FINAL REPORT ON THE SCIENTIFIC NETWORK
SKEW PRODUCT DYNAMICS AND MULTIFRACTAL ANALYSIS

T. JÄGER

1. General information

Main applicant: Prof. Dr. Tobias Oertel-Jäger (publishing as T. Jäger).

Topic: Skew product dynamics and multifractal analysis.

General area: Mathematics, Ergodic Theory and Dynamical Systems.

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Resources: Funding of €40.000 was provided for the organisation of workshops and other network activities.

Network members:

- Vasso Anagnostopoulou (TU Dresden/Imperial College London)
- Gabriel Fuhrmann (TU Dresden/Friedrich-Schiller-Universität Jena)
- Katrin Gelfert (Universidade Federal do Rio de Janeiro)
- Maik Gröger (Universität Bremen/Friedrich-Schiller-Universität Jena)
- Johannes Jaerisch (Shimane University)
- Tobias Jäger (TU Dresden/Friedrich-Schiller-Universität Jena)
- Kurt Falk (Universität Bremen)
- Gerhard Keller (Friedrich-Alexander-Universität Erlangen)
- Marc Keßeböhmer (Universität Bremen)
- Arne Mosbach (Universität Bremen)
- Christian Oertel (TU Dresden/Friedrich-Schiller-Universität Jena)
- Atsuya Otani (Friedrich-Alexander-Universität Erlangen)
- Anthony Samuel (Universität Bremen/California Polytechnic State University)
- Malte Steffens (Universität Bremen)
- Bernd Stratmann (Universität Bremen)
- Roland Zweimüller (Universität Wien)

2. Summary

The aim of the scientific network "Skew product dynamics and multifractal analysis" was to bring together experts on dynamical systems theory and fractal geometry in order to work on the interface of the two topics and to advance the state of the field, with a particular focus on the fractal strucutes of invariant curves and graphs-like repellers in skew product systems.

A key element for the interaction within the network was a workshop program that has been implemented as a part of the project, with four major workshops organised on an annual basis in the years 2012–2015. Due to the partial school-like character of these events, they specifically fostered the development of several young scientists participating in the network and allowed them to acquire profound background knowlegde on various issues of high topicality. At the same time, the meetings allowed for a stimulating exchange on the ongoing research of the participants, and numerous publications were initiated at or benefited from these events.

During the project, substantial progress was made on several important aspects of the topic. for instance, a 25 year old conjecture on the fractal structure of so-called strange non-chaotic attractors has been confirmed, and methods have been established to determine the dimensions of such attractors in broad classes of systems. Further, members of the network have taken an active role in recent advances in the study of the graph of the classical Weierstrass function. The latter had been introduced by Weierstrass already around the

beginning of the 20th century as an example of a continuous, but nowhere differentiable curve. However, a precise description of its fractal structure and dimensions has only been completed very recently.

Further topics on which several advances were made include the study of bifurcation pattern and blowout bifurcations in forced systems and the fractal structure of so-called hyperbolic graphs, which occur in skew product systems with chaotic base dynamics and contracting fibres. Finally, surprising connections have been discovered between skew product dynamics and the mathematical theory of quasicrystals, and a substantial and fruitful transfer of ideas and methods between the two fields has been initiated.

3. Progress report

3.1. **Aims and objectives of the project.** The purpose of the scientific network *Skew product dynamics and multifractal analysis* was to bring together a group of experts working at the interface between dynamical systems and fractal geometry in order to advance the state of knowledge in this area and trace out new perspectives for future research. A particular focus should be given to skew product dynamics and the dimension-theoretic and fractal-geometric problems that arise in their study. A series of initial research topics was described in the proposal: *dimensions of attractors in quasiperiodically forced systems*, *dimensions of Weierstrass graphs*, *the Kaplan-Yorke conjecture for coupled systems*, *multifractal analysis of blowout bifurcations in chaotically driven systems* and *bifurcation pattern in skew product systems*. Beyond these specific projects, the aim was to obtain a detailed overview of the state of the art and to devise strategies for further and more systematic investigations.

As a key element of the envisaged interaction, the plan was to carry out three workshops with a school-like character and participation of invited international speakers and three smaller work meetings focusing more specifically on the ongoing research projects. These events were meant to foster the interaction between the involved scientists and research groups and to provide training, in particular for the younger scientists, allowing them to acquire a solid background on central topics and current developments in the area. The main aims of the scientific network can be highlighted as follows.

- (A1) To implement the above-mentioned workshop program.
- (A2) To carry out specific research projects both within and in collaboration between the different research groups.
- (A3) To identify future research objectives and a strategy for further systematic developments of the field, and ideally to give a written account of the findings.
- (A4) To build up strong research potential as a prerequisite for future coordinated research projects and grant applications in the field of Ergodic Theory and Dynamical Systems.

A detailed discussion will be given in Sections 3.3–3.6 below.

3.2. **Unexpected developments, modifications of the project plan and other comments.** The most severe and also tragic turn of events came with the unexpected death of Bernd Stratmann, who fell ill shortly after the project's start and passed away in August 2015. We deeply regret this loss of a great colleague and enthusiastic mathematician. His expertise and ideas were certainly missed and surely would have been a considerable additional benefit to the project. As a specific consequence, the research on the Kaplan-Yorke conjecture in coupled map lattices, for which Bernd Stratmann was supposed to act as PI, could not not be carried out. Similarly, the work on Weierstrass functions was slowed down, but eventually this research line was taken over by other network members (Keller, Otani).

