TOEPLITZ FLOWS AND MODEL SETS

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Abstract. We show that binary Toeplitz flows can be interpreted as Delone dynamical systems induced by model sets and analyse the quantitative relations between the respective system parameters. This has a number of immediate consequences for the theory of model sets. In particular, we use our results in combination with special examples of irregular Toeplitz flows from the literature to demonstrate that irregular proper model sets may be uniquely ergodic and do not need to have positive entropy. This answers questions by Schlottmann and Moody.

1. Introduction

Toeplitz flows have played an important role in the development of ergodic theory, since they provide a wide class of minimal dynamical systems that may exhibit a variety of exotic properties. Accordingly, they have often been employed to clarify fundamental questions on dynamical systems and to provide examples for particular combinations of dynamical properties, like strict ergodicity and positive entropy or minimality and absence of unique ergodicity [14, 17, 25].

The aim of this article is to make this rich source of examples available in the context of aperiodic order — often referred to as the mathematical theory of quasicrystals — and more specifically for the study of (general) cut and project sets. Cut and project sets, introduced by Meyer [18] in a somewhat different context, are arguably the most important class of examples within the theory of quasicrystals. With a focus on proper (but not necessarily regular) model sets their investigation has become a cornerstone of the theory, see e.g. the survey papers [19, 20]. Recently, also the more general classes of repetitive Meyer sets [1, 15] (see the survey [2] as well) and of weak model sets [4, 16] have been studied in some detail.

It is an elementary observation that any two-sided repetitive (or almost periodic) sequence $\xi = (\xi_n)_{n\in\mathbb{Z}}$ of 0's and 1's can be identified with a Delone subset of $\mathbb{Z}$, for instance the one given by $\mathcal{D}(\xi) = \{n \in \mathbb{Z} \mid \xi_n = 1\}$. Our main result, Theorem 1 in Section 2, shows that if ξ is a Toeplitz sequence, then $\mathcal{D}(\xi)$ can always be interpreted as a model set arising from a cut and project scheme (CPS). Not surprisingly, the internal group of this CPS is chosen as the odometer associated with the Toeplitz sequence.

Moreover, we provide a quantitative analysis and show that the regularity and the scaling exponents of the Toeplitz sequence can be computed either in terms of the measure of the boundary of the windows in the CPS (in the case of irregular Toeplitz flows, Section 2) or in terms of the box dimension of this boundary (for regular Toeplitz flows, Section 3). This ties together the principal quantities of both system classes and immediately allows to answer a number of questions on model sets, which were – to the best of our knowledge – still open.

In particular, we thus obtain that a positive measure of the boundary of the window of a cut and project set does neither imply positive topological entropy, nor the existence of multiple ergodic measures (Section 4). Moreover, using Toeplitz examples due to Downarowicz and Lacroix [10], we can then demonstrate that model sets may have any given countable subgroup of $\mathbb{R}$ as their dynamical spectrum, provided it contains infinitely many rationals. In particular, irrational eigenvalues may occur despite the fact that the underlying odometer only has rational point spectrum, and this further implies the existence of non-continuous eigenfunctions.

It is worth mentioning that the model sets provided by our construction are minimal and satisfy the additional regularity feature of properness. In particular, they fall into both the class of repetitive Meyer sets and the class of weak model sets mentioned above.

Our construction below deals with binary Toeplitz sequences, as this suffices to provide the desired counterexamples. However, clearly, similar results can be achieved for Toeplitz sequences over larger alphabets than $\{0,1\}$. Indeed, for the purposes of the present paper it is not hard to reduce the case of arbitrary Toeplitz sequences to binary sequences by identifying all letters but one.

2. Binary Toeplitz sequences and model sets

Let us start by discussing Toeplitz sequences and flows. For background and references we refer the reader to [9]. Let $\varSigma = \{0,1\}^{\mathbb{Z}}$ and denote by $\sigma\colon \varSigma \longrightarrow \varSigma$ the left shift. Suppose $\xi \in \varSigma$ is a Toeplitz sequence, which means that for all $k \in \mathbb{Z}$ there exists $p \in \mathbb{N}$ such that $\xi_{k+np} = \xi_k$ for all $n \in \mathbb{Z}$. Then, the shift orbit closure $\varSigma_\xi = \overline{\{\sigma^n(\xi) \mid n \in \mathbb{Z}\}}$ is a minimal set for the shift σ, and $(\varSigma_\xi, \sigma)$ is called the *Toeplitz flow* generated by ξ. Elements of $\varSigma_\xi$ which are not Toeplitz sequences themselves are called *Toeplitz orbitals*. The latter necessarily exist in any $\varSigma_\xi$ that is built from a non-periodic Toeplitz sequence ξ; compare [3, Cor. 4.2]. In the remainder of this paper, we shall restrict our attention to non-periodic Toeplitz sequences.

