified power entropy.*

There is, of course, a standard measure-theoretic analogue of modified power entropy, introduced in [Fer97, KT97] (see also [HK02]), which is bounded above by topological modified power entropy. However, since the only structural property that is needed is the invariance under isomorphisms, we do not need to state any detail here and omit these for the sake of brevity.

We also note that we understand 'variational principle' in the sense that the topological quantity equals the supremum over all measure-theoretic ones, where the supremum is taken over all *invariant* measures. For the standard notion of entropy it suffices to consider only ergodic measures due to the linearity of measure-theoretic entropy, but in general this can make a big difference (see also [HK02, Section 4.4b and page 81]).

Proof of Corollary 5.2. Suppose that h^* is a real-valued function of pairs (f,μ), where f is a continuous map on some (compact) metric space. Suppose for a contradiction that $h_{\mathrm{mod}}(f) = \sup_\mu h^*(f,\mu)$, where the supremum is taken over all invariant measures μ of f.

In the example in Theorem 5.1, there exist exactly two ergodic measures μ_0 and μ_1, which are the one-dimensional Lebesgue measures on the two circles $\mathcal{T}_0 = \{0\} \times \mathbb{T}^1$ and $\mathcal{T}_1 = \{1\} \times \mathbb{T}^1$, and the restriction of f to these circles is just the rotation by ρ. Any invariant measure μ is a convex combination of μ_0 and μ_1 and obviously isomorphic to $(f_{|\mathcal{T}_0\cup\mathcal{T}_1}, \mu_{|\mathcal{T}_0\cup\mathcal{T}_1})$. However, as f restricted to $\mathcal{T}_0 \cup \mathcal{T}_1$ is an isometry we have $h_{\mathrm{mod}}(f_{|\mathcal{T}_0\cup\mathcal{T}_1}) = 0$ and hence $h^*(f,\mu) = 0$ by the assumed variational principle. This means $\sup_\mu h^*(f,\mu) = 0$, whereas $\underline{h}_{\mathrm{mod}}(f) \geq 1/2$, which yields the required contradiction. □

We also note that in the situation of Theorem 5.1 we have $\underline{h}_{\mathrm{mod}}(f) > \underline{h}_{\mathrm{mod}}(f_{|\Omega(f)})$, where $\Omega(f)$ denotes the set of non-wandering points of f. This shows that modified power entropy is sensitive to transient dynamics. Since all invariant measures are supported on the non-wandering set, this is equally not compatible with a variational principle. We turn to the construction of the example.

Proof of Theorem 5.1. We first construct the diffeomorphism $\tau : I \to I$. To that end, let $I_n = [2^{-n}, 3\cdot 2^{-(n+1)}]$ and $I'_n = [2^{-n}, 5\cdot 2^{-(n+2)}]$ where $n \geq 3$. Then, we choose a $\mathcal{C}^1$-function $\alpha : I \to I$ with the following properties:

(i) $|\alpha'(x)| < 1$ for all $x \in I$;

(ii) $\alpha(0) = \alpha(1) = 0$;

(iii) $\alpha(x) > 0$ for all $x \in (0,1)$;

(iv) $\alpha_{|I_n} = 2^{-(3n+4)}$;

Further, we let $\beta(x) = x$ if $x \in [0, 7/8]$ and extend this differentiable to all of I in such a way that $\beta_{|B_\varepsilon(1)} = 0$ for some $\varepsilon > 0$. Note that since $d(I_n, I_{n+1}) = 2^{-(n+2)}$, condition (iv) does not contradict the differentiability of α. Due to (i) and (ii) the map $\tau : I \to I,\ x \mapsto x + \alpha(x)$ is a $\mathcal{C}^1$-diffeomorphism of I with unique fixed points 0 and 1. Moreover, due to (iii) we have $\lim_{n\to\infty} \tau^n(x) = 1$ for all $x \in (0,1]$.