On the scientific side, there have been a number of positive developments and surprises. First of all, the research line on strange non-chaotic attractors in quasiperiodically forced systems has developed very well and went well beyond what was envisaged in the proposal (see Section 3.4.1). In the study of Weierstrass graphs, there has been a rapid progress over the last years, including in particular the solution to a long-standing conjecture on their Hausdorff dimension. While these advances were not initiated from within the network, its members G. Keller and A. Otani took an active part in these developments and made substantial contributions to the topic (see Section 3.4.2). The most important and unexpected development, however, was the realisation of a substantial transfer of methods and ideas from skew product dynamics to the theory of quasicrystals and aperiodic order. This issue will be discussed in detail in Sections 3.4.6 and 3.6.

A modification of the workshop program had to be made due to a significant budget cut, since only about 60% of the requested funding had been granted. This meant that only 4 workshops could be organised with the provided funds (instead of 3+3). The role of the smaller meetings, which were supposed to ensure the continuity of the interaction, was taken by other events that were organised by different network members and funded by other means. Most importantly, a series of winter schools focusing on selected topics at the interface between dynamical systems, probability theory and fractal geometry was established at the University of Bremen (see Section 3.3).

Finally, there was also some fluctuation of network members over the course of the project. Due to the focus on the support of young scientists, it was natural to include a number of new doctoral researchers, namely A. Mosbach, M. Steffens and C. Oertel, that started their PhD studies during the project in one of the participating research groups. Conversely, we also had to witness a certain brain drain, in that several network members took either temporary positions (V. Anagnostopoulou, now Marie Curie Fellow at Imperial College London) or permanent positions abroad (J. Jaerisch, now lecturer at Shimame University, and T. Samuel, now assistant professor at California Polytechnic State University), and therefore could not take part in the network's activities as frequently as before.

3.3. **Workshop program.** As mentioned, four workshops and conferences have been organised in the context of the scientific network.

- Skew product dynamics and multifractal analysis, 1st-5th October 2012, Luisenthal, Germany (organisers: M. Gröger and T. Jäger).
- Complexity and dimension theory of skew product systems, 16th-20th September 2013, Erwin-Schrödinger-Institute Vienna, Austria (organisers: H. Bruin, T. Jäger, and R. Zweimüller).
- Dynamical systems and dimension theory, 8th-12th September 2014, Wöltingerode, Germany (organisers: G. Fuhrmann and T. Jäger).
- Fractals, dynamics and quasicrystals, 5th-10th October 2015, Wöltingerode, Germany (organisers: K. Gelfert, M. Gröger, and T. Jäger).

Workshops 1,3 and 4 mainly involved the network members, invited speakers and some additional, mostly younger researchers and had around 20-25 participants. The second workshop, which was organised at the Erwin-Schrödinger Institute in Vienna in collaboration with the ergodic theory group in Vienna and the Vienna-Budapest-Seminar, had a wider scope with around 40 participants and focused in particular on the scientific exchange with the Eastern European dynamics community. During each of these workshops, two lecture series and/or mini-courses allowed to realise a substantial transfer of knowledge and specifically addressed the younger network members, allowing them to acquire a broad scope of specific expertise in their fields of research. The topics of the given courses were *skew product dynamics* (A.J. Homburg and T. Jäger), *multifractal analysis* (J. Schmeling), *topological complexity* (P. Oprocha), *thermodynamic formalism for skew product systems* (M. Stadlbauer), *dimensions of Weierstrass graphs* (G. Keller), *quasicrystals and aperiodic order* (D. Lenz), *regularity of 'nonlinear' Weierstrass-type functions* (D. Todorov) and *spectral aspects of aperiodic order* (M. Baake). The courses were complemented by a large number of research talks and problem sessions, which fostered the lively interaction within the network and beyond and often directly resulted in new research activities and publications.

As mentioned above, a series of winter schools (and symposiums) established at the University of Bremen helped to ensure the continuity of the interaction. The different events focused on selected topics at the interface between dynamical systems, probability theory and fractal geometry.

- Winter school on multifractals and number theory, 18th-22nd March 2013 (organisers: M. Gröger and T. Samuel).
- Winter school on Kleinian groups and transcendental dynamics, 7th-11th April 2014 (organisers: K. Falk and A. Zielicz).

- Winter school and symposium: diffusion on fractals and non-linear dynamics, 24th March-2nd April 2015 (organisers: K. Falk, M. Koch, K. Lorenz, J. Rademacher, and T. Samuel).
- Winter school and symposium: dynamics, chaos and applications, 14th-18th March 2016 (organisers: M. Kirsebom, I. Ovsyannikov, J. Rademacher, and M. Kesseböhmer).

Related publications

[1] Diffusion on fractals and non-linear dynamics. Guest editors: K. Falk, T. Jäger, M. Keßeböhmer, J. Rademacher, and T. Samuel. *Discrete Contin. Dyn. Syst. Ser. S*, 10(2), 2017.

