A dynamical decomposition of the torus into pseudo-circles

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To the memory of Dmitri V. Anosov.

Abstract

We build an irrational pseudo-rotation of the 2-torus which is semiconjugate to an irrational rotation of the circle in such a way that all the fibres of the semi-conjugacy are pseudo-circles. The proof uses the well-known *'fast-approximation method'* introduced by Anosov and Katok.

1 Introduction

It is well known that continua (connected compact metric spaces) with complicated structure naturally appear in smooth surface dynamics. A striking example is provided by the *pseudo-circle*, introduced by Bing [Bi1] and characterized by Fearnley [Fe]. It is a continuum which:

– can be embedded in $\mathbb{S}^2$ and separates,

– is circularly chainable: it admits coverings into compact subsets $(A_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ whose diameter are arbitrarily small, such that $A_i \cap A_j \neq \emptyset$ if and only if if $i = j \pm 1$ or $i = j$,

– is indecomposable: it cannot be written as the union of two proper continua,

– and whose non-trivial proper subcontinua are indecomposable, homogeneous (any point can be sent on any other point by some homeomorphism) and all homeomorphic to the same topological space (called the *pseudo-arc*).

Handel [Ha] has built a smooth diffeomorphism of $\mathbb{S}^2$ preserving a minimal invariant set homeomorphic to the pseudo-circle. Later, Prajs [Pr] has constructed a partition of the annulus into pseudo-arcs, and likewise his method could be used to produce partitions of the torus into pseudo-circles. It was not known, however, if such a pathological foliation could be 'dynamical', that is, invariant under the dynamics of a torus homeomorphism or diffeomorphism that permutes the leaves of the foliation. Conversely, if a homeomorphism of the two-torus is semiconjugate to an irrational rotation of the circle, one may wonder whether most, or at least some, of the fibres of the semi-conjugacy must have a simple structure or even be topological circles. We give a positive answer to the first and a negative to the second of these questions. Denote by $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus and by $\mathrm{Diff}^{\omega}_{\mathrm{vol},0}(\mathbb{T}^2)$ the space of real-analytic diffeomorphisms of $\mathbb{T}^2$ that are isotopic to the identity and preserve the canonical volume.

Theorem. *There exists a minimal diffeomorphism $f \in \mathrm{Diff}^{\omega}_{vol,0}(\mathbb{T}^2)$ whose rotation set is reduced to a unique totally irrational*[1] *vector and which preserves a partition $\mathcal{C}$ of $\mathbb{T}^2$ into pseudo-circles.*

Moreover there exists a continuous map $\pi\colon \mathbb{T}^2 \to \mathbb{T}^1$ which semi-conjugates f to an irrational rotation. The elements of $\mathcal{C}$ are the pre-images $\pi^{-1}(x)$.

This result has implications for a number of questions that naturally come up in the rotation theory on the torus and, more specifically, the dynamics of totally irrational pseudo-rotations. We discuss these issues in more detail in Section 4, alongside with the uniqueness of the semi-conjugacy.

Idea of the construction. The diffeomorphism f is obtained as limit of a sequence of diffeomorphisms f_n that are conjugated to rational rotations R_{α_n} by diffeomorphisms H_n isotopic to the identity, following the celebrated Anosov-Katok method, see [FK1]. As a side effect, this means that its dynamics can be made uniquely ergodic, although we will not expand on this. Note that most of the constructions using this method deal with the C^∞ category; some cases, as [FK2] allow to work in the real-analytic category. The sequence $f_n = H_n^{-1} \circ R_{\alpha_n} \circ H_n$ is obtained inductively. At stage n, the diffeomorphism f_n preserves the foliation $\mathcal{V}_n$ which is the pre-image under H_n of the foliation by vertical circles $\{x\} \times \mathbb{T}^1$. The main requirement is to have the circles of the foliation $\mathcal{V}_{n+1}$ arbitrarily close to the circles of $\mathcal{V}_n$ in the Hausdorff topology, but more crooked. Then the partitions $\mathcal{V}_n$ will converge to the partition into pseudo-circles. The foliation $\mathcal{V}_{n+1}$ will be the pre-image of the foliation $\mathcal{V}_n$ under a new homeomorphism h_{n+1}. In order to obtain $\mathcal{V}_{n+1}$, one first builds a leaf of $H_n(\mathcal{V}_{n+1})$, which is crooked with respect to the vertical circles (recall that these vertical circles are the leaves of the foliation $H_n(\mathcal{V}_n)$), and transverse to a green periodic linear flow φ_{n+1}. The complete foliation $H_n(\mathcal{V}_{n+1})$ is obtained by pushing this initial leaf by the flow φ_{n+1} (see Figure 1). The homeomorphism h_{n+1} is chosen so that it maps the foliation $H_n(\mathcal{V}_{n+1})$ to the foliation into vertical circles. Since we then let $H_{n+1} = h_{n+1} \circ H_n$, this ensures that the foliation $H_{n+1}(\mathcal{V}_{n+1})$ is again the foliation given by vertical circles.