In order to prove $\underline{h}_{\mathrm{mod}}(f) > 0$, fix $n \geq 3$ and choose $x_1^n < x_2^n < \ldots < x_{2^n}^n \in I'_n$ with $x_{i+1}^n - x_i^n = 2^{-(2n+2)}$. By (iv) and the choice of the intervals I'_n and I_n, we have $\tau^k(x_j^n) \in I_n$ for all $j = 1, \ldots, 2^n$ and $k = 0, \ldots, 2^{2n+2}$. Since α is constant on I_n, this means that the points x_j^n remain at equal distance for the first 2^{2n+2} iterations. If we consider the 2^n points $(x_j^n, 0) \in I \times \mathbb{T}^1$, then for $l, m = 1, \ldots, 2^n$ the vertical distance after n steps is

$$d\left(\pi_2 \circ f^k(x_l^n, 0), \pi_2 \circ f^k(x_m^n, 0)\right) \;=\; d'\left(k \cdot (l-m) \cdot 2^{-(2n+2)}, 0\right) \;.$$

Here d' denotes the canonical distance on $\mathbb{T}^1$. An easy computation yields for $l \neq m$

$$\hat{d}^f_{2^{2n+2}}((x_l^n, 0), (x_m^n, 0)) \;\geq\; \frac{1}{2^{2n+2}} \sum_{k=0}^{2^{2n+2}-1} d'\left(k \cdot (l-m) \cdot 2^{-(2n+2)}, 0\right) \;=\; \frac{1}{4} \;.$$

This means that the set $\{x_1^n, \ldots, x_{2^n}^n\}$ is $\frac{1}{4}$-separated with respect to $\hat{d}^f_{2^{2n+2}}$. We thus obtain $\hat{S}_n(f, 1/4) \geq 2^n$ and hence

$$\underline{h}_{\mathrm{mod}}(f) \;\geq\; \varliminf_{n\to\infty} \frac{\log \hat{S}_n(f,\delta)}{\log n} \;\geq\; \lim_{k\to\infty} \frac{\log(2^k)}{\log(2^{2k+2})} \;=\; \frac{1}{2} \;. \qquad \square$$

In order to conclude this section, we want to discuss why the above example also shows that there is no direct relation, in terms of an inequality, between modified power entropy and amorphic complexity. To that end, let us first look at some trivial examples.

Since Morse-Smale systems have a finite set of fixed or periodic points and these attract all other orbits, it is easy to see that they have zero amorphic complexity. This shows that one may have $\mathrm{ac}(f) < h_{\mathrm{pow}}(f)$. On the other hand, consider $f : \mathbb{T}^2 \to \mathbb{T}^2,\ (x,y) \mapsto (x, x+y)$. Then any two points with different x-coordinate rotate with different speed in the vertical direction, and it is therefore easy to see that they are ν-separated with respect to ν_δ^f if $\nu, \delta > 0$ are chosen sufficiently small. Therefore, $S_\nu^*(f,\delta) = \infty$ and thus $\mathrm{ac}(f) = \infty$. At the same time, it is easy to check that $h_{\mathrm{pow}}(f) \leq 1$ (see [FGJ15, Section 3.7]). Hence, we may have $\mathrm{ac}(f) > h_{\mathrm{pow}}(f)$ (and thus also $\mathrm{ac}(f) > h_{\mathrm{mod}}(f)$). The only remaining direction is therefore to show that $h_{\mathrm{mod}}(f) > \mathrm{ac}(f)$ is possible as well. However, we claim that this is the case in the example constructed above. In order to see this, the following basic observation is helpful.

Lemma 5.3 ([FGJ15, Lemma 3.11]). *Suppose that X is a compact metric space, f is a continuous map and $A \subseteq X$ is a forward invariant subset such that for all $x \in X \setminus A$ there exists $y_x \in A$ such that $\lim_{n\to\infty} d(f^n(x), f^n(y_x)) = 0$. Then $\mathrm{ac}(f) = \mathrm{ac}(f_{|A})$.*

In the above situation, we say that f has the *unique target property* with respect to A. If f is the example constructed in the proof of Theorem 5.1, then this assumption is satisfied for $A = \Omega(f)$. This is a direct consequence of the fact that $\beta_{|B_\varepsilon(1)} = 0$ (and the reason for including this condition, which has not been used otherwise). Note here that all orbits outside of $\mathcal{T}_0$ converge to the circle $\mathcal{T}_1$ upon forward iterations, and once they enter $B_\varepsilon(1) \times \mathbb{T}^1$ the rotation in the second coordinate is always equal to ρ. For this reason, all these orbits have a unique 'target orbit' in $\mathcal{T}_1$. Since $f_{|\mathcal{T}_0 \cup \mathcal{T}_1}$ is an isometry and therefore has amorphic complexity zero, the above Lemma 5.3 yields $\mathrm{ac}(f) = 0 < \underline{h}_{\mathrm{mod}}(f)$.

It remains to point out that since the example constructed above has the unique target property with respect to the non-wandering set, the transient dynamics causing the positive modified power entropy should still be considered as rather 'tame'. Amorphic complexity is equally sensitive to transient dynamics, but these have to 'mix up' orbits arbitrarily close to the non-wandering set. An example similar to the one above is given in [FGJ15, Section 3.5].