3.4. **Scientific output.** We summarise the most important scientific progress and the related publications resulting from the network activities in the following subsections, corresponding to the different focal topics of the network. In most cases, we consider skew product maps of the general form

$$T : \Xi \times X \to \Xi \times X \quad , \quad (\xi, x) \mapsto (\tau(\xi), T_\xi(x)) \ , \tag{3.1}$$

where Ξ is called the *base space*, $\tau : \Xi \to \Xi$ the *forcing transformation* and the maps $T_\xi : X \to X$ the *fibre maps*. Usually both Ξ and X are metric spaces, which we equip with the corresponding Borel σ-algebras. An *invariant graph* of T is a measurable function $\varphi : \Xi \to X$ that satisfies

$$T_\xi(\varphi(\xi)) \ = \ \varphi(\tau(\xi)) \tag{3.2}$$

either for all $\xi \in \Xi$, or for μ-a.e. $\xi \in \Xi$, where μ is a τ-invariant measure on Ξ that is chosen depending on the context. In the latter case, if X is one-dimensional[1] and the fibre maps T_ξ are differentiable, then the *Lyapunov exponent* of φ with respect to a τ-invariant measure μ is given by

$$\lambda_\mu(\varphi) \ = \ \int_\Xi \log \left| T'_\xi(\varphi(\xi)) \right| \ d\xi \ . \tag{3.3}$$

An invariant graph is called *attracting* (with respect to a given measure) if the Lyapunov exponent $\lambda_\mu(\varphi)$ is negative. It is called *uniformly attracting* if the Lyapunov exponents with respect to all the invariant measures are strictly negative. Depending on the properties of the forcing transformation τ and the fibre maps f_ξ, a variety of different dimension-theoretical concerning the structure of attracting invariant graphs come up in different contexts.

3.4.1. ***Dimensions of strange non-chaotic attractors in quasiperiodically forced systems.*** A *quasiperiodically forced (qpf) map* (with discrete time) is a skew product map T of the form (3.1), with $\Xi = \mathbb{T}^d$ a d-dimensional torus and τ irrational rotation on Ξ. Since the d-dimensional Lebesgue measure on $\mathbb{T}^d$ is the only τ-invariant measure in this case, its serves as the natural reference measure. In this situation, the structure of continuous attracting graphs is easy to understand, since these will always be as smooth as the system itself [SS00]. However, a phenomenon that has attracted considerable interest in this context is the widespread occurrence of *strange non-chaotic attractors* (SNA) [GOPY84, PNR01, FKP06]. If the fibres are one-dimensional, these correspond to non-continuous invariant graphs with negative Lyapunov exponents. The existence and structure of these objects has been investigated in a great number of numerical studies throughout the 1980's and 1990's (e.g. [DGO89b]–[DSS$^+$90], see [PNR01] and [FKP06] for a comprehensive overview), but rigorous results remained rare for a long time.

Concerning the dimensions of SNA, a conjecture made by Ding, Grebogi and Ott in 1989 [DGO89a], based on numerical evidence, states that there is a discrepancy between the box dimension and the information dimension (which is closely related to the pointwise and the Hausdorff dimension). For systems with one-dimensional fibres, the conjectured value for the box dimension was two, whereas that for the information dimension was one.

[1]That is, either $X \subseteq \mathbb{R}$ is an interval or $X = \mathbb{T}^1$ is the circle.

In the context of the project, we have been able to confirm this conjecture and to establish efficient methods and results to determine the dimensions of SNA in broad classes of qpf one-dimensional maps. A crucial step was an initial result for so-called pinched systems, which have been introduced by Grebogi and coworkers in [GOPY84] and later described rigorously by Keller in [Kel96]. An example is shown in Figure 3.1. In [2], we have proved that the unique invariant measure supported on the SNA is rectifiable and exact-dimensional with pointwise dimension one. This immediately entails that both information and Hausdorff dimension equal one as well. At the same time, previous results on the topological structure of the attractors in [Jäg07] allowed to show that the box dimension equals two.

While pinched systems must rather be considered as toy models and have some special features which greatly simplify their analysis (including the a priori existence of an invariant continuous curve and the non-invertibility of some fibre maps), they show essentially the same geometric mechanisms that are also present in more natural system classes. Based on the intuition from [2], we have been able to carry over the results to SNA created by the collision of invariant tori (*non-smooth saddle-node bifurcations*) [4] and to extend them also for flows generated by qpf scalar vector fields [5]. Analogous results for qpf circle maps have been established in [6].

Of course, it is now a natural question to ask for similar results on SNA in systems with higher-dimensional fibres. This is an issue of high topicality, since over the last years SNA have been reported for a variety of models in a broad scope of applications, including epidemics [BSPM16], conceptual climate models [MCA15, MA14, MC16], electronic oscillators [SPT13, RRM15] or astronomy [LKK+15]. However, so far the mathematical tools for the description of these systems have not been developed, and this task was well beyond the scope of this project and must be addressed in more comprehensive future studies. For this reason, the comprehensive description that has been obtained for one-dimensional systems can be considered and optimal outcome, with respect to what could realistically be expected. In particular, it goes far beyond the original aims formulated in the proposal, which were restricted to pinched systems only.

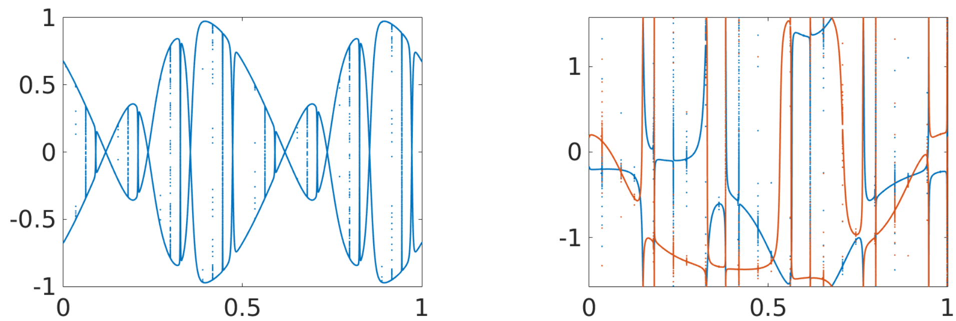

FIGURE 3.1. *Left:* SNA discovered by Grebogi *et al* [GOPY84] in the qpf interval map $(\xi, x) \mapsto (\xi + \gamma \bmod 1, \tanh(\alpha x) \cdot \sin(2\pi\xi))$ with $\alpha = 4$. *Right:* SNA and repeller in the Harper map (the projective action of a quasiperiodic Schrödinger cocycle, associated to a discrete one-dimensional Schrödinger operator with quasiperiodic potential (see [HP06]). In both cases, γ is the golden mean.