In line with the standard literature, we call

$$\mathrm{Per}(p,\xi) \;=\; \{k \in \mathbb{Z} \mid \xi_k = \xi_{k+np} \;\text{ for all }\; n \in \mathbb{Z}\}$$

the *p-skeleton* of ξ and refer to its elements as *p-periodic positions* of ξ. A $p \in \mathbb{N}$ is an *essential period* of ξ if $\mathrm{Per}(p',\xi) \neq \mathrm{Per}(p,\xi)$ for all $p' < p$. A *period structure* for ξ is a sequence $(p_\ell)_{\ell\in\mathbb{N}}$ of essential periods such that, for all $\ell \in \mathbb{N}$, p_ℓ is an essential period of ξ that divides $p_{\ell+1}$ and that, together, they satisfy $\bigcup_{\ell\in\mathbb{N}} \mathrm{Per}(p_\ell,\xi) = \mathbb{Z}$. Such a sequence always exists and can be obtained, for example, by defining p_ℓ as the multiple of all essential periods occuring for the positions in $[-\ell,\ldots,\ell]$. The *density* of the p-skeleton is defined as

$$D(p) \;=\; \#\bigl(\mathrm{Per}(p,\xi) \cap [0,p-1]\bigr)/p.$$

A Toeplitz sequence and the asociated flow are called *regular*, if $\lim_{\ell\to\infty} D(p_\ell) = 1$ and *irregular* otherwise. This distinction turns out to be independent of the choice of the period structure.

Given a period structure $(p_\ell)_{\ell\in\mathbb{N}}$, we let $q_\ell = p_\ell/p_{\ell-1}$ for $\ell \geq 1$, with the convention that $p_0 = 1$. Then, the compact Abelian group $\Omega = \prod_{\ell\in\mathbb{N}} \mathbb{Z}/q_\ell\mathbb{Z}$ equipped with the addition defined according to the carry over rule is called the *odometer group* with *scale* $(q_\ell)_{\ell\in\mathbb{N}}$. We denote the Haar measure on Ω by μ and let $\tau\colon\ \Omega \longrightarrow \Omega$ with $\omega \mapsto \omega+(1,0,0,\ldots)$ denote the canonical minimal group rotation on Ω. We call (Ω,τ) the *odometer associated*[1] *to* (Σ_ξ,σ). This odometer (Ω,τ) coincides with the maximal equicontinuous factor (MEF) of (Σ_ξ,σ), and the factor map $\beta\colon\ \Sigma_\xi \longrightarrow \Omega$ can be defined by

$$\beta(x) = \omega \quad :\Longleftrightarrow \quad \mathrm{Per}\big(p_\ell, \sigma^{k(\ell,\omega)}(x)\big) = \mathrm{Per}(p_\ell,\xi), \quad \text{for all } \ell\in\mathbb{N},$$

where $k(\ell,\omega) = \sum_{i=1}^{\ell} \omega_i p_{i-1}$; compare [10] for details. Given $w \in \prod_{i=1}^{\ell} \mathbb{Z}/q_i\mathbb{Z}$, we also let $k(\ell,w) = \sum_{i=1}^{\ell} w_i p_{i-1}$.

Remark 1. There is an alternative equivalent description of the odometer, which is actually closer to considerations in the quasicrystal literature, as e.g. in [7]. As this is instructive in the context of our construction below we shortly discuss this: Let Ω' be the inverse limit of the system $(\mathbb{Z}/p_\ell\mathbb{Z})_{\ell\geq 0}$, so the elements of Ω' are the sequences $(x_\ell)_{\ell\geq 0}$ with $x_\ell \in \mathbb{Z}/p_\ell\mathbb{Z}$ and $x_{\ell-1} = \pi_\ell(x_\ell)$ for each $\ell\in\mathbb{N}$. Here, $\pi_\ell\colon\ \mathbb{Z}/p_\ell\mathbb{Z} \longrightarrow \mathbb{Z}/p_{\ell-1}\mathbb{Z}$ is the canonical projection. Then, Ω' is an Abelian group under componentwise addition, and the map

$$\Omega \longrightarrow \Omega',\ \omega \mapsto (k(\ell,\omega))_{\ell\geq 0},$$

provides an isomorphism of topological groups. Under this ismomorphism, the map τ on Ω corresponds to addition of 1 in each component of elements of Ω'. $\Diamond$