6 Modified power entropy of regular almost 1-1 extensions

Given two compact metric spaces $(X, d), (\Xi, \rho)$ and two continuous maps $f : X \to X$, $\tau : \Xi \to \Xi$, we say (X, f) is a *(topological) extension* of (Ξ, τ) if there exists a continuous onto map $h : X \to \Xi$ such that $h \circ f = \tau \circ h$. In this situation, h is called a *factor map* or *semi-conjugacy* from f to τ and (Ξ, τ) is called a *(topological)* factor of (X, f). For the sake of brevity, we will sometimes omit the spaces and say τ is a factor of f. An extension is called *almost 1-1* if the set $\Omega = \{\xi \in \Xi \mid \#h^{-1}(\xi) = 1\}$ is generic in the sense of Baire (that is, a residual set). Note that if f and τ are invertible, then the set Ω is τ-invariant. Moreover, if in addition τ is minimal, then it suffices to require that there exist a single ξ with $\#h^{-1}(\xi) = 1$.

From now on, we assume for the remainder of this section that f and τ are invertible and τ is minimal. Further, we suppose that τ is *almost periodic* (that is, equicontinuous). In this case, there exists a unique τ-invariant probability measure μ on Ξ, which is necessarily ergodic (unique ergodicity). We say that the extension (X, f) is *regular* if $\mu(\Omega) = 1$ and *irregular* if $\mu(\Omega) = 0$. Note that by invariance of Ω and ergodicity of μ, one of the two always holds. We refer to [Aus88] for a comprehensive exposition.

A regular almost 1-1 extension of an equicontinuous minimal system is always uniquely ergodic and isomorphic to its factor. For this reason, the topological entropy is zero in this case. However, if the extension is not everywhere 1-1 (that is, there exists $\xi \in \Xi$ such that $\#h^{-1}(\xi) > 1$), then f cannot be equicontinuous.[5] Thus, there is a break of equicontinuity when going from equicontinuous minimal systems to their almost 1-1 extensions, but at the same time this does not lead beyond the regime of zero entropy. It is therefore a natural question to ask how a topological invariant for low-complexity systems behaves during this bifurcation. In particular, it is one possible task for such a slow entropy to detect this change in the qualitative behaviour. We will discuss a positive result in this direction for amorphic

[5]From now on, whenever speaking of extensions we will assume implicitly that the factor map is not injective.

complexity and power entropy further below. Modified power entropy, however, does not respond to this transition.

Theorem 6.1. *Suppose $f : X \to X$ is a regular almost 1-1 extension of a minimal equicontinuous homeomorphism $\tau : \Xi \to \Xi$. Then $\sup_{n\in\mathbb{N}} \hat{S}_n(f,\delta) < \infty$ for all $\delta > 0$ and in particular $h_{\mathrm{mod}}(f) = 0$.*

For the proof, the following statement will be useful.

Lemma 6.2 ([FGJ15, Lemma 2.6])**.** *Let $h : X \to \Xi$ be the factor map of an almost 1-1 extension and define*

$$E_\delta \;=\; \{\xi \in \Xi \mid \mathrm{diam}(h^{-1}(\xi)) \geq \delta\} \ .$$

Then for all $\delta > 0$ and $\varepsilon > 0$ there exists $\eta_\delta(\varepsilon) > 0$ such that if $x, y \in X$ satisfy $d(x,y) \geq \delta$ and $\rho(h(x),h(y)) < \eta_\delta(\varepsilon)$, then $h(x)$ and $h(y)$ are both contained in $B_\varepsilon(E_\delta)$.

Proof of Theorem 6.1. By going over to an equivalent metric, we may assume without loss of generality that τ is an isometry. Fix $\delta > 0$ and choose

$$\nu \;<\; \frac{\delta}{2\mathrm{diam}(X) - \delta} \ . \tag{3}$$

Then, choose $\varepsilon > 0$ such that $\mu(A) < \nu$ where $A = \overline{B_\varepsilon(E_{\delta/2})}$. Let $\eta = \eta_{\delta/2}(\varepsilon)$ be as in Lemma 6.2. Due to the Uniform Ergodic Theorem we can find $M \in \mathbb{N}$ such that for all $n \geq M$ and $\xi \in \Xi$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1} \mathbf{1}_A \circ \tau^i(\xi) \;<\; \nu \ .$$