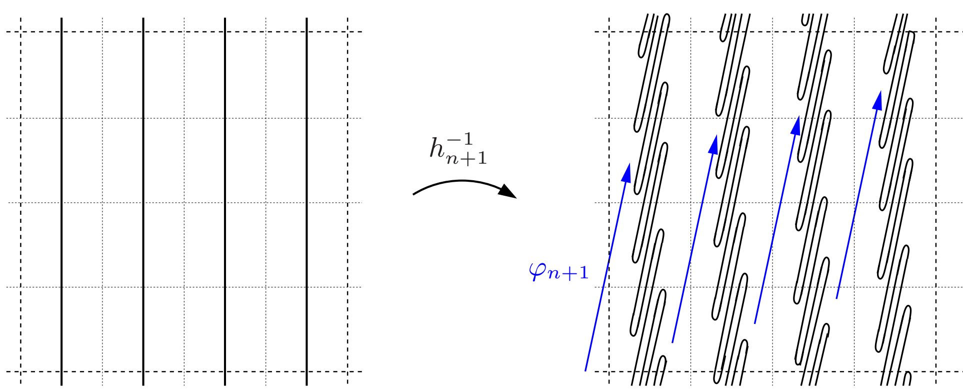

Figure 1: *The inverse h_{n+1}^{-1} maps straight lines (solid lines on the left) to crooked leaves of $\mathcal{V}_{n+1}$ (on the right). The latter are translates of each other along the flow lines of φ_{n+1}. Since the foliation into vertical circles equals $H_n(\mathcal{V}_n)$, this implies that the leaves of $\mathcal{V}_{n+1}$ are crooked with respect to $\mathcal{V}_n$.*

[1] A vector $(a, b) \in \mathbb{R}^2$ is called *totally irrationnal* if 1, a and b are independent over $\mathbb{Q}$.

We note that A. Avila has announced recently the construction of an element of $\mathrm{Diff}^{\omega}_{\mathrm{vol},0}(\mathbb{T}^2)$ whose rotation set is a non-trivial compact interval contained in a line with irrational slope and which does not contain any rational point (a counter-example to one case of a conjecture by Franks and Misiurewicz [FM]). His construction may somewhat be compared to ours: the diffeomorphism is obtained as the limit of diffeomorphisms acting periodically on the leaves of a foliation by circles. In his case however the homotopy class of the leaves is modified at each stage of the construction.

2 A criterion for the existence of a partition into pseudo-circles

2.a – Crooking. The construction of the pseudo-circle uses the following notions.

Definitions. A *circular chain* is a finite family of sets $\mathcal{D} = \{D_\ell, \ell \in \mathbb{Z}/N\mathbb{Z}\}$ such that D_k intersects D_ℓ if and only if $k - \ell \in \{-1, 0, +1\}$. A circular chain $\mathcal{D}' = \{D'_i, i \in \mathbb{Z}/N'\mathbb{Z}\}$ said to be *crooked inside* another circular chain $\mathcal{D} = \{D_\ell, \ell \in \mathbb{Z}/N\mathbb{Z}\}$ if there exists a map $\ell \colon \mathbb{Z}/N'\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ with the following properties.

- $D'_i \subset D_{\ell(i)}$ for each $i \in \mathbb{Z}$;
- if $i < j$ are such that for all $i < k < j$ the element $\ell(k)$ belongs to the same closed interval bounded by $\ell(i)$ and $\ell(j)$ (either positively or negatively oriented in $\mathbb{Z}/N\mathbb{Z}$) and the length of this interval is greater than 4, then there exists u, v with $i < u < v < j$ such that $d(\ell(u), \ell(j)) \leq 1$ and $d(\ell(v), \ell(i)) \leq 1$. (Here d denotes the canonical distance on $\mathbb{Z}/N\mathbb{Z}$.)

The pseudo-circle can then be obtained as follows. For the sake of consistency with the later sections, we work in the torus instead of $\mathbb{R}^2$ and require that the circular chains – and thus the resulting pseudo-circle – are homologically non-trivial.

Theorem ([Bi1, Fe]). *Consider a sequence $(\mathcal{D}_n)_{n\geq 0}$ of circular chains of open topological disks in $\mathbb{T}^2$. Assume that*

- *$\mathcal{D}_{n+1}$ is crooked inside $\mathcal{D}_n$ for each n,*
- *the closure of $\bigcup_{D\in\mathcal{D}_{n+1}} D$ is contained in $\bigcup_{D\in\mathcal{D}_n} D$ for every n,*
- *the maximal diameter of the elements of $\mathcal{D}_n$ goes to zero as $n \to +\infty$.*
- *the union $\bigcup_{D\in\mathcal{D}_n} D$ contains homotopically non-trivial loops of a unique homotopy type $v \in \mathbb{Z}^2 \setminus \{0\}$.*

Then the compact set $X := \bigcap_{n\geq 0} \bigcup_{D\in\mathcal{D}_n} D$ is homeomorphic to the pseudo-circle. Moreover, X is an annular continuum of homotopy type v (see [JP, JT]).

At some point later on, we will have to speak about lifts of circular chains in the torus to the universal covering $\mathbb{R}^2$, and similarly about lifts of circular chains of intervals in the circle to $\mathbb{R}$. Suppose that $\mathcal{D}$ is a circular chain of topological disks in $\mathbb{T}^2$ as above and denote by $p : \mathbb{R}^2 \to \mathbb{T}^2$ the canonical projection. Note that for each $D_\ell \in \mathcal{D}$, the preimage $p^{-1}(\mathcal{D})$ consists of a countable number of connected components, each of which is a topological disk homeomorphic to D_ℓ and disjoint from all its integer translates. Suppose in addition that $\mathrm{Diam}(D_\ell) < 1/4$ for all $\ell \in \mathbb{Z}/N\mathbb{Z}$, so that none of the unions $D_\ell \cup D_{\ell+1}$ is essential in the torus (contains a homotopically non-trivial loop).

Definitions. A *lift* $\widehat{\mathcal{D}}$ of $\mathcal{D}$ is a sequence of topological disks $(\widehat{D}_\ell)_{\ell\in\mathbb{Z}}$ of $\mathbb{R}^2$ such that

- for all $\ell \in \mathbb{Z}$ the disk $\widehat{D}_\ell$ is a connected component of $p^{-1}(D_\ell)$;
- $\widehat{D}_k$ intersects $\widehat{D}_\ell$ if and only if $k-\ell \in \{-1,0,1\}$.

The disc $\widehat{D}_{k+N}$ is the image of $\widehat{D}_k$ by translation by a vector $v \in \mathbb{Z}^2$ which does not depend on the lift, nor on k, and is called the *homotopy type of* $\mathcal{D}$.