Let us now turn to cut and project schemes (CPS) and model sets, where we refer to [3] and references therein for background and general notions. In general, a CPS $(G,H,\mathcal{L})$ is given by a pair of locally compact Abelian groups G,H together with a discrete co-compact subgroup $\mathcal{L}$ of $G\times H$ such that $\pi : G\times H \to G$ is injective on $\mathcal{L}$ and $\pi_{\mathrm{int}} : G\times H \to H$ maps $\mathcal{L}$ to a dense subset of H. Given any subset (*window*) W of H, such a CPS produces a subset of G, called a *cut and project set*, given by

$$\curlywedge(W) = \pi\left((G\times W)\cap\mathcal{L}\right).$$

Such a set is called a *model set* when W is relatively compact with non-empty interior. In this case, $\curlywedge(W)$ is always a Delone set. When, in addition, the boundary ∂W of the window has zero measure in H, the model set is called *regular*. As a standard case, one considers compact windows W that satisfy $\varnothing \neq W = \overline{\mathrm{int}(W)}$, in which case they are called *proper*.

[1] We note that period structures of Toeplitz sequences are not uniquely defined, but all the odometers with scales corresponding to different period structures of the same Toeplitz sequence are isomorphic.

Since we consider Toeplitz sequences as weighted subsets of $\mathbb{Z}$, the easiest way to describe them as model sets works for a CPS of the form $(G, H, \mathcal{L})$ where $G = \mathbb{Z}$ or $\mathbb{R}$ and $H = \varOmega$ is the odometer from above. So, we consider the situation summarised in the following diagram,

$$
\begin{array}{ccccc}
G & \xleftarrow{\;\pi\;} & \mathbb{Z} \times H & \xrightarrow{\;\pi_{\text{int}}\;} & H \\
\cup & & \cup & & \cup \;\text{dense} \\
\mathbb{Z} & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\quad \star \quad\qquad} & & L^{\star}
\end{array}
$$

Further, $\mathcal{L}$ is a lattice in $G \times H$ that emerges as a diagonal embedding,

$$\mathcal{L} \,:=\, \{(n, n^{\star}) \mid n \in \mathbb{Z}\}, \tag{1}$$

where $n^{\star} := \tau^{n}(0)$ defines the so-called $\star$-map $\star \colon\; G = \mathbb{Z} \longrightarrow H$. Clearly, the restriction of π to $\mathcal{L}$ is one-to-one and the restriction of π_{int} has dense range. Using the $\star$-map, a *cut and project set* for $(G, H, \mathcal{L})$ and window W can equally be written as

$$\curlywedge(W) \,=\, \{x \in L \mid x^{\star} \in W\}\ .$$

Using these ingredients, our main result now reads as follows.

Theorem 1. *Let $\varSigma = \{0,1\}^{\mathbb{Z}}$ and suppose $\xi \in \varSigma$ is a non-periodic Toeplitz sequence with period structure $(p_{\ell})_{\ell \in \mathbb{N}}$. Let $(\varOmega, \tau)$ be the associated odometer. Then,*

$$\mathcal{D}(\xi) \,=\, \{n \in \mathbb{Z} \mid \xi_n = 1\}$$

is a model set for the CPS $(\mathbb{Z}, \varOmega, \mathcal{L})$ or $(\mathbb{R}, \varOmega, \mathcal{L})$, with the lattice $\mathcal{L}$ of Eq. (1) and the $\star$-map defined above. Moreover, the corresponding window $W \subseteq \varOmega$ is proper and satisfies

$$\mu(\partial W) \,=\, 1 - \lim_{\ell \to \infty} D(p_{\ell}).$$

Proof. We discuss the case $(\mathbb{Z}, \varOmega, \mathcal{L})$ with $\mathcal{L}$ as in (1); the other case is analogous, because we view $\mathcal{D}(\xi)$ as a subset of $\mathbb{Z}$, so that the $\mathbb{R}$-action emerges from the $\mathbb{Z}$-action by a simple suspension with a constant height function. In order to derive the window W for the CPS, we denote cylinder sets in $\varOmega$ either by $[w] = [w_1, \ldots, w_{\ell}] = \{\omega \in \varOmega \mid \omega_i = w_i \text{ for } 1 \leqslant i \leqslant \ell\}$ with $w \in \prod_{i=1}^{\ell} \mathbb{Z}/q_i\mathbb{Z}$ or, given $\omega \in \varOmega$ and $\ell \in \mathbb{N}$, by $[\omega]_{\ell} = [\omega_1, \ldots, \omega_{\ell}]$. Note that $\mu\bigl([w]\bigr) = 1/p_{\ell}$.