Therefore, given two points $x, y \in X$ with $\rho(h(x),h(y)) < \eta$ and $n \geq M$, the fact that τ is an isometry together with Lemma 6.2 implies

$$\begin{aligned}\hat{d}^f_n(x,y) \;&\leq\; \frac{\mathrm{diam}(X)}{n}\cdot\left(\sum_{i=0}^{n-1}\mathbf{1}_A\circ\tau^i(h(x))\right) + \frac{\delta}{2n}\cdot\left(n - \sum_{i=0}^{n-1}\mathbf{1}_A\circ\tau^i(h(x))\right)\\ &\leq\; \mathrm{diam}(X)\cdot\nu + \frac{\delta}{2}\cdot(1-\nu) \overset{(3)}{<} \delta \ .\end{aligned}$$

Thus, independent of $n \geq M$, two points x, y can only be δ-separated with respect to $\hat{d}^f_n$ if $h(x)$ and $h(y)$ have distance greater than η in Ξ. Hence, any set $S \subseteq X$ which is δ-separated with respect to $\hat{d}^f_n$ projects to a set which is η-separated with respect to the metric in Ξ. Since Ξ is compact, there exists an upper bound $K(\eta)$ on the maximal cardinality of an η-separated set. We obtain $\hat{S}_n(f,\delta) < K(\eta)$ for all $n \geq M$ and consequently $h_{\mathrm{mod}}(f) = 0$ as claimed. □

Theorem 6.1 is in contrast to the following result in [FGJ15], which shows that the asymptotic separation numbers involved in the definition of amorphic complexity are sensitive to the break of equicontinuity in the above situation.

Theorem 6.3 ([FGJ15, Theorem 2.10])**.** *Let $f : X \to X$ be a minimal almost 1-1 extension of an equicontinuous homeomorphism $\tau : \Xi \to \Xi$. Then there is $\delta > 0$ such that $\sup_{\nu>0} S^*_\nu(f,\delta) = \infty$.*

We note that this result does not guarantee polynomial growth rates, but as discussed in Section 3 one can obtain positive amorphic complexity with respect to a suitably chosen scale function. In order to obtain a similar result for power entropy, it suffices to use the following elementary observation.

Lemma 6.4. *If* $\sup_{\nu>0} S^*_\nu(f,\delta) = \infty$, *then* $\sup_{n\in\mathbb{N}} S_n(f,\delta) = \infty$.

Proof. Suppose that S is a set which is ν-separated with respect to ν^f_δ. Then there exists $n > 0$ such that for all $x \neq y \in S$ we have $\#\{i = 0,\ldots,n-1 \mid d(f^i(x), f^i(y)) \geq \delta\}/n \geq \nu/2 > 0$. This immediately implies $d^f_n(x,y) \geq \delta$ for all $x \neq y \in S$. Hence, S is a δ-separated set with respect to d^f_n and therefore $S_n(f,\delta) \geq \#S$. The statement now follows easily. □

We note that the following direct consequence is also contained in a more general result by Blanchard, Host and Maass [BHM00, Proposition 2.2].

Corollary 6.5. *Suppose $f : X \to X$ is a minimal almost 1-1 extension of an equicontinuous homeomorphism $\tau : \Xi \to \Xi$. Then there exists $\delta > 0$ such that* $\sup_{n\in\mathbb{N}} S_n(f,\delta) = \infty$.

7 An example of an irregular Toeplitz flow of low complexity

Since modified power entropy does not detect the difference between equicontinuous minimal systems and their regular almost 1-1 extensions, one could hope that instead it responds to the transition from regular to irregular almost 1-1 extensions. The aim of this section is to demonstrate that this is not the case either. To that end, we construct an example of an irregular Toeplitz sequence, leading to an irregular almost 1-1 extension f of the corresponding odometer, which has modified power entropy zero and even bounded separation numbers $\hat{S}_n(f,\delta)$ for all $\delta > 0$. We assume some acquaintance with the theory of odometers and Toeplitz flows and refer to the excellent survey [Dow05] or classical papers by Jacobs and Keane [JK69], Eberlein [Ebe71] and Williams [Wil84] for the relevant details.

We let $\Sigma = \{0,1\}^{\mathbb{Z}}$ and equip it with the metric

$$d(\omega,\tilde{\omega}) \ = \ \sum_{\omega_k \neq \tilde{\omega}_k} 2^{-|k|} \ ,$$

where $\omega,\tilde{\omega} \in \Sigma$ and the index k in the sum runs over all of $\mathbb{Z}$, to make it a compact metric space. By σ we denote the left shift on Σ. Given $\omega \in \Sigma$, we let Σ_ω be the shift orbit closure of ω.