Note that if $\mathcal{D}$ and $\mathcal{D}'$ are circular chains of topological disks with homotopy type $v \in \mathbb{Z}^2\setminus\{0\}$ as above, $\mathcal{D}'$ is crooked inside $\mathcal{D}$ and $\widehat{\mathcal{D}}, \widehat{\mathcal{D}}'$ are lifts in the above sense, then there exists a function $\hat{\ell}: \mathbb{Z} \to \mathbb{Z}$ (to which we refer as a lift of $\ell : \mathbb{Z}/N'\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$) such that

- $\widehat{D}'_{i+N'} = \widehat{D}'_i + v$ and $\widehat{D}_{i+N} = \widehat{D}_i + v$;
- $\hat{\ell}(i+N') = \hat{\ell}(i) + N$;
- $\widehat{D}'_i \subseteq \widehat{D}_{\ell(i)}$ for all $i \in \mathbb{Z}$;
- if $i < j < i+N$ are such that for all $i<k<j$ the integer $\hat{\ell}(k)$ belongs to the interval bounded by $\hat{\ell}(i)$ and $\hat{\ell}(j)$ and $|\hat{\ell}(j)-\hat{\ell}(i)| > 4$, then there exists u,v with $i<u<v<j$ such that $|\hat{\ell}(u)-\hat{\ell}(j)| \le 1$ and $|\hat{\ell}(v),\hat{\ell}(i)| \le 1$.

All these remarks apply in an analogous way to circular chains of intervals in the circle and their lifts to $\mathbb{R}$.

During the construction, we will also use another notion of the crooking.

Definition. For $\varepsilon > 0$, a continuous map $g\colon I \to \mathbb{R}$ on the interval I is *ε-crooked* if for any $a<b$ in I, there are $a<c<d<b$ such that $|g(d)-g(a)| < \varepsilon$ and $|g(c)-g(b)| < \varepsilon$.

Note that ε-crooked maps exist for any ε (see [Bi2]).

2.b – Elements of the construction. Let $p_1\colon \mathbb{T}^2 \to \mathbb{T}^1$ be the projection on the first coordinate. Let $\mathcal{B}(N)$ be the covering of the circle by N open intervals defined as follows

$$\mathcal{B}(N) = \{B_i, i \in \mathbb{Z}/N\mathbb{Z}\} \quad \text{where} \quad B_i = \left(\frac{i-5/4}{N}, \frac{i+1/4}{N}\right). \tag{2.1}$$

We will build inductively:

- a sequence of integers $(N_n)_{n\ge 0}$,
- a sequence of positive real numbers $(\varepsilon_n)_{n\ge 0}$,
- a sequence of conjugating diffeomorphisms $(H_n)_{n\ge 0}$ in $\mathrm{Diff}^{\omega}_{\mathrm{vol},0}(\mathbb{T}^2)$,
- a sequence of rational rotations $(R_{\alpha_n})_{n\ge 0}$ of $\mathbb{T}^2$.

To $N_n, \varepsilon_n, H_n, R_{\alpha_n}$, we will associate:

- for each $x \in \mathbb{T}^1$, the annulus $A_{n,x}$ which is the image under H_n^{-1} of the vertical annulus $(x-\varepsilon_n, x+\varepsilon_n) \times \mathbb{T}^1$,

– for each $x \in \mathbb{T}^1$, the covering $\mathcal{D}_{n,x}$ of the annulus $A_{n,x}$ defined by

$$\mathcal{D}_{n,x} = \left\{ H_n^{-1}((x-\varepsilon_n, x+\varepsilon_n) \times B_i) \ \mid \ B_i \in \mathcal{B}(N_n) \right\}$$

(note that $\mathcal{D}_{n,x}$ is a circular chain with N_n elements),

– the projection $\pi_n = p_1 \circ H_n : \mathbb{T}^2 \to \mathbb{T}^1$ (where $p_1 : \mathbb{T}^2 \to \mathbb{T}^1$ is the projection to the first coordinate),

– the diffeomorphism $f_n = H_n^{-1} \circ R_{\alpha_n} \circ H_n \in \mathrm{Diff}^{\omega}_{\mathrm{vol},0}(\mathbb{T}^2)$.

We will denote by $\left(\frac{r_n}{q_n}, \frac{s_n}{q_n}\right)$ the coordinates of α_n, with $r_n, s_n \in \mathbb{Z}$ and $q_n \in \mathbb{N} - \{0\}$.

2.c – Inductive properties. The torus $\mathbb{T}^2$ is embedded as the subset $\{(z_1, z_2) \in \mathbb{C}^2,\ |z_1| = |z_2| = 1\}$ of $\mathbb{C}^2$. One will consider homeomorphisms f of $\mathbb{T}^2$ such that both f and f^{-1} extend as holomorphic functions defined on a neighborhood of $\Delta = \left\{(z_1, z_2) \in \mathbb{C}^2,\ |z_1|, |z_2| \in \left[\frac{1}{2}, 2\right]\right\}$. One then introduces the supremum norm $\|.\|_\Delta$ on Δ and the metric

$$d_0(f, f') = \max(\|f, f'\|_\Delta, \|f^{-1}, f'^{-1}\|_\Delta)\ .$$

The sequences (N_n), (ε_n), (H_n), (R_{α_n}) will be constructed inductively so that the following properties hold.