Consider

$$A(\ell, s) \,=\, \Bigl\{ w \in \prod_{i=1}^{\ell} \mathbb{Z}/q_i\mathbb{Z} \Bigm| k(\ell, w) \text{ is a } p_{\ell}\text{-periodic position of } \xi \text{ and } \xi_{k(\ell,w)} = s \Bigr\}$$

with $s \in \{0,1\}$. Then, define $U_{\ell} = \bigcup_{w \in A(\ell,1)} [w]$ and $V_{\ell} = \bigcup_{w \in A(\ell,0)} [w]$. Clearly, $U_{\ell} \subset U_{\ell+1}$ and $V_{\ell} \subset V_{\ell+1}$ hold for any ℓ. Set $U = \bigcup_{\ell \in \mathbb{N}} U_{\ell}$ and $V = \bigcup_{\ell \in \mathbb{N}} V_{\ell}$. Now, we let $W = \overline{U}$ and claim that this window W satisfies the assertions of our theorem.

First, we show that our CPS $(\mathbb{Z}, \Omega, \mathcal{L})$ together with the window W produces the Delone set $\mathcal{D}(\xi)$ as its model set, that is,

$$\curlywedge(W) := \{n \in \mathbb{Z} \mid \tau^n(0) \in W\} = \mathcal{D}(\xi).$$

In order to do so, fix $k \in \mathbb{Z}$ and suppose $\xi_k = 1$, so that $k \in \mathcal{D}(\xi)$. Let ℓ be the least integer such that k is a p_ℓ-periodic position of ξ. Then, there exists a unique $k' \in [0, p_\ell - 1]$ such that $k' = k + np_\ell$ for some $n \in \mathbb{Z}$. This k' is a p_ℓ-periodic position as well and we also have $\xi_{k'} = 1$. Further, we have $k' = k(\ell, w)$ for a unique $w \in \prod_{i=1}^{\ell} \mathbb{Z}/q_i\mathbb{Z}$. However, this means that $[w] \subseteq U \subseteq W$ by construction. Since $\tau^m(0) \in [w]$ for all $m \in k(\ell, w) + p_\ell\mathbb{Z}$ (note that $\tau^{k(\ell,w)}(0) = (w, 0, 0, \ldots)$ and any cylinder of length ℓ is p_ℓ-periodic for τ), we in particular have that $\tau^k(0) \in W$, so that $k \in \curlywedge(W)$. In a similar way, we obtain that $\xi_k = 0$ implies $\tau^k(0) \in V$ and thus $k \notin \curlywedge(W)$ (note here that V is open, so that $V \subseteq \Omega \setminus W$). This proves $\curlywedge(W) = \{k \in \mathbb{Z} \mid \xi_k = 1\} = \mathcal{D}(\xi)$.

Next, we determine the measure of ∂W (and obtain properness as a byproduct). We have

$$\#\big(A(\ell, 0) \cup A(\ell, 1)\big) = \#\{k \in [0, p_\ell - 1] \mid k \text{ is a } p_\ell\text{-periodic position}\} = D(p_\ell) \cdot p_\ell.$$

This means that $\mu(U_\ell \cup V_\ell) = D(p_\ell)$, and thus

$$\mu(U \cup V) = \lim_{\ell \to \infty} \mu(U_\ell \cup V_\ell) = \lim_{\ell \to \infty} D(p_\ell).$$

Thus, it suffices to show that $\partial W = \Omega \setminus (U \cup V)$. By openess and disjointness of U and V, this is equivalent to $(W =)\, \overline{U} = \Omega \setminus V$ and $\overline{V} = \Omega \setminus U$. As the situation is symmetric, we restrict to prove $W = \Omega \setminus V$. The inclusion $W \subset \Omega \setminus V$ is clear. It remains to show the opposite inclusion. To that end, fix $\omega \in \Omega \setminus V$ and $\kappa \in \mathbb{N}$. We are going to show that U intersects every cylinder neighbourhood $[\omega]_\kappa$ of ω, so that $\omega \in \overline{U} = W$.