Theorem 7.1. *There exists an irregular Toeplitz sequence ω such that the corresponding Toeplitz flow (Σ_ω,σ) satisfies $\sup_{n\in\mathbb{N}} \hat{S}_n(\sigma_{|\Sigma_\omega},\delta) < \infty$ for all $\delta > 0$, and in particular $h_{\mathrm{mod}}(\sigma_{|\Sigma_\omega}) = 0$.*

Note that conversely irregular Toeplitz flows may have positive entropy [BK92], in which case the modified power entropy is infinite by Lemma 4.1.

Before we turn to the proof, we first need to address some technical issues. Given $\omega,\tilde{\omega} \in \Sigma$ and $n \in \mathbb{N}$, we let

$$D_n(\omega,\tilde{\omega}) \ = \ \frac{1}{2n+1} \sum_{i=-n}^{n} |\omega_i - \tilde{\omega}_i| \ .$$

Further, we denote by $R_n(\Sigma_\omega,\delta)$ the largest cardinality of a set $R \subseteq \Sigma_\omega$ which is δ-separated with respect to D_n. The following statements allow to relate $R_n(\Sigma_\omega,\delta)$ to $\hat{S}_n(\sigma_{|\Sigma_\omega},\delta)$.

Lemma 7.2. *Let $n \in \mathbb{N}$. We have $\hat{d}_n^\sigma(\omega,\tilde{\omega}) \leq 9D_{2n}(\omega,\tilde{\omega}) + 2^{-(n-1)}$ for all $\omega,\tilde{\omega} \in \Sigma$. In particular, if $\hat{d}_n^\sigma(\omega,\tilde{\omega}) \geq \delta$ and $2^{-(n-1)} \leq \delta/2$, then $D_{2n}(\omega,\tilde{\omega}) \geq \delta/18$ and hence $R_{2n}(\Sigma_\omega,\delta/18) \geq \hat{S}(\sigma_{|\Sigma_\omega},\delta)$.*

Proof. We have

$$\begin{aligned}
\hat{d}_n^\sigma(\omega,\tilde{\omega}) &= \frac{1}{n}\sum_{i=0}^{n-1} d(\sigma^i(\omega),\sigma^i(\tilde{\omega})) \;=\; \frac{1}{n}\sum_{i=0}^{n-1}\sum_{\omega_{k+i}\neq\tilde{\omega}_{k+i}} 2^{-|k|} \\
&= \frac{1}{n}\sum_{\omega_k\neq\tilde{\omega}_k}\sum_{i=0}^{n-1} 2^{-|k-i|} \;\leq\; \frac{1}{n}\left(\sum_{|k|\leq 2n:\omega_k\neq\tilde{\omega}_k} 3 \;+\; \sum_{|k|>2n:\omega_k\neq\tilde{\omega}_k} 2^{-(|k|-n)}\right) \\
&\leq \frac{3(2n+1)}{n} D_{2n}(\omega,\tilde{\omega}) + 2^{-(n-1)} \;\leq\; 9D_{2n}(\omega,\tilde{\omega}) + 2^{-(n-1)} \; .
\end{aligned}$$

□

Corollary 7.3. *If $\sup_{n\in\mathbb{N}} R_n(\Sigma_\omega,\delta) < \infty$ for all $\delta > 0$, then $\sup_{n\in\mathbb{N}} \hat{S}_n(\sigma_{|\Sigma_\omega},\delta) < \infty$ for all $\delta > 0$.*

Thus, it suffices to consider the pseudometrics D_n in the proof of Theorem 7.1, which are easier to handle than the Hamming metrics in this context.

For the particular case of Toeplitz flows, there is a further simplification, which is due to the fact that for a Toeplitz sequence ω the space Σ_ω consists precisely of those sequences which have exactly the same subwords as ω. This leads to the following elementary observations. In order to specify finite subwords of $\omega \in \Sigma$, we let $\omega_j^{m,n} = \omega_{m+j}$ for $j = -n,\ldots,n$, so that $\omega^{m,n}$ is the subword of ω with length $2n+1$ and center position m. In order to count the number of mismatches between two subwords of the same length, we let

$$D_n(\omega^{m,n},\omega^{m',n}) \;=\; D_n(\sigma^m(\omega),\sigma^{m'}(\omega)) \;=\; \frac{1}{2n+1}\sum_{j=-n}^{n} |\omega_j^{m,n} - \omega_j^{m',n}| \; .$$

Further, we denote by $\tilde{R}_n(\omega,\delta)$ the largest cardinality of a family of subwords of ω of length $2n+1$ which are δ-separated with respect to D_n.