1. For each $x \in \mathbb{T}^1$, the circular chain $\mathcal{D}_{n+1,x}$ is crooked inside the circular chain $\mathcal{D}_{n,x}$ (in particular, the annulus $A_{n+1,x}$ is contained in the interior of the annulus $A_{n,x}$), and the supremum of the diameter of the elements of the coverings $\mathcal{D}_{n+1,x}$ is less than $\frac{1}{n+1}$,
2. The angle α_{n+1} is close, but not equal, to α_n. More precisely: both $|\frac{r_{n+1}}{q_{n+1}} - \frac{r_n}{q_n}|$ and $|\frac{s_{n+1}}{q_{n+1}} - \frac{s_n}{q_n}|$ are smaller than $1/(2^{n+2}q_n)$ and the orbits of $R_{\alpha_{n+1}}$ are $\frac{1}{2^{n+1}}$-dense in $\mathbb{T}^2$.
3. Every orbit of the diffeomorphism f_{n+1} is $\frac{1}{n+1}$-dense in $\mathbb{T}^2$.
4. The diffeomorphism f_{n+1} is (very) close to f_n. More precisely, both f_{n+1}, H_{n+1} and their inverses extend holomorphically to $(\mathbb{C} \setminus \{0\})^2$ and satisfy:
   (a) $d_0(f_{n+1}^i, f_n^i) < \min(\frac{1}{2} d_0(f_n^i, f_{n-1}^i), \frac{1}{n})$ for $i = 1, \ldots, q_n$;
   (b) $d_0(f_{n+1}, f_n) < \frac{\eta_n}{2}$ where η_n is choosen such that, for every homeomorphism g in the ball (for d_0) centered at f_n of radius η_n, the rotation set of g is contained in the ball centered at α_n of radius $\frac{1}{n}$.
5. The projection π_{n+1} is close to π_n for the C^0-topology. More precisely: $d_0(\pi_{n+1}, \pi_n) < \frac{1}{2^n}$.

Remarks. The existence of the real number η_n used in property 4.b is a consequence of the upper semi-continuity of the rotation set $\rho(F)$ with respect to F [MZ, Corollary 3.7].

Property 1 (more precisely, the fact that $\mathcal{D}_{n+1,x}$ is crooked inside $\mathcal{D}_{n,x}$) implies that the sequence of conjugating diffeomorphisms (H_n) will necessarily diverge. Nevertheless, the sequence of diffeomorphisms $(f_n) = (H_n^{-1} \circ R_{\alpha_n} \circ H_n)$ will converge (Property 4). This convergence is obtained by using the well-known ingredients of the Anosov-Katok method:

– one first chooses a conjugating diffeomorphism H_{n+1} of the form $H_{n+1} = h_{n+1} \circ H_n$, where h_{n+1} might be very wild, but commutes with the rotation R_{α_n} ; this implies that $f_n = H_{n+1}^{-1} \circ R_{\alpha_n} \circ H_{n+1}$;

– then, choosing α_{n+1} close enough to α_n is enough to ensure that $f_{n+1} = H_{n+1}^{-1} \circ R_{\alpha_{n+1}} \circ H_{n+1}$ is close to $f_n = H_{n+1}^{-1} \circ R_{\alpha_n} \circ H_{n+1}$.

A specific point in our construction is that, although the sequence of diffeomorphisms (H_n) will diverge, we require that the sequence of maps $(p_1 \circ H_n)$ converges (Property 5). Indeed, we want that the fibers of $p_1 \circ H_n$ converge to pseudo-circles "foliating" $\mathbb{T}^2$.

2.d – Proof of the theorem. One can easily check that the theorem follows from the inductive properties stated above. Properties 1 imply that, for every $x \in \mathbb{T}^1$, the sequence of annuli $(A_{n,x})$ decreases and converges in Hausdorff topology to a pseudo-circle $\mathcal{C}_x$. Moreover, the collection of pseudo-circles $\mathcal{C} = \{\mathcal{C}_x\}_{x\in\mathbb{T}^1}$ is a partition of $\mathbb{T}^2$; this follows from the following fact:

– for every n, the collection of annuli $\{A_{n,x}, x \in \mathbb{T}^1\}$ covers $\mathbb{T}^2$,

– for any $x \neq x'$, the annuli $A_{n,x}$ and $A_{n,x'}$ are disjoint if n is large enough.

Property 4.a implies that the sequence (f_n) converges to an holomorphic function f on the interior of Δ. The same holds for (f_n^{-1}). Consequently, the restriction of f to $\mathbb{T}^2$ is a real-analytic diffeomorphism. Since each f_n is volume-preserving, f belongs to $\mathrm{Diff}^{\omega}_{\mathrm{vol},0}(\mathbb{T}^2)$.

Given $x \in \mathbb{T}^1$, let $x' := x + p_1(\alpha_n)$. Then $f_n(A_{n,x}) = A_{n,x'}$. From property 1, one deduces that $A_{n+1,x}$ is mapped by f_n inside $A_{n,x'}$ and $A_{n+1,x'}$ is mapped by f_n^{-1} inside $A_{n,x}$. Hence, the annulus $f(A_{n+1,x})$ is contained in the $d_0(f, f_n)$-neighbourhood of the annulus $A_{n,x'}$ and the annulus $f^{-1}(A_{n+1,x'})$ is contained in the $d_0(f, f_n)$-neighbourhood of the annulus $A_{n,x}$. Since $d_0(f, f_n)$ tends to 0 as n goes to infinity, this implies that f preserves the partition into pseudo-circles $\mathcal{C} = \{\mathcal{C}_x\}_{x\in\mathbb{T}^1}$.

Property 2 implies that the sequence (α_n) converges towards some $\alpha \in \mathbb{R}^2$. It also implies that the q_n first iterates of R_α are $1/2^{n+1}$-close to those of $R_{\alpha_{n+1}}$, hence are $\frac{1}{2^n}$-dense in $\mathbb{T}^2$. Consequently α is totally irrational.

Properties 4 imply that the rotation set of f is reduced to $\{\alpha\}$ (indeed, they imply that $d_0(f, f_n) < \eta_n$ and therefore the rotation set of f is contained in the ball of radius $\frac{1}{n}$ centered at α_n for every n). Hence f is a totally irrational pseudo-rotation, *i.e.* the rotation set of f is reduced to a single totally irrational vector.