Related publications

[2] T. Jäger and M. Gröger. Dimensions of attractors in pinched skew products. *Comm. Math. Phys.*, 320(1):101-119, 2013.

[3] R. Hric and T. Jäger. A construction of almost automorphic minimal sets. *Israel J. Math.*, 204(1):373-395, 2014.

[4] G. Fuhrmann, M. Gröger, and T. Jäger. Non-smooth saddle-node bifurcations II: dimensions of strange attractors. To appear in *Ergodic Theory Dyn. Syst.*

[5] G. Fuhrmann. Non-smooth saddle-node bifurcations III: Strange attractors in continuous time. *J. Diff. Eq.*, 261(3):2109-2140, 2016.

[6] G. Fuhrmann and J. Wang. Rectifiability of a class of invariant measures with one non-vanishing Lyapunov exponent. *Preprint* `arXiv:1606.04787`, 2016.

3.4.2. ***Dimensions of Weierstrass graphs.*** The Weierstrass function

$$\varphi(\xi) \;=\; \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n \xi) \tag{3.4}$$

with parameters $\lambda \in (0,1)$ and $b \in \mathbb{N}$ with $\lambda b > 1$ has been introduced by Weierstrass as an example of a continuous, but nowhere differentiable function, a fact that was later proven for all parameters by Hardy [Har16]. The box dimension of the graph of this function has long been known and equals

$$\dim_B(\varphi) \;=\; 2 + \log\lambda / \log b \tag{3.5}$$

(see [KMPY84, Fal07]). The Hausdorff dimension was expected to take the same value, but the confirmation of this conjecture turned out to be highly difficult and became a long-standing problem in fractal geometry that had not been solved until very recently.

Soon after the start of the scientific network, a partial solution to this problem (with some restrictions on the parameters) was given by Baransky, Barany and Romanowska (published in [BBR14], with the `arXiv` preprint posted in September 2013). They combined Ledrappier-Young theory [LY85a, LY85b, Led92] with work of Tsujii on fat solenoidal attractors [Tsu01] in order to confirm the equality between box and Hausdorff dimension of the Weierstrass graph. A substantial contribution was then made by Keller [7], who provides an alternative proof that by-passes the heavy machinery of Ledrappier-Young theory and replaces it with an intricate, but elementary telescoping argument. This makes the proof more accessible and also allows one to avoid using certain results by Ledrappier, whose proofs are only sketched [Led92]. Hence, while the results from [7] do not improve on that of [BBR14], the simplified methods should be instrumental for future extensions of these results to broader system classes. First extensions and more detailed studies in this context were then made by Otani [8, 9]. For the classical Weierstrass graph, an extension of the results in [BBR14, 7] to all parameter values was given by Shen [She15].

Related publications

[7] G. Keller. An elementary proof for the dimension of the graph of the classical Weierstrass function. *Ann. Inst. H. Poincaré Probab. Statist.*, 53(1):169-181, 2017.

[8] A. Otani. An entropy formula for a non-self-affine measure with application to Weierstrass-type functions. *Preprint* `arXiv:1503.06451`, 2015.

[9] A. Otani. Hausdorff spectra of the local Hölder exponent of Weierstrass-type functions. *Preprint* `arXiv:1603.03954`, 2016.

3.4.3. ***Blowout bifurcations in chaotically driven systems.***

While Weierstraß graphs (and related ones) occur as global attractors in skew product systems with additively forced strictly contracting 1D affine fibre maps over a hyperbolic base, this research line focusses on one-dimensional concave or sigmoidal fibre maps over a hyperbolic base, which allow coexistence of positive and negative Lyapunov exponents in the fibre direction. Publications [10] and [11] study multiplicatively forced concave fibre maps. Their global attractor is bounded from below by the graph of the constant zero function and from above by some upper semicontinuous invariant graph determined by a pullback construction. The most interesting phenomena occur when there are orbits with negative fibre Lyapunov exponent on the zero graph, although Lebesgue almost points on that graph have a positive fibre Lyapunov exponent. In that case the two invariant graphs coincide on a residual set of Lebesgue measure zero. The Hausdorff and packing dimension of this set was determined in [10], and the stability index of the regions above and below the “pullback graph” in [11], both are characterized via thermodynamic formalism. Quite surprisingly, techniques borrowed from queuing theory form an essential ingredient of [11].

Publications [12] and [13] deal with sigmoidal fibre maps (technically speaking: fibre maps with negative Schwarzian derivative). The global attractor of such systems is bounded from above and below by semicontinuous invariant graphs, and there is a third measurable invariant graph in between them. Some or all of these graphs may coincide over some set P of base points. The most interesting phenomena occur when there are orbits with negative fibre Lyapunov exponent on the central graph, although Lebesgue almost points on that graph have a positive fibre Lyapunov exponent. The Hausdorff dimension of the set P was determined in thermodynamic terms in [13]. It is noteworthy that the proof is not just a generalization of the one in [10], because now the central invariant graph can a priori be highly irregular and not admit the application of thermodynamic formalism. A basic idea from the analysis of Weierstraß graphs, also used in [7] , finally helped to overcome this problem. In [12] the fibre maps were chosen such that the upper and lower bounding graphs are constant so that the central invariant graph separates the two basins of (fibrewise) attraction of the constant invariant graphs. An irregular central graph thus leads to intermingled basins, and the stability index for this situation was characterized in [13] through thermodynamic formalism and ideas from queuing theory again.