Corollary 7.4. *If $\omega \in \Sigma$ is a Toeplitz sequence, then we have $\tilde{R}_n(\omega,\delta) = R_n(\Sigma_\omega,\delta)$ for all $n \in \mathbb{N}$, $\delta > 0$. In particular, if for each $\delta > 0$ we have $\sup_{n\in\mathbb{N}} \tilde{R}_n(\omega,\delta) < \infty$, then $\sup_{n\in\mathbb{N}} \hat{S}_n(\sigma_{|\Sigma_\omega},\delta) < \infty$ for all $\delta > 0$.*

We can now turn to the

Proof of Theorem 7.1. Our construction is a classical one which has been used in similar form by many authors [Oxt52, Wil84, BK90, BK92, Dow05]. The difficulty lies in controlling the separation numbers with respect to the Hamming metrics.

We first fix $a_1 \in \mathbb{N}$ and a sequence $(b_n)_{n\in\mathbb{N}}$ of integers ≥ 2, specified further below, and let $a_{n+1} = 2b_n a_n$ for all $n \geq 1$. Further, we let

$$A_n \;=\; \{-a_n,\ldots,a_n\} + a_{n+1}\mathbb{Z} \quad , \quad B_n \;=\; \bigcup_{i=1}^{n} A_n \quad \text{and} \quad C_n \;=\; B_n \setminus B_{n-1} \; .$$

Intervals of the form $\{-a_n,\dots,a_n\}+\ell a_{n+1}$ with $\ell\in\mathbb{Z}$ will be called *n-blocks*. Note that since the a_n converge to ∞, we have $\biguplus_{n\in\mathbb{N}} C_n=\mathbb{Z}$. Now, we can define a Toeplitz sequence $\omega\in\Sigma$ by

$$\omega_k \;=\; \begin{cases} 0 & \text{if} \quad k\in C_n \text{ with } n \text{ odd;} \\ \\ 1 & \text{if} \quad k\in C_n \text{ with } n \text{ even.} \end{cases}$$

If $k\in C_n$, then by definition k is an a_{n+1}-periodic position, that is, $\omega_{k+\ell a_{n+1}}=\omega_k$ for all $\ell\in\mathbb{Z}$. By construction, all positions $k\in\mathbb{Z}$ are periodic for some a_n in this sense, so that by definition ω is a Toeplitz sequence with periodic structure $(a_n)_{n\in\mathbb{N}}$. Further, if $k\in C_{n+1}$, then there exists $\ell\in\mathbb{Z}$ with $k+\ell a_{n+1}\in C_{n+2}\setminus C_{n+1}$. Hence, we have $\omega_{k+\ell a_{n+1}}\neq\omega_k$, so that k is not an a_{n+1}-periodic position. Therefore, we obtain that the set $\mathrm{Per}(\omega,a_{n+1})$ of a_{n+1}-periodic positions equals C_n. Since $C_n=\bigcup_{i=1}^n A_i$, we obtain that this set has density

$$\mathcal{D}(a_{n+1}) \;\leq\; \sum_{i=1}^n \frac{2a_i}{a_{i+1}} \;=\; \sum_{i=1}^n \frac{1}{b_i}\ .$$

If we choose the b_i's such that $\sum_{i=1}^\infty b_i^{-1}<1$, then $\lim_{n\to\infty}\mathcal{D}(a_n)<1$. This means, by definition, that the Toeplitz sequence ω with periodic structure $(a_n)_{n\in\mathbb{N}}$ is *irregular* and thus (Σ_ω,σ) is an irregular almost 1-1 extension of a corresponding odometer (see [Dow05]).

In order to show that $\sup_{n\in\mathbb{N}}\tilde{R}_n(\omega,\delta)<\infty$ for all δ, we fix $\delta>0$ and $j_0\in\mathbb{N}$ with $2^{-j_0}<\delta/4$. Since $\tilde{R}_n(\omega,\delta)\leq\tilde{R}_{n'}(\omega,\delta/2)$ if $n\leq n'\leq 2n$, it suffices to show that

$$\sup_{j\in\mathbb{N}}\tilde{R}_{2^j}(\omega,\delta) \;<\; \infty \tag{4}$$

for all $\delta>0$. To that end, choose $s\in\mathbb{N}$ with $\sum_{i=s+1}^\infty b_i^{-1}<\delta/8$, so that the asymptotic density of $B_n\setminus B_s$ is smaller than $\delta/4$ for all $n>s$. Given $N=2^j$ with $j\geq j_0$, we will define a partition $\{A^\iota_{p,q}\}^\iota_{p,q}$ of $\mathbb{Z}$ with the property that if m,m' belong to the same element of the partition, then $\omega^{m,N}$ and $\omega^{m',N}$ cannot be δ-separated with respect to D_n. This implies immediately that $\tilde{R}(\omega,\delta)$ does not exceed the number of partition elements. Since the latter will be independent of N, this will prove (4).