Consider a point $z \in \mathbb{T}^2$. For every n, according to property 3, the orbit of z under f_n is $\frac{1}{n}$-dense in $\mathbb{T}^2$. Using property 4.a, we obtain that the orbit of z under f remains at distance less than $\frac{2}{n}$ of the orbit of z under f_n for a time q_n. Combined with property 3, this means that the orbit of z under f is $\frac{3}{n}$-dense in $\mathbb{T}^2$. (Recall here that f_n is q_n-periodic.) Since n is arbitrary, f is minimal.

Property 5 implies that the sequence of maps (π_n) converges in topology C^0 towards a continuous map π. For each n, the map π_n semi-conjugates f_n to the rotation of $\mathbb{T}^1$ with angle $p_1(\alpha_n)$. Passing to the limit, it follows that the map π semi-conjugates f to the rotation of angle $p_1(\alpha)$. So we get all the conclusions of the theorem. □

3 Inductive construction

Now we explain how to construct a sequence of integers $(N_n)_{n\geq 0}$, a sequence of real numbers $(\varepsilon)_{n\geq 0}$, a sequence of conjugating diffeomorphisms $(H_n)_{n\geq 0}$ and a sequence of vectors $(\alpha_n)_{n\geq 0}$, so that properties 1…5 are satisfied. We assume that the sequences have already been constructed up to rank n. We will construct $N_{n+1}, \varepsilon_{n+1}, H_{n+1}, \alpha_{n+1}$.

3.a – Preliminary constructions. Recall that we denote by $\left(\frac{r_n}{q_n}, \frac{s_n}{q_n}\right)$ the coordinates of α_n. We introduce a periodic linear flow $\varphi_{n+1} : (t,(x,y)) \mapsto (x,y) + t\cdot\alpha_n + t\cdot(0, r_n b_{n+1})$ on $\mathbb{T}^2$ where b_{n+1} is an integer which will be specified below. Observe that the time 1 map of this flow is the rotation R_{α_n}. The first return map of φ_{n+1} on the vertical circle $\{x\}\times\mathbb{T}^1$ is the time $\frac{q_n}{r_n}$ map of φ_{n+1}. Denote by m_n the period of this first return map, and observe that m_n depends on α_n, but does not depend on the choice of the integer b_{n+1}.

We also introduce a C^ω map $\theta_n\colon \mathbb{R}\to\mathbb{R}$ satisfying the following properties.

- $x\mapsto \theta_n(m_n x) - m_n x$ is a trigonometric polynomial (hence $\theta_n - \mathrm{Id}$ is $1/m_n$-periodic),
- $\theta_n(0) = 0$, $\theta_n\left(\frac{1}{2m_n}\right) = 2$, $\theta_n\left(\left[0, \frac{1}{2m_n}\right]\right) = [0,2]$ and θ_n is $\frac{1}{4N_n}$-crooked on $\left[0, \frac{1}{2m_n}\right]$,
- $\theta_n\left(\frac{1}{m_n}\right) = \frac{1}{m_n}$, $\theta_n\left(\left[\frac{1}{2m_n}, \frac{1}{m_n}\right]\right) = \left[\frac{1}{m_n}, 2\right]$ and θ_n is $\frac{1}{4N_n}$-crooked on $\left[\frac{1}{2m_n}, \frac{1}{m_n}\right]$.

Indeed by [Bi2], there exists ε-crooked maps of the interval for any $\varepsilon > 0$. This allows to build a map satisfying the two last items above and which is the sum of the identity with a $1/m_n$-periodic function. Being croocked is an open property. The density of the trigonometric polynomials inside the space of periodic functions allows to get the first item as well.

Note that $\theta_n - \mathrm{Id}$ induces a function on $\mathbb{T}^1$ which extends holomorphically to $\mathbb{C}\setminus\{0\}$. Recall that $\mathcal{B}(N)$ denotes the covering of the circle by N compact intervals defined by (2.1). Moreover, θ itself induces a degree one map on the circle, which we denote by θ again for simplicity.

Claim. *If N_{n+1} is large enough, then, for any $\omega\in\mathbb{T}^1$, the circular chain of intervals $\{\theta_n(B-\omega)+\omega, B\in\mathcal{B}(N_{n+1})\}$ is crooked inside the circular chain $\mathcal{B}(N_n)$.*

Proof. We work with a lift of the family $\mathcal{B}(N)$, in the sense discussed in Section 2, obtained as a covering $\widehat{\mathcal{B}}(N)$ of the real line by intervals of the form:

$$\widehat{B}_i = \left(\frac{i-5/4}{N}, \frac{i+1/4}{N}\right), \quad i\in\mathbb{Z}.$$

If N_{n+1} is large enough, each interval $\theta_n(\widehat{B}_i - \omega)+\omega$ with $\widehat{B}_i\in\widehat{\mathcal{B}}(N_{n+1})$ has lenght strictly less than $\frac{1}{2N_n}$ and therefore is contained in an interval $\widehat{B}'_{\hat\ell(i)}$ of the family $\widehat{\mathcal{B}}(N_n)$. Since θ_n has degree 1, one can choose the function $\hat\ell$ such that $\hat\ell(i+N_{n+1}) = \hat\ell(i)+N_n$.

Let $i<j$ be two integers such that $\hat\ell(k)$ belongs to the interval bounded by $\hat\ell(i)$ and $\hat\ell(j)$ for each $i<k<j$ and such that $4 < |\hat\ell(j)-\hat\ell(i)| < N_n$. Let us choose $i\le i' < j' \le j$ such that $\hat\ell(i') = \hat\ell(i)$, $\hat\ell(j') = \hat\ell(j)$ and some points $a\in\widehat{B}_{i'}$ and $b\in\widehat{B}_{j'}$. One considers $a\le a' < b' \le b$ such that a, a' (resp. b, b') have the same image by $x\mapsto\theta_n(x-\omega)$. One can assume that $a'-\omega, b'-\omega$ belong to the same interval $I_k := \left[\frac{k}{2m_n}, \frac{k+1}{2m_n}\right]$: indeed, for each k, the image under θ_n of I_k has length >1 and the point $\theta_n\left(\frac{k}{2m_n}\right)$ is an end point of $\theta_n(I_{k-1}\cup I_k)$.