As $\bigcup_{\ell \in \mathbb{N}} \mathrm{Per}(p_\ell, \omega) = \mathbb{Z}$, there exists a least integer ℓ such that $k = k(\kappa, \omega_1, \ldots, \omega_\kappa)$ is a p_ℓ-periodic position. First, suppose that $\ell \leq \kappa$. Then, k is a p_κ-periodic position, and we have $[\omega]_\kappa \subseteq U_\kappa$ if $\xi_k = 1$ and $[\omega]_\kappa \subseteq V_\kappa$ if $\xi_k = 0$. The latter is not possible, since we assume $\omega \notin V$. Hence, we have that $[\omega]_\kappa \subseteq U$. Secondly, suppose that $\ell > \kappa$. Then, k cannot be a p_κ-periodic position, and hence there exists $n \in \mathbb{Z}$ such that $k' = k + np_\kappa$ satisfies $\xi_{k'} = 1$. Choose the least $\ell' \in \mathbb{N}$ such that k' is a $p_{\ell'}$-periodic position and let $v \in \prod_{i=1}^{\ell'} \mathbb{Z}/q_i\mathbb{Z}$ be such that $k' = k(\ell', v) \bmod p_{\ell'}$. Set $k'' = k(\ell', v)$. By construction, we have $[v] \subseteq U_{\ell'} \subseteq U$. At the same time, we have $v_i = \omega_i$ for all $1 \leqslant i \leqslant \kappa$, since $k'' = k + np_\kappa + mp_{\ell'}$ for some $m \in \mathbb{Z}$. Hence, we have that $[v] \subseteq [\omega]_\kappa$ and thus $U \cap [\omega]_\kappa \neq \varnothing$.

As mentioned already, the argument is completely symmetric with respect to U and V, and we also obtain $\overline{V} = \Omega \setminus U$. This then implies that $U = \mathrm{int}(W)$, and since $W = \overline{U}$ by definition, we obtain that W is proper. $\square$

3. Regular Toeplitz flows

For the case of *regular* Toeplitz sequences, more information is available to relate the scaling behaviour of $D(p_\ell)$ to the box dimension of the boundary of the window. In order to state the result, we assume that d is a metric on Ω that generates the product topology and is invariant under the group rotation τ. Note that since cylinder sets in Ω are mapped to cylinder sets, and τ is transitive on Ω, all cylinder sets of a given level ℓ have the same diameter d_ℓ. The choice of the sequence d_ℓ defines the metric and is more or less arbitrary, as long as it is decreasing in ℓ. The box dimension of Ω depends on this choice and is given by

$$\mathrm{Dim}_B(\Omega) \;=\; \lim_{\ell\to\infty} \frac{\log(p_\ell)}{\log(d_\ell)}.$$

If this limit does not exist, then one defines upper and lower box dimension $\overline{\mathrm{Dim}}_B(\Omega)$ and $\underline{\mathrm{Dim}}_B(\Omega)$ by using the limit superior, respectively inferior. We also note that the canonical choice for the metric d is given by $d_\ell = p_{\ell+1}^{-1}$, but our statement is valid in general.

Theorem 2. *Suppose that, in the situation of Theorem* 1*, we have* $\lim_{\ell\to\infty} D(p_\ell) = 1$*. Then, the window* W *can be chosen such that*

$$\overline{\mathrm{Dim}}_B(\partial W) \;=\; \left(1 + \overline{\lim_{\ell\to\infty}} \frac{\log(1-D(p_\ell))}{\log(p_\ell)}\right) \overline{\mathrm{Dim}}_B(\Omega)$$

and

$$\underline{\mathrm{Dim}}_B(\partial W) \;=\; \left(1 + \underline{\lim_{\ell\to\infty}} \frac{\log(1-D(p_\ell))}{\log(p_\ell)}\right) \underline{\mathrm{Dim}}_B(\Omega).$$

Proof. The construction in the proof of Theorem 1 is completely independent of the value of $\lim_{\ell\to\infty} D(p_\ell)$, and in particular applies also to the regular case. Hence, we may choose the same window W as above and only need to determine the box dimension of ∂W.