The partition elements $\mathcal{A}^\iota_{p,q}$ depend on three parameters. The parameter p describes the position of the subwords with respect to the s-blocks. Given $p\in\{0,\dots,a_{s+1}-1\}$, we let

$$\mathcal{A}_p \;=\; \{m\in\mathbb{Z}\mid m=\nu a_{s+1}+p \text{ for some } \nu\in\mathbb{Z}\}$$

and write $p(m)=p$ if $m\in\mathcal{A}_p$. The second parameter q describes the position with respect to the nearest n-block, where n is chosen such that $a_n/2<N\leq a_{n+1}/2$, and the third parameter ι determines the local configuration of symbols around the n-block. Both are somewhat more subtle to define.

Suppose first that the interval $I^{m,N}=\{m-N,\dots,m+N\}$ does not intersect any n-block. Then we let $q(m)=0$ and $\iota(m)=0$. Otherwise, $I^{m,N}$ intersects exactly one n-block $B(m)=\{-a_n,\dots,a_n\}+\ell a_{n+1}$. In this case, we define $j(m)=\ell a_{n+1}-m$ and let

$$q(m) \;=q \quad \text{if} \quad j(m)\in\left(\frac{qN}{2^{j_0}},\frac{(q+1)N}{2^{j_0}}\right]$$

Note that $j(m)$ lies between $-N-a_n$ and $N+a_n$ and thus $q(m)$ ranges only from at least $-3\cdot 2^{j_0}$ to at most $3\cdot 2^{j_0}-1$.

In order to define ι for the case $I^{m,N}$ intersects an n-block $B(m)$, we denote by $B^-(m)$ and $B^+(m)$ the two nearest n-blocks to the left, respectively, right of $B(m)$. Further, we denote by $J^-(m)$ the interval between $B^-(m)$ and $B(m)$ and by $J^+(m)$ the interval between $B(m)$ and $B^+(m)$. Note that by construction, there exist unique integers $n^\pm$ such that $J^\pm(m) \setminus A_n \subseteq C_{n^\pm}$. Hence, ω_k remains constant on each of the sets $J^\pm(m) \setminus A_n$. We assume that n is even, such that $\omega_k = 0$ for all $k \in B(m) \setminus A_{n-1} = B(m) \cap C_n$. Then there are four possibilities:

(1) $\omega_k = 0$ for all $k \in (J^-(m) \cup J^+(m)) \setminus A_n$;

(2) $\omega_k = 0$ for all $k \in J^-(m) \setminus A_n$ and $\omega_k = 1$ for all $k \in J^+(m) \setminus A_n$;

(3) $\omega_k = 1$ for all $k \in J^-(m) \setminus A_n$ and $\omega_k = 0$ for all $k \in J^+(m) \setminus A_n$;

(4) $\omega_k = 1$ for all $k \in (J^-(m) \cup J^+(m)) \setminus A_n$;

We let $\iota(m) = \iota$ if case (ι) applies. Now, given $q \in \{-3 \cdot 2^{j_0}, \ldots, 3 \cdot 2^{j_0} - 1\}$ and $\iota \in \{0, \ldots, 4\}$ we let

$$\mathcal{A}^\iota_{p,q} \;=\; \{m \in \mathcal{A}_p \;\mid\; q(m) = q \text{ and } \iota(m) = \iota\} \;,$$

where we set $\mathcal{A}^0_{p,q} = \emptyset$ if $q \neq 0$. This defines the required decomposition of $\mathbb{Z}$ into at most $30 \cdot 2^{j_0} \cdot a_{s+1}$ partition elements. It remains to show that given $m, m' \in \mathcal{A}^\iota_{p,q}$, the words $\omega^{m,N}$ and $\omega^{m',N}$ cannot be δ-separated with respect to D_N. Thus, we need to estimate the maximal number of mismatches that can appear between two such words.

First, since the position of the words with respect to the a_{s+1}-periodic set A_s is identical, we have that $j + m \in A_s$ if and only if $j + m' \in A_s$, and in this case $\omega^{m,N}_j = \omega^{m',N}_j$.