Now the points $a' - \omega$ and $b' - \omega$ belong to the same interval $I_k = \left[\frac{k}{2m_n}, \frac{k+1}{2m_n}\right]$. Since θ_n is $(4N_n)^{-1}$-crooked on this interval, this implies that there exists $a' < c < d < b'$ such that

– $\theta_n(a - \omega) = \theta_n(a' - \omega)$ and $\theta_n(d - \omega)$ are $(4N_n)^{-1}$-close,

– $\theta_n(b - \omega) = \theta_n(b' - \omega)$ and $\theta_n(c - \omega)$ are $(4N_n)^{-1}$-close.

Hence d is contained in an interval $\widehat{B}_v$ such that $|\hat{\ell}(v) - \hat{\ell}(i)| \leq 1$. Similarly, c is contained in an interval $\widehat{B}_u$ such that $|\hat{\ell}(u) - \hat{\ell}(i)| \leq 1$. Since $a < c < d < b$ and $2 < |\hat{\ell}(j) - \hat{\ell}(i)|$ one gets $i < u < v < j$.

In the projection to $\mathbb{T}^1$, the above shows that the circular chain $\{\theta_n(B - \omega) + \omega, B \in \mathcal{B}(N_{n+1})\}$ is crooked inside the circular chain $\mathcal{B}(N_n)$ as announced. □

Since m_n only depends on α_n, one can fix the map θ_n but choose b_{n+1} and N_{n+1} arbitrarily large later in the construction.

3.b – Construction of H_{n+1}. Consider the map $\Theta_{n+1} : \{0\} \times \mathbb{T}^1 \to \mathbb{T}^1$ defined by $\Theta_{n+1} : (0, y) \mapsto \theta_n(y) - y$. Since $\theta_n - \mathrm{Id}$ is $1/m_n$-periodic, and since the period of the return map associated to the linear flow φ_{n+1} on $\{0\} \times \mathbb{T}^1$ is equal to m_n, this maps extends to a map $\Theta_{n+1} : \mathbb{T}^2 \to \mathbb{T}^1$ which is constant along the orbits of φ_{n+1}. We define H_{n+1} by setting $H_{n+1} := h_{n+1} \circ H_n$, where

$$h_{n+1}(x, y) = \varphi_{n+1}\left(-\frac{\Theta_{n+1}(x, y)}{r_n b_{n+1}}, (x, y)\right) = \Big(x, y - \Theta_{n+1}(x, y)\Big) - \frac{\Theta_{n+1}(x, y)}{q_n b_{n+1}}\left(1, \frac{s_n}{r_n}\right).$$

Since Θ_{n+1} is constant along the orbits of φ_{n+1}, one has

$$h_{n+1}^{-1}(x, y) = \Big(x, y + \Theta_{n+1}(x, y)\Big) + \frac{\Theta_{n+1}(x, y)}{q_n b_{n+1}}\left(1, \frac{s_n}{r_n}\right). \tag{3.1}$$

Clearly, h_{n+1} (and hence H_{n+1}) belongs to $\mathrm{Diff}^{\omega}_{\mathrm{vol},0}(\mathbb{T}^2)$. Note also that both h_{n+1} and h_{n+1}^{-1} extend holomorphically to the domain $(\mathbb{C} \setminus \{0\})^2$.

3.c – Choice of b_{n+1}, ε_{n+1} and N_{n+1}. Now, we explain how to fix the values of b_{n+1}, ε_{n+1}, and N_{n+1} so that properties 1 and 5 hold. From (3.1), if b_{n+1} is large enough, the image under h_{n+1}^{-1} of every point $z = (x, y) \in \mathbb{T}^2$ is arbitrarily close to $(x, y + \Theta_{n+1}(x, y))$. Now observe that, for each $x \in \mathbb{T}^1$, there exists $\omega_x \in \mathbb{T}^1$ such that

$$\Theta_{n+1}(x, y) = \theta_n(y - \omega_x) - y + \omega_x.$$

As a consequence, if b_{n+1} is large enough, the image under h_{n+1}^{-1} of every point $z = (x, y) \in \mathbb{T}^2$ is arbitrarily close to $(x, \theta_n(y - \omega_x) + \omega_x)$. Hence, if b_{n+1} is large enough and ε_{n+1} is small enough, the image under h_{n+1}^{-1} of a rectangle $(x - \varepsilon_{n+1}, x + \varepsilon_{n+1}) \times B$ is contained in an arbitrary small neighbourhood of the square $\{x\} \times (\theta_n(B - \omega_x) + \omega_x)$. Using the claim above, this implies that the family $\mathcal{D}_{n+1,x}$ is crooked inside $\mathcal{D}_{n,x}$, provided N_{n+1} is large enough. By continuity of H_{n+1}^{-1}, the diameter of the elements of the covering $\mathcal{D}_{n+1,x}$ is less than $\frac{1}{n+1}$ if b_{n+1}, N_{n+1} are large enough and ε_{n+1} is small enough. We have thus checked that property 1 holds, provided that b_{n+1}, N_{n+1} are chosen large enough and ε_{n+1} is chosen small enough.

By definition of π_n, π_{n+1}, and H_{n+1}, in order to check that property 5 is satisfied, it is enough to check that $p_1 \circ h_{n+1}$ is close to p_1 for the C^0-topology. This is a direct consequence of the definition of h_{n+1}, provided that b_{n+1} is chosen large enough.