To that end, note that $\partial W = \Omega \setminus (U \cup V)$ and $U_\ell \cup V_\ell \subseteq (U \cup V)$, so that $\Omega \setminus (U_\ell \cup V_\ell)$ contains ∂W. However, we have that

$$\Omega \setminus (U_\ell \cup V_\ell) \;=\; \bigcup_{w\in\mathbb{Z}^\ell \setminus A(\ell,0)\cup A(\ell,1)} [w] \;,$$

so that this set is a union of $N(\ell) = (1 - D(p_\ell)) \cdot p_\ell$ cylinders of order ℓ. Moreover, it is not possible to cover ∂W with a smaller number of such cylinders, so that $N(\ell)$ is the least number of sets of diameter d_ℓ needed to cover ∂W. Hence, we obtain

$$\overline{\mathrm{Dim}}_B(\partial W) \;=\; \overline{\lim_{\ell\to\infty}} \frac{\log((1-D(p_\ell))\cdot p_\ell)}{\log d_\ell} \;=\; \left(1 + \overline{\lim_{\ell\to\infty}} \frac{\log(1-D(p_\ell))}{\log p_\ell}\right) \cdot \overline{\mathrm{Dim}}_B(\Omega) \;.$$

The analogous computation yields the relation for the lower box dimensions. □

Remark 2. Let us point out some further properties and directions as follows.

(a) As a consequence of our main theorem, a Toeplitz sequence is regular if and only if the associated model set is regular.

(b) As it is well-known, regular Toeplitz flows are almost one-to-one extensions of their MEF; compare [10]. In particular, they are uniquely ergodic and have pure point dynamical spectrum with continuous eigenfunctions, which separate almost all points. Therefore, the general characterisation of dynamical systems coming from regular models sets provided in [6] directly applies to provide a model set construction for these systems.

(c) As regular Toeplitz flows have pure point dynamical spectrum, they also exhibit pure point diffraction by the general equivalence theorem, see [5] and references therein for details and background. As they are uniquely ergodic, each individual sequence of the flow then exhibits the same pure point diffraction. So, the method of [7] can be used to provide a CPS for a regular Toeplitz sequence. This leads to an alternative, but equivalent, way in the spirit of Remark 1, as can be seen from the example of the period doubling chain in [7]. $\Diamond$

4. Consequences for irregular Toeplitz flows

There has been quite some speculation on connections between irregularity of the model set and occurrence of positive entropy or failure of unique ergodicity for the associated dynamical systems. Indeed, Schlottmann asks whether irregularity of the model set implies failure of unique ergodicity [24] and Moody has suggested that irregularity is related to positive entropy. The suggestion of Moody is recorded in [23] (later subsumed in [22]) and is also discussed in the introduction to [12]. When combined with examples of Toeplitz systems studied in the past, our main theorem allows us to answer these speculations by presenting model sets with various previously unknown features (such as irregularity combined with unique ergodicity and zero entropy). This is discussed in this section. Let us emphasise that all Toeplitz flows are minimal and that the model sets presented below are even proper (as Theorem 1 provides proper model sets).

Arguably among the most interesting examples in our present context are irregular Toeplitz flows with zero entropy [21], as these immediately imply the following statement.

Corollary 1. *Positive measure of the boundary of a window of a CPS is not a sufficient criterion for positive topological entropy of the associated Delone dynamical system.* $\square$

We note that there even exist irregular Toeplitz flows (and thus irregular model sets) whose word complexity is only linear [11]. A number of further interesting examples are available in the literature. Amongst these are the following.

- Irregular Toeplitz flows may be uniquely ergodic [25], so that a window for a CPS with a boundary of positive measure does *not* contradict unique ergodicity of the resulting Delone dynamical system. Moreover, the set of ergodic invariant measures of an irregular Toeplitz flow may have any cardinality.
- Any countable subgroup of $\mathbb{R}$ that contains infinitely many rationals can be the dynamical spectrum of a Toeplitz flow [10], and thus of the Delone dynamical system arising from a CPS. As all continuous eigenvalues of a Toeplitz flow are rational, this gives in particular examples of model sets with (many) measurable eigenvalues (see [8] as well for a recent study of Toeplitz systems of finite topological rank with measurable eigenvalues).

- Oxtoby's original example of a Toeplitz flow with zero entropy in [21] is not uniquely ergodic, and the same is true of the example in [11]. However, there also exist uniquely ergodic irregular Toeplitz flows both with and without positive entropy [25]. In particular, there exist irregular model sets with uniguely ergodic minimal dynamical systems with zero entropy.

Remark 3. There is an emerging theory of weak model sets, see for instance [4, 12, 16], dealing with irregular windows for a given CPS. If the arising model sets satisfies a maximality condition for its density, the associated dynamical systems will generally not be minimal. Thus, the irregular Toeplitz sequences are, in this sense, never weak model sets of maximal density. They rather provide a versatile class of examples to explore the possibilities that emerge from missing out on the maximality property. $\Diamond$

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