Secondly, if either $j + m \in A_{n-1} \setminus A_s$ or $j + m' \in A_{n-1} \setminus A_s$, then this might result in a mismatch. However, since

$$\frac{\#\left(I^{m,N} \cap A_{n-1}\right)}{\# I^{m,N}} \;\leq\; 2 \sum_{i=s+1}^{n-1} b_i^{-1} \;<\; \delta/4$$

and likewise for $I^{m',N}$, we have that the contribution of such mismatches to $D_N(\omega^{m,N}, \omega^{m',N})$ is at most $\delta/2$.

Finally, it remains to count the possible mismatches with $j+m, j+m' \notin A_{n-1}$. If $j(m) = j(m')$, then there are no such mismatches, since the intervals $B(m) - m$ and $B(m') - m'$ as well as $J^\pm(m) - m$ and $J^\pm(m') - m'$ coincide and case (ι) above applies to both m and m'. Otherwise, there are possible overlaps between non-corresponding intervals, but since $|j(m) - j(m')| \leq N/2^{j_0}$ these overlaps concern at most $2N/2^{j_0} < 2N \cdot (\delta/4)$ positions. Again, this results in a contribution to $D_N(\omega^{m,N}, \omega^{m',N})$ of at most $\delta/2$. Altogether, this yields $D_N(\omega^{m,N}, \omega^{m',N}) < \delta$ as required and thus completes the proof. □

Remark 7.5.

(a) Using similar, but simpler arguments, it is possible to show that the power entropy of the above example equals 1.

(b) At the same time, it can be shown that the amorphic complexity of the example is infinite, and even $S^*_\nu(f, \delta) = \infty$ for sufficiently small $\nu, \delta > 0$. This is, in fact, a consequence of a much more general statement. It is possible to prove that the asymptotic separation numbers $S^*_\nu(f,\delta)$ of a minimal action of a homeomorphism f on a compact metric space are all finite if and only if the system is Weyl mean equicontinuous. By a recent result of Downarowicz and Glasner, this holds if and only if the system is an isomorphic extension of its maximal equicontinuous factor [DG15, Theorem 2.1]. In particular, it needs to be uniquely ergodic, which is not true for our example due to the fluctuating symbol frequencies. Since these issues will be explored further in [FGJ], we do not go into further detail here.

(c) A construction which is very similar to the above one, but results in a uniquely ergodic irregular Toeplitz flow, can be found in [DK15]. Analogous arguments can be applied to show that this example also has modified power entropy zero.

8 Conclusions and open questions

It may seem, admittedly, that this note has a somewhat negative touch, since the presented results are mostly negative ones. We mainly showed that modified power entropy does not satisfy a variational principle, is not independent of transient dynamics, does not respond to the transition from equicontinuous systems to their almost 1-1 extensions and cannot be used either to distinguish between regular and irregular extensions of minimal equicontinuous systems. However, as we discussed in Section 2 already, these issues do not have to be seen as disadvantages of the notion itself. As said before, the existence of a variational principle and the insensitivity to transient effects are not necessarily positive features of a slow entropy, since this depends very much on the purpose one has in mind. Thus, the presented facts should rather be understood as clarifications and simply imply that for the specific aspects we concentrate on other topological invariants have to be identified in order to fulfill the respective tasks or requirements. We also note that it follows from results of Ferenczi that modified power entropy does detect the transition from uniquely ergodic isomorphic to non-isomorphic extensions of compact group rotations [Fer97].

As we have seen, the transition from equicontinuous minimal systems to their almost 1-1 extensions can be detected by means of a-entropy or amorphic complexity (with suitably chosen scale functions). In the other cases, however, this leads to the following open questions.

(a) Is there a topological invariant h for continuous maps on (compact) metric or topological spaces that gives meaningful information about zero entropy systems, but at the same time satisfies $h(f) = h(f_{|\Omega(f)})$?

(b) Is there such a topological invariant that satisfies a variational principle with respect to a suitable measure-theoretic analogue?

We note that Kong and Chen [KC14] recently introduced a slow entropy which satisfies a 'non-standard' variational principle, in which the supremum is taken over all probability measures on the phase space (and not just the invariant ones).

(c) Is there a meaningful topological invariant for dynamical systems which is zero for all regular almost 1-1 extensions of equicontinuous systems, but strictly positive for all irregular almost 1-1 extensions of such systems?

Some progress on closely related questions has recently been made by Li, Tu and Ye [LTY14] and Downarowicz and Glasner [DG15].

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