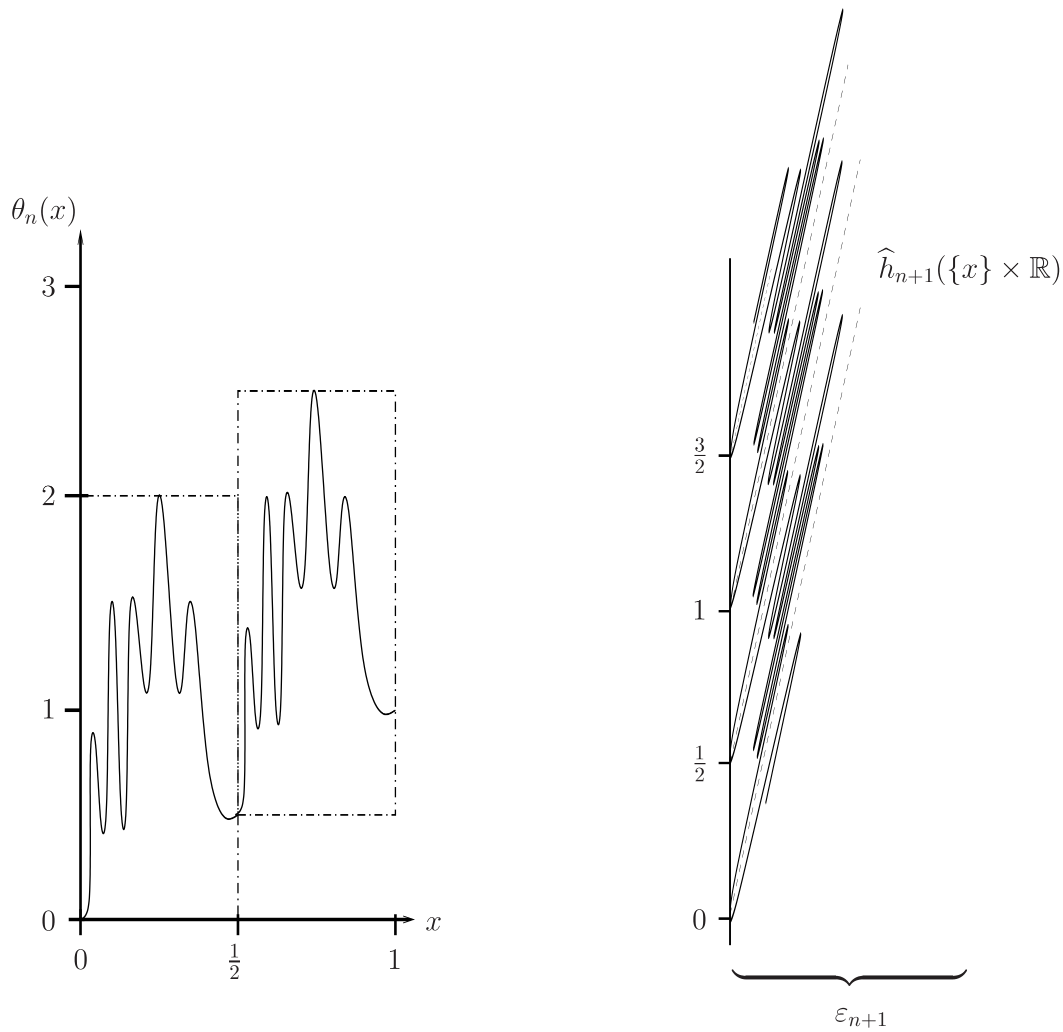

Figure 2: *Choice of the function θ (on the right) and the image of a vertical line $\{x\} \times \mathbb{R}$ under the lift $\widehat{h}_{n+1}$ of h_{n+1} (schematic picuture with $m_n = 2$). Note that the preimages of vertical lines under h_{n+1} have a similar* 'crookedness'*, which is the fact that is needed for our construction (see next section). The horizontal size of images (and preimages) of vertical lines under h_{n+1} is small compared to ε_{n+1}. This size is controlled by the flow lines of φ_{n+1} (dashed lines), whose direction is given by the almost vertical vector $\left(\frac{1}{b_{n+1}q_n+(s_n/r_n)}, 1\right)$.*

3.d – Choice of α_{n+1}. Clearly, one can choose α_{n+1} arbitrarily close to α_n so that property 2 holds. By uniform continuity of H_{n+1} and H_{n+1}^{-1}, there exists η so that the orbits of $f_{n+1} = H_{n+1} \circ R_{\alpha_{n+1}} \circ H_{n+1}^{-1}$ are $\frac{1}{n+1}$-dense in $\mathbb{T}^2$ provided that the orbits of the rotation $R_{\alpha_{n+1}}$ are η-dense in $\mathbb{T}^2$. One can thus choose α_{n+1} arbitrarily close to α_n so that property 3 is satisfied. By construction both f_{n+1} and f_{n+1}^{-1} extend holomorphically to $(\mathbb{C} \setminus \{0\})^2$. Moreover the diffeomorphism h_{n+1} commutes with the flow φ_{n+1}, hence with the rotation R_{α_n}. Consequently $f_n = H_{n+1}^{-1} \circ R_{\alpha_n} \circ H_{n+1}$. This shows that f_{n+1} is arbitrarily close to f_n when α_{n+1} is chosen arbitrarily close to α_n. In particular, property 4 holds provided that α_{n+1} is chosen close enough to α_n.

4 Uniqueness of the semi-conjugacy, non-existence of wandering curves and further remarks

The aim of this last section is to discuss, somewhat informally, the implications of our construction for some questions arising in the context of dynamics and rotation theory on the two-torus. By a *totally irrational pseudo-rotation of* $\mathbb{T}^2$ we mean a homeomorphism of $\mathbb{T}^2$ whose rotation set is reduced to a single totally irrational vector. Throughout this section, we assume f is a totally irrational pseudo-rotation of $\mathbb{T}^2$ with an invariant foliation of pseudo-circles, consisting of the fibres of a semi-conjugacy $\pi : \mathbb{T}^2 \to \mathbb{T}^1$ to a rigid rotation R_α. Moreover, we will freely add further assumptions on f if these can easily be ensured in the preceeding Anosov-Katok-construction.