Related publications

[10] G. Keller and A. Otani. Bifurcation and Hausdorff dimension in families of chaotically driven maps with multiplicative forcing. *Dyn. Syst.*, 28(2):123-139, 2013.

[11] G. Keller. Stability index for chaotically driven concave maps. *J. Lond. Math. Soc. (2)*, 89(2):603-622, 2014.

[12] G. Keller and A. Otani. Chaotically driven sigmoidal maps. *Preprint* `arXiv:1610.10010`, 2016.

[13] G. Keller. Stability index, uncertainty exponent, and thermodynamic formalism for intermingled basins of chaotic attractors. *Discrete Contin. Dyn. Syst. Ser. S*, 10(2):313-334, 2017.

3.4.4. ***Bifurcation pattern in skew product systems.*** The first branch of this research line concentrated on saddle-node bifurcations in quasiperiodically forced systems and the creation of strange non-chaotic attractors in this setting. A general description of this bifurcation pattern for a broad class of quasiperiodically forced one-dimensional maps has been given by Fuhrmann in [14], with analogous results on continuous-time systems in [5]. This work provides a rigorous confirmation for the wide-spread existence of SNA in these systems classes. Moreover, these results provided the basis for the study of dimensions of such attractors [4, 6], see Section 3.4.1 above.

Models for a non-autonomous two-step version of the classical Hopf bifurcation have been established in [15], building on preliminary work on random semi-uniform ergodic theorems in [16]. Investigations into the fractal geometric properties of the corresponding attractors have not been carried out yet, which is partly due to the transition of V. Anagnostopoulou to the Imperial College London and a resulting shift in her research priorities, and partly due to the stronger focus that has been given to the study of quasicrystals in the final stages of the network. However, an important preliminary result in this direction is given by the study of SNA in quasiperiodic $\mathrm{SL}(2,\mathbb{R})$-cocycles in [6], which occur as the two-dimensional projections of SNA in the Hopf bifurcation models in [15]. This sets the stage for more detailed studies of the attractors in [15] in the future.

Related publications

[14] G. Fuhrmann. Non-smooth saddle-node bifurcations I: existence of an SNA. *Ergodic Theory Dyn. Syst.*, 36(4):1130-1155, 2016.

[15] V. Anagnostopoulou, T. Jäger, and G. Keller. A model for the non-autonomous Hopf bifurcation. *Nonlinearity*, 28(7):2587-2616, 2015.

[16] T. Jäger and G. Keller. Random minimality and continuity of invariant graphs in random dynamical systems. *Trans. Am. Math. Soc.*, 368(9):6643-6662, 2016.

3.4.5. ***Regularity and box dimension of hyperbolic graphs***. A setting that has emerged a the starting point of more systematic investigations is that of skew product systems of the form (3.1) with a hyperbolic base transformation τ, one-dimensional fibres and uniformly attracting fibre maps. The latter means that there exists a global constant $\lambda \in (0,1)$ such that

$$d(T_\xi(x), T_\xi(y)) \;\leq\; \lambda d(x,y) \tag{3.6}$$

for all $\xi \in \Xi$ and $x, y \in X$. Using an invertible extension of the base transformation, both Weierstrass-type systems discussed in Section 3.4.2 and the solenoidal attractors studied by

Tsujii in [Tsu01] can be transformed to systems of this type. The existence of a unique attracting invariant graph that serves as a global attractor of the system is then easily established. The regularity and fractal dimensions of these attractors have been studied by several authors in different contexts [KMPY84, Bed89, PW94, HNW02, Wal07].

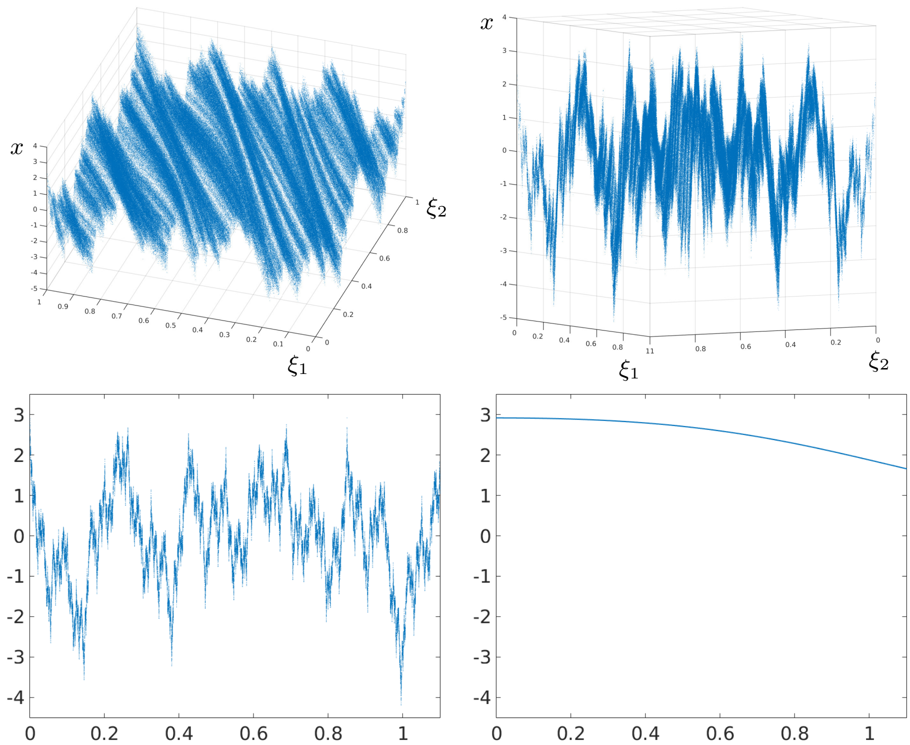

FIGURE 3.2. The global attractor (attracting invariant graph) for the map $(\xi_1, \xi_2, x) \mapsto (\tau(\xi_1, \xi_2), \lambda x + \sin(2\pi\xi_1)\sin(2\pi\xi_2) + \cos(4\pi\xi_2))$, where τ is the cat map on $\mathbb{T}^2$ and $\lambda = 2/3$. The full attractor from two different angles is shown on top. The bottom left shows a slice of the attractor over a stable manifold in the base, the bottom right a slice over an unstable manifold. The fact that the global attractor is smooth over the unstable manifolds is well-known [HNW02, 17]. The box dimension of the attractor in this example is $3 + \log\lambda/\log\kappa$, where $\kappa = (3+\sqrt{5})/2$ is the maximal Lyapunov exponent of τ.

The first interesting phenomenon in this context is *critical regularity*. Under some mild assumptions, there exists a dichotomy for the attracting invariant graph, which states that the latter is either Lipschitz continuous or has a maximal Hölder exponent $\gamma \in (0,1)$ that only depends on the expansion and contraction rates in the base and the fibres. In this second case, the box dimension is bounded above by $\min\{D+1-\gamma, D/\gamma\}$, where D denotes the box dimension of Ξ.[2] Using stronger assumptions on the uniform hyperbolicity of the base transformation, it is further possible to derive explicit formulas for the box dimension in terms of certain topological pressure functions. For expanding circle maps in the base, such formulas have been established by Bedford in [Bed89]. Similar results for hyperbolic surface diffeomorphism in the base have been announced in [Wal07]. However, a careful review of the latter work has revealed serious gaps and flaws in the respective proofs. Moreover, in the case of invariant graphs over Cantor sets of small dimension, the statements in [Wal07] are incorrect and contradict the natural upper bound D/γ on box dimension given by the Hölder continuity. One of the main basic errors that is made in [Wal07] in this context is that the Intermediate Value Theorem is applied to continuous functions defined on Cantor

[2]Note here that $D+1-\gamma \leq D/\gamma$ for all $\gamma \in [0,1]$ whenever $D \geq 1$, so that the second bound D/γ only plays a role when the restriction of the attracting graph to invariant Cantor sets of small dimension is considered.

sets, for which it is clearly not valid. The attempt to close this gap in the literature lead to the current preprint [17], which combines ideas of Bedford [Bed89] with the concept of blenders [BD96] in order to provide correct proofs of the respective Bowen-Ruelle-type pressure formulas. The article also reviews critical regularity and includes proofs of this and other basics facts. Hence, although it contain a large proportion of original research, it partly has the nature of a survey article and should provide a good starting point and sound basis for further investigations in the field.

Related publications

[17] L.J. Díaz, K. Gelfert, M. Gröger, and T. Jäger. Hyperbolic graphs: critical regularity and box dimension. *Preprint* `arXiv:1702.06416`, 2017.

3.4.6. ***Quasicrystals and aperiodic order.***

One of the most important outcomes of the project was the discovery of close connections and links between skew product dynamics and the theory of quasicrystals and the realisation of a substantial transfer of ideas and methods between the two fields. In order to describe this in detail, we first provide some background knowledge.

In the mathematical theory of quasicrystals and aperiodic order, one of the major constructions of aperiodic structures is the so-called *cut and project method*. A *cut and project scheme* (CPS) consists of a triple $(G, H, \mathcal{L})$, consisting of two locally compact abelian groups G (called *external group*) and H (*internal group*) together with a co-compact discrete subgroup $\mathcal{L} \subseteq G \times H$ such that the projection of $\mathcal{L}$ to G is injective and that to H is dense. Figure 3.3 shows these ingredients for the case where $G = H = \mathbb{R}$ and $\mathcal{L}$ is an irrationally rotated version of $\mathbb{Z}^2$. Given a *window* $W \subseteq H$ as an additional input, one obtains a *model set*[3] $\Lambda(W)$ by first taking the intersection of the lattice $\mathcal{L}$ with the strip $G \times W$ (*cut*) and then *projecting* this set down to G, that is, $\Lambda(W) = \pi_G(\mathcal{L} \cap (G \times W))$.

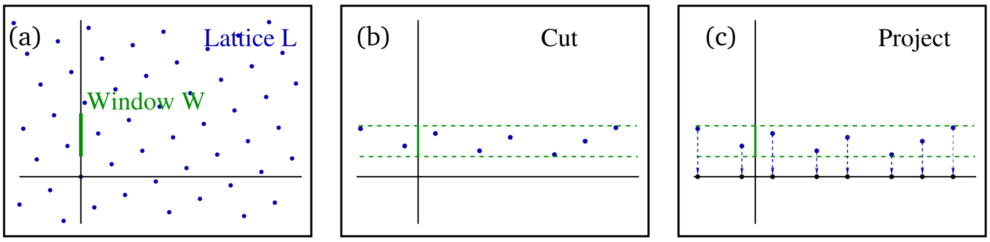

FIGURE 3.3. Schematic illustration of the cut-and-project method.