We first note that the semi-conjugacy in our theorem is unique, modulo post-composition by rotations.

Proposition. *The semi-conjugacy π in our main theorem is uniquely determined: any continuous map π' which is homotopic to p and semi-conjugates f to the same circle rotation as p can be written as $\pi' = R \circ \pi$ where R is a rotation of the circle.*

This follows directly by combining [JP, Corollary 4.3] (uniqueness of the semi-conjugacy provided the non-wandering set is externally transitive) with the minimality of the map f. In fact, by [Po, Theorem A], any irrational pseudo-rotation is externally transitive on its non-wandering set, so that minimiality is not strictly required for the above statement.

The uniqueness of the semi-conjugacy further allows to see that f does not admit any loop which is wandering (i.e. disjoint from all its iterates) and has the same homotopy type as the pseudo-circles (that is, homotopy vector $v_2 = (0, 1)$ in our construction). The reason for this is the fact that the existence of such a loop Γ would allow to construct a semi-conjugacy $\tilde{p}$ to the rotation R_α such that Γ is contained in a single fibre of the semi-conjugacy. Details of this construction can be found in [JP, Lemma 3.2] (the fact that Γ is contained in a single fibre is not mentioned explicitly, but is obvious from the proof). Due to the uniqueness of the semi-conjugacy (modulo rotations) and the fact that none of the pseudo-circles of the foliation contains any non-degenerate curves, this yields a contradiction.

More generally, it is even possible to show that f does not admit any loops disjoint of all its iterates, regardless of the homotopy type. This is slightly more subtle, and we only sketch the argument. The crucial observation is the fact that we may construct f such that

$$\sup |\langle F^n(z) - z - n\rho, v\rangle| \; < \; \infty \quad \text{iff} \quad v = (1, 0) \; , \tag{4.1}$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of f and $\rho \in \mathbb{R}^2$ is the corresponding rotation vector. Now, if there exists a wandering loop of homotopy type $w \in \mathbb{Z}^2 \setminus \{0\}$, then it is not hard to see that

$$\sup \left| \left\langle F^n(z) - z - n\rho, w^\perp \right\rangle \right| \; < \; \infty \; .$$

However, according to (4.1) this is only possible if $w = (0, 1)$, and this is exactly the homotopy type of the pseudo-circles which was excluded before. Homotopically trivial wandering loops cannot exist by minimality, and therefore no loop of any homotopy type can be wandering.

Roughly speaking, in order to prove (4.1) one has to use the fact that since the leaves of the foliations $\mathcal{V}_k$ are increasingly crooked, connected fundamental domains of these circles in the lift become arbitrarily large in diameter. Since an iterate of f_k acts as a rotation on these leaves, this allows to see that for suitable integers n_k the vertical deviations $\left|\left\langle F_k^{n_k}(z) - z - n_k\rho, w^\perp\right\rangle\right|$ become arbitrarily large. If the f_k converge to f sufficiently fast, then this carries over to the limit and yields unbounded vertical deviations for f. At the same time horizontal deviations (that is, $v = v_1 = (1,0)$ in (4.1)) are bounded due to the existence of the semi-conjugacy (e.g. [JT, Lemma 3.1]). Together, these two facts yield unbounded deviations for all $v \neq v_1$.

The fact that f does not admit any wandering loop is of some interest in the context of the Arc Translation Theorem due to Kwapisz [Kw, BCL], which asserts that given an irrational pseudo-rotation and any integer n, there exist essential loops which are disjoint from their first n iterates. It is natural to ask under what additional assumptions this statement can be strengthened by passing from a finite number to all iterates. A natural obstruction is certainly to have unbounded deviations in all directions, as discussed above. However, our example shows that even if the deviations are bounded in some direction, the existence of a wandering curve is not guaranteed. Thus, in this sense the statement of the Arc Translation Theorem is optimal, and essential loops have to be replaced by more general classes of essential continua in order to obtain results in infinite time.

Finally, we want to mention a loose connection of our construction to the Franks-Misiurewicz Conjecture [FM]. The latter asserts that if the rotation set of a torus homeomorphism is a line segment of positive length, then either it contains infinitely many rational points, or it has a rational endpoint. As mentioned in the introduction, Avila has recently announced a counterexample to this conjecture for the case where the rotation segment has irrational slope, but the line it defines does not pass through a rational point. Conversely, Le Calvez and Tal have announced the first positive partial result on the conjecture: if the rotation set is a segment with irrational slope, it cannot contain a rational point in its relative interior. One case which is still completely open, however, is whether the rotation set can be a line segment with rational slope, but without rational points – for example, of the form $\{\alpha\}\times[a,b]$ with $\alpha \in \mathbb{R}$ irrational and $a < b$. Now, in the situation where there exists a semi-conjugacy π, homotopic to p_1, to the rotation R_α on the circle, the rotation set has to be contained in the line $\{\alpha\}\times\mathbb{R}$ [JT, Lemma 3.1]. Hence, this is a natural class of maps to look for counterexamples to this subcase of the Franks-Misiurewicz Conjecture. However, it is known that in order to have a non-degenerate rotation interval, the fibres of the semi-conjugacy need to have a complicated structure – more precisely, they need to be indecomposable [JP]. Our example shows that such a rich fibre structure is possible in principle. Yet, whether a non-degenerate rotation interval can be achieved remains open. Here, the fact that the Anosov-Katok method typically leads to uniquely ergodic examples, resulting in unique rotation vectors, suggests that a different approach would be needed to produce such examples, if these exists at all.

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