Dynamics then come into play by considering the *dynamical hull* of $\Lambda(W)$, which is the closure of the translation orbit $\{\Lambda(W) - t \mid t \in G\}$ with respect to a suitable metric.[4] The group G acts on $\Omega(\Lambda(W))$ again by translation, which yields a topological dynamical system $(\Omega(\Lambda(W)), \varphi)$ with G-action $\varphi : (t, \Gamma) \mapsto \Gamma - t$. The key to the analysis of (weak) models sets are close relations between the properties of W, the geometry of $\Lambda(W)$, the dynamics on the hull $\Omega(\Lambda)$ and the resulting diffraction spectra. We discuss some of these relations, which may for example be found in [Sch00, LP03, BG13].

If W is proper, then $\Lambda(W)$ is a Delone set.[5] The dynamics on the hull are minimal if and only if the corresponding point set is repetitive, that is, every finite pattern of points (*patch*) occurs infinitely often and the distances between occurrences are bounded. This happens whenever ∂W does not intersect $\pi_H(\mathcal{L})$. Further, the dynamics are uniquely ergodic if and only if every patch occurs with a well-defined frequency. The dynamical and diffraction properties of model sets are particularly well-understood if $|\partial W| = 0$, where $|A|$ denotes the Haar measure of a subset $A \subseteq H$. The underlying reason is a strong connection between

[3]In fact, one usually speaks of a model set only if W is *proper* ($\overline{\mathrm{int}(W)} = W$), and calls $\Lambda(W)$ a *weak model set* otherwise.

[4]If $G = \mathbb{R}^d$, one may take the Hausdorff metric applied to the stereographic projections of point sets onto $\mathbb{S}^d$.

[5]Given a metric d on G, $\Lambda \subseteq G$ is called *Delone* if it is *uniformly discrete* ($\exists r > 0 : \#(B_r(x) \cap \Lambda) \leq 1\ \forall x \in G$) and *relatively dense* ($\exists R > 0 : (B_R(x) \cap \Lambda) \geq 1\ \forall x \in G$). If G is not metric, one uses topological analogues of these notions.

the dynamics on the hull and a canonical G-action on the quotient group ($\mathbb{T} = G \times H$), given by $\omega_t[g,h]_{\mathcal{L}} = [g+t,h]_{\mathcal{L}}$.[6] Namely, there exists a flow morphism β between φ and ω (*torus parametrization*), and if $|\partial W| = 0$ then β is almost surely one-to-one [Sch00]. This implies that φ is uniquely ergodic and isomorphic to ω. It therefore has pure point dynamical spectrum, which further entails pure point diffraction [RJ07, BL04] and zero entropy [BLR07]. The flow morphism and the corresponding existence of an almost periodic factor $(\mathbb{T}, \omega)$ presents the key analogy between the dynamics of model sets and skew products over a quasiperiodic base. This connection allowed for a fruitful transfer of ideas and methods in several directions.

First, by using techniques from the study of strange non-chaotic attractors, irregular model sets with positive entropy arising from Euclidean CPS have been constructed in [19]. Ongoing research now concentrates on the question whether Euclidean CPS also allow to produce irregular model sets with zero entropy, a problem that goes back to Moody (see [PH13]). In a more general (non-Euclidean) setting, this problem has already been solved in [18], where a correspondence principle between model sets and symbolic Toeplitz flows has been established. The latter allows to interpret Toeplitz sequences as quasicrystals (the positions with symbol 1 corresponding to the positions of atoms) and to relate the measure of the boundary of the respective window to the scaling behaviour of the Toeplitz sequence. The vast literature on Toeplitz flows then provides examples of model sets which are irregular, but still have zero entropy. The significance of these contributions lies in the fact that the dynamics of irregular model sets are not very well-understood so far, and one of the main problems in their study has so far been the lack of broader classes of examples, which can now be provided through the above constructions.

Secondly, concepts from the theory of minimally forced interval maps in [Sta03, JS06, FJJK05] have been transferred to the study of weak model sets in [20, 21]. In particular, this lead to a new characterisation of the so-called Mirski measure, a canonical ergodic measure that arises in this context, and allowed to prove its pure point dynamical spectrum. Further ongoing studies concentrate on so-called $\mathcal{B}$-free systems, which arise in number theory the context of the Möbius function and can be interpreted as weak model sets [KPLW15, BKPLK15].

Finally, concepts of multifractal analysis and the study of Sturmian subshifts and subshifts arising from groups of intermediate growth have been generalised and applied to the classification of aperiodic structures in [22, 23].

The links and analogies that are at the heart of all these advances are expected to carry much further, and more systematic investigations of these connections and their consequences are ongoing. Ideally, such studies should be carried out in a structured research program that also involves experts from the quasicrystal side. This issue will be discussed further in Section 3.6.

Related publications

[18] M. Baake, T. Jäger, and D. Lenz. Toeplitz flows and model sets. *Bull. Lond. Math. Soc.*, 48(4):691-698, 2016.

[19] T. Jäger, D. Lenz, and C. Oertel. Model sets with positive entropy in Euclidean cut and project schemes. *Preprint* `arXiv:1605.01167`, 2016.

[20] G. Keller and C. Richard. Dynamics on the graph of the torus parametrization. Published online in *Ergodic Theory Dyn. Syst.*, 2015.

[21] G. Keller. Maximal equicontinuous generic factors and weak model sets. *Preprint* `arXiv:1610.03998`, 2016.

[22] M. Gröger, M. Keßeböhmer, A. Mosbach, T. Samuel, and M. Steffens. A classification of aperiodic order via spectral metrics & Jarník sets. *Preprint* `arXiv:1601.06435`, 2016.

[23] F. Dreher, M. Kesseböhmer, A. Mosbach, T. Samuel, and M. Steffens. Regularity of aperiodic minimal subshifts. *Preprint* `arXiv:1610.03163`, 2016.

[6] Note that in the situation of Figure 3.3, this is simply an irrational Kronecker flow on the two-torus.

3.4.7. ***Further topics***. There is a substantial variety of further projects that are more loosely connected to the core topics of the network, but have nevertheless profited significantly from the interaction in the network and have been initiated and/or presented at the various network events. A notable example is the work of Diaz, Gelfert and coauthors on step skew products, which were studied as simple models of invariant sets in partially hyperbolic systems [26]–[29]. We refrain from giving a detailed discussion of these and other works and just include the following list.

[24] J.P. Boroński and J. Kupka. New chaotic planar attractors from smooth zero entropy interval maps. *Adv. Difference Equ.*, 2015:232,11pp., 2015.
[25] J.P. Boroński and P. Oprocha. Rotational chaos and strange attractors on the 2-torus. *Math. Z.*, 279(3):689-702, 2015.
[26] L.J. Díaz, K. Gelfert, and M. Rams. Almost complete Lyapunov spectrum in step skew-products. *Dyn. Syst.*, 28(1):76-110, 2013.
[27] L.J. Díaz, K. Gelfert, and M. Rams. Abundant phase transitions in step skew products. *Nonlinearity*, 27(9): 2255-2280, 2014.
[28] L.J. Díaz, K. Gelfert, and M. Rams. Nonhyperbolic step skew-products: Ergodic approximation. Published online in *Ann. Inst. H. Poincar Anal. Non Linaire*, 2016.
[29] L.J. Díaz, K. Gelfert, and M. Rams. Nonhyerbolic step skew-products: Entropy spectrum of Lyapunov exponents. *Preprint* `arXiv:1610.07167`, 2016.
[30] G. Fuhrmann, M. Gröger, and T. Jäger. Amorphic complexity. *Nonlinearity*, 29(2):528-565, 2016.
[31] M. Gröger and T. Jäger. Some remarks on modified power entropy. In *Dynamics and Numbers, Cont. Math.*, 669:105-122, 2016.
[32] J. Jaerisch, M. Keßeböhmer, and S. Munday. A multifractal analysis for cuspidal windings on hyperbolic surfaces. *Preprint* `arXiv:1610.05827`, 2016.
[33] J. Kautzsch, M. Keßeböhmer, T. Samuel, and B.O. Stratmann. On the asymptotics of the α-Farey transfer operator. *Nonlinearity*, 28(1):143-166, 2015.
[34] J. Kautzsch, M. Keßeböhmer, and T. Samuel. On the convergence to equilibrium of unbounded observables under a family of intermittent interval maps. *Ann. Henri Poincaré*, 17(9):2585-2621, 2016.
[35] M. Keßeböhmer and S. Zhu. On the quantization for self-affine measures on Bedford-McMullen carpets. *Math. Z.*, 283(1-2):39-58, 2015.
[36] T. Samuel, N. Snigireva, and A. Vince. Embedding the symbolic dynamics of Lorenz maps. *Math. Proc. Camb. Phil. Soc.*, 156(3):505-519, 2014.

3.5. **Review article and scientific wiki.** One of the expressed aims of the project was to produce a survey article or an equivalent document that summarises important findings of the network and provides a basis for further systematic investigations. As mentioned in Section 3.4.5, the setting of skew products with hyperbolic base and uniformly attracting fibres has emerged as being central in this context, since it comprises well-known classical examples like Weierstrass-type systems and solenoidal attractors (both after some canonical modifications), leaves ample space for further investigations and has multifaceted ramifications into other topics like the study of general hyperbolic sets, partially hyperbolic systems and fractal geometry (non-conformal iterated function systems).

As discussed before, while trying to obtain an overview of the state of the art and collecting material for a broader survey, it became apparent that a key contribution to this topic – the pressure formulas for box dimensions in [Wal07] – still lacked a rigorous proofs. Substantial efforts were undertaken, notably by Díaz, Gelfert, Gröger and Jäger to close this gap in the literature and to provide a mostly self-contained accessible account and proofs of the respective statements. As a consequence of this unexpected obstacle, the resulting article [17] is a sort of mixture between a survey and an original research article, but we still believe that it serves the envisaged purpose well.

In parallel, we have also experimented with a new form of communication by setting up a scientific wiki on the topic, which will give a much broader overview and eventually also allow to involve further authors from a broader community. However, at this moment the wiki is still under construction and password restricted.

3.6. **Further initiatives.** One of the expressed aims of the scientific networks was to obtain a critical mass and momentum that would allow to initiate further coordinated research projects and lead to a much-needed strengthening of ergodic theory and dynamical systems in Germany. The mentioned links to the theory of quasicrystals and the involvement of highly renowned experts like M. Baake, D. Lenz, C. Richard and others from this area provided a welcome opportunity to pursue this goal further. As a result of concerted effort, a large-scale research program on aperiodic order and topological dynamics has been set up, and an application for a DFG research group was made in October 2016. However, unfortunately this initiative did not obtain the approval of the DFG review board. Since this decision is very recent, it is not yet clear to what extent feasible alternatives can be found to sustain the positive development that has been initiated by the networks activities